EXTREMUM PROPERTIES OF HEXAGONAL PARTITIONING AND THE UNIFORM DISTRIBUTION IN EUCLIDEAN LOCATION

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ABSTRACT

We consider a zero-sum game with a maximizer who selects a point x in given polygon R in the plane and a minimizer who selects K points $c_1, c_2, \ldots, c_K$ in the plane; the payoff is $\min_{1 \le i \le K} \|x - c_i\|$, or any monotonically nondecreasing function of this quantity. We derive lower and upper bounds on the value of the game by considering, respectively, the maximizer's strategy of selecting a uniformly distributed random point in R and the minimizer's strategy of selecting K members of a (uniformly) randomly positioned grid of centers that induces a covering of R by K congruent regular hexagons. Our analysis shows that these strategies are asymptotically optimal (for $K \to \infty$).

For Euclidean location problems with uniformly distributed customers, our results imply that hexagonal partitioning of the region is asymptotically optimal, and that the uniform distribution is asymptotically the worst possible.

KEY WORDS: Facility Location, K-Median Problem, Location Games, Planar Partitioning

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1. Introduction

Let $x_1, x_2, \ldots, x_N$ be a given set of points in the plane R^2 and consider the problem of finding K other points ("centers", "medians") $c_1, c_2, \ldots c_K$ so as to minimize

$$\sum_{i=1}^{N} \min_{1 \leq j \leq K} \|x_i - c_j\| \tag{1.1}$$

where $\|x_i - c_j\|$ is the Euclidean distance between x_i and c_j. This problem, which is known as the Multi-Source Weber Problem ([KS]) or as the Euclidean K-median problem[1] ([HM]), generalizes naturally to the problem of finding a set $C = \{c_1, c_2, \ldots, c_K\}$ so as to minimize

$$D(w,C) \equiv \int \min_{c \in C} \|x - c\| \, dw(x). \tag{1.2}$$

In this expression, w is a finite positive Borel measure with bounded support on R^2. We will refer to w as the <u>demand</u>. Let S(w) denote the <u>support of w</u> and let $|w| \equiv w(R^2)$ be the <u>total demand</u>. (The choice $w = \sum_{i=1}^{N} \delta_{x_i}$, where for any x_i and T, $\delta_{x_i}(T)$ is defined to be 1 if $x_i \in T$ and 0 otherwise, will return us to (1.1).)

The generalized problem is of interest, in part, because continuous distributions often provide computationally useful approximations of large ensembles of points (see [Ha3] for further details). When normalized so that $w(R^2) = \int dw = 1$, w may be viewed as a probability measure on R^2 and D(w,C) as the associated expected distance between a random point and the point of C closest to it. Of particular interest is the case in which

[1] Note that the variant of the K-median problem considered here does not require c_j's to be chosen from among the x_j's.

w is uniform on a region R of the plane, i.e., $w = \bar{m}\,\mu_R$ where $\bar{m}$ is a positive constant and μ_R is the restriction of μ, Lebesgue measure, to R.

Now let $D_K(w)$ denote the minimal value of D(w,C) when C is restricted to contain no more than K points. Theorem 1 of [Ha2] implies that there is a positive constant γ_2 satisfying

$$\lim_{K\to\infty} K^{1/2} D_K(w) = \gamma_2 \left(\int m^{2/3} d\mu\right)^{3/2} \tag{1.3}$$

for all w having m as the density of its absolutely continuous part. Moreover, the author shows that for any measurable set T, as $K \to \infty$ the number of points in $C \cap T$ is proportional to $\int_T m^{2/3} d\mu$. This result is rather general and is not tied to many of the particular characteristics of the 2-dimensional Euclidean K-median problem. Using these characteristics, in this paper we are able to consider finer details of the (asymptotic) solution structure. We show in Section 2 that if w is uniform on some bounded region R, then, as $K \to \infty$, the minimizing $c_1, c_2, \ldots, c_K$ tend to be configured like the centers of the hexagonal cells in a honeycomb covering R. As a consequence, we find that

$$\gamma_2 = \sqrt{2/(3\sqrt{3})}\,(1/3 + 1/4\,\ell n 3) \simeq 0.377 \tag{1.4}$$

which is the average distance between the center and a (uniformly distributed) random point in a regular hexagon with unit area.

Consider now another, seemingly remote, problem. What is the largest possible value of $D_K(w)$, given that the support S(w) of w is contained in R and that $|w| = w(R) = 1$; moreover, what demand w yields this maximum value? Such maximin problems, which are the subject of [HaM], seems quite difficult for $K > 2$. By randomizing the positioning of an hexagonal center grid, in Section 3 we show, however, that there is a positioning of such a grid for which the cost $D(w,C)$ (and therefore $D_K(w)$) is smaller than the D_K obtained for a demand w that is uniform on a hexagonal cover of R and that is only slightly larger than R (when the perimeter is finite the area of the difference is $O(K^{-1/2})$. This result helps to establish that

$$\lim_{K\to\infty} K^{1/2} \sup_w \{D_K(w): \ |w| = 1, \ S(w) \subset R\} = \gamma_2 \mu(R)^{1/2} \qquad (1.5)^{2}$$

(where γ_2 is given by (1.4), as well as the fact that as K grows, the uniform distribution becomes the worst possible (i.e., cost maximizing) demand.

The results contained in Section 2 originated in [HA1]. They are similar to those established independently (and simultaneously) by Papadimitriou [Pa] and, as it turns out, quite earlier by Fejes-Toth [F1], [F2].[3] Our results in this section differ from Papadimitriou's in two respects. First, he relies on a computer-aided proof to establish an essential convexity property (see Lemma 2 of this paper) underlying the analysis. On the other hand, our development is completely analytic. Furthermore, our proof provides more geometric insight and actually establishes a far more general result: any monotone function f of Euclidean distances can be used in the objective function (1.2) in place of the Euclidean distances themselves. Although not explicitly

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2 Note that (1.5) is stronger than $\sup_w \{\lim K^{1/2} D_K(w): |w| = 1, s(w) \subset R\} = \gamma_2 \mu(R)^{1/2}$ which (using Jensen's inequality) is an immediate consequence of (1.3) (see Section 3).

3 We are grateful to J.M. Steele for bringing this early work to our attention.

stated as such, Fejes-Toth's proof, which differs from our own, also implies this generalization.

Finally, it is pleasing to note that the maximum result (1.5) to be considered in Section 3 also generalizes in this same fashion.

2. Hexagonal Partitioning and Lower Bounds for Uniform Demand

A Preview:

As motivation for the honeycomb[4] form of the asymptotic solution, let us consider the following version of the problem. Suppose that for any particular K, we were free to move the demand points and choose any shape and size for the single facility subregions, provided only that the total area were equal to A. Then as one could easily verify, each subregion would be circular with area A/K. Unfortunately, such a partition of R is not possible in general (it is never possible for more than one value of K). Intuitively, a close approximation to the K-circle partition seems to consist of K disjoint congruent circles of maximal size packed inside the region R and with the remainder of R divided between the circles so that each point is assigned to the center closest to it. Asymptotically this solution is equivalent to partitioning to K congruent regular hexagons.

Suppose, instead, that we consider partitioning R into congruent regular polygons. Figure 1 illustrates partitions into equilateral triangles, into squares, and into hexagons. Partitioning into congruent regular polygons with more than 6 edges is impossible (see Lemma 4 to follow).

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[4] Regular hexagonal partitioning structures are common in nature (see On Growth and Form by D'Arcy Thompson [Th]). They arise for a variety of reasons, sometimes as a solution to extremum problems. For example, they minimize (asymptotically) the total length of the walls of a partition of a planar region into K equal area subregions, thus providing an economic (wax saving) rationale to the hexagonal structure of honeycombs.

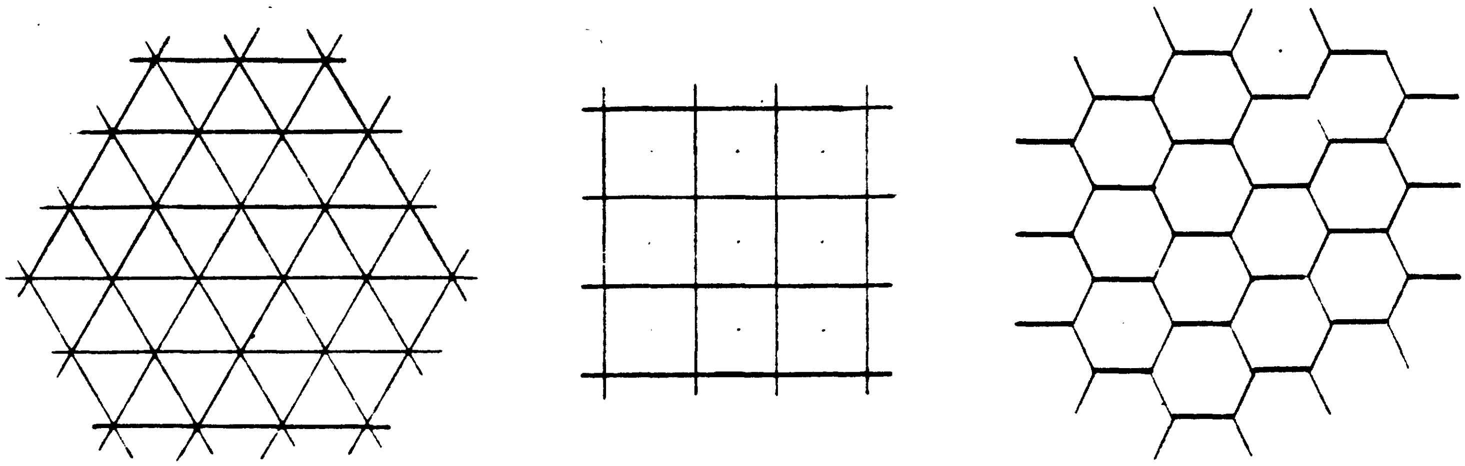


FIGURE 1: The regular partitioning schemes

Consequently, from among all regular partitions, the natural choice for approximating the nonachievable ideal circular single facility service regions would appear to be the hexagonal partitioning scheme.

Of course, it is not clear that a non-regular polygonal partitioning having some cells with more than 6 edges and some with fewer than 6 edges might not be preferable. As frequently is the case on such occasions, showing that the cost in an "average" cell (that is, one with an average number of edges and an average area) is lower than the average cost per cell, involves application of a convexity argument. Indeed, our formal proof relies on such an argument.

Problem Setting:

As we will see, the hexagonal partitioning property applies to problems with cost structure broader than that of the K-median problem, namely whenever the cost is monotonically increasing (not necessarily linearly, as in the K-median problem) in the Euclidean customer to facility distance.

Let $f: [0,\infty) \rightarrow [0,\infty]$ be monotonically increasing and consider the following generalizations of $D(w,C)$ and $D_K(w)$:

$$D^f(w,C) = \int f(\min_{c\varepsilon C} ||x - c||)dw$$

and

$$D_K^f(w) = \min \{D^f(w,C): |C| \leq K\} .$$

D^f and D_K^f coincide with D and D_K whenever f is the identity, i.e., $f(r) = r$.

We shall require some additional notational conventions:

- Let $T(\alpha,A)$ denote the right angle triangle with area A, with a vertex of acute angle α at the origin, and with one edge on the horizontal axis.
- Let $\sigma_f(\alpha,A) = D^f(\mu_{T(\alpha,A)},\{0\}) = \int_{T(\alpha,A)} f(||x||)d\mu$.
- For $y > 2$ and $A \geq 0$, let $\phi_f(y,A) = 2y \cdot \sigma_f(\frac{\pi}{y},\frac{A}{2y})$.

  Note that for $y = 3,4,5\ldots$, $\phi_f(y,A) = \int_{P_y(A)} f(||x||)d\mu$ is the cost of serving a regular y-gon $P_y(A)$ of area A by a facility located at its center.
- Let $\phi(y) = \phi_I(y,1)$, where I is the identity (i.e., $I(r) = r$).

  That is, $\phi(y) = 2y \int_{T(\frac{\pi}{y},\frac{1}{2y})} ||x||d\mu$.

Calculation yields:

$$\phi(y) = \frac{1}{3} (y \tan^3 \frac{\pi}{y})^{-1/2}(\sec \frac{\pi}{y} \tan \frac{\pi}{y} + \ln (\sec \frac{\pi}{y} + \tan \frac{\pi}{y})) .$$

Note that for y = 3,4,5,...,$\phi(y)$ is the average distance to the center in a regular unit area y-gon.

Results:

We are ready to state the main result of this section:

THEOREM 1: Let f: $[0,\infty] \to [0,\infty]$ be monotonically increasing and let R be an n-gon of area A. Then for all $K \geq 1$,

$$D_K^f(\mu_R) \geq K\phi_f(6 + \frac{n-6}{K}, \frac{A}{K}).$$

In particular (when f is the identity),

$$D_K(\mu_R) \geq \phi(6 + \frac{n-6}{K}) \, A \, (\frac{A}{K})^{\frac{1}{2}}.$$

Remarks:

- If $n \leq 6$ (e.g., R is rectangular), we may substitute $\phi_f(6,\frac{A}{K})$ and $\phi(6)$ for $\phi_f(6 + \frac{n-6}{K}, \frac{A}{K})$ and $\phi(6 + \frac{n-6}{K})$ in the theorem.
- If K = 1, the right-hand side of these expressions become $\phi_f(n,A)$ and $\phi(n) \, A^{3/2}$. Thus, for the location of a single center, the regular n-gon gives the smallest cost from among all n-gons.
- If R is disconnected and has ℓ components and h "holes", but still has a piecewise linear boundary, the theorem remains valid with $6 + \frac{n - 6(\ell - h)}{K}$ in place of $6 + \frac{n-6}{K}$.

The asymptotic optimality of hexagonal partitions is an obvious corollary of Theorem 1. Observing that ϕ_f is continuous in both arguments, that $6 + \frac{n-6}{K} \to 6$ as $K \to \infty$, and that R can be covered by

K congruent regular hexagons of an area that converges to A as $K \to \infty$, we have

<u>COROLLARY 1.1</u> <u>If</u> R <u>is a polygon</u>, <u>and</u> f <u>is monotone</u>, <u>then</u>,

$$\lim_{K\to\infty} \frac{D_K^f(\mu_R)}{K\phi_f(6,\frac{\mu(R)}{K})} = 1.$$

In particular, $\lim_{K\to\infty} \frac{D_K(\mu_R)}{K^{-\frac{1}{2}}\ \phi(6)\mu(R)^{3/2}} = 1$ and thus, recalling Theorem 1 of [Ha2] (see also (1.3), we conclude that $\gamma_2 = \phi(6)$, i.e.,

<u>COROLLARY 1.2</u> <u>For any demand distribution</u> w <u>in the plane</u>,

$$\lim_{K\to\infty} K^{\frac{1}{2}}\ D_K(w) = \phi(6)(\int m^{2/3}\ d\mu)^{3/2}$$

<u>where</u> $\phi(6) = \sqrt{\frac{2}{3\sqrt{3}}}(\frac{1}{3} + \frac{1}{4}\ \ell n\ 3)$ <u>is the average distance of points from the center of a regular unit area hexagon</u>.

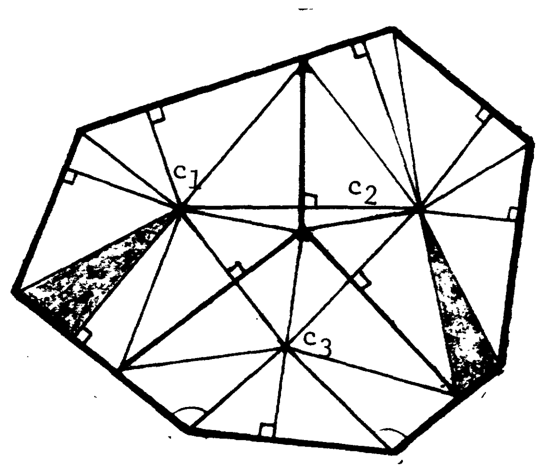
The rest of this section is devoted to the proof of Theorem 1. The proof is based on the convexity of σ_f, the (decreasing) monotonicity of ϕ_f in its first argument, y, and the fact that the average number of edges in a single facility service cell is no more than $6 + \frac{n-6}{K}$. We establish each of these facts as intermediate results. The proof can be viewed as constructive in the sense that it is based on a sequence of mappings of R (and C) that reshape R while (i) preserving its area, and (ii) not increasing the distance of any point in R to the closest center in C.

Proof of Theorem 1

Though the assertion holds as stated, we restrict ourselves to situations in which R is convex. (The extension to the nonconvex case is tedious and not very illuminating.) We may assume that $C \subset R$, since the projection on R of any point $c \in C-R$ lies closer than c to any point in R, and thus reduces $D^f(\mu_R, C)$.

Let $R_j = \{x \in R: ||x - c_j|| \leqq ||x - c_i|| \text{ for all } i = 1, 2, \ldots, K\}$. $R_1, R_2, \ldots, R_K$ constitutes a partition[5] of R into K polygons. Draw lines from each center c_j to the vertices of its associated polygon R_j and draw perpendicular from each c_j onto the edges of R_j (provided they lie inside R_j). The result is a "center" partition of R into Triangles $T_1, T_2, \ldots, T_m$. Each triangle T_i has one of the centers as a vertex and has another vertex incident to a right or obtuse angle (see Figure 2).

[5]Strictly speaking, these sets do not give a proper partition, since the common boundaries (that have a null area) of the polygons are counted more than once.



number of centers K = 3

number of triangles m = 29

FIGURE 2: Partition of R into triangles

For i = 1, 2, ..., m, let A_i denote the area of triangle T_i, and let α_i denote the acute angle of T_i that is adjacent to a center.

Consider the following intermediate results:

<u>LEMMA 1</u>: $D^f(\mu_{T_i}, C) \geqq \sigma_f(\alpha_i, A_i)$ <u>for</u> <u>all</u> i = 1, ..., m.

<u>Proof</u>: If T_i is a right angle triangle, then by definition and by the invariance of Euclidean distances and of areas under displacement $D^f(\mu_{T_i}, C) = \sigma_f(\alpha_i, A_i)$. If, on the other hand, T_i has an obtuse angle then, as depicted in Figure 3, we compare it to a right angle triangle with the same angle at the "center" vertex.

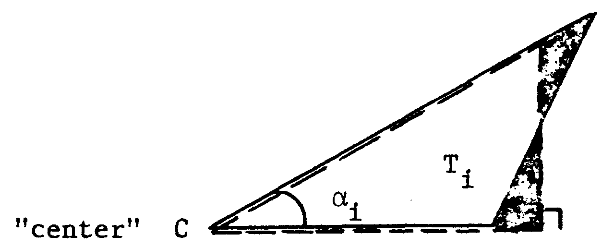


FIGURE 3: Comparison between obtuse and right angle triangles

Since any point in the lower shaded area in Figure 3 except d is closer to the "center" vertex than any point in the upper shaded area, we concluded that $D^f(\mu_{T_i}, C) > \sigma_f(\alpha_i, A_i)$. □

LEMMA 2: $\sigma_f(\beta, B)$ is convex on the region $(0, \frac{\pi}{2}) \times [0, \infty)$.

Proof: Since σ_f is continuous, it suffices to show that $\sigma_f(\beta_1, B_1) + \sigma_f(\beta_2, B_2) \geqq 2\sigma_f(\frac{\beta_1 + \beta_2}{2}, \frac{B_1 + B_2}{2})$ whenever $0 < \beta_1, \beta_2 < \frac{\pi}{2}$ and $B_1, B_2 \geqq 0$. We compare a pair of right angle triangles with areas B_1, B_2 and angles β_1, β_2 at a "center" vertex, with a pair of congruent right angle triangles each with area $\frac{1}{2}(B_1 + B_2)$ and an angle $\frac{1}{2}(\beta_1 + \beta_2)$ at the "center" vertex (see Figure 4).

The comparison, depicted in Figure 4, is carried out in two stages. We first compare the original pair of right angle triangles (leftmost in Figure 4) to an intermediate pair of right angle triangles (middle of

Figure 4) of the same total area and total angle at the "center" vertex, and with a common hypotenuse. Then, we compare the intermediate pair to the pair of congruent right angle triangles (rightmost in Figure 4) with the same total area and total angle at the "center" vertex.

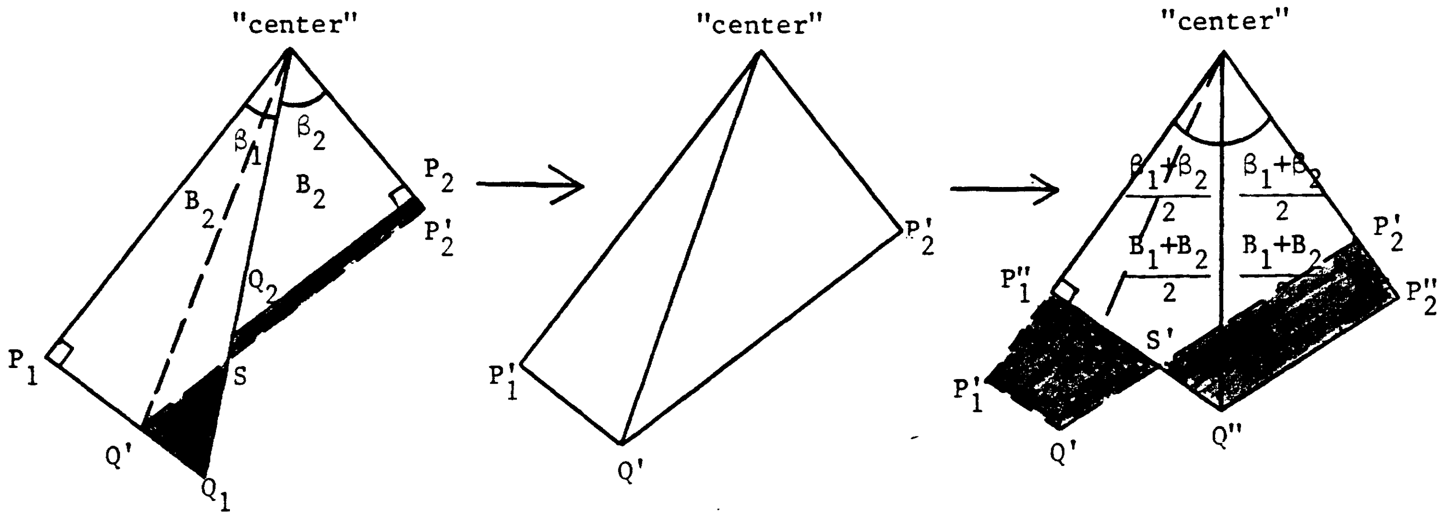


FIGURE 4: Comparison between pairs of right angle triangles

It is enough to show that, when we move from left to right in Figure 4, the area that is at distance r or more from the center is decreasing.

That this assertion is true for the first comparison follows simply from the fact that every point in the trapezoid $Q_2P_2P_2'S$ other than S is closer to the center than any point in the (equivalent in area) triangle Q_1Q_2S. (This argument is similar to that employed in the proof of Lemma 1.)

The situation in the second comparison is not so straightforward. It is not the case that every point in the trapezoid $S'P_2'P_2''Q''$ lies closer to the center than any point in the trapezoid $S'P_1''P_1'Q'$. We can, however, carry out a point by point comparison or precisely speaking, demonstrate an area (i.e., Lebesgue measure) preserving and distance (to center) decreasing mapping of $S'P_1''P_1'Q'$ into $S'P_2'P_2''Q''$. One such mapping can be constructed as follows (see Figure 5): Slice trapezoid $S'P_1''P_1'Q'$ into very thin (almost rectangular) strips, using lines parallel to the basis $S'P_1''$. Then, select the first strip (that is adjacent to $S'P_1''$) and "stretch" it so that its length expands to the length of $S'P_2''$, while its width shrinks so as to preserve the area. Place the stretched strip in the trapezoid $S'P_2'P_2''Q''$, so that the point that was adjacent to P_1'' is now adjacent to P_2'. To complete the mapping, we proceed in the same manner, selecting the next strip, "stretching" it and placing it along the previously transformed strip and so on until the whole trapezoid is transformed. More precisely, if X is a point in $S'P_1''P_2'Q'$, then there is $0 \leqq \lambda(X) \leqq 1$ so that $X = \lambda\underline{X} + (1 - \lambda)\overline{X}$ where $\underline{X}$ lies on $\overline{P_1''P_1'}$, $\overline{X}$ lies on $\overline{S'Q'}$ and the segment $\overline{\underline{X}\overline{X}}$ (passing through X) is parallel to $\overline{P_1''S'}$. Let A(X) be the area of the sub-trapezoid $S'P_1''\underline{X}X$. Next construct a sub-trapezoid $S'P_2'\underline{Y}\overline{Y}$, of $S'P_2'P_2''Q''$ (see Figure 5) that has the same area A(X) and let $Y = \lambda\underline{Y} + (1 - \lambda)\overline{Y}$. Consider the 1-1 mapping of $S'P_1''Q_1'Q'$ into $S'P_2'P_2''Q''$ defined by the correspondence $X \to Y$. The mapping is obviously area (measure) preserving, and it is a straightforward exercise (see Lemma 5 in the Appendix) to verify that it is distance decreasing.

We conclude, then, that for all monotonically increasing f, $2\sigma_f(\frac{\beta_1 + \beta_2}{2}, \frac{B_1 + B_2}{2}) \geqq \sigma_f(\beta_1,B_1) + \sigma_f(\beta_2,B_2)$ or, that σ_f is convex in $(0,\frac{\pi}{2}) \times [0,\infty)$. □

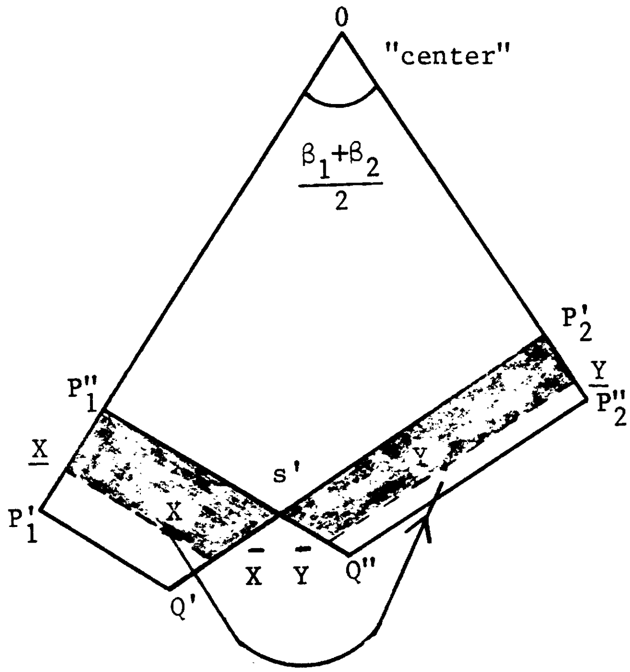


FIGURE 5: Area preserving, distance decreasing mapping

LEMMA 3: *For fixed A and f, $\phi_f(y,A)$ is monotonically decreasing in y over the interval $(2,\infty)$.*

Proof: By Lemma 2, we know that σ_f is convex, and thus, for any positive scalar $\lambda, \sigma_f(\alpha,\lambda\alpha)$ is convex in α. Furthermore, $\sigma_f(0,\lambda 0) = \sigma_f(0,0) = 0$. Consequently (by the 3-chord lemma, for example), $\frac{\sigma_f(\alpha,\lambda\alpha)}{\alpha}$ is monotonically increasing in α for $0 < \alpha < \frac{\pi}{2}$. Recalling that $\phi_f(y,A) = 2\pi \frac{\sigma_f(\frac{\pi}{y}, \frac{A}{2\pi}\cdot\frac{\pi}{y})}{\pi/y}$ completes the proof. □

LEMMA 4: *The number of triangles m in the "center" partition of R does not exceed $2K(6 + \frac{n-6}{K})$.*

Proof: By our construction, $m \leqq 2 \sum_{j=1}^{K} n_j$ where n_j is the number of edges of the polygon R_j.

Consider the vertices of the polygons R_i, R_2, ..., R_K; there are three types of vertices:

(1) those that are vertices of R,

(2) those that lie on the relative interior of edges of R, and

(3) those that lie in the interior of R,

There are (by definition) n type 1 vertices. Let N_i for i = 2 and 3 denote the number of type i vertices.

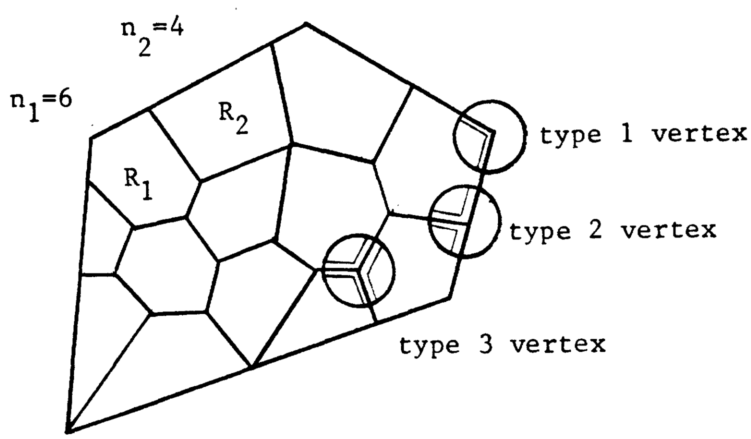


FIGURE 6: Partition of R into polygonal service regions

Each type 2 vertex is a vertex of at least two polygons and each type 3 vertex is adjacent to at least 3 polygons. Furthermore, it is easy to see that each such vertex must be a vertex of all the polygons adjacent to it. (That is, no two edges adjacent to a type 3 vertex are colinear.)

Consequently, as depicted in Figure 6, the number of subregional edges adjacent to each type 1 [respectively type 2 and 3] vertex is at least 2 [respectively 4 and 6]. (Note that any edge of two adjacent subregions is counted twice.) Since the total number of edges in R_j is n_j, summing over all the vertices gives the subregional edge count

$$\sum_{j=1}^{K} n_j \geq \tfrac{1}{2}(2n + 4N_2 + 6N_3). \qquad (2.1)$$

The division by 2 of the right hand side reflects the fact that each edge is counted twice in the vertex adjacent count, once for each of its two vertices.

Summing the angles over the vertices and over the polygons yields the equation

$$\sum_{\substack{j=1 \\ R_j \neq \phi}}^{K} \pi(n_j - 2) = \pi(n - 2) + \pi N_2 + 2\pi N_3 \qquad (2.2)$$ [6]

which implies that $N_3 = \tfrac{1}{2}(\sum_{j=1}^{K} n_j - N_2 - n - 2(K - 1))$. Substituting this expression in (2.1) and rearranging gives

[6] This equation could be obtained from Euler's Lemma for planar connected graphs.

$$\sum_{j=1}^{K} n_j \leqq 6(K - 1) + n - N_2 \leqq 6(K - 1) + n = K(6 + \frac{n - 6}{K})$$

and, thus, $m \leqq 2 \sum_{j=1}^{K} n_j \leqq 2K(6 + \frac{n - 6}{K})$. □

We are now ready to complete the proof of Theorem 1. Using the fact that $\sum_{i=1}^{m} \alpha_i = 2\pi K$[7] and $\sum_{i=1}^{m} A_i = A$, and Lemmas 1-4, we have

$$D^f(\mu_R, C) = \sum_{i=1}^{m} D^f(\mu_{T_i}, C) \geqq \sum_{i=1}^{m} \sigma_f(\alpha_i, A_i) \geqq m\sigma_f(\frac{\sum_{i=1}^{m} \alpha_i}{m}, \frac{\sum_{i=1}^{m} A_i}{m})$$

$$= m\sigma_f(\frac{2\pi K}{m}, \frac{A}{m}) = K\phi_f(\frac{m}{2K}, \frac{A}{K}) \geqq K\phi_f(6 + \frac{n - 6}{K}, \frac{A}{K}).$$

The three inequalities in this expression follow, respectively, from Lemma 1, Lemma 2, and Lemma 3 together with Lemma 4. Since this result is valid for any C, the theorem is valid. □

To conclude this section, we note that the proof of Theorem 1 actually establishes (by a sequence of measure preserving, distance decreasing transformations) a stochastic dominance property of hexagonal partitioning. Let F_H denote the cumulative distribution of distances from a (uniformly) random point in a hexagon with area $\mu(R)/K$ to its center. This distance distribution is achievable for

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[7] This equality is obvious when all centers are in the interior of R. However, if c_j lies on an edge or a vertex of R, one can add degenerate null area triangles to justify $\Sigma\, \alpha_i = 2\pi$ at that center.

the distances of points to their nearest center if the region R can be partitioned into K congruent regular hexagons. (This "ideal" partition is in general achievable approximately, with negligible boundary effects, only for large values of K.) Next, suppose, for simplicity, that the region R is a square and that C is any specified center set of K points. Then as shown in Figure 7, F_H stochastically dominates the cumulative distribution of distances of points in R to their nearest center in C. This dominance relationship is, incidentally, the form of Theorem 1 as stated by Fejes-Toth [F2].

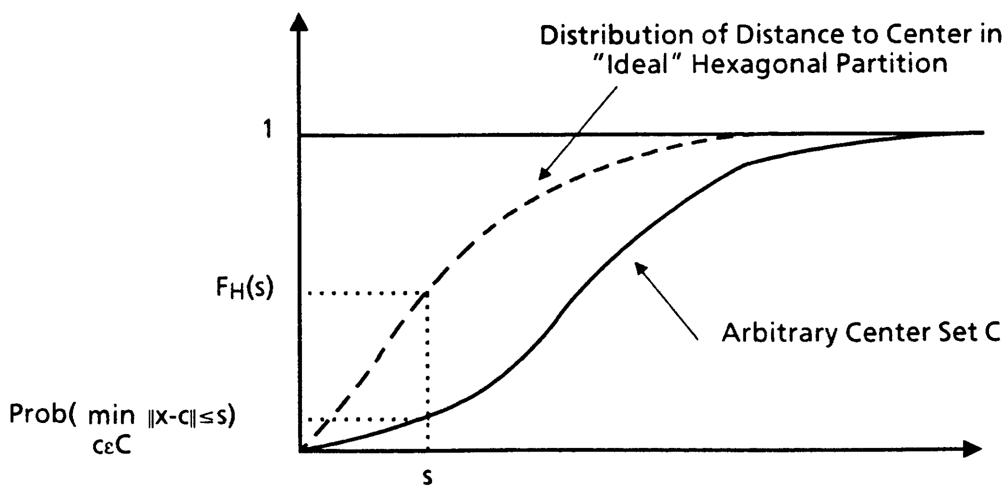

FIGURE 7: Stochastic Dominance of Hexagonal Partitions

The proof of Theorem 1 implicitly uses this dominance property to show that hexagonal partitioning minimizes $\int \min_{1\leq j\leq K} f(||x-c_j||)d\mu(x)$ for any monotone function f. In addition, the dominance property also implies that hexagonal partitioning is asymptotically cost minimizing when other cost functionals are used in place of the integral. For example, hexagonal partitioning asymptotically minimizes

$\max_{x\varepsilon R} \min_{1\leq j\leq K} ||x-c_j||$, as shown by Zemel [Z1].

3. Uniform Demand and Upper Bounds for Hexagonal Partitioning

In this section we derive an upper bound on $D_K^f(w)$ (and $D_K(w)$) for demand distributions w whose support is contained in a bounded region R with a rectifiable boundary. The bound depends only on the area A and the perimeter p of R. In contrast to Theorem 1, though, the bound is not restricted to uniform demand.

This upper bound and the lower bound of Theorem 1 imply that the uniform distribution on a region R is in some "strong" asymptotic sense (that will be explained) the worst possible (i.e., cost maximizing) distribution in that region.

THEOREM 2: *Let R be a connected region in the plane with finite area A and with perimeter p. Then for any demand w with support in R and for any measurable real-valued function f defined on* $[0,\infty)$,

$$D_K^f(w) \leqq |w| \; \phi_f(6, \frac{\bar{A}}{K})(\frac{\bar{A}}{K})^{-1}.$$

In particular, if f is the identity, then

$$D_K(w) \leqq \phi(6) \cdot |w| (\frac{\bar{A}}{K})^{\frac{1}{2}}.$$

In these expressions,

$$\bar{A} \equiv A(\frac{1}{1 - aK^{-1}} + \frac{2bK^{-1/2}}{1 - aK^{-1}} \sqrt{\frac{1}{1 - aK^{-1}} + (\frac{b}{1 - aK^{-1}})^2})$$

where $a \equiv \frac{8\pi}{3\sqrt{3}}$ *and* $b \equiv \sqrt{\frac{2}{3\sqrt{3}} \frac{p^2}{A}}$.

Since ϕ_f is continuous in its second argument and since $\bar{A} \to A$ as $K \to \infty$ ($\bar{A} = A + o(K^{-\frac{1}{2}})$), we immediately have (in conjunction with Theorem 1)

COROLLARY 2.1: *Assume the conditions of Theorem 2. Then,*

$$\lim_{K\to\infty} (\mathrm{Sup}_{w} \{\frac{D_K^f(w)}{|w|\phi_f(6, \frac{A}{K})(\frac{A}{K})^{-1}} : S(w) \subset R,\ w \neq 0\}) = 1.$$

Proof: That the limit is at most 1 follows immediately from Theorem 2. Selecting w as uniform over R and invoking Theorem 1, we conclude that the limit is at least 1. □

Remark: Corollary 2.1 as well as the subsequent Corollary 2.2 hold for any Lebesgue measurable bounded R with $\mu(R) = A$, but we will not pursue this matter in this paper.

Recalling Theorem 1 (or Corollary 1.1), we have

COROLLARY 2.2: *Assume the conditions of Theorem 2 and let* $\bar{m} = |w|/A$ (*i.e.*, $\bar{m}\mu(R) = |w|$). *Then,*

$$\lim_{K\to\infty} (\mathrm{Sup}_{w} \{\frac{D_K^f(w)}{D_K^f(\bar{m}\mu_R)} : S(w) \subset R,\ w \neq 0\}) = 1.$$

That is, among all demand distributions in a given region with some fixed total demand, the uniform distribution is asymptotically worst (i.e., cost maximizing).

It is interesting to compare this last property with a slightly weaker asymptotic maximality result. Consider the equality

$$\lim_{K\to\infty} K^{\frac{1}{2}} D_K(w) = \phi(6)(\int m^{2/3}\, d\mu)^{3/2}$$

of Corollary 1.2. Applying Jensen's inequality to the convex function $g(x) = x^{3/2}$ gives

$$\left(\frac{1}{\mu(R)} \int m^{2/3} \, d\mu\right)^{3/2} \leqq \frac{1}{\mu(R)} \int (m^{2/3})^{3/2} \, d\mu$$

that is

$$\left(\int m^{2/3} \, d\mu\right)^{3/2} \leqq \left(\int m d\mu\right) \mu(R)^{\frac{1}{2}} \leqq |w| \mu(R)^{\frac{1}{2}}$$

with equalities if and only if m is constant almost everywhere in R, and equal to $\bar{m} \equiv \frac{|w|}{\mu(R)}$. In other words,

$$\operatorname*{Sup}_{w} \left\{ \lim_{K\to\infty} \frac{D_K(w)}{D_K(\bar{m}\mu_R)} : S(w) \subset R,\ w \neq 0 \right\} = 1. \qquad (*)$$

This result is substantially weaker than Corollary 2.2, in the sense that here we maximize with a single demand for the whole tail of the sequence, where in Corollary 2.2 we are free to maximize with different demands for the individual K's. (Recall that in general Sup(lim(·)) ≦ lim(Sup(·)).) It is interesting to note that if we consider a variation of the K-median problem that restricts the centers to lie within the support of the demand (i.e., $C \subset S(w)$), then (*) will be still valid ([Ha 2], Section 4), while Corollary 2.2 will not ([HaM], Section 7).

The proof of Theorem 2, which concludes this section, is based on a randomized hexagonal covering argument. This randomized covering can be interpreted as a randomized strategy in a zero-sum game with payoff $\min_{c \in C} f(||x - c||)$. The maximizer in this game chooses $x \in R$ and the minimizer chooses K points C from R^2. With this interpretation, w is a randomized strategy for the maximizer (note that in our original problem there is no loss of generality in assuming that $|w| = 1$), while the randomized hexagonal covering is a randomized strategy for the minimizer. Our conclusion is that asymptotically (as $K \to \infty$) the uniform distribution for x together with a uniformly randomized positioning of a regular hexagonal cover constitute a saddle point of this game. A companion paper [HaM] treats this result and other so-called "location games" more thoroughly.

Proof of Theorem 2:

Partition the plane into congruent regular hexagons (reference hexagons) each with area $\bar{A}/K$. Take the grid of the centers of the hexagons and, without changing its orientation, position it (as a whole) randomly so that each single center is distributed uniformly in a regular hexagon of area $\bar{A}/K$.

By Lemma 6 of the Appendix, R is covered, for any positioning of this grid, by no more than K of the hexagons; that is, we may assume that for any positioning of the grid, every demand point in R is served by the grid point closest to it.

Consider now an arbitrary point $x \in R$ and any particular random grid G of centers generated as indicated above. Draw a regular hexagon of area $\bar{A}/K$ around x (Figure 8). This hexagon always contains exactly one grid point from G, which is the center serving the demand point x from this grid of centers. By our choice of randomization, this center has a uniform distribution in this hexagon. Thus, if there is discrete demand mass $\Delta w(x)$ at point x, the average (expected) cost paid by the demand there is

$$\frac{\Delta w(x)}{\bar{A}/K}\, \phi_f(6, \frac{\bar{A}}{K}).$$

(Recall that $\frac{1}{\bar{A}/K}\, \phi_f(6, \frac{\bar{A}}{K}) = \frac{1}{\mu(H)} \int_H f(\|x\|)d\mu$, where H is a regular hexagon of area $\bar{A}/K$, is the average cost per unit demand at our point.)

Summation (or integration in the non-discrete case) over all randomized grid solutions yields an average total cost $|w|\phi_f(6, \frac{\bar{A}}{K})(\frac{\bar{A}}{K})^{-1}$. Since each of these randomized grid solutions is feasible (i.e., use K or fewer centers), the value of the minimum cost solution must be no more than $|w|\phi_f(6, \frac{\bar{A}}{K})(\frac{\bar{A}}{K})^{-1}$ as well. □

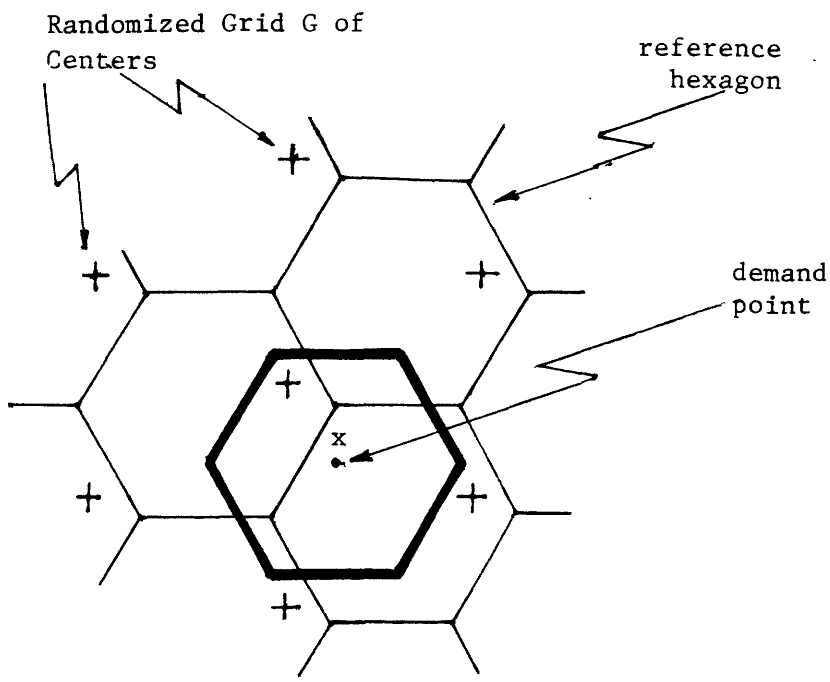

FIGURE 8: A single customer amid a randomized center grid.

4. Summary

We have derived a lower bounds on the optimal cost of K-median problems (or similar monotone-in-distance K-median problems) with demand uniformly distributed in a given bounded region R. On the other hand, using a randomly positioned hexagonal grid we have established an upper bound for the optimal cost that is valid for all possible demand distributions in R. These two bounds converge (in ratio) as K tends to infinity to the cost for demand uniformly distributed over K regular hexagons of area $\mu(R)/K$ each with their centers chosen for $c_1, c_2, \ldots, c_K$. Therefore, as $K \to \infty$, if the demand is distributed uniformly, a hexagonal partitioning scheme is asymptotically optimal.

As mentioned in [HaM], we have essentially found a saddlepoint of a zero-sum game in which the maximizer chooses a point in R, the minimizer chooses K points in R and the payoff is the distance between the maximizer's choice to the closest of the minimizer's choices (or some nondecreasing function of this distance).

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APPENDIX

LEMMA 5: In Figure 5, $\overline{OY}$ is shorter than $\overline{OX}$. (The construction of Figure 5 is described in Lemma 3 in Section 2.)

Proof: (Let $\overline{MN}$ denote the length of segment MN and let $\measuredangle$ MNL denote the angle MNL.) Since the trapezoids $S'P_1''\underline{X}\overline{X}$ and $S'P_2'\underline{Y}\overline{Y}$ are equal in area, while the bases of the first one are shorter than those of the second one, their altitudes are inversely ordered, i.e., $\overline{P_1''\underline{X}} \geqq \overline{P_2'\underline{Y}}$. Noting now that $\measuredangle P_1''S'\overline{X} = \measuredangle P_2'S'\overline{Y}$ we get $\overline{S'\overline{X}} \geqq \overline{S'\overline{Y}}$. This result together with the fact $\measuredangle OS'\overline{X} \geqq \measuredangle OS'\overline{Y}$ imply $\overline{O\overline{X}} \geqq \overline{O\overline{Y}}$.

Also, $\psi_1 \equiv \measuredangle O\overline{X}\underline{X} \leqq \frac{\beta_1 + \beta_2}{2} \leqq \measuredangle O\overline{Y}\underline{Y} \equiv \psi_2$. Consequently,

$$\overline{OY} = \overline{O\overline{Y}}\sqrt{\text{Cos}^2\psi_2 + \lambda^2\text{Sin}^2\psi_2} = \overline{O\overline{Y}}\sqrt{1 - (1 - \lambda^2)\text{Sin}^2\psi_2} \leqq \overline{O\overline{X}}\sqrt{1 - (1 - \lambda^2)\text{Sin}^2\psi_1}$$

$$= \overline{OX}\ . \qquad \square$$

LEMMA 6: Let R be a connected region in the plane with finite area A and perimeter p, then any partition of the plane into congruent regular hexagons, each with area $\bar{A}/K$, where

$$\bar{A} = A\left(\frac{1}{1 - aK^{-1}} + \frac{2bK^{-1/2}}{1 - aK^{-1}}\sqrt{\frac{1}{1 - aK^{-1}} + \left(\frac{b}{1 - aK^{-1}}\right)^2}\right) \qquad \text{(A.1)}$$

$$a = \frac{8\pi}{3\sqrt{3}} \qquad \text{and} \qquad b = \sqrt{\frac{2}{3\sqrt{3}}\frac{p^2}{A}}$$

will use no more than K hexagons to cover R.

(Remark: If R is not connected, but has ℓ components and h "holes," then the result still holds with $a = \frac{8\pi}{3\sqrt{3}}(\ell - h)$.)

Proof: Consider any partition of the plane into congruent regular hexagons of diameter d (i.e., edges each have length d/2). Then, the set $\overline{R}(d) \equiv \{x + y: x \in R, ||y|| \leqq d\}$ contains any hexagon that covers some point in A. Hence, if

$$\mu(\overline{R}(d)) \leqq K \frac{3\sqrt{3}}{8} d^2 \tag{A.2}$$

($\frac{3\sqrt{3}}{8} d^2$ being the area of a single hexagon), then there are no more than K such hexagons.

By Lemma 7 to follow, we know that

$$\mu(\overline{R}(d)) \leqq A + pd + \pi d^2 \tag{A.3}$$

(with $(\ell - h)\pi d^2$ instead of πd^2 if R is not connected and has ℓ components and h holes).

So, to guarantee no more than K hexagons in a cover, it is sufficient to have

$$A + pd + \pi d^2 \leqq K \frac{3\sqrt{3}}{8} d^2 . \tag{A.4}$$

So, let $\bar{d}$ be the smallest solution of (A.4), that is the positive solution of $(K \frac{3\sqrt{3}}{8} - \pi)d^2 - pd - A = 0$, and let $\bar{A} = K \frac{3\sqrt{3}}{8} \bar{d}^2 = A + p\bar{d} + \pi\bar{d}^2$. A simple calculation yields the expression (A.1) for $\bar{A}$. □

LEMMA 7: Let R be a connected region in the plane, with a finite perimeter p, then for $d \geqq 0$

$$\mu(\bar{R}(d)) \leqq \mu(R) + pd + \pi d^2 \quad (\bar{R}(d) \equiv \{x + y : x \in R, \|y\| \leqq d\},$$

(**Remarks**: This inequality becomes an equality if R is convex. If R is not connected and has ℓ components and h "holes", the inequality holds with $(\ell - h)\pi d^2$ instead of πd^2.)

Proof: We consider only a polygonal set R as depicted in Figure A1 (the validity for other sets follows from simple continuity arguments).

Construct external rectangles of width d on the sides of R. In the case of a vertex with corner angle $\alpha < \pi$, we must add a circular sector of area $\frac{1}{2}(\pi - \alpha)d^2$ to complete the turn of the strip around the corner, while if $\alpha > \pi$, we have an overlapping of two rectangles on a rhombus of area $\tan(\frac{\alpha - \pi}{2})d^2 > \frac{1}{2}(\alpha - \pi)d^2$. We may conclude then that the area of the strip is at most $p \cdot d + \frac{1}{2}\sum_i(\pi - \alpha_i)d^2$ (the summation is over all vertices, outer and inner).

Observe now that in general $\sum_i(\pi - \alpha_i) = 2\pi(\ell - h)$, where ℓ is the number of connected components of R and h is the number of holes in R. And since $\ell = 1$, we have $\sum_i(\pi_i - \alpha_i) \leqq 2\pi$, and the area of the strip is at most $p \cdot d + \pi d^2$. □

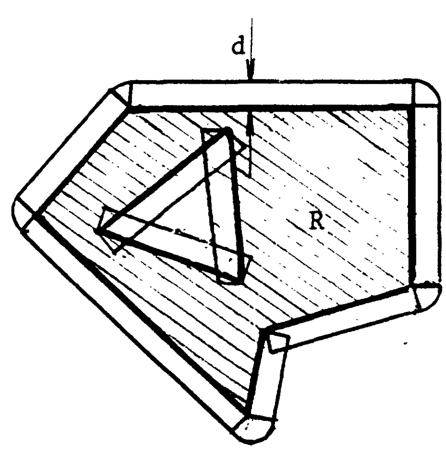

FIGURE A1: A d-wide boundary strip