LOCATION GAMES AND BOUNDS FOR MEDIAN PROBLEMS

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ABSTRACT

We consider a two-person zero-sum game in which the maximizer selects a point in a given bounded planar region, the minimizer selects K points in that region, and the payoff is the distance from the maximizer's location to the minimizer's location closest to it. In a variant of this game, the maximizer has the privilege of restricting the game to any subset of the given region. We evaluate/approximate the values (and the saddle point strategies) of these games for K = 1 as well as for $K \to \infty$, thus obtaining tight upper bounds (and worst possible demand distributions) for K-median problems.

KEY WORDS: Location Games, K-Median Problem, Euclidean Lo-ation

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1. Introduction

In the preliminary design of geographically distributed service systems, it is quite useful to have simple bounds on the optimal cost that depend only on crude problem data, like the total number of customers, the area of the region in which they are distributed and the number of facilities allowed. The purpose of this paper is to derive such bounds for the K-median problem.

Let $X = \{x_1, x_2, \ldots, x_N\}$ be a given set of customer locations in the plane R^2, and let $C = \{c_1, c_2, \ldots, c_K\}$ be a set of facility locations to be determined. The value of the free K-median problem is

$$D_K(X) \equiv \min_{\substack{C \subset R^2 \\ |C| \leq K}} \sum_{x \varepsilon X} \min_{c \varepsilon C} \|x-c\| .$$

The value of the restricted K-median problem is

$$\bar{D}_K(X) \equiv \min_{\substack{C \subset X \\ |C| \leq K}} \sum_{x \varepsilon X} \min_{c \varepsilon C} \|x-c\|$$

where $|C|$ denotes the cardinality of the set C and where $\|x-c\|$ is the distance from x to c.

Let R be a given compact planar subset; how large can $D_K(X)$ and $\bar{D}_K(X)$ become, if $X \subset R$? A sharp answer to this question requires the evaluation of

$$V_K^N(R) \equiv \max_{\substack{X \subset R \\ |X|=N}} D_K(X) \quad ; \quad \bar{V}_K^N(R) \equiv \max_{\substack{X \subset R \\ |X|=N}} \bar{D}_K(X) .$$

If instead of a finite number of (equally weighted) customers, we have a demand distribution, w, that is a positive regular Borel measure with a bounded support S(w), we may, appropriately, by abuse of notation, set

$$D_K(w) \equiv \min_{\substack{C\subset R^2 \\ |C|\leqq K}} \int \min_{c\varepsilon C} ||x-c||dw(x)$$

$$\bar{D}_K(w) \equiv \min_{\substack{C\subset S(w) \\ |C|\leqq K}} \int \min_{c\varepsilon C} ||x-c||dw(x)$$

as the values of the free and restricted versions of the K-median problem. Analogously to V_K^N and $\bar{V}_K^N$, we define

$$V_K(R) \equiv \max_{w(R)=w(R^2)=1} D_K(w) \quad ; \quad \bar{V}_K(R) \equiv \max_{w(R)=w(R^2)=1} \bar{D}_K(w). \qquad (1)$$

Clearly,

$$V_K(R) \geqq \frac{1}{N} V_K^N(R) \quad ; \quad \bar{V}_K(R) \geqq \frac{1}{N} \bar{V}_K^N(R)$$

and

$$V_K(R) = \lim_{N\to\infty} \frac{1}{N}V_K^N(R) \quad ; \quad \bar{V}_K(R) = \lim_{N\to\infty} \frac{1}{N}\bar{V}_K^N(R) .$$

In Section 2, we introduce zero-sum games associated with the maximin problems (1). It is quite easy to see (Section 3) that $V_1(R)$ is the radius, r(R), of the (smallest) circle circumscribing R and that any three (two, if R lies on a line) points of R lying on that circle are enough to support a worst possible demand distribution. The evaluation of $\bar{V}_1(R)$, however, is more involved. To this end, in Section 4 we consider matrix games in which one of the players, say the maximizer, selects as the payoff matrix any (square) symmetric submatrix from a given symmetric matrix. We obtain necessary conditions for the solution of such games. These conditions enable us, for example, to show, in Section 5, that all the points of R lying on the circumscribing circle are in the support of a worst possible demand distribution, and allow us to evaluate the (unique) worst possible distribution and $\overline{V}_1(R)$ in cases where all the extreme points of R lie on a circle. Consequently, $\overline{V}_1(R) \leqq \frac{4}{\pi}r(R)$ with equality if and only if the boundary of R coincides with that of a circle.

Finally in Section 6, we present results concerning $V_K(R)$ and $\overline{V}_K(R)$ and the associated worst demand distributions for K>>1. We show that there are constants η and $\overline{\eta}$ to which $K^{\frac{1}{2}}V_K(R)$ and $K^{\frac{1}{2}}\overline{V}_K(R)$ converge respectively for all unit area R's. We have shown elsewhere ([HaM], Theorem 2), that $\eta = (\frac{1}{3} + \frac{1}{4}\ell n3)\sqrt{\frac{2}{3\sqrt{3}}}$ (= $\int_H ||x||d\mu$ where H is a unit area regular hexagon centered at the origin). In this paper, we show that $\frac{2}{3}\sqrt{\frac{2}{3\sqrt{3}}} \leqq \overline{\eta} < 2\eta$.[1] These bounds complement the asymptotic formulas in [Ha1] and [Ha2] which are based on specific statistical assumptions about the demand distributions.

2. K-location Games

The values of $V_K(R)$ and $\overline{V}_K(R)$ of the maximin problems in (1) will, clearly, not change if we allow the "minimizer" to randomize his choice of centers. Note also that the demand measure w (when $w(R^2) = 1$ as in (1)) may be viewed as a probability measure used by the "maximizer" to randomize the location of x. In this setting, it is quite natural to consider the following zero-sum games.

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1) We conjecture that $\overline{\eta}$ actually coincides with the lower bound.

The <u>free</u> <u>K-location</u> <u>game</u> contains two players, one of which, the maximizer, selects a point x in R while the other, the minimizer, selects a K-tuple[2] $C = (c_1, c_2, \ldots, c_K)$ of points in R^2. The payoff (paid to the maximizer by the minimizer) is the distance $d(x,C) = \min_{1 \leqq j \leqq K} ||x-c_j||$ between x and the closest c_j. To mix (randomize) their strategies, the maximizer uses the probability measure w on R and the minimizer uses the probability measure λ on R^{2K} (the K-fold cartesian product of R^2).[3] The expected payoff is

$$D(w,\lambda) = \int_{R \times R^{2K}} d(x,C)d(w \times \lambda) = \int_{R^{2K}} D(w,C)d\lambda(C) = \int_R D(x,\lambda)dw(x) \qquad (2)$$

where, by abuse of notation, $D(w,C) = \int_R d(x,C)dw(x)$ and $D(x,\lambda) = \int_{R^{2K}} d(x,C)d\lambda(C)$ and where $w \times \lambda$ denotes the product of w and λ[4].

Now since the Kernel d(x,C) of the game is continuous in both its arguments and since R is compact and since so is the K-fold cartesian product of its convex hull (to which the support of λ can be actually restricted), we know from a simple adaptation of the celebrated minimax principle (e.g., Theorem IV.6.1 of [Ow]) that there is a saddle point w^*, λ^* of (optimal) strategies satisfying $D(x,\lambda^*) \leqq D(w^*,\lambda^*) \leqq D(w^*,C)$ for all $x \in R$ and all $C \in R^{2K}$. It is straightforward to show that the

2) The order between the elements of C is not important, but for notational convenience, we will regard C as a K-tuple rather than as an unordered set.

3) We assume the usual Borel σ-algebras on R and on R^{2K}.

4) Existence and finiteness of all these integrals as well as the equality between the iterated integrals and the double integral is elementary.

value $v^* = D(w^*,\lambda^*)$ is equal to the tight upper bound $V_K(R)$ and that an optimal strategy for the maximizer is also a worst possible demand distribution for the free K-median problem in R and vice versa.[5,6]

In the <u>restricted K-location game</u>, the maximizer also chooses a (closed) subset of R, from which the players may select x and $c_1, c_2, \ldots, c_K$. This feature of the problem is equivalent to restricting the support of λ to be contained in the K-fold cartezian product of the support of w. The minimax principle holds in this setting only locally. For every support the maximizer might choose, there is a saddle pair of strategies. It is possible to show that there exist an optimal (maximizing) support[7] and an associated pair of optimal strategies w^*,λ^*. Here too, it is not hard to show that the value $v^* = D(w^*,\lambda^*)$ is equal to the tight upper bound $\overline{V}_K(R)$ and that an optimal strategy for the

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5) Clearly $D_K(w) = \min_{|C|\leq K} D(w,C) \leq \int D(w,C)d\lambda^*(C) = D(w,\lambda^*) \leq D(w^*,\lambda^*)$ $\leq D(w^*,C)$ for all w with support in R and for all $C \in R^{2K}$. Thus, $V_K(R)$ $\leq D(w^*,\lambda^*) \leq \min_{|C|\leq K} D(w^*,C) = D_K(w^*)$. Thus, $V_K(R) = D(w^*,\lambda^*)$ and w^* is a worst possible demand. Conversely, if $\bar{w}^*$ is a worst possible demand, then $D(\bar{w}^*,C) \geq \min_{|\hat{C}|\leq K} D(w^*,\hat{C}) = D(\bar{w}^*,\lambda^*)$ for all $C \in R^{2K}$ and thus $\bar{w}^*$ is optimal for the maximizer.

6) Actually to have full correspondence between the game and the free K-median problem, we should not have restricted the minimizer's to choose points from R. This restriction is however meaningless when R is convex or when $K \to \infty$.

7) Consider the topology induced by the Hausdorff metric ([Du], p. 205) on F(R), the nonempty closed subsets of R. Then, F(R) is compact in that topology ([Du] p. 253), while the value of the game (as a function of the support ε F(R)) is continuous in that topology.

maximizer is also a worst possible demand distribution for the restricted K-median problem in R and vice versa.[8]

3. Free Single Median Problems

As a point of departure for our analysis, we first consider the simple case of the free 1-median problem. The associated game has a kernel $d(x,c) = ||x-c||$ which is strictly[9] convex in both arguments. It is therefore possible to show (following, for example, the proof of Theorem IV.4.2 of [Ow], and noting that the minimizer's choice will always be in the compact convex hull of R) that the minimizer will have a pure (non-randomized) strategy. As a consequence, we have the following elementary result:

Proposition: The value of the free 1-location game is

$$r(R) = \min_{c \in R^2} \max_{x \in R} ||x-c||,$$

i.e., the radius of the smallest circle containing R. The minimizer will choose the center of this circle while the randomization of the maximizer may be restricted to any three (or two) extreme points of R that lie on the boundary of the circle, and whose convex hull contains the center.

Consequently, $V_1(R) = r(R)$ and any three (or two) such points on the circumscribing circle suffice to support a worst possible demand distribution. The distribution of demand within such a triple can be found by a well-known construction from location theory, the inverse construction

8) Following footnote 5, we know that an optimal maximizer strategy w^* is a worst possible distribution on its support $S(w^*)$, while a worst possible distribution $\bar{w}^*$ is an optimal maximizer strategy on its support $S(\bar{w}^*)$. Obviously, w^* and $\bar{w}^*$ must yield the same value.

9) Unless x-c is restricted to some fixed line. In this case, it is just linear.

of the corresponding Weber triangle ([W], [K]); namely, the demands are proportional to the length of the edges in a triangle with edges parallel to the radii from the center of the bounding circle, to the corresponding demand points.

4. Restricted Symmetric Games

We next consider the restricted 1-median problem and its associated game of location. We show that the points in the support of the best strategy for the maximizer are all equally costly to the minimizer, and at least as costly to him as any other point in R. In other words, from among all the points in R, only the most costly are feasible for the minimizer.

We proceed now with the formal statement and proof of this assertion. Note that the restricted 1-location game on R can be approximated to any degree of accuracy by a restricted 1-location game on a finite subset of R. We consider, then, as a first step, only finite R, leaving to Theorem 2(iii) (and footnote 14) to follow the technical details for the extension of the subsequent results to the general case.

For $R = \{x_1, x_2, \dots, x_n\}$, we simplify our notation somewhat. Let $d_{ij} = d(x_i,x_j)$, $w_i = w(\{x_i\})$, $\lambda_j = \lambda(\{x_j\})$, $D(w,j) = D(w,x_j)$ and $D(i,\lambda) = D(x_i,\lambda)$, i.e., $D(w,j) = \sum_{i=1}^{n} w_i d_{ij}$, $D(i,\lambda) = \sum_{j=1}^{n} d_{ij}\lambda_j$. Thus, $D(w,\lambda) = \sum_{j=1}^{n} D(w,j)\lambda_j = \sum_{i=1}^{n} w_i D(i,\lambda) = \sum_{i,j=1}^{n} w_i d_{ij} \lambda_j$. As before, we let $S(w) = \{1 \leqq i \leqq n: w_i > 0\}$ denote the support of w. Similarly, we let $S(\lambda)$ denote the support of λ. Consider, then, the maximin problem

(P) $$\max_{w} \min_{j \in S(w)} D(w,j).$$

Note that $v(w) = \min_{j \varepsilon S(w)} D(w,j)$ is piecewise linear, but discontinuous. This function is upper-semi-continuous, though, and therefore attains its maximum over the (compact) n-dimensional simplex, $\{w \varepsilon R^n\colon w_i \geqq 0$ for all i and $\sum_{i=1}^{n} w_i = 1\}$. Our main result, regarding (P) is

<u>THEOREM 1</u>: <u>Let</u> w^* <u>solve</u> (P) <u>with</u> <u>value</u> v^*, <u>then</u>

(i) $D(w^*,j) \leqq v^*$ <u>for</u> <u>all</u> $j = 1, 2, \ldots, n$

<u>and</u> (ii) $D(w^*,j) = v^*$ <u>if</u> $w_j^* > 0$.

At first, it might appear as though a simple perturbation argument, involving shift of weights (probabilities) from points $j \varepsilon S(w)$ with low $D(w,j)$ into points $j \varepsilon \{1, 2, \ldots, n\}$ with high $D(w,j)$, will suffice to prove the proposition. Consider, however, the simple example in Figure 1, where $D(w,4) > D(w,1) = D(w,2) = D(w,3)$ and $\{1, 2, 3\} \subset S(w)$. Obviously, w does not satisfy the conditions of the proposition. Note, however, from the geometry of Figure 1, that any single shift of weight from 1, 2 or 3 into 4 will decrease the cost $D(w,j)$ not only at $j = 4$, but

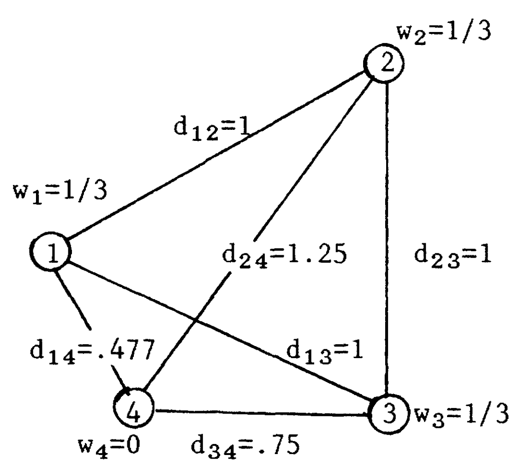

Figure 1: Increasing Cost by shifting weights

at some other points as well. One must consider (as in the following proof) more involved perturbations of w to show that it is not optimal.

Proof (of Theorem 1)

We argue by contradiction. Suppose there are solutions of (P) that violate (i). Among such solutions, let w^* be one with a support of minimum cardinality. Let $S^* = S(w^*) \cup \{1 \leqq j \leqq n: D(w^*,j) > v^*\}$. Consider the free 1-location game in S^*[10]. Its value is equal to v^* since a) $D(w^*,j) \geqq v^*$ for all $j \in S^*$ and b) the existence of w such that $D(w,j) > v^*$ for all $j \in S^*$ contradicts the assumption that v^* is the value of (P). Hence, w^* is an optimal strategy for the maximizer in the modified game[11], and moreover there is a strategy λ^* (with support in S^*) for the minimizer such that $\lambda^*_j = 0$ if $D(w^*,j) > v^*$, i.e., $S(\lambda^*) \subset S(w^*)$ and

$$D(w^*,j) \geqq D(w^*,\lambda^*) = v^* \geqq D(i,\lambda^*) = D(\lambda^*,i) \text{ for all } i,j \in S^* \quad (3)$$

where the right most equality follows from the symmetry of the distances ($d_{ij} = d_{ji}$ for all i,j).

Now if $\lambda^* = w^*$, then (3) implies $D(w^*,j) = v^*$ for all $j \in S^*$ which contradicts the supposition that w^* violates (i).

If $\lambda^* \neq w^*$, then consider $\bar{w} = w^* + \theta(w^*-\lambda^*)$ where $\theta > 0$ is chosen so that $S(\bar{w}) \subset S(w^*) - \{q\}$ for some $q \in S(w^*)$. Such θ and q exist since

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10) Free in the limited sense that the minimizer can choose any point of the given domain (not R^2) independently of the maximizer's support within S^*.

11) To be more precise, the statement holds for the restriction of w^* to S^* rather than for w^* itself. The difference, though, is merely semantic since $S(w^*) \subset S^*$.

$S(\lambda^*) \subset S(w^*)$. By (3) we have, for all $j \varepsilon S^*$, $D(\bar{w},j) = D(w^* + \theta(w^*-\lambda^*),j) = D(w^*,j) + \theta[D(w^*,j) - D(\lambda^*,j)] \geqq D(w^*,j)$. Hence, $\bar{w}$ solves (P) and violates (i) if w^* does. The support of $\bar{w}$, however, is smaller than that of w^* which contradicts our minimality assumption. Consequently, (i) is proven. (ii) is an immediate corollary of (i). □

Corollary 1.1: If w^* solves (P) then

(i) w^* is an optimal strategy for both players in the restricted 1-location game in R.

(ii) w^* also solves

$$(P^*) \quad \max_{w} \min_{j \varepsilon S(w^*)} D(w,j).$$

(iii) Any $\hat{w}$ with $S(\hat{w}) \subset S(w^*)$ such that for some $\hat{v} \varepsilon R$ $D(\hat{w},j) = \hat{v}$ for all $j \varepsilon S(w^*)$, solves (P) as well.

Proof:

(i) That w^* is optimal for the maximizer and that v^* is the value of the game is obvious. Now w^* is feasible for the minimizer in that game, while from Theorem 1 and the symmetry of the distances we have that $D(i,w^*) = D(w^*,i) \leqq v^*$ for all $1 \leqq i \leqq n$. Consequently, w^* is optimal for the minimizer as well.

(ii) Moreover, the maximizer gains nothing if the rules of the game are changed so that the minimizer is unilaterally restricted to $S(w^*)$, while the maximizer is free to use all R. That is, if the minimizer uses w^*, the maximizer cannot do better than v^* and may as well adhere to w^*. Consequently, w^* solves (P^*) as well.

(iii) $\hat{w}$ is feasible for both players in the free 1-location game in $S(w^*)$[10]. Hence, $\hat{v} = v^*$ and $\hat{w}$ solves (P). □

<u>Remark 1</u>.

Note that the proofs of Theorem 1 and Corollary 1.1 used only the symmetry property of the distances (i.e. $d_{ij} = d_{ji}$ for all i,j). The d_{ij}'s need not satisfy the triangle inequality and may even be negative. Moreover, the diagonal elements d_{ii} need not be zero. Consequently, <u>Theorem 1 and Corollary 1.1 hold for all real symmetric matrices</u> $\{d_{ij}\}_{i,j=1}^{n}$. We may apply these results, then, to similar location games defined on general undirected weighted graphs. To point out a possible area of application of these results in problems without geometric structure, consider the following reliability design problem.

There are n candidate locations (nodes) among which some commodity (or activity) is to be distributed. Let w_i be the fraction of the commodity stored at location i. In the case that there is only one indivisible unit of the commodity, w_i may represent the fraction of time or the probability that it is stored at i. If $w_i > 0$, we say that i is "active".

Suppose that one of the active locations might be subject to a catastrophic event (an accident or a hostile attack). Let d_{ij} be the probability of survival at i when a catastrophe occurs at j, and assume that $d_{ij} = d_{ji}$. In this setting problem (P) may be interpreted as follows: How should the commodity be distributed if we want to maximize the expected fraction that survives (or in the indivisible case, the probability of survival)?

As an example one might consider the distribution of explosives between close storage locations, when a spontaneous explosion in one "active" location may trigger explosions in other locations. The probability of more than one spontaneous explosion is assumed to be negligible, and the probability of more than one induced explosion is also assumed negligible (for example if the cross survival probabilities d_{ij} for $i \neq j$, are close enough to 1). □

Remark 2.

Whether or not the necessary conditions provided by Theorem 1 are also sufficient remains an open question, at least for Euclidean distances. For general d_{ij}'s, however, they are not sufficient, as the following counterexample illustrates:

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} = \begin{bmatrix} 0 & 10 & 1 & 1 \\ 10 & 0 & 1 & 1 \\ 1 & 1 & 0 & 20 \\ 1 & 1 & 20 & 0 \end{bmatrix}$$

Note that for $w = (\frac{1}{2}, \frac{1}{2}, 0, 0)$, we have

$$D(w,j) = \begin{cases} 5 & \text{for } j \in S(w) = \{1,2\} \\ 1 & \text{for } j \notin S(w) \end{cases}$$

while for $w = (0, 0, \frac{1}{2}, \frac{1}{2})$, we have

$$D(w,j) = \begin{cases} 10 & \text{for } j \in S(w) = \{3,4\} \\ 1 & \text{for } j \notin S(w). \end{cases}$$

Thus, both candidate solutions satisfy the conditions of Theorem 1, but only the second is optimal. □

5. Restricted Single median Problems

We may extend the results and derive further properties of w^* to general (not necessarily finite) compact sets R, using the convexity of the distance function.

THEOREM 2: *Let* v^* *be the value of the restricted 1-location game and let* w^* *be an optimal strategy for the maximizer. Then*

(i) w^* *is concentrated on the extreme points of* R, *i.e.*, $S(w^*) \subset R^*$

(ii) $D(w^*,c) \leqq v^*$ *for all* $c \in R$

(iii) w^* *is optimal for the minimizer as well.*

Proof:

Note first that if the result holds for the convex hull of R, then it will hold for R. Assume, then, without loss of generality that R is convex, and for simplicity let R have only a finite number of extreme points, i.e., R is a ℓ-gon.

The set $T = \{c \in R^2: D(w^*,c) < v^*\}$ is convex since $D(w^*,c)$ is convex in c. The (nonconvex) compact set R - T has finitely many (at most 3ℓ) extreme points consisting of the vertices of R that are in the exterior of T and of the (at most 2ℓ) points of $\partial R \cap \partial T$ (where ∂T and ∂R, respectively are the boundaries of T and R).

As in the proof of Theorem 1, we know that w^* is also optimal in the free[10] 1-location game in R - T. Let λ^* be the minimizer's strategy in this game. Since $D(x,\lambda^*)$ is convex in x, it assumes its maximum over R -T at extreme points of R - T. Moreover, we claim that unless R - T is on a line segment (in which case it is easy to show that w^* should be evenly distributed between the two extreme points), then $D(x,\lambda^*)$ assumes its maximum in R - T only at extreme points. Suppose not; then $D(x,\lambda^*)$ should be constant ($=v^*$) on the line segment between some pair of the extreme points at which $D(x,\lambda^*)$ is maximized. It is possible to show that this result is possible only if λ^* is evenly distributed between these two extreme points[12]. By the assumption that R - T is not on a line segment, it has another extreme point, and by the triangle inequality, $D(x,\lambda^*)$ is larger on that point than on the pair of the supposedly maximizing extreme points, which is a contradiction. We may conclude, then, that the support of w^*, $S(w^*)$ is contained in the finite set $(R - T)^*$ of extreme points of R - T.

W^* is clearly optimal, then, also for the restricted 1-location game in $(R - T)^*$, and by Theorem 1, we have $D(w^*,c) \leqq v^*$ for all $c \in (R - T)^*$. Together with convexity of $D(w^*,c)$, this fact implies that $D(w^*,c) = v^*$ for all $c \in R - T$. Recalling that, by definition, $D(w^*,c) < v^*$ for all $c \in T$, we may conclude that $D(w^*,c) \leqq v^*$ for all $c \in R$, which is the result (ii). No point of R lies, then, in the exterior of T, and since both R and T are convex $(R - T)^* \subset R^*$[13] and thus $S(w^*) \subset R^*$, which is the result (i). (iii) follows from (ii) exactly as (i) of Corollary 1.1 follows from Theorem 1.

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12) The strictness of the triangle inequality for non colinear vectors implies that if $D(x,\lambda^*)$ is constant on a line segment, then the support of λ^* must lie on the line containing that segment. It is also possible to show that for any point in that segment, the mass of λ^* strictly to its right (left) is no more than ½ because otherwise $D(x,\lambda^*)$ will decrease when moving to the right (left). Hence, λ^* must have mass ½ at the two end points of the segment.

13) By (ii), $R \subset T \cup \partial T$ and thus $R - T \subset \partial T$. Since T is convex, R - T consists of vertices of R and/or of complete edges of R. Consequently, $(R - T)^* \subset R^*$.

The technicalities involved in extending the proof for the case when R has infinitely many extreme points are beyond the scope of the current discussion.[14] □

In general, the support of the worst possible demand (= best maximizer's strategy), w^*, need not necessarily include all of the extreme points of R as is illustrated by the simple counterexample depicted in Figure 2.

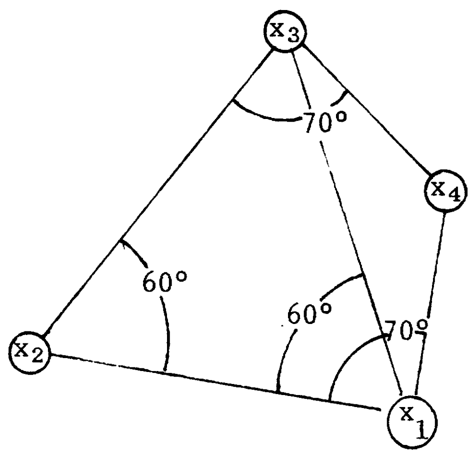

Figure 2: An example where $S(w^*) \neq R^*$

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14) To outline: for every m there is a polygon $R_m \subset R$, with $R_m^* \subset R^*$, such that $\max_{x \varepsilon R} \min_{y \varepsilon R} ||x-y|| \leqq 1/2m$. Let v_m^* be the value of the restricted 1-location game on R_m; then $v^* - 1/m \leqq v_m^* \leqq v^*$, i.e., $v_m^* \to v^*$ as $m \to \infty$. Furthermore, if w_m^* is the associated strategy (which is concentrated on $R_m^* \subset R^*$), then by the weak* compactness of the set of regular Borel probability measures on R^* (or R) in the dual of the space of continuous functions over R^* (or R), and by the continuity of $d(x,i)$ in x, there is a w^* with support in R^* satisfying $D(w^*,c) = \lim_{m\to\infty} D(w_m^*,c) \leqq \lim_{m\to\infty} v_m^* = v^*$ for all $c \varepsilon R$. By additional arguments, one can show that $D(w^*,c) = v^*$ for all $c \varepsilon S(w^*)$ and thus that it is optimal.

Some straightforward though tedious arguments show that the worst possible demand on R of this example is evenly distributed among the 3 extreme points x_1, x_2, x_3 (i.e., $w_1^* = w_2^* = w_3^* = 1/3$); the fourth extreme point x_4 does not belong to $S(w^*)$.

The following result specifies a class of cases for which the support $S(w^*)$ of the worst possible demand/best maximizer strategy includes all of the extreme points of R.

THEOREM 3: *If R^*, the set of extreme points of R, lies on the boundary of some circle then $S(w^*) = R^*$. Moreover, the worst possible demand, w^*, is unique.*

Proof:

Without loss of generality, assume that $R^* = \{x_1, x_2, \ldots, x_\ell\}$. The assertion is trivially true for $\ell \leqq 2$, so we consider only the cases where $\ell \geqq 3$.

From Theorem 2, we have $S(w^*) \subset R^*$. Assume now that $S(w^*) \neq R^*$. Without loss of generality, assume that $x_3 \notin S(w^*)$ and that x_1 and x_2 are the two points of $S(w^*)$ that are next to x_3 on the circumference of the circle (Obviously, there are at least 2 points in $S(w^*)$), as is depicted in Figure 3.

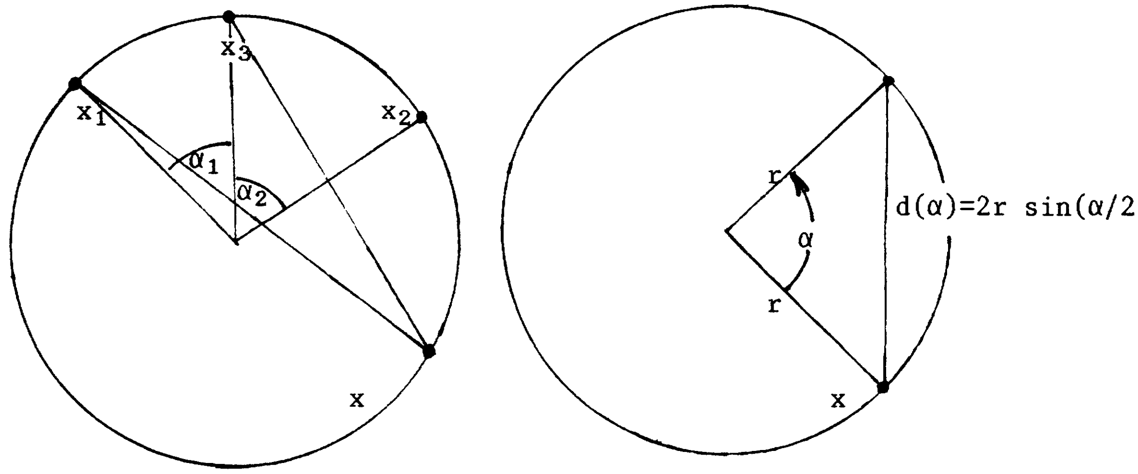

Figure 3: Extreme points on a boundary of a circle

By our choice of x_1 and x_2, $S(w^*)$ lies entirely within the circular arc between x_1 and x_2 that does not include x_3. Now for any point c on that arc (and therefore also for any point $c \in S(w^*)$), we have

$$||c - x_3|| > \frac{\alpha_1}{\alpha_1 + \alpha_2} ||c - x_1|| + \frac{\alpha_2}{\alpha_1 + \alpha_2} ||c - x_2|| \qquad (4)$$

where α_1 and α_2 are the circular angles corresponding to the arcs between x_3 and x_1, and between x_3 and x_2 as shown in Figure 3. This inequality follows from the fact that the length of a chord $d(\alpha)$ is a strictly concave function of the circular angle α corresponding to it ($d(\alpha) = 2r \sin \alpha/2$ is strictly concave for $0 \leqq \alpha \leqq 2\pi$).

Now set $\delta = \min (w^*(\{x_1\}), w^*(\{x_2\}))$ and consider the demand $\bar{w}$ defined by

$$\bar{w}(\{x_j\}) = \begin{cases} \delta & \text{if } j = 3 \\ w^*(\{x_j\}) - \dfrac{\alpha_j}{\alpha_1 + \alpha_2}\,\delta & \text{if } j = 1,2 \\ w^*(\{x_j\}) & \text{otherwise.} \end{cases}$$

The inequality (4) implies that $D(\bar{w},c) > D(w^*,c)$ for all $c \in S(w^*)$, which contradicts part (ii) of Corollary 1.1. Consequently, the assumption $S(w^*) \neq R^*$ is false, and $S(w^*) = R^*$ is proven.

Assume now that there are two worst possible demands w^* and $\bar{w}$. From the previous discussion, we know that $S(w^*) = S(\bar{w}) = R^*$. Thus, $D(w^*,c) = D(\bar{w},c) = v^*$ for all $c \in R^*$. Clearly, there is a $\theta > 0$ such that $\hat{w} = w^* + \theta(w^* - \bar{w})$ is a demand satisfying $S(\hat{w}) \subsetneq R^*$. And, by the linearity of $D(w,c)$ with respect to w, we have $D(w,c) + v^*$ for all c (w^*). But this conclusion is impossible by the previous result. Consequently, w^* is unique. □

Remark 3

An interesting consequence of Theorem 3 and of part (iii) of Corollary 1.1 is

The matrix of distances $D = \{d_{ij}\}_{i,j=1}^{n}$ $d_{ij} = ||x_i - x_j||$ generated by n points located on a circle is nonsingular. [Furthermore, $D^{-1}e$ (where $e = \text{col}(1, 1, \ldots, 1)$) is positive and $e'D^{-1}e = \sum_{i,j=1}^{n} d_{ij}^{(-1)}$ (where $D^{-1} = \{d_{ij}^{(-1)}\}_{i,j=1}^{n}$) as a function of the set of points is monotonically decreasing (with respect to inclusion) on the subsets of the boundary of a circle.]

This result follows from the fact that for any vector $u \in R^n$, there exist $\gamma, \delta \in R$ and $z \in R^n$ such that $u = \gamma w^* + \delta z$. Thus, $Du = \gamma D w^* + \delta Dz = \gamma v^* e + \delta Dz$. Assume now that $Du = 0$; then $Dz = (-\frac{\gamma v^*}{\delta})e$. The condition $|-\frac{\gamma v^*}{\delta}| > v^*$ is impossible because then by slightly perturbing w^* (which is strictly positive) in the z or -z direction, we can increase the (optimal) cost beyond v^* which is assumed maximum already. The condition $|-\frac{\gamma v^*}{\delta}| = v^*$ is impossible as well because then such a perturbation yields non-uniqueness in the maximizing w. Finally, the condition $|-\frac{\gamma v^*}{\delta}| < v^*$ is impossible since then the new w will violate part (iii) of Corollary 1.1. Therefore, $Du \neq 0$ for all $u \in R^n$, and hence D^{-1} in nonsingular. (The rest of this result follows from the observation that $w^* = v^* D^{-1} e$ and that $\frac{1}{v^*} = e'D^{-1} e = \sum_{i,j=1}^{n} d_{ij}^{(-1)}$). □

An intuitively appealing consequence is

COROLLARY 3.1: (i) *If R is a regular n-gon, then w* is evenly distributed among the vertices of R.* (ii) *If R is a circle, then w* is uniformly distributed on its circumference.*

As a consequence, we obtain the following:

COROLLARY 3.2: $\overline{V}_1(R) \leq \frac{4}{\pi} r(R)$, *where r(R) is the radius of the (smallest) circle containing R.*

Proof: For a unit demand uniformly distributed over the circumference of the bounding circle, we get $\overline{D}_1(w) = \frac{1}{\pi} \int_0^{\pi} 2r \sin \alpha \, d\alpha = \frac{4}{\pi} r$ □

This result should be compared to the proposition in Section 3 which showed that $V_1(R) = r(R)$.

6. Multiple Medians $(K \to \infty)$

One might expect the characterization of the maximizer's strategy (worst case demand distribution) for the 1-location games (1-median problems) to be generalizable to $K \geq 2$. We found, however, that this extension is not straightforward. Even the free K-location game, which is trivial for K=1, becomes, in general, quite involved when $K \geq 2$. We will not attempt, then, to evaluate or characterize the saddle point values and strategies for K-location games for any finite $K \geq 2$. Instead, we consider their asymptotic behavior. Surprisingly, the free K-location game that was easy for K = 1, and then hard for $K \geq 2$, becomes tractable again as $K \to \infty$. We are not so fortunate, however, with the restricted K-location game. Before we evaluate (estimate) the asymptotic values and saddlepoints, we introduce the following convergence property which is a simple application of the results in [Hal] (Lemma 3).

THEOREM 4: (i) There are constants η and $\bar{\eta}$ so that for every bounded measurable set R with area $\mu(R) = 1$ with null area boundary (i.e., $\mu(\partial R) = 0$),

$$\lim_{k\to\infty} K^{1/2} V_K(R) = \eta$$

and

$$\lim_{k\to\infty} K^{1/2} \bar{V}_K(R) = \bar{\eta}.$$

Proof: The functionals V_K and $\bar{V}_K$ share the following properties (spelled out here only for V_K):

$$V_{K_1+K_2}(R_1 \cup R_2) \leq V_{K_1}(R_1) + V_{K_2}(R_2)$$

$$V_K(\lambda R) = \lambda V_K(R) \qquad \text{(where } \lambda R \equiv \{\lambda x : x \varepsilon R\}\text{)}$$

$V_K(R+y) = V_K(R)$ (where $R+y \equiv \{x+y : x \varepsilon R\}$)

$K_1 \leq K_2$ implies $V_{K_2}(R) \leq V_{K_1}(R)$

$V_1([0,1]^2) < \infty$.

These properties coincide essentially with properties (P1), (P2), (P4), (P5) and (P6) of $D_K(w)$ as given in Lemma 2 of [Hal], regardless of the fact that V_K is defined on planar subsets while D_K is defined on measures. Note also that if one allowed $w(R) = m\mu(R)$ rather than $w(R) = 1$ in the definition (1) of V_K(or $\overline{V}_K$), we would have an additional property of proportionality with respect to the additional parameter m; this property corresponds to property (P6) in Lemma 2 of [Hal]. Lemmas 3 and 3* of [Hal] that are stated for a uniform measure with density m can be applied here since the situation is completely identical (rather than producing an isomorphic proof). From Lemma 3* of [Hal], then, we obtain the desired result by substituting $m = \frac{1}{\mu(R)}$ (so as to get $w(R) = 1$). □

What are the asymptotic value constants η and $\overline{\eta}$ and what are the associated asymptotic saddlepoint strategies? We have only a partial solution.

<u>THEOREM 5</u>:

(i) [HaM] $\eta = (\frac{1}{3} + \frac{1}{4} \ell n 3)\sqrt{\frac{2}{3\sqrt{3}}}$.

The associated asymptotic saddlepoint strategies[15] are

15) By <u>asymptotic saddlepoint</u> strategy, we mean a fixed strategy that tends to optimality (for MAX or for MIN) as $K \to \infty$. So, this a statement about convergence in the optimality sense and no other. (Although it can be shown to imply here other modes of convergence).

- maximizer: Randomize uniformly over R
- minimizer: Cover R by K cells of a regular hexagonal tesselation so as to minimize the cell areas. Now while keeping the tesselation in place, randomize the position of its center grid so that every center is uniformly distributed in its cell.

(ii) $\frac{2}{3}\sqrt{\frac{2}{3\sqrt{3}}} \leq \bar{\eta} < 2\eta$.

(a) The maximizer strategy associated with the lower bound is to uniformly randomize over a set of 3K points placed in R so as to maximize the minimum distance between any 2 of them (this choice is (asymptotically) equivalent to a grid of centers in a hexagonal tesselation or to 3K "dense packed" points, see Figure 4).

(b) The minimizer strategy associated with the upper bound is to apply the same strategy as in part (i) of the Theorem with the difference that the K points chosen are not the randomized grid points themselves, but those points in the support of the maximizer that are closest to them.

Proof:

(i) see [HaM]

(ii) The lower bound that follows from the strategy described in part (iia) is at least $K^{1/2}\cdot\frac{2}{3}\cdot a$ where a is the distance between adjacent points. This bound is valid because 2K out of 3K (i.e., 2/3) of the maximizer points will be at distance at least a from the K minimizer points. Now since every grid point is adjacent to 6 adjacent equilateral triangles (see Figure 4) with side a and each such triangle is adjacent, of course, to 3 points, we have (neglecting boundary effects) $2\cdot(3K)$ triangles that add up to area 1, i.e., $6K(\frac{1}{2}a\cdot\frac{\sqrt{3}}{2}a) = 1$ implying $a = \sqrt{\frac{2}{3\sqrt{3}}}\,K^{-1/2}$ which when substituted into the expression $K^{1/2}\,\frac{2}{3}\,a$ gives the lower bound.

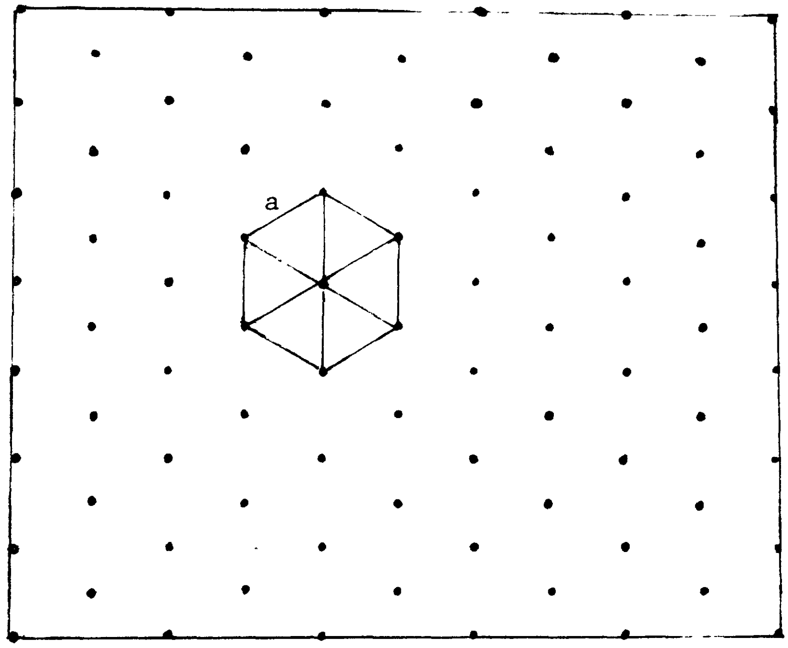

FIGURE 4: 3K "Densed Packed" Grid Points

The upper bound is a corollary of (i) and Lemma 6 to follow. □

LEMMA 6: $\bar{D}_K(w) < 2D_K(w)$ for all w and $K \geq 1$

Proof: We prove this result first for K = 1. Let c* be an optimal non-restricted center, i.e., $D(w,\{c^*\}) = D_1(w)$. Let $\bar{c}^*$ be the point in S(w) that is closest to c^*. Clearly, $D_1(w) \geq |w| \cdot \|\bar{c}^* - c^*\|$, and by the triangle inequality $\|x-\bar{c}^*\| \leq \|x - c^*\| + \|\bar{c}^* - c^*\|$ implying that

$$\bar{D}_1(w) \leq D(w, \{\bar{c}^*\}) \leq D(w,\{c^*\}) + |w|\cdot\| c^* - \bar{c}^*\| \leq 2D_1(w).$$

Now if $\bar{D}_1(w) > D_1(w)$, then $\bar{c}^* \neq c^*$ and there is, therefore, a neighborhood of $\bar{c}^*$ (containing some nonzero demand) where $\|x-\bar{c}^*\| < \|x-c^*\|$, implying that the middle inequality in the last displayed expression is strict.

If K > 1, then $D_K(w) = \sum_{j=1}^{K} D_1(w_j)$

where w_j is the part of the demand served by the j^{th} center in some optimal solution. Consequently, $\bar{D}_K(w) \leq \sum_{j=1}^{K} \bar{D}_1(w) < \sum_{j=1}^{K} 2D_1(w_j) = 2D_K(w)$ □

We conclude with an open problem.

Conjecture: $\bar{\eta} = \frac{2}{3}\sqrt{\frac{2}{3\sqrt{3}}}$

i.e., the lower bound is essentially the asymptotic value and the 3K cluster strategy described is optimal for the maximizer.

The pattern of the points in this conjectured maximal arrangement for the K-median problem is not dissimilar from other conjectured maximal arrangements of points for the Traveling Salesman problem, ([F1], [F2]) , for Steiner's road network problem and for the minimum spanning tree problem (and even for the value ratio of the two last problems [GP],[P],[CH]) which, as this partial reference list indicates, have eluded researchers for some time and to the best of our knowledge are still open.

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