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ASSUMPTIONS FOR A MARKET SHARE THEOREM

by

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ABSTRACT

Many marketing models use variants of the relationship: market share equals marketing effort divided by total marketing effort. Usually, share is defined within a customer group presumed to be reasonably homogeneous and overall share is obtained by weighting for the number in the group. Although the basic relationship can be assumed directly, certain insight is gained by deriving it from more fundamental assumptions as follows: For the given customer group, each competitive seller has a real-valued "attraction" with the following properties: (1) attraction is non-negative; (2) the attraction of a set of sellers is the sum of the attractions of the individual sellers; and (3) if the attractions of two sets of sellers are equal, the sellers have equal market shares in the customer groups.

It is shown that, if the relation between share and attraction satisfies the above assumptions, is a continuous function, and is required to hold for arbitrary values of attraction and sets of sellers, then the relation is: Share equals attraction divided by total attraction. Insofar as various factors can be assembled into an attraction function that satisfies the assumptions of the theorem, the method for calculating share follows directly.

1: INTRODUCTION

Marketing model builders frequently use relationships of the form (us)/(us and them) to express the effects of "us variables on purchase probability and market share. For example, Hlavac and Little [1] hypothesize that the probability a car buyer will purchase his car at a given dealer is the ratio of the dealer's attractiveness (which depends on various dealer characteristics) to the sum of the same quantities over all dealers. Urban [2], in his new product model SPRINTER, makes the sales rate of a brand in a store depend on the ratio of a function of certain brand variables to the sum of such functions across brands. Kuehn and Weiss [3] make use of (us)/(us and them) formulations in a marketing game model, as does Kotler [4] in a market simulation. Mills [5] and Friedman [6] employ models of this form in game-theoretic analyses of competition. Urban [7] and Lambin [8] fit similar models to empirical data, Urban to a product sold in supermarkets and Lambin to a gasoline market.

In all these cases a competitive effect is introduced by a simple normalization. That is, a quantity is defined that relates only to the marketing actions and uncontrolled variables of a specific selling entity. Then, by adding over sellers and using the sum as a denominator, a market share is obtained for each seller. (Time lags, market segmentation or other phenomena may also be introduced to complicate matters, but we focus here on the basic formulation.) The result is a competitive model, since any seller's market share depends on the actions of every other seller.

Normalized attraction models of this type can be postulated directly, but it is of interest to examine them more closely and ask what more basic assumptions can be used to derive them. The purpose of this paper is to examine the mathematics of the situation and argue that under certain conditions such a normalization is required.

2: FORMAL DEVELOPMENT

Consider a finite set, $\mathcal{S}$, of sellers which includes all sellers from whom a given customer group makes its purchases. Suppose that the customer group's inclination toward each seller can be expressed as a real-valued "attraction". Let

$\mathcal{S} = \{s_1, \ldots, s_n\}$ = the set of sellers

$a(S)$ = the attraction of a subset S of sellers

$A = \sum_{i=1}^{n} a(s_i)$ = sum of attractions of all individual sellers

$m(s_i)$ = market share of seller s_i in the customer group

We assume

A1: Attraction is non-negative

$$a(s_i) \geq 0 \qquad s_i \in \mathcal{S}$$

A2: The attraction of a set, S, of sellers is finite and is the sum of the attractions of the individual sellers

$$a(S) = \sum_{s_i \in S} a(s_i) \qquad S \subset \mathcal{S}$$

A3: If two sellers have equal attractions, their market shares are equal

$$a(s_i) = a(s_j) \Rightarrow m(s_i) = m(s_j)$$

A4: Market share, as a function of attraction, is continuous.

Our goal is to find a functional relation between share and attraction that satisfies these assumptions. In doing so, we shall require the relationship to be general in the sense that, if the value of an attraction is increased or a seller added or other arbitrary change made in the system, the same function will hold.

Theorem If a market share is assigned to each seller in such a way that assumptions A1 - A4 are satisfied and such that the function relating attraction to share holds for arbitrary values of attraction and sets of sellers, then market share is given by

$$m(s_i) = a(s_i) \Big/ \sum_{j=1}^{n} a(s_j)$$

Proof Consider a particular case with a set of sellers, $\mathcal{S}$, and attraction values $\{a(s_i)\}$. Any market share function for the situation, $m : \mathcal{S} \rightarrow [0, 1]$ must satisfy

(i) $\sum_{i=1}^{n} m(s_i) = 1$

(ii) $a(s_i) = a(s_j) \Rightarrow m(s_i) = m(s_j)$

Consider a candidate class of functions, F, for mapping attraction into share. For $f \in F$, $f : [0,\infty) \rightarrow [0,1]$. First we observe that

$$\sum_{i=1}^{n} f[a(s_i)] = 1 \qquad (2.1)$$

since market shares must add to 1.

Further by A2 and A3

$$f[a(S)] = f[\sum_{s_i \varepsilon S} a(s_i)]$$

and in particular, for $S = \mathscr{S}$,

$$f[a(\mathscr{S})] = f[A] = 1 \qquad (2.2)$$

Suppose that $a(s_k) = 0$

Let $S = \{s_i \mid i \neq k\}$ Then

$$a(S) = \sum_{s_i \varepsilon S} a(s_i) = A$$

so that $f[a(S)] = 1$. However, since shares must add to 1,

$$f[a(S)] + f[a(s_k)] = 1$$

and so $f[0] = 0$ (2.3)

A 1 - 1 correspondence can be set up between f - functions and m - functions in the sense than an m - function can be defined from any f - function and vice-versa as follows:

Given m define

$$\begin{aligned} f[x] &= m(s_i) && \text{if } x = a(s_i) \\ &= x/A && \text{otherwise} \end{aligned}$$

Given f, define

$$m(s_i) = f[a(s_i)]$$

Notice that except for 0 and A, we have not yet defined f on any of the points of interest, i.e., on $a(s_i)$. We are requiring that f satisfy A1 - A3 for arbitrary $\{a(s_i)\}$. Consider the following sequence of cases:

$$({}^A/_2, {}^A/_2, 0, \ldots\ldots, 0)$$

$$({}^A/_3, {}^A/_3, {}^A/_3, 0, \ldots, 0)$$

$$\cdots\cdots\cdots\cdots$$

$$({}^A/_n, {}^A/_n, \ldots\ldots, {}^A/_n)$$

In the k^{th} case A3 and (2.1) require

$$k\, f[{}^A/_k] + (n - k)\, f\,[0] = 1$$

or $$f[{}^A/_k] = {}^1/_k \qquad k = 1, 2, \ldots, n$$

Now consider the case

$$(rA/k, A/k, \ldots\ldots, A/k, 0, \ldots\ldots, 0) \qquad r < k$$

then

$$f[rA/k] + (k - r)\, f\,(A/k) + (n - k + r - 1)\, f\,[0] = 1$$

or $$f[rA/k] = r/k \qquad \forall\ 0 \leq r \leq k \leq n$$

Since this is to hold for any finite n and f is continuous by A4, ${}^{r}/_{k}$ can be replaced by $y \varepsilon [0, 1]$ or, letting $x = yk$

$$f[x] = {}^{x}/_{A} \qquad\qquad x \varepsilon [0, A]$$

In particular

$$m(s_i) = f[a(s_i)] = a(s_i) \,/\, \sum_{i=1}^{n} a(s_i)$$

as was to be shown.

3. DISCUSSION

The key point of the mathematical analysis is that, if a quantity (e.g., "attraction") is declared to be additive and to be related to another which is constrained in its sum (e.g., Σ share = 1), then mathematical consistency pushes the functional relation into the normalized form. In the present case A2 establishes the additivity, A3 establishes a connection between the two quantities, and the constraint arises from the definition of market share. The role of A4 is to avoid the mathematical embarrassment of having the function defined only on the rational numbers or of devising some other circumlocution.

The important point for the model builder is that the simplest type of model, namely, one which focuses on the efforts of a single seller and assumes additivity, is converted immediately into a fully competitive model by the simple device of normalization. In practice, one must be cautious about assuming additivity but usually it is valid over a range and is the first step on the road to more complicated structures.

It is instructive to point out an appealing method that cannot be used to deduce the normalization model. At first glance it appears that, since market share is, by definition, the ratio of sales to total sales, it would be sufficient to assume that sales are proportional to the seller's attraction function. Calculation of share immediately gives the normalization model.

However, this will only be valid in a totally non-competitive market where the marketing activities of one seller do not influence the sales of another. If, for example, the market is of fixed size in total sales, individual sales cannot be linear with the attraction function. Furthermore, sales cannot be independent of competitive attraction.

It should be pointed out that we have not deduced any specific results about market behavior, but rather some mathematical rules of the game. Thus, if someone asserts that attraction = x^2 where x is advertising, and the right relationship is really x^3, the calculation of market shares will be wrong.

4. APPLICATION TO PROBABILITY OF PURCHASE

The statement of the assumptions and results was in terms of market share but the term "probability of purchase" could clearly be substituted without affecting the mathematical development. Notice that the results refer to probability of purchase from a seller given that a purchase will be made. In other words, the sum of the purchase probabilities is presumed to be 1. Obviously, the probability of no purchase can be introduced as an extension of the model, as can be multiple purchases, and so on. We here concentrate on this single aspect.

5. RELATIONSHIP TO PROBABILITY THEORY

Assumptions A1 and A2 are two of the three axioms of finite sample space probability theory. (See, for example, Parzen [9].) The third axiom is that the probability of a certain event is 1. Thus A1 and A2 are assumptions that make a(.) an unnormalized probability function on a set of sellers.

The choice of these assumptions is, of course, deliberate and brings out the close mathematical connection between the attraction theory being presented and probability. Market share, on the other hand, satisfies all the axioms and so, mathematically speaking, is a probability function defined on the set of sellers. The role of A3 is to connect attraction and share,which it does by forcing a normalization that maps the attraction function into an ordinary probability function.

The fact that market share has the mathematical properties of a probability can be helpful in other ways. For example, if several customer groups or market segments are identified, the concept of conditional market share becomes useful.

Let

$$C = \{c_1, \; . \; . \; . \; . \; . \; , c_r\} = \text{a set of } r \text{ customer groups}$$

$$a(s_i|c_j) = \text{attraction of seller groups } s_i \text{ given customer group } j$$

$$A(c_j) = \sum_{i=1}^{n} a(s_i|c_j)$$

$$p(c_j) = \text{proportion of total sales coming from customer group } c_j$$

Then assuming that A1 - A4 hold for each customer group,

$$m(s_i|c_j) = a(s_i|c_j) / \sum_{i=1}^{n} a(s_i|c_j)$$

and, in any case,

$$m(s_i) = \sum_{j=1}^{r} m(s_i|c_j) \; p \; (c_j) \; .$$

By partitioning the population into groups or segments a complex model can be built up from simple elements. Different marketing variables, say, price, promotion, advertising, and distribution, may impinge differently on different segments, which may, in turn, respond differently. The responses would define relative attraction function which would then be assembled as shown above.

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