ANALYZING MULTI-OBJECTIVE LINEAR AND MIXED INTEGER PROGRAMS BY LAGRANGE MULTIPLIERS

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Abstract

A new method for multi-objective optimization of linear and mixed programs based on Lagrange multiplier methods is developed. The method resembles, but is distinct from, objective function weighting and goal programming methods. A subgradient optimization algorithm for selecting the multipliers is presented and analyzed. The method is illustrated by its application to a model for determining the weekly re-distribution of railroad cars from excess supply areas to excess demand areas, and to a model for balancing cost minimization against order completion requirements for a dynamic lot size model.

Key Words: Programming: linear and integer, multiple criteria, relaxation/subgradient.

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Introduction

In this paper, we present a new approach to analyzing multi-objective linear and mixed integer programs based on Lagrangean techniques. The approach resembles classical methods for using non-negative weights to combine multiple objectives into a single objective function. The weights in our construction, however, are Lagrange multipliers whose selection is determined iteratively by reference to targeted values for the objectives. Thus, the Lagrangean approach also resembles goal programming due to the central role played by the target values (goals) in determining the values of the multipliers. The reader is referred to Steuer [1986] for a review of weighting and goal programming methods in multi-objective optimization.

The plan of this paper is as follows. In the next section, we formulate the multi-objective LP model as the LP Existence Problem. We then demonstrate how to convert the LP Existence Problem to an optimization problem by constructing an appropriate Lagrangean function. The dual problem of minimizing the Lagrangean is related to finding a solution to the LP Existence Problem, or proving that there is no feasible solution. In the

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** The approach developed in this paper was suggested by Fred Shepardson.

following section, we provide an economic interpretation of efficient solutions generated by the method. In the section after that, we present a subgradient optimization algorithm for analyzing the LP Existence Problem by optimizing the dual problem. This algorithm provides a sequence of solutions which converge to a solution to the LP Existence Problem, if the Problem has a solution. A summary of the computational methods is then given. In the following section, we extend the analysis to the multi-objective MIP model which is reformulated as the MIP Existence Problem.

The paper continues with two illustrative examples taken from actual planning situations where the methods could be applied. One is the railroad car redistribution problem (Maiwand [1989]) which has been modeled as a multi-objective transportation model. The other application is cost vs. order completion optimization of dynamic lot size problems. This application is an example from a family of production and distribution problems of increasing interest to practitioners. The paper concludes with a brief discussion of areas of future research and applications.

Statement of the Linear Programming Existence Problem and Lagrangean Formulation

We formulate the multi-objective linear programming model as the LP Existence Problem: Does there exist an $x \varepsilon R^n$ satisfying

$$Ax \leq b \tag{1}$$

$$c_k x \geq g_k \quad \text{for } k = 1, 2, \ldots, K \tag{2}$$

$$x \geq 0 \tag{3}$$

In this formulation, the matrix A is $m x n$, each c_k is a $1 x n$ objective function vector, and each g_k is a target value for the kth objective function. We assume for convenience that the set

$$X = \{x \mid Ax \leq b, \ x \geq 0\} \tag{4}$$

is non-empty and bounded. We let x_r for r=1, ..., R denote the extreme points of X . For future reference, we define the Kxn matrix

$$C = \begin{pmatrix} c_1 \\ \bullet \\ \bullet \\ \bullet \\ c_K \end{pmatrix}$$

and the Kx1 vector g with coefficients g_k . We say that the vector g in the LP Existence Problem is <u>attainable</u> if there is an $x \varepsilon X$ such that $Cx \geq g$; otherwise, the vector g is <u>unattainable</u>.

Letting $\pi = (\pi_1, \pi_2, ..., \pi_K) \geq 0$ denote Lagrange multipliers associated with the K objectives, we price out the constraints (2) to form the Lagrangean

$$\begin{aligned} L(\pi) = -\pi g + \text{maximum } \pi Cx \\ \text{Subject to} \quad Ax \leq b \\ x \geq 0 \end{aligned} \tag{5}$$

We let $x(\pi)$ denote an optimal solution to (5). The following definition and result, which is well known and therefore stated without proof, characterizes these solutions.

DEFINITION: The solution $x \varepsilon X$ is <u>efficient</u> (undominated) if there does not exist a $y \varepsilon X$ such that $Cy \geq Cx$ with strict inequality holding for at least one component.

THEOREM 1: Any solution $x(\pi)$ that is optimal in the Lagrangean (5) is efficient if $\pi_k > 0$ for k=1, 2, ..., K .

We say that the solution $x(\pi)$ <u>spans</u> the target vector $Cx(\pi)$; if π has all positive components, $x(\pi)$ is an efficient solution for the Existence Problem with this target vector.

The two possible outcomes for the LP Existence Problem (the Problem is either feasible or infeasible) can be analyzed simultaneously by optimizing the Multi-Objective LP Dual Problem

$$D = \text{minimize } L(\pi)$$

$$\text{Subject to} \qquad \pi \geq 0 \qquad (6)$$

THEOREM 2: If the LP Existence Problem has a feasible solution, $L(\pi) \geq 0$ for all $\pi \geq 0$ and $D = 0$. If the LP Existence Problem has no feasible solution, there exists a $\pi^* \geq 0$ such that $L(\pi^*) < 0$ implying $L(\theta\pi^*) \rightarrow -\infty$ as $\theta \rightarrow +\infty$ and therefore $D = -\infty$.

Proof:

It is easy to show that $L(\pi) \geq 0$ for all $\pi \geq 0$ when there exists an $\hat{x} \geq 0$ satisfying $A\hat{x} \leq b$ and $C\hat{x} \geq g$. For then, we have $\pi(C\hat{x} - g) \geq 0$ for all $\pi \geq 0$ which implies $L(\pi) \geq 0$ since $L(\pi) \geq \pi(C\hat{x} - g)$.

To complete the proof, we consider the phase one LP for evaluating the LP Existence Problem

$$W = \text{minimize} \sum_{k=1}^{K} s_k$$

$$\text{Subject to} \qquad Ax \leq b \qquad (7)$$

$$c_k x + s_k \geq g_k \qquad \text{for } k=1, \ldots, K$$

$$x \geq 0, \; s_k \geq 0 \qquad \text{for } k=1, \ldots, K \; .$$

The linear programming dual to this problem is

$$W = \text{maximize } -\sigma b + \pi g$$

$$-\sigma A + \pi C \leq 0 \tag{8}$$

$$0 \leq \pi_k \leq 1 \qquad \text{for } k=1, \ldots, K$$

$$\sigma \geq 0 \;,\; \pi \geq 0$$

Let x^*, s^*, denote an optimal solution to the primal problem (7) found by the simplex method, and let σ^*, π^* denote an optimal solution to the dual problem (8).

By LP duality, we have

$$W = -\sigma^* b + \pi^* g \tag{9}$$

$$\sigma^* \left(b - Ax^*\right) = 0 \tag{10}$$

$$\left(-\sigma^* A + \pi^* C\right) x^* = 0 \tag{11}$$

Thus,

$$\begin{aligned} W &= -\sigma^* A\, x^* + \pi^* g \\ &= -\pi^* C\, x^* + \pi^* g \\ &= -\pi^* \left(C\, x^* - g\right) \end{aligned}$$

where the first equality follows from (9) and (10), and the second equality from (11). If the LP Existence Problem has a feasible solution, we have $W = 0$ and $L\left(\pi^*\right) = \pi^* \left(C\, x^* - g\right) = 0$. This completes the first part of the proof because π^* must be optimal in (6) and $D = 0$.

If the LP Existence Problem does not have a solution, we have $W > 0$, or

$$\pi^*(Cx^* - g) = -W < 0 \tag{12}$$

Our next step is to show $L(\pi^*) = -W$.

To this end, consider any $\hat{x} \geq 0$ satisfying $A\hat{x} \leq b$. We have

$$(-\sigma^* A + \pi^* C)\hat{x} \leq 0$$

since $-\sigma^* A + \pi^* C \leq 0$, implying

$$\pi^* C\hat{x} \leq \sigma^* A\hat{x} \leq \sigma^* b$$

where the second inequality follows because $\sigma^* \geq 0$. Adding $-\pi^* g$ to both sides of the inequality, we obtain

$$\pi^*(C\hat{x} - g) \leq \sigma^* b - \pi^* g = -W = \pi^*(Cx^* - g) \tag{13}$$

Thus, $L(\pi^*) = \pi^*(Cx^* - g) < 0$. Moreover, (13) implies for any $\theta > 0$ that

$$L(\theta\pi^*) = -\theta W = (\theta\pi^*)(Cx^* - g) \geq \theta\pi^*(C\hat{x} - t) .$$

This establishes the desired result in the case when the LP Existence Problem is infeasible. ■

It is well known and easy to show that L is a piecewise linear convex function that is finite and therefore continuous on R^K. Although L is not everywhere differentiable, a generalization of the gradient exists everywhere. A K-vector τ is called a subgradient of L at π if

$$L(\hat{\pi}) \geq L(\pi) + (\hat{\pi} - \pi)\tau \qquad \text{for all } \hat{\pi}$$

A necessary and sufficient condition for π to be optimal in the Multi-Objective LP Dual Problem is that there exist a subgradient τ of L at π such that

$$\begin{aligned} \tau_k &= 0 \qquad \text{if } \pi_k > 0 \\ \tau_k &\geq 0 \qquad \text{if } \pi_k = 0 \end{aligned} \tag{14}$$

Algorithms for determining an optimal π are based in part on exploiting this condition characterizing optimality. The K-vectors $Cx(\pi) - g$ are the subgradients with which we will be working.

It is clear from the definition of the Lagrangean that $L(\lambda\pi) = \lambda L(\pi)$ for any $\pi \geq 0$ and any $\lambda \geq 0$; that is, L is homogeneous of degree one. Equivalently, for each extreme point $x_r \in X$, the set

$$\{\pi \mid \pi \geq 0 \text{ and } L(\pi) = \pi(Cx_r - t)\}$$

is a cone. The geometry is depicted in Figure 1 where $\tau_r = Cx_r - t$ for $r = 1, 2, 3, 4$.

The implication of this structure to our analysis of the LP Existence Problem is that we could restrict the vectors π to lie on the simplex

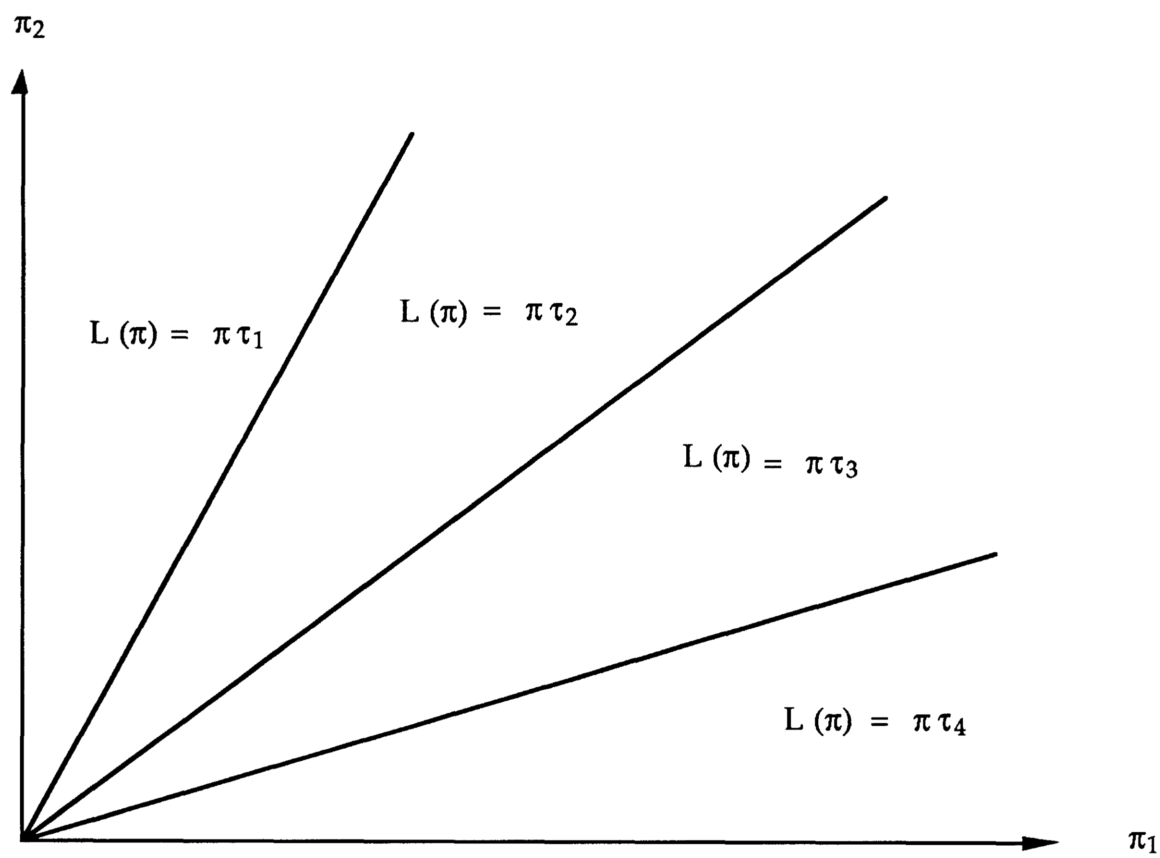
$$S = \left\{ \pi \,\middle|\, \sum_{k=1}^{K} \pi_k = 1, \pi_k \geq 0 \right\}.$$

Theorem 2 can be re-stated as

Corollary 1: Suppose the multipliers π are chosen to lie on the simplex S. If the LP Existence Problem has a feasible solution, $L(\pi) \geq 0$ for all $\pi \in S$ and

$D = 0$. If the LP Existence Problem has no feasible solution, there exists a $\pi^* \in S$ such that $L(\pi^*) < 0$.

For technical reasons, we choose not to explicitly add this constraint. When reporting results, however, we will sometimes normalize the π_k so that $\sum_{k=1}^{K} \pi_k = 1$. The normalization makes it easier for the decision-maker to compare solutions.



Conic Structure of the Lagrangean
Figure 1

Of course, the reader may have asked him(her)self why we need the Lagrangean method when we can test feasibility of the LP Existence Problem simply by solving the phase one linear program (7). The answer is that the decision-maker is usually unsure about the specific target values g_k that he (she) wishes to set as goals. The Lagrangean formulation and the algorithm discussed in the next section allow him (her) to interactively generate efficient solutions (assuming all the $\pi_k > 0$) spanning target values in a neighborhood of given targets g_k if these given targets are attainable. If the targets prove unattainable, undesirable, or simply uninteresting, the decision-maker can adjust them and re-direct the exploration to a different range of efficient solutions.

Economic Interpretation of Efficient Solutions

Frequently, one of the objective functions in the LP Existence Problem, say c_1, refers to money (e.g., maximize net revenues). In such a case, each efficient solution generated by optimizing the Lagrangean function (5) lends itself to an economic interpretation. Consider π^* with $\pi_k^* > 0$ for all k, and let x^* denote an optimal solution to (5). Furthermore, let $g^* = Cx^*$. It is easy to show that x^* is optimal in the linear program

$$\max \quad c_1 x \tag{15.1}$$

$$\text{s. t.} \quad c_k x \geq g_k^* \qquad k = 2, \ldots, K \tag{15.2}$$

$$Ax \leq b \tag{15.3}$$

$$x \geq 0 \tag{15.4}$$

THEOREM 3: The quantities

$$\hat{\pi}_k = \frac{-\pi_k^*}{\pi_1^*} \qquad k = 2, \ldots, K \tag{16}$$

are optimal dual variables for the constraints (15.2) in the linear program (15).

Proof: The proof is similar to the proof of Theorem 1 and is omitted.

Thus, when the objective function c_1 refers to revenue maximization or some other money quantity, each time the Lagrangean is optimized with all $\pi_k^* > 0$, the quantities $\hat{\pi}_k$ for $k = 2, \ldots, K$ given by (16) have the usual LP shadow price interpretation. Namely, the rate of increase of maximal revenue with respect to increasing objective k at the value g_k^* spanned by the efficient solution x^* is approximated by

$$\frac{-\pi_k^*}{\pi_1^*}$$

The quantity is only approximative because the $\hat{\pi}_k$ may not be unique optimal dual variables in the linear program (15) (see Shapiro [1979; pp. 34-38] for further discussion of this point).

<u>Subgradient Optimization Algorithm for Selecting Lagrange Multipliers</u>

Subgradient optimization generates a sequence of K-vectors $\{\pi_w\}$ according to the rule:

1. If $L(\pi_w) < 0$ or π_w and τ_w satisfy the optimality conditions (14), stop. In the former case, the LP Existence Problem is infeasible. In the latter case π_w is optimal in the Multi-Objective LP Dual Problem and $L(\pi_w) = 0 = D$. Otherwise, go to Step 2.

2. For $k = 1, \ldots, K$

$$\pi_{w+1,k} = \text{maximum} \left\{ 0, \pi_{w,k} - \frac{\sigma_w L(\pi_w)}{\|\tau_w\|^2} \tau_{w,k} \right\} \tag{18}$$

where τ_w is any subgradient of L at π_w, $\varepsilon_1 < \sigma_w < 2 - \varepsilon_2$ for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, and $\| \; \|$ denotes Euclidean norm. The subgradient typically chosen in this method is $\tau_w = Cx_w - g$ where x_w is the computed optimal solution to the Lagrangean at π_w.

At each π_w, the algorithm proceeds by taking a step in the direction $-\tau_w$; if the step causes one or more components of π to go negative, rule (18) says set that component to zero. The only specialization of the standard subgradient optimization algorithm is the assumption in the formula (18) that the minimum value $D = 0$, and therefore that the step length is determined by the value $L(\pi_w)$ which is assumed to be positive.

The following theorem characterizes convergence of the algorithm as we have stated it. The proof is a straightforward extension of a result by Polyak [1967] (see also Shapiro [1979]); the reader is referred to those references for the proof.

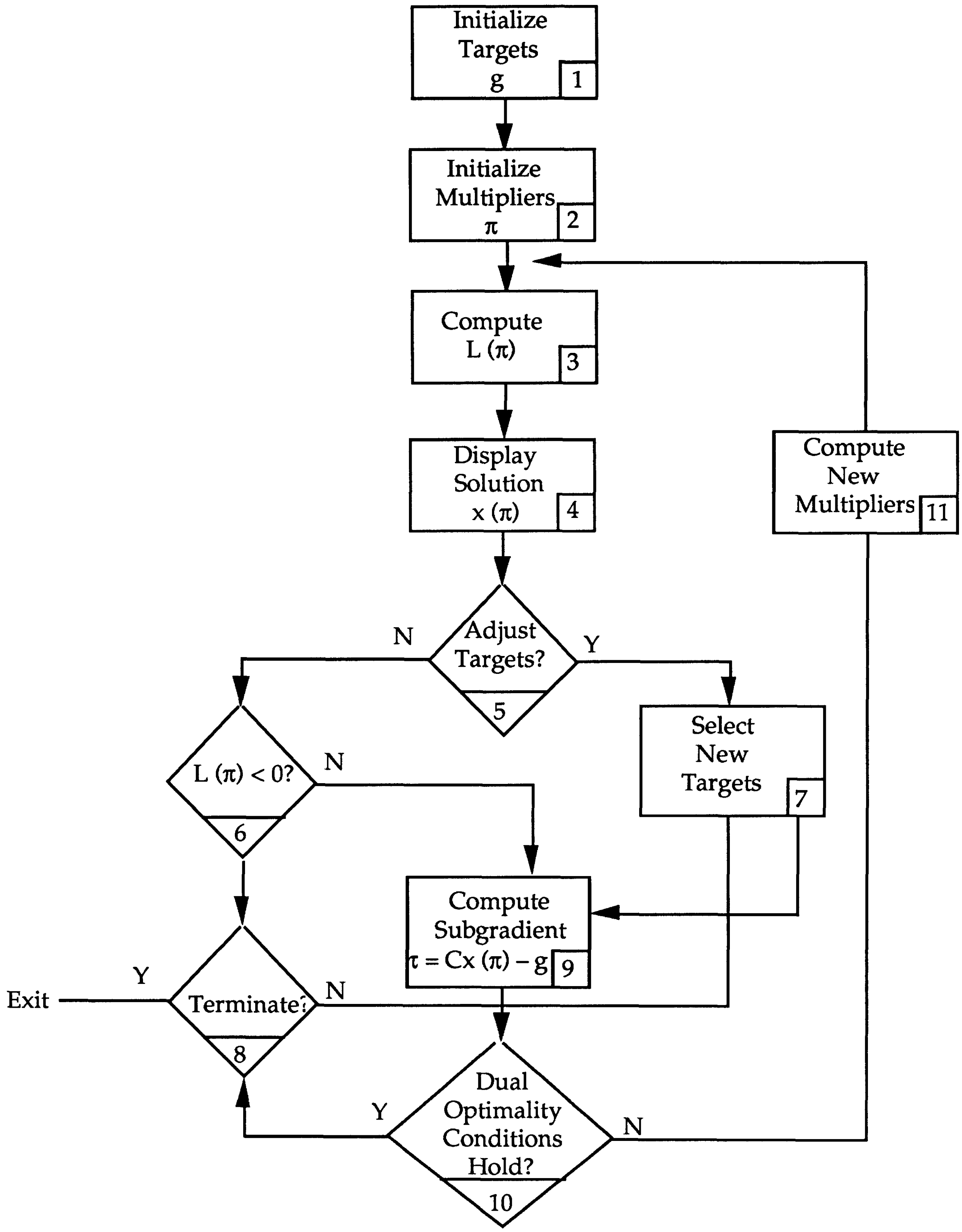
THEOREM 4: If the LP Existence Problem is feasible (g is attainable), the subgradient optimization algorithm will converge to a π^* such that $L(\pi^*) = 0$. If the LP Existence Problem is infeasible (g is unattainable), the algorithm will converge to a π^* such that $L(\pi^*) \leq 0$.

If the LP Existence Problem is infeasible, the subgradient algorithm may not converge finitely. In this case, the algorithm may generates subsequences converging to π^{**} such that $L(\pi^{**}) = 0$, and the algorithm fails to indicate infeasibility. Of course, it is likely that the procedure will terminate finitely by finding π_w such that $L(\pi_w) < 0$. Alternatively, if we knew that the LP Existence problem were infeasible, we could replace the term $L(\pi_w)$ in (18) by $L(\pi_w) + \delta$ for $\delta > 0$ and the algorithm would converge to π^{**} such that $L(\pi^{**}) \leq -\delta$. In effect, this is equivalent to giving the subgradient optimization algorithm a target of $-\delta < 0$ as a value for L. The danger is that if we guess wrong and the Existence Problem is infeasible, then the algorithm will ultimately oscillate and fail to converge.

Another drawback of the subgradient optimization algorithm is that attainment of the optimality conditions (14) at an optimal solution π may require convexifying subgradients computed at that point. The generalized primal dual simplex algorithm is an alternative approach to subgradient optimization for selecting the π vectors that does not suffer from such a deficiency. This issue is discussed again at the end of the section below on the application of the method to a railroad car redistribution problem.

Summary of Computational Methods

A computational scheme based on our constructions thus far are shown in Figure 2. Step 3, Compute $L(\pi)$, is an LP optimization. Except for the first such optimization, an advance starting basis is available for reducing the computation time. Step 4, Display Solution, is a point at which the user can exert control. After viewing the solution, he/she is asked if he/she would like to adjust the targets. If not, and $L(\pi) < 0$, the user is faced with the alternative of exiting with the information that the LP Existence Problem is infeasible (and with a number of efficient solutions), or continuing by adjusting the targets, even though he/she was not previously inclined to do so.



Computational Scheme
Figure 2

On the other hand, if $L(\pi) \geq 0$, the method proceeds (Step 9) by computing a subgradient of L at π based on the optimal solution found in Step 3. Then the optimality conditions (14) are checked (Step 10). If the optimality conditions hold, the user is again faced with the alternative of exiting, this time with the information that the LP Existence Problem is feasible (and with a number of efficient solutions), or continuing by adjusting the targets. If the optimality conditions do not hold, the method proceeds (Step 11) by computing new multipliers based on the formula (18). Then the process is repeated.

Extension to Multi-Objective Mixed Integer Programs

The MIP Existence Problem is: Does there exist x, y, with $x \in R^{n_1}$, $y \in R^{n_2}$ satisfying

$$\begin{aligned} Ax + Qz &\leq b \\ c_k x + f_k z &\leq g_k \qquad \text{for } k = 1, \dots, K \qquad (19) \\ x \geq 0, \quad z_j &= 0 \text{ or } 1 \end{aligned}$$

Let $Z = \{(x, y) \mid Ax + Qz \leq b,\ x \geq 0,\ z_j = 0 \text{ or } 1\}$. For each zero-one $z = \tilde{z}$, we assume the set

$$Z(\tilde{z}) = \{x \mid Ax \leq b - Q\tilde{z},\ x \geq 0\}$$

is bounded. Moreover, for at least one $\tilde{z}$, the set is also non-empty. Let C denote the $k x n_1$ matrix with rows c_k, and F denote the $k x n_2$ matrix with rows f_k.

For $\pi \geq 0$, we define the Lagrangean

$$M(\pi) = -\pi g + \max(\pi C)x + (\pi F)y$$

$$\text{s.t.} \quad Ax + Qz \le b \tag{20}$$

$$x \ge 0, \quad z_j = 0 \text{ or } 1$$

Let $(x(\pi), z(\pi))$ denote an optimal solution to (20). As before, we say the solution $(x, y) \in Z$ is efficient if there does not exist $(\tilde{x}, \tilde{y}) \in Z$ such that $C\tilde{x} + F\tilde{z} \ge Cx + Fz$ with strict inequality holding for at least one component. If $\pi_k > 0$ for $k = 1, \dots, K$, any solution $(x(\pi), z(\pi))$ in (20) is efficient. Unlike the LP case, however, there may be efficient MIP solutions for which no π exists such that they are optimal in (20) (see Bitran [1977]). This is not a serious drawback to our analysis in most cases since we wish merely to systematically sample the set of efficient solutions, rather than generate them all.

The Multi-Objective MIP Dual Problem is

$$E = \text{minimize } M(\pi)$$

$$\text{Subject to} \quad \pi \ge 0 \tag{21}$$

The following theorem, which we state without proof since it draws on well known results about the relationship of a mixed integer program to its Lagrangean relaxation, generalizes Theorem 2. First, we state the MIP Relaxed Existence Problem: Does there exist x, y with $x \in R^{m_1}$, $x \in R^{m_2}$ satisfying

$$Ax + Qy \le b$$

$$c_k x + f_k z \ge g_k \qquad \text{for } k = 1, \dots, K$$

$$x \ge 0, \quad 0 \le z_j \le 1$$

THEOREM 5: If the MIP Existence Problem has a feasible solution, $M(\pi) \geq 0$ for all $\pi \geq 0$ and $E = 0$. If the MIP Existence Problem has no solution but the MIP Relaxed Existence Problem does, $M(\pi) \geq 0$ for all $\pi \geq 0$ and $E = 0$. Finally, if the MIP Existence Problem and the MIP Relaxed Existence Problem both have no feasible solution, there exists a $\pi^* \geq 0$ such that $M(\pi^*) < 0$ implying $M(\theta\pi^*) \to -\infty$ as $\theta \to +\infty$ and therefore $E = -\infty$.

The subgradient optimization algorithm given above for optimizing the Multi-Objective LP Dual Problem can be applied to iteratively select the multipliers for optimizing the Multi-Objective MIP Dual Problem. By Theorem 5, however, we run the risk that $M(\pi) \geq 0$ for all $\pi \geq 0$ but the MIP Existence Problem is infeasible. A resolution of this difficulty is to imbed the entire procedure in a branch-and-bound scheme, an approach that would add another worthwhile dimension of user control to the exploration process.

Railroad Car Redistribution Problem

We illustrate the Lagrangean method for multi-optimization of linear programs with a specific example drawn from the railroad industry (Maiwand [1989]). A railroad company wishes to minimize the cost of relocating its railroad cars for the coming week. Distribution areas 1 through 6 are forecasted to have a surplus (supply exceeds demand) of cars whereas distribution areas 7 through 14 are forecasted to have a deficit (demand exceeds supply). Unit transportation costs are shown in Table 1, surpluses for distribution areas 1 through 6 in Table 2, and deficits for distribution areas 7 through 14 in Table 3. Storage of excess cars at each of the 14 locations are limited to a maximum of 20.

	7	8	9	10	11	12	13	14
1	58	86	150	100	130	110	80	85
2	77	58	62	90	92	114	110	125
3	170	142	112	114	100	97	127	128
4	160	130	72	140	120	145	150	175
5	160	135	55	75	60	75	90	103
6	150	130	141	92	94	70	80	58

Unit Transportation costs c_{ij}

Table 1

Distribution Area	1	2	3	4	5	6
Surpluses	102	85	60	25	78	44

Surpluses S_i

Table 2

Distribution Area	7	8	9	10	11	12	13	14
Deficits	48	31	30	6	27	25	44	39

Deficits D_i

Table 3

Management is also concerned with other objectives for the week's redistribution plan. First, they would like to *minimize the flow on the link from location 2 to location 8* because work is scheduled for the roadbed. Second, they would like to *maximize the flow to locations 7* because they anticipate added demand there.

The relocation problem can be formulated as the following multi-objective linear program.

Indices:

i = 1, ..., 6

j = 7, ..., 14

Decision Variables:

x_{ij} = number of cars to be transported from distribution area i to distribution area j.

E_i = number of excess cars at distribution area i.

E_j = number of excess cars at distribution area j.

Constraints:

$$\sum_{j=7}^{14} x_{ij} + E_i = S_i \qquad \text{for } i = 1, \ldots, 6$$

$$\sum_{i=1}^{6} x_{ij} - E_j = D_j \qquad \text{for } j = 7, \ldots, 14$$

$$0 \leq E_i, E_j \leq 20 \qquad \text{for } i = 1, \ldots, 6;\ j = 7, \ldots, 14.$$

$$x_{ij} \geq 0 \qquad \text{for } i = 1, \ldots, 6;\ j = 7, \ldots, 14.$$

Objective functions:

- Minimize cost

$$Z_1 = \sum_{i=1}^{6} \sum_{j=7}^{14} c_{ij} x_{ij}$$

- Minimize flow on link (2, 8)

$$Z_2 = x_{28}$$

- Maximize flow to distribution area 7

$$Z_3 = \sum_{i=1}^{6} x_{i7}$$

We begin our analysis by optimizing the model with respect to the first objective function. The result is

Objective: Minimize Z_1 (cost)

Solution:

$Z_1^* = 19222$

$Z_2 = 51$

$Z_3 = 48$

This data is used by the decision-maker in setting reasonable targets on the three objectives:

$$\sum_{i=1}^{6} \sum_{j=7}^{14} c_{ij} x_{ij} \leq 20000$$

$$x_{28} \leq 30$$

$$\sum_{i=1}^{7} x_{i7} \geq 58$$

Taking into account that the cost and flow objectives are minimizing ones of the form $c_k x \leq g_k$, we multiply by −1 to put the Existence Problem in standard form. In addition, to enhance computational efficiency and stability, we scale the cost targets and objective function by .001 to make them commensurate with the other two. We now form the Lagrangean as detailed in the previous section and apply the subgradient optimization algorithm.

(Actually, we applied a modified and heuristic version of the algorithm outlined above that ensures π vectors with positive components are generated at each iteration. For further details, see Ramakrishnan [1990]).

The results of nine steps are given in Table 4. Each row corresponds to a solution. The first column gives the value of $L(\pi)$ and the next three columns contain Z_1 (cost), Z_2 (flow on link (2, 8)) and Z_3 (flow to area 7) respectively. The percentage increase over minimal cost (PIC) is also provided to help the decision-maker's evaluation.

No.	$L(\pi)$	Z_1	Z_2	Z_3	π_1	π_2	π_3	PIC
1	12.976	21029	0	68	0.334	0.333	0.333	9.4
2	6.569	21029	0	68	0.516	0.113	0.371	9.4
3	4.175	19556	35	68	0.608	0.001	0.391	1.7
4	7.347	21029	0	68	0.654	0.228	0.118	9.4
5	3.711	21029	0	68	0.757	0.103	0.140	9.4
6	1.878	21029	0	68	0.809	0.040	0.151	9.4
7	1.919	19572	31	68	0.835	0.008	0.157	1.8
8	1.937	21029	0	68	0.887	0.086	0.027	9.4
9	0.979	21029	0	68	0.914	0.053	0.033	9.4
10	0.721	19572	31	68	0.928	0.036	0.036	1.8

RR Car Redistribution Problem

Table 4

The results in Table 4 point out a deficiency of the subgradient optimization algorithm for minimizing the piecewise linear convex function L. Among the ten distinct dual vectors π that were generated during the descent, we find only three distinct efficient solutions to the Existence Problem. This is partially, but not entirely, the result of the small size of the illustrative example.

An alternative descent algorithm for MODP that would largely eliminate this deficiency is one based on a generalized version of the primal-dual simplex algorithm (see Shapiro [1979]). The generalized primal-dual is a local descent algorithm that converges finitely and monotonically to a π^* optimal in MODP. Moreover, it easily allows the constraints

$$\sum_{k=1}^{K} \pi_k = 1$$

to be added to MODP. It has, however, two disadvantages: (1) it is complicated to program; and (2) for MODP's where L has a large or dense number of piecewise linear segments, the algorithm would entail a large number of small steps. Given the intended exploratory nature of the multi-objective proceeding, it appears preferable to use the subgradient optimization algorithm and present only distinct solutions to the decision-maker.

The generalized primal-dual algorithm can be viewed as a constructive procedure for finding a subgradient satisfying the optimality conditions (14) by taking convex combinations of the subgradients derived from extreme points to X. Indeed, we may only be able to meet all our targets by taking such a combination of extreme point solutions. This suggests a heuristic for choosing an appropriate convex combination of the last two solutions in Table 4. For example, if we weight the solution on row 9 by .032 and the solution on row 10 by .968, we obtain a solution satisfying all three targets with the values

$$Z_1 = 19619$$
$$Z_2 = 30$$
$$Z_3 = 68$$

Taking the same convex combination of the multipliers associated with solutions 9 and 10, and applying the result of Theorem 3, we obtain $\hat{\pi}_2 = \$48.30$ as the (approximate) rate of increase of minimal cost with respect to decreasing the flow on link (2, 8) at a flow level of 30, and $\hat{\pi}_3 = \$37.50$ as the (approximate) rate of increase of minimal cost with respect to increasing the flow to distribution area 7 at a delivered flow level of 68.

Cost vs. Order Completion Optimization of Dynamic Lot-Size Problems

The classic dynamic lot-size model is concerned with achieving an optimal balance of set-up costs against inventory carrying costs in the face of non-stationary demand for a single item over a multiple period planning horizon (Wagner and Whitin [1958]). Through the years, this model and numerous extensions have been studied by many researchers. A review is given in Shapiro [1990].

Recognizing that demand is frequently composed of a number of smaller, individual orders, a production manager has the option of adjusting demand, and thereby reducing costs, by deciding which orders to complete in a given period. Such decision options can be explicitly incorporated in the classic models, but require a multi-objective approach to reconcile the opposing objectives of cost and customer service. Models for this type of analysis are timely since measuring the impact of customer service (and quality) on product pricing strategies has recently become a topic of increased managerial interest (CRA [1991], Shycon [1991]).

In this section, we apply our Lagrange multiplier method to a single item dynamic lot-size model to which the order completion decisions are added. Our main purpose is to illustrate the multi-objective approach to a potentially important class of problems and associated models. Research is underway to extend the approach to multi-item, dynamic lot-size problems, and to distribution planning problems.

First, we state the classic model. After that, we discuss its extension to incorporate order completion decisions, and how the Lagrange multiplier method can be used to evaluate them. The section concludes with a numerical example.

Parameters:

f = set-up cost (\$)

h = inventory carrying cost (\$/unit)

r_t = demand in period t (units)

M_t = upper bound on production in period t (units)

$\bar{y}_T$ = lower bound on inventory at the end of the planning horizon

Decision Variables:

y_t = inventory at the end of period t

x_t = production during period t

$$\delta_t = \begin{cases} 1 & \text{if production occurs during period } t \\ 0 & \text{otherwise} \end{cases}$$

CLASSIC DYNAMIC LOT-SIZE MODEL

$$v = \min \sum_{t=1}^{T} \{f\,\delta_t + h\,y_t\} \tag{22}$$

Subject to

$$\left.\begin{aligned} y_t &= y_{t-1} + x_t - r_t \\ x_t - M_t\,\delta_t &\le 0 \end{aligned}\right\} \quad \text{for } t=1, \ldots, T \tag{23}$$

$$y_0 \text{ given, } y_T \ge \bar{y}_T \tag{24}$$

$$y_t \ge 0,\; x_t \ge 0,\; \delta_t = 0 \text{ or } 1 \tag{25}$$

In this formulation, the objective function (22) includes production set-up costs and inventory carrying costs, but not direct production costs since they cannot be avoided. The constraints (23) are the usual inventory balance and fixed charge constraints. Since $y_t \geq 0$, no backlogging is allowed. In (24), note that a lower bound (which may be zero) on inventory at the end of the planning horizon is included. Selecting this lower bound is actually a third objective that could be analyzing by our Lagrange multiplier methods.

Suppose now that demand consists entirely of orders of size w_j for $j = 1, ..., N$, where each order has a promised completion (shipping) date t_j. Suppose further than every order must be 100% complete before it is shipped. Finally, suppose an order is allowed to be completed one or two periods after the due date t_j. Management is concerned with limiting late deliveries, or somewhat equivalently, with penalizing late shipments.

We extend the classic model and its optimization as follows.

Let

$$J_t = \{j \mid t_j = t\}$$

This is the set of orders that the company promised to ship in period t. Our assumption is that

$$r_t = \sum_{j \in J_t} w_j \qquad \text{for all } t$$

Let β_{j0}, β_{j1}, β_{j2} denote zero-one variables that take on values of one only if order j is completed in periods t_j, $t_j + 1$, $t_j + 2$, respectively. Let g_0 and g_1 denote customer service targets for shipping orders on-time, and one period late, respectively. The g_k integers are less than or equal to $|J|$, the total number of orders. We would expect $g_0 \leq g_1$.

The classic model becomes

COST VS. ORDER COMPLETION DYNAMIC LOT-SIZE MODEL

$$v \;=\; \min \sum_{t=1}^{T} \left\{ f\,\delta_t \;+\; h\,y_t \right\} \tag{26}$$

Subject to

$$\left.\begin{aligned} y_t &= y_{t-1} + x_t - \left(\sum_{j \in J_t} w_j \beta_{j0} + \sum_{j \in J_{t-1}} w_j \beta_{j1} + \sum_{j \in J_{t-2}} w_j \beta_{j2} \right) \\ & x_t - M_t \delta_t \le 0 \end{aligned}\right\} \quad \text{for } t = 1, \ldots, T \tag{27}$$

$$\beta_{j0} + \beta_{j1} + \beta_{j2} = 1 \qquad \text{for } j = 1, \ldots, N \tag{28}$$

$$\sum_{l=0}^{k} \sum_{j=1}^{N} \beta_{jl} \ge g_k \qquad \text{for } k = 0, 1 \tag{29}$$

$$y_0 \text{ given}, \quad y_T \ge \bar{y}_T \tag{30}$$

$$y_t \ge 0, \quad x_t \ge 0, \quad \delta_t = 0 \text{ or } 1, \quad \beta_{jk} = 0 \text{ or} \tag{31}$$

In this formulation, we have modified the inventory balance equation in (27) to incorporate the effective demand satisfied. This is the term in parentheses on the right expressing, for each t, orders promised in period t that were shipped on-time, orders promised in period $t-1$ that were shipped one period late, and orders promised in period $t-2$ that were shipped two periods late. The multiple choice constraints (28) determine the timeliness of each order j. The inequalities (29) describe the customer service requirements.

For example, if $|J| = 25$ and $g_0 = 20$ and $g_1 = 23$, the customer service requirement is at least 80% of the orders shipped on-time, and at least 92% of the orders shipped more than one period late.

Clearly, the model just described is one of many possible formulations describing cost vs order completion decisions. One could consider, for example, delaying some orders for more than two weeks. The customer service requirements (29) could be expressed in terms of total quantity shipped, rather than by numbers of orders shipped, since some orders will be much larger (and more important) than others. Or, there may be priority classes of orders, with different customer service requirements for each. Anyone of these generalizations, and many others as well, could be modeled and analyzed in much the same manner.

As we have stated it, the Cost vs. Completion Dynamic Lot-Size Model is not a multi-objective problem. We propose to treat it as such because the customer service levels g_k are somewhat arbitrary. Moreover, we can presume that the production manager would like to investigate the tradeoff between cost and customer service, rather than impose rigid service levels.

To this end, we elect to dualize the target constraints (29). The Lagrangean is

$$M(\pi_0, \pi_1) = \sum_{k=0}^{1} \pi_k g_k$$

$$+ \min \left[\sum_{t=1}^{T} \left\{ f\,\delta_t + h\,y_t \right\} - \sum_{j=1}^{N} \left(\pi_0 + \pi_1 \right) \beta_{j0} - \sum_{j=1}^{N} \pi_1\, \beta_{j1} \right]$$

Subject to $y_t, x_t, \delta_t, \beta_{jk}$ satisfy (27), (28), (30), (31)

The π_0, π_1, are selected iteratively by the subgradient optimization algorithm described above.

We illustrate the approach with a numerical example. Consider production of a single item with $f = 3750$, $h = 7$, $y_0 = 300$. Demand over the next ten periods by promised shipping period for 25 orders is given by Table 5. Inventory at the end of the 10 periods is constrained to be at least 200.

In formulating the model, to avoid end effects, we eliminated the option of being two weeks late for orders 10, 24, and 25 since they fall at the end of the planning horizon. We chose as our targets the quantities $g_0 = 20$, $g_1 = 23$.

Period	Order No. - Size	Total Demand
1	1–35; 11–100	135
2	2–50; 12–40; 13–75	165
3	3–90; 14–125; 15–60	275
4	4–60; 16–30	90
5	5–300; 17–100	400
6	6–25; 18–150; 19–40	215
7	7–75; 20–30	105
8	8–130; 21–50; 22–100	280
9	9–150; 23—35	185
10	10–20; 24–60; 25–80	160

Orders by Periods
Table 5

Table 6 contains the results of 6 Lagrangean optimizations at the indicated multiplier values. Recall that these solutions are optimal for the customer service levels they span. The sequence of values of π_0 and π_1 were selected by taking steps in subgradient directions.

Solution Number	1	2	3	4	5	6
π_0	100	340	460	720	600	780
π_1	100	220	340	580	460	400
Solution Cost	13905	4400	14400	17655	15855	17655
Lagrangean Value	15105	16080	16560	16355	16600	16595

Order No.	Order Qty	1	2	3	4	5	6
1	35						
2	50						
3	90	1 wk					
4	60						
5	300	2 wks	1 wk	1 wk	1 wk	1 wk	1 wk
6	25	1 wk					
7	75						
8	130	2 wks	2 wks	2 wks		1 wk	
9	150	2 wks	2 wks	2 wks	1 wk		1 wk
10	20						
11	100						
12	40						
13	75						
14	125	1 wk					
15	60	1 wk					
16	30		2 wks	2 wks	2 wks	2 wks	2 wks
17	100	2 wks	1 wk	1 wk	1 wk	1 wk	1 wk
18	150	1 wk					
19	40	1 wk					
20	30						
21	50	2 wks	2 wks	2 wks		1 wk	
22	100	2 wks	2 wks	2 wks		1 wk	
23	35	1 wk	1 wk	1 wk			
24	60						
25	80					1 wk	

Lagrangean Analysis
Table 6

Several points are worth noting. The initial values of π_0 and π_1 are insufficient to induce a strategy near the desired customer service levels. By solution 4, however, the values of π_0 and π_1 have increased significantly to yield a strategy that exceeds the prescribed levels. In that strategy, 21 orders are shipped on-time (the target is 20) and 24 orders are shipped no later than one period late (the target is 23). This solution costs $17,655 compared to the cost $22,455 associated with the solution in which all orders must be shipped on their promised dates, a reduction of 21.4%.

At the next iteration, solution 5, the subgradient optimization algorithm selects lower values of π_0 and π_1. This solution violates the first goal by a little, namely 18 orders are shipped at their promised times rather than 20, but satisfies the second goal, namely 24 orders are shipped no later than one period late as opposed to the target of 23. Moreover, the cost of strategy 5 is $15,860., or 10.2% lower than strategy 4.

The analysis is continued for one additional iteration to strategy 6, which is the same as strategy 4. One could argue that strategies 4 and 5 represent two attractive alternatives for the manager. He/She can save 20.4% of his/her avoidable costs by the relative mild slippage in the promised shipping schedule given by strategy 4, or he/she can save an additional 10.2% by selecting strategy 5 which allows the number of orders shipped one period late to increase from 3 to 6.

Finally, we note that maximization of the Lagrangean E appears to occur around 16600. For the purposes of comparison, we formulated and optimized the MIP model (26) to (31), of which maximization $M(\pi)$ is a relaxation, and found its value to be 17760. Thus, the duality gap appears to be on the order of 6%, a level to be expected for this type of MIP model.

Future Directions

We envision several directions of future research and development for the Lagrangean approach to multi-objective optimization developed in this paper. The approach needs testing on live industrial applications. In this regard, the railroad car distribution example presented above is an actual application where we hope the technique will one day be used. The cost vs. order completion optimization example was stimulated by planning problems faced by production managers in the food and forest products industries.

The technique was successfully applied in the construction of a pilot optimization model for allocating budgets to acquire, install, and maintain new systems for the U.S. Navy submarine fleet (Manickas [1988]). For this class of problems, the multiple objectives were various measures of submarine performance with and without specific system upgrades. Unfortunately, the project did not continue beyond the pilot stage to implementation of an interactive system for supporting decision making in this area at the pentagon. The method was also applied to multi-objective optimization of personnel scheduling problems (Shepardson [1990]).

For effective practical use, the models and methods discussed here should be imbedded in decision support systems with graphical user interfaces for displaying efficient solutions and soliciting human interaction. Korhonen and Wallenius [1988] report on the successful implementation of a pc-based interactive system for a multi-objective linear programming model used to manage sewage sludge, and for other applications. Interaction in this system is based on a "Pareto Race" method that allows the decision maker to freely search the efficient frontier by controlling the speed and direction of motion. A reconciliation of the Pareto race method with our Lagrangean and subgradient method is being studied.

Interactive analysis of multi-objective models would allow the underlying preference structure, or utility function, of the decision-maker to be assessed by asking him/her to compare the most recently generated efficient solution with each of the previously generated ones. The information about preferences gleaned from these comparisons could be represented as constraints on the decision vector x (see Zionts and Wallenius [1983] or Ramesh et al [1989]). Alternatively, we could apply the method of cojoint analysis developed by Srinivasan and Shocker [1973] to identify the decision-maker's ideal target vector g from the pairwise preferences.

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