RETURN ON INVESTMENT ANALYSIS FOR FACILITY LOCATION

by

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ABSTRACT: *We consider how the optimal decision can be made if the optimality criterion of maximizing profit changes to that of maximizing return on investment for the general uncapacitated facility location problem. We show that the inherent structure of the proposed model can be exploited to make a significant computational reduction.*

Key words: Return on investment, facility location, fractional programming.

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1. Introduction

Most of the facility location models, especially private sector ones typically have the objective of maximizing the profit. When a model assumes given fixed demands to be satisfied, this objective can be presented as that of minimizing the total cost. In business decision-making environments, however, this decision criterion is not necessarily the best one. In fact, most of financial decisions are based not so much on the absolute value of the profit as on the efficiency of investment. For example, when we select the most desirable investment among several possible alternatives, the main concern is the value of each investment. Therefore, when we consider a business opportunity which involves facility location, we might need to figure out the efficiency of investment for doing that project.

The efficiency of investment is frequently measured by the ratio of the total revenue to the total cost, which is referred to as *return on investment*(ROI). If we stick to the assumption that demands are given fixed and must be satisfied, an ROI maximizing model is the same as a profit maximizing model. However, for a more realistic general setting in which demands are influenced by a decision maker's pricing policy as in Erlenkotter[1] and Hansen and Thisse[3], the optimal decisions for the two criteria do not coincide.

In this paper, we consider how the optimal decision can be made if the criterion of maximizing ROI is used in place of that of maximizing the profit for the above-mentioned general setting of a simple plant location problem(SPLP). We also require that the optimal decision guarantee the minimum required net profit. This is because if we simply maximize ROI, the optimal decision sometimes yields extraordinarily small profit under the circumstances of diminishing marginal return.

The ROI model for the SPLP has a fractional objective function, and thus it is more difficult to solve the model than a profit maximizing version. The former one usually requires solving iteratively the latter ones. Here we show that if we use the inherent structure of the proposed ROI model, we can obtain significant computational reduction when successively solving a profit maximizing model. In Section 2, we formulate an ROI model for a general pricing-location version of the SPLP. The structural properties of the model and their algorithmic implications are discussed in Section 3. Section 4 provides an illustrative example for demonstrating the procedure developed in Section 3. Finally, some concluding remarks are given in the last section.

2. Model Formulation

To formulate the proposed model, we use the following notation: $I = \{1, \cdots, m\}$ is the set of potential sites for facilities; $J = \{1, \cdots, n\}$ is the set of customers; s_{ij} is the amount supplied to customer j from facility i; S_j is the total amount supplied to customer j, i.e., $\sum_{i \in I} s_{ij} = S_j$; $R_j(S_j)$ is the total revenue accrued to customer j from the supplied amount S_j; $\pi_o(> 0)$ is the minimum required profit level; $y_i = 1$ if facility i is established and 0 otherwise; t_{ij} is the nonnegative variable production and transportation cost per unit of customer j's demand supplied from facility i; f_i is the positive fixed cost for establishing facility i; and M is the sufficiently large number. We assume that the revenue functions $R_j(\cdot)$ are concave, differentiable, and bounded above with $R_j(0) = 0$. Those revenue functions are closely related with the demand functions and a pricing policy. For more details, refer to [1,3,4].

The ROI model of the SPLP we consider can be formulated as the following fractional nonlinear mixed 0-1 integer programming problem:

$$z_R = \min \quad \frac{TC}{TR} \tag{1}$$

$$\begin{aligned}
\text{s.t.} \quad & TR - TC \geq \pi_o & & (2)\\
& \sum_{i \in I} s_{ij} = S_j, \qquad j \in J & & (3)\\
& \sum_{j \in J} s_{ij} \leq M y_i, \qquad i \in I & & (4)\\
& s_{ij} \geq 0, \qquad i \in I, j \in J & & (5)\\
& y_i = 0 \text{ or } 1, \qquad i \in I & & (6)
\end{aligned}$$

where

$$\begin{aligned}
TC &= \sum_{i \in I} \sum_{j \in J} t_{ij} s_{ij} + \sum_{i \in I} f_i y_i\\
TR &= \sum_{j \in J} R_j(S_j).
\end{aligned}$$

Consider the following problem:

$$(P(\lambda)) \quad z(\lambda) = \min \quad TC - \lambda TR$$

$$\text{s.t.} \quad (2), (3), (4), (5), \text{ and } (6).$$

As is usual in the fractional programming problem, $z(\lambda)$ is nonincreasing for $\lambda \geq 0$ and the following relation exists between the two problems.

Lemma 1 $z(\lambda^*) = 0$ *if and only if* $\lambda^* = z_R$.

Since we assume π_o to be positive, such λ^* always exists between 0 and 1 as far as the problem has a feasible solution. From now on, we assume that there exists a feasible solution for the problem. Even if not, the forthcoming proposed solution procedure could check the problem feasibility at an early stage of its solution process without any extra efforts.

3. Solution Method

In this section, we develop a procedure to derive λ^* with $z(\lambda^*) = 0$. We first present the outline of our solution method, and show some structural properties of $(P(\lambda))$ and its relaxation, and the algorithmic implications of those properties.

3.1. Outline of the solution method

The main steps of our procedure follow a usual scheme adopted in fractional programming. The outline of our solution procedure is as follows:

Initialization. Set $t = 0$ and $\lambda^0 = 1$.

Iterative Step. Solve $(P(\lambda^t))$. Set y^t as the optimal y-vector of $(P(\lambda^t))$.

Termination. If $z(\lambda^t) = 0$, stop. Otherwise, increase t by 1, calculate λ^t at which the value of $z(\lambda^t)$ with y-vector fixed at y^{t-1} becomes zero, and go to the next iterative step.

Note that $z(\lambda)$ is nonincreasing and λ^* exists between 0 and 1. If the problem is feasible, $z(\lambda^0) \leq -\pi_o$. As already mentioned, even if the original problem is infeasible, we could check it at the very first iterative step of our solution process. The process of calculating the succeeding λ^t is simple and easy, which will be clarified later on.

Our basic strategy for solving $(P(\lambda^t))$ is to use a Lagrangean relaxation of $(P(\lambda))$ which dualizes constraint (2). Let u be the corresponding Lagrangean multiplier, then the relaxed problem for the given u is as follows:

$$(LR_\lambda(u)) \quad z^{\lambda}_{LR}(u) = \min \quad u\pi_o + (u+1)\{TC - \frac{u+\lambda}{u+1}TR\}$$

$$\text{s.t.} \quad (3), (4), (5), \text{ and } (6).$$

And to obtain a good lower bound, we need to solve the following Lagrangean dual problem:

$$(LD_\lambda) \quad z_D(\lambda) = \max_{u \geq 0} z_{LR}^{\lambda}(u).$$

$(P(\lambda))$ is quite similar to Erlenkotter's quasi-public model[1]. He also used a Lagrangean relaxation by dualizing constraint (2) for his problem. We can simply take his method for solving $(P(\lambda))$. However, although his algorithm works well, it still requires fairly large computation since $(P(\lambda))$ itself is a very difficult problem. In fact, in order to solve it, we must solve a number of SPLP's to obtain an optimal Lagrangean multiplier, and sometimes carry out even a branch and bound process when a duality gap exists. Moreover, we need to solve $(P(\lambda))$ for a succession of λ values. Thus a cleverer scheme is strongly desired to make the problem more amenable.

3.2. Structural analysis

Here we probe for the special structure of the problem to exploit when solving $(P(\lambda))$ for a succession of λ values. Consider the following problem which plays the role of a basic module in our solution procedure.

$$(P_1(k)) \quad z_1(k) = \min \quad TC - kTR \tag{7}$$

$$\text{s.t.} \quad (3), (4), (5), \text{ and } (6).$$

This problem is the same as $(P(\lambda))$ without constraint (2). Moreover, to solve $(LR_\lambda(u))$ is just to solve $(P_1(k))$ with $k = \frac{u+\lambda}{u+1}$.

As is shown in [1], $(P_1(k))$ can be easily transformed to the following equivalent SPLP:

$$
\begin{array}{lll}
\min & \sum_{i\in I}\sum_{j\in J}\left[t_{ij}D_{ij}^k - kR_j(D_{ij}^k)\right]q_{ij} + \sum_{i\in I} f_i y_i & \\
\text{s.t.} & \sum_{i\in I} q_{ij} \le 1, & j \in J \\
& q_{ij} \le y_i, & i \in I, j \in J \\
& q_{ij} \ge 0, & i \in I, j \in J \\
& y_i = 0 \text{ or } 1, & i \in I
\end{array}
$$

where parameters D_{ij}^k are determined as follows:

$$
\begin{array}{ll}
D_{ij}^k = 0, & \text{if } dR_j(0)/dS_j \le t_{ij}/k, \\
dR_j(D_{ij}^k)/dS_j = t_{ij}/k, & \text{otherwise.}
\end{array}
$$

For more details about the transformation procedure, refer to [1]. The advantage of this transformation is in that we can solve it easily using an existing code for the SPLP such as Erlenkotter's DUALOC [2].

Now we shall derive the two main theorems on which our solution method is based. For that we first need the following lemma. Let x denote a vector which consists of two kinds of elements, s_{ij} and y_i, and also let $SL(x) = \pi + TC - TR$. Then the following holds.

Lemma 2 *For any k_1 and k_2 such that $0 \le k_1 < k_2 \le 1$, let x_1 and x_2 be optimal solutions of $P_1(k_1)$ and $P_1(k_2)$, respectively. Then $SL(x_1) \ge SL(x_2)$.*

Proof. Suppose that $SL(x_1) < SL(x_2)$. From the optimality assumption of x_1 and x_2,

$$
\begin{array}{ll}
k_1\{TR(x_2) - TR(x_1)\} & \le TC(x_2) - TC(x_1), \qquad (8) \\
k_2\{TR(x_2) - TR(x_1)\} & \ge TC(x_2) - TC(x_1). \qquad (9)
\end{array}
$$

And from the assumption that $SL(x_1) < SL(x_2)$,

$$
TR(x_2) - TR(x_1) < TC(x_2) - TC(x_1). \qquad (10)
$$

First, if $TR(x_2) - TR(x_1) > 0$, then $k_2 > 1$ from (9) and (10), and this fact contradicts the assumption of the lemma. Second, if $TR(x_2) - TR(x_1) = 0$, then (8), (9), and (10) are not consistent. Finally, if $TR(x_2) - TR(x_1) < 0$, then $k_2 \leq k_1$ from (8) and (9), which also contradicts the assumption of the lemma. $\square$

Likewise we have the following corollary.

Corollary 1 *For any λ_1 and λ_2 such that $0 \leq \lambda_1 < \lambda_2 < 1$, let x_1 and x_2 be optimal solutions of $P(\lambda_1)$ and $P(\lambda_2)$, respectively. Then $SL(x_1) \geq SL(x_2)$.*

Proof. Note that the feasible region of $(P_1(k))$ has not been referred to when proving Lemma 2. So the proof of Lemma 2 is still effective for this corollary. $\square$

We are now in a position to state the following two important facts.

Theorem 1 *If an optimal solution of $(P(\widehat{\lambda}))$ for some $0 < \widehat{\lambda} < 1$ satisfies (2) as an equality, then it is also optimal for all $(P(\lambda))$ with $0 \leq \lambda < \widehat{\lambda}$.*

Proof. Let $\widehat{x}$ be an optimal solution of $P(\widehat{\lambda})$ satisfying (2) as an equality, i.e., $SL(\widehat{x}) = 0$. Suppose $\widehat{x}$ is not optimal for $P(\widetilde{\lambda})$ with $0 \leq \widetilde{\lambda} < \widehat{\lambda}$. Then there exists $\widetilde{x}$ satisfying

$$\widetilde{\lambda}(TR(\widehat{x}) - TR(\widetilde{x})) < TC(\widehat{x}) - TC(\widetilde{x}). \qquad (11)$$

And also from Corollary 1, $SL(\widetilde{x}) = 0$. From the fact that $SL(\widehat{x}) = SL(\widetilde{x}) = 0$, $TR(\widehat{x}) - TR(\widetilde{x}) = TC(\widehat{x}) - TC(\widetilde{x})$. In addition, from (11) and the fact that $\widetilde{\lambda} < 1$, $TR(\widehat{x}) - TR(\widetilde{x}) > 0$. Therefore, since $0 \leq \widetilde{\lambda} < \widehat{\lambda} < 1$,

$$\widehat{\lambda}(TR(\widehat{x}) - TR(\widetilde{x})) < TC(\widehat{x}) - TC(\widetilde{x}).$$

This contradicts the assumption that $\hat{x}$ is optimal for $\hat{\lambda}$. $\square$

Corollary 2 *If there exists an optimal solution for $(P_1(k'))$ satisfying $SL = 0$ for some $0 \leq k' < 1$, then it is also optimal for $(P(\lambda))$ with $0 \leq \lambda \leq k'$.*

From Lemma 2, we also obtain the following relation between λ and the corresponding optimal Lagrangean multiplier of (LD_λ), denoted by $u^*(\lambda)$.

Theorem 2 *Suppose $(P_1(k))$ doesn't have an optimal solution with $SL = 0$ for $0 \leq k \leq 1$. Let k^* be defined such that $(P_1(k^*))$ has at least two optimal solutions, one with $SL > 0$ and the other with $SL < 0$. Then such k^* uniquely exists between 0 and 1. Moreover, the following holds:*

$$u^*(\lambda) = \begin{cases} 0, & \text{if } k^* \leq \lambda \leq 1, \\ \dfrac{k^* - \lambda}{1 - k^*}, & \text{if } 0 \leq \lambda < k^*. \end{cases}$$

Proof. Let $X(k)$ be the set of all the optimal vectors of $(P_1(k))$. And also let $SL_k^l = \min_{x \in X(k)} SL(x)$ and $SL_k^u = \max_{x \in X(k)} SL(x)$. Since $X(0)$ contains only the null vector, $SL_0^l > 0$. And from the feasibility assumption of $(P(\lambda))$, $SL_1^u < 0$. Therefore, if k^* exists, then $0 < k^* < 1$ and k^* is unique by Lemma 2. Suppose that k^* doesn't exist. Then there must exist $0 < k' < 1$ such that $SL_{k'}^u < 0$ and $SL_{k'-\epsilon}^l > 0$ for any $\epsilon > 0$. This contradicts the fact that $z_1(k)$ is continuous. The proof of the former statement is thus completed.

For the remaining statement, first note that every optimal solution of $(LR_\lambda(u))$ is also optimal for $(P_1(k))$ with $k = \frac{u+\lambda}{u+1}$, and that each of the corresponding SL's provides a subgradient of $z_{LR}^\lambda(u)$ at that value of u. Since $z_{LR}^\lambda(u)$ contains a zero subgradient at $u = u^*(\lambda)$, it trivially holds from the theory of Lagrangean duality that $u^*(\lambda)$ becomes an optimal Lagrangean multiplier of (LD_λ) for each λ. $\square$

Theorem 2 renders the following

Corollary 3 *If $(P_1(k))$ doesn't have an optimal solution with $SL = 0$ for $0 \leq k \leq 1$, then a duality gap exists between $z(\lambda)$ and $z_D(\lambda)$ for all $0 \leq \lambda < k^*$ where k^* is defined as in Theorem 2.*

Proof. For any $0 \leq \lambda < k^*$, $u^*(\lambda) > 0$ from Theorem 2 and $SL(x) > 0$ for any optimal solution of $(LR_\lambda(u^*(\lambda)))$ from Lemma 2. □

3.3. Modular description of the algorithm

Here we show how the properties presented in Section 3.2 can be used to construct an algorithm. Suppose that we are to solve $(P(\lambda^t))$ at iteration t of the solution process outlined in Section 3.1. Recall that λ^t is set at the value making the objective value zero for given y^{t-1}. This corresponds to a feasible solution of $(P(\lambda^t))$ having the objective value of zero, so $z(\lambda^t) \leq 0$. By Theorem 1, if an optimal solution of $(P(\lambda^{t-1}))$ satisfied (2) as an equality, we also obtain an optimal solution of the problem. Otherwise, we must solve $(P(\lambda^t))$. From Theorem 2, $u^*(\lambda^{t-1})$ must be equal to either 0 or $\dfrac{k^* - \lambda^{t-1}}{1 - k^*}$. In the latter case, we can calculate $u^*(\lambda^t)$ without any extra efforts, due to Theorem 2 as well as the fact that k^* doesn't depend on λ. In the former case, we first check whether $u^*(\lambda^t) = 0$. This is done by solving $(LR_{\lambda^t}(0))$ i.e., $(P_1(k))$ with $k = \lambda^t$. Note that if $(LR_{\lambda^t}(0))$ provides an optimal solution with $SL \leq 0$, $u^*(\lambda^t) = 0$. Otherwise, we calculate k^*.

Now consider how to derive k^*. Let $I^+(k)$ denote the index set of open facilities at the optimal solution of $(P_1(k))$, and $\Pi(I^+, k)$ be the optimal objective value of (7) for given I^+ and k. Suppose that we try to calculate k^* at the t^{th} step and let x_1 and x_2 be optimal solutions of $(P_1(\lambda^{t-1}))$ and $(P_1(\lambda^t))$, respectively. Then by the nature of our procedure, $SL(x_1) < 0$ and $SL(x_2) > 0$. We initially set k^{min} as λ^t and k^{max} as λ^{t-1}. Then we find k' such that $\Pi(I^+(k^{min}), k') = \Pi(I^+(k^{max}), k')$, and solve $(P_1(k'))$. If $I^+(k^{min})$ and

$I^+(k^{max})$ are also optimal for k', then $k^* = k'$. Otherwise, update k^{min} or k^{max} depending on the SL value for the obtained optimal solution of $(P_1(k'))$, and continue the above process. Of course, if $(P_1(k'))$ provides an optimal solution with $SL = 0$, we can immediately terminate the whole procedure by Corollary 2. Moreover, we update an upper bound of $z(\lambda^t)$, if possible. Among a number of upper-bounding strategies varying in updating frequency, ours is to update the upper bound whenever k^* is calculated. This is simply done by calculating $z(\lambda^t)$ with y-vector fixed such that $y_i = 1$ for $i \in I^+(k^*)$ and 0 otherwise.

So far we have considered solving $(P(\lambda))$ for a number of successive λ values. All the developments hitherto made can still be applied for solving partially restricted $(P(\lambda))$'s where some of y variables are fixed at 0 or 1. This saves a significant amount of computation when performing a number of branch and bound (BB) processes, one for solving each single $(P(\lambda^t))$. In fact by Corollary 3, once a BB process is conducted to solve $(P(\lambda^t))$ at some λ^t, a BB process is required thereafter for each subsequent $(P(\lambda^t))$. For this, we keep the information regarding the BB tree, especially those BB nodes at which k^* has been calculated. Note that each BB node is associated with the corresponding BB subproblem, i.e., the corresponding partial restriction of $(P(\lambda))$. So when solving a BB subproblem during a BB process for some $(P(\lambda))$, we first check whether k^* has been obtained for the corresponding partial restriction of $(P(\lambda))$ (BB node) in the preceding BB process. If found, solving the present BB subproblem, and thus updating the lower bound, can be done without any extra effort.

Now we explain why it is relatively easy to calculate $z_1(k)$ and $z(\lambda)$ for varying values of k and λ when the values of y_i variables are fixed. First consider the case of calculating $z_1(k)$. For given y-vector, the optimal values of x_{ij}'s in a transformed SPLP are simply constructed as follows: for each j, if the minimum objective coefficient of x_{ij} among $i \in I^+ = \{i \in I | y_i = 1\}$ is nonnegative, every x_{ij} for $i \in I$ is zero. Otherwise, only the x_{ij} corresponding to the minimum coefficient is equal to one, and all other x_{ij}'s are zero. Note that for different values of k, the ordering of x_{ij} according to the object coefficients never changes and is the same as that according to t_{ij}. Therefore, for given y-vector, we can easily calculate $z_1(k)$ for varying values of k. Even in the case of calculating $z(\lambda)$, the whole process is the same except considering one more inequality. Moreover Corollary 1 and Theorem 1 are still valid for this restricted problem.

4. An illustrative example

In this section, we provide an example to demonstrate the solution procedure developed in Section 3. We assume a quadratic revenue function denoted by $a_j S_j - b_j S_j^2$. Data for an example problem are given in Table 1. The required profit level, π_o, is 5000.

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Table 1 here

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(i) First we solve $(P(\lambda))$ at $\lambda = 1$. Let $u = 0$ for (LD_λ) and solve $(P_1(1))$. The resulting optimal solution has $I^+ = \{1, 2, 4\}$ and negative SL, and thus it is also optimal for $(P(1))$. We then calculate λ which gives a zero objective value of $z(\lambda)$ for given I^+. The calculated value is 0.4179.

(ii) We obtain an optimal solution of $(P_1(0.4179))$ which provides $I^+ = \{4\}$ and positive SL. So we calculate k^* by initially setting $k^{min} = 0.4179$ and $k^{max} = 1$. The obtained k^* is 0.4409 and we obtain a new upper bound with $I^+ = \{2, 4\}$ whose objective value is -462. Then we calculate $u^*(0.4179)$ and the resulting $z_D(\lambda)$, which are 0.0412 and -512, respectively. Since there exists a duality gap at this step, we must initiate a branch and bound process. The summarizing result for a branch and bound process is shown in Figure 1. Since an optimal solution doesn't satisfy (2) as an equality, we calculate new λ from an optimal open facility set, $I^+ = \{2, 4\}$. The new λ is 0.3692.

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Figure 1 here

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(iii) From Corollary 3, we know that there also exists a duality gap for $\lambda = 0.3692$. So we directly carry out a branch and bound process using the branch and bound tree produced in the previous step. For the subproblem where y_2 is fixed closed, we can simply calculate a lower bound using previously obtained $k^* = 0.54007$. Since a lower bound for this problem is 5.82, this branch is fathomed. For the subproblem where y_2 is fixed open, we calculate again $(P_1(0.3692))$ since we didn't get k^* in the previous calculation. This subproblem provides a lower bound with zero and is fathomed. So $z(0.3692) = 0$, and an optimal solution is found.

5. Conclusion

In this paper, we have developed an ROI maximizing model for a general SPLP where locational and pricing decisions are to be determined simultaneously. Here we have dealt with the discriminatory pricing environment which assumes different pricing for different customers. Even under a different environment such as a uniform price system [4], an ROI model can also be applied to it. However, the model becomes more difficult to solve. Another possible extension of our model is to impose the capacity restriction on each facility. However, this model is also difficult to solve, since it doesn't allow the reformulation of the model to a fixed demand model.

We can also consider the two different objective criteria, maximizing net profit and maximizing ROI, in other settings. For example, we can maximize the profit while maintaining the minimum required ROI level. In this case, it can easily be shown that optimal solutions are always obtained at the minimum ROI level. Therefore, this model can be transformed to a standard maximizing profit model. A bi-objective model may be a good alternative to ours, for which the properties found here can also be used in finding non-dominant solutions.

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Table 1.
Data for Example

i\ j	t_{ij} 1	2	3	4	f_i
1	20	60	80	140	570
2	60	20	40	100	1000
3	80	60	20	60	1500
4	120	100	40	20	1000
a_j	80	180	100	220	
b_j	1	2	1	2	

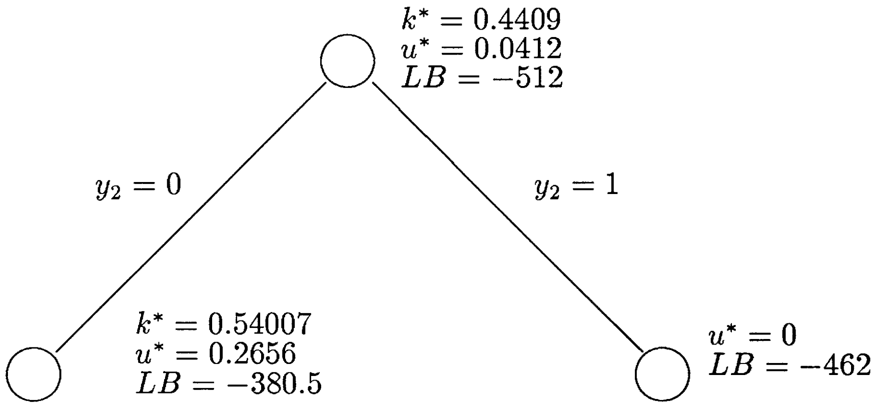

Figure 1. Branch and bound tree for $(P(\lambda))$ when $\lambda = 0.4179$