Heuristics, LPs, and Trees on Trees: Network Design Analyses

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Abstract

We study a class of models, known as overlay optimization problems, with a "base" subproblem and an "overlay" subproblem, linked by the requirement that the overlay solution be contained in the base solution. In some telecommunication settings, a feasible base solution is a spanning tree and the overlay solution is an embedded Steiner tree (or an embedded path). For the general overlay optimization problem, we describe a heuristic solution procedure that selects the better of two feasible solutions obtained by independently solving the base and overlay subproblems, and establish worst-case performance guarantees on both this heuristic and a LP relaxation of the model. These guarantees depend upon worst-case bounds for the heuristics and LP relaxations of the unlinked base and overlay problems. Under certain assumptions about the cost structure and the optimality of the subproblem solutions, both the heuristic and the LP relaxation of the combined overlay optimization model have performance guarantees of 4/3. We extend this analysis to multiple overlays on the same base solution, producing the first known worst-case bounds (approximately proportional to the square root of the number of commodities) for the uncapacitated multicommodity network design problem. In a companion paper, we develop heuristic performance guarantees for various new multi-tier, survivable network design models that incorporate both multiple facility types or technologies and differential node connectivity levels.

Introduction

This paper considers a general class of models, which we call *overlay optimization problems*, that combines two sets of decisions: the choice of activity levels to provide a basic level of service to all customers, and decisions regarding which of these activities to enhance to meet more stringent service requirements for subsets of important customers. The activity levels might represent facility installation and sizing decisions, with the basic and enhanced activities representing two levels of technology that differ in speed, capacity, or functionality. We treat the installation of higher grade facilities as "overlaying" or upgrading the base facilities at extra cost. Overlay optimization has potential applications in logistics and infrastructure planning including the design of telecommunications, transportation, electric power, and pipeline networks. For instance, in transportation planning, customers correspond to cities and towns. Basic service represents providing access to every town via a paved road, but certain important cities must be interconnected by, say, all-weather highways. Likewise, in telecommunications planning, the base decisions model the installation of switching and transmission facilities that can accommodate basic voice and data services (e.g., DS1 services), while overlay variables represent upgrading certain facilities to high capacity, broadband (e.g., fiber optic) systems between selected locations.

We analyze the worst-case performance of a generic heuristic strategy for solving overlay optimization problems, and also characterize the gap between the optimal value and the linear programming relaxation value, assuming that the original model can be formulated as a (mixed) integer program. By extending these results to the multi-overlay case, we develop the first known worst-case bounds for heuristics and linear programming relaxations for the uncapacitated, fixed-charge network design problem.

1. Problem definition and overview

The overlay optimization problems that we address in this paper have the following general form. Let $x = \{x_1, x_2, ..., x_m\}$ denote the vector of m **base** activity levels. For each base activity x_j, we consider K **overlay** variables y_j^k for $k = 1, 2, ..., K$ and let $y^k = \{y_1^k, y_2^k, ..., y_m^k\}$. The base and overlay variables x_j and y_j^k have nonnegative per unit costs of a_j and b_j^k, respectively. We refer to a_j as the **base cost**, and b_j^k as the **incremental** or **overlay cost** for activity k. Using this notation, the overlay optimization problem is:

Problem [P]

$$Z^* = \min \quad a\,x + \sum_{k=1}^{K} b^k y^k \tag{1.1}$$

subject to

$$y^k \in Y^k \qquad \text{for } k = 1, 2, ..., K, \tag{1.2}$$

$$y^k \le x, \qquad \text{for } k = 1, 2, ..., K \text{ and} \tag{1.3}$$

$$x \in X. \tag{1.4}$$

In this model, X and Y^k for k = 1, 2, ..., K represent problem-specific constraints (e.g., network configuration requirements) on the base and overlay decisions. We assume that $X \subseteq Y^k$. Typically, X and Y^k are discrete sets in R^m_+. The **linking constraints** (1.3) link each of the K overlay solutions to the base solution; when x_j and y^k_j are binary variables, these inequalities resemble the familiar forcing constraints of the uncapacitated plant location model.

The overlay optimization model [P] generalizes the hierarchical network design problem (Current, Revelle, and Cohon [1986]), the multi-weighted Steiner tree problem (also called the two-level network design problem; see Duin and Volgenant [1991] and Balakrishnan, Magnanti, and Mirchandani [1992]), and the uncapacitated network design problem (Magnanti and Wong [1984]). All these problems are known to be NP-hard even for certain special cost structures that we consider. For one of the special cases, the two-level network design problem, Balakrishnan et al. [1992] analyzed the worst-case performance of a heuristic strategy based upon solving a Steiner tree subproblem and a minimum spanning tree subproblem, and constructed examples to show that the performance bound is tight. This paper generalizes and extends these previous results.

Section 2 considers the "single overlay" version of formulation [P], i.e., overlay optimization problems with K = 1. For this class of problems, we analyze a composite heuristic solution procedure that chooses the best solution obtained by two embedded heuristics. We develop worst-case performance guarantees on the cost of this composite heuristic solution and on the value of a linear programming relaxation of the overlay problem using bounds on heuristic methods and linear programming formulations of its two subproblems—a base subproblem and an overlay subproblem—obtained by ignoring the linking constraints. We consider two alternative cost structures: the *proportional costs* case in which the ratio of overlay to base costs is the same for all activities, and the *unrelated costs* case that permits this ratio to vary by activity. Our results imply, for instance, that problems with proportional costs have worst-case bounds of 4/3 rds for both the composite heuristic and the LP relaxation when we solve the base and overlay subproblems optimally and their LP relaxations are exact (i.e., have optimal integer solutions). Researchers have not previously studied how and why linear programming gaps arise for the general overlay optimization problem or its special cases. For two-level network design problems with triangular costs, we provide an example showing that our worst-case bound on the LP relaxation value is tight. For the hierarchical network design special case, the LP gap for our example is only about half the worst-case bound; we conjecture that our linear programming bound for this problem is not tight.

Section 3 analyzes the worst-case performance of the composite heuristic strategy for a multicommodity, uncapacitated network design problem. We can view this problem as an "multi-overlay" optimization problem in which we first select a basic design (incurring fixed edge costs) and then superimpose multiple paths, one connecting each origin-destination pair; the overlay cost for each path is the routing cost for satisfying the demand between its origin and destination. We present two alternative overlay interpretations—a simultaneous overlay of all origin-to-destination paths, and a recursive framework that sequentially overlays one path at a time. These two interpretations motivate two slightly different heuristic strategies. Our analysis considers two different modeling assumptions: one requiring the network design to be connected and another permitting arbitrary (nonconnected) designs. We also consider several different cost assumptions. For each model, we develop performance bounds on the composite heuristic. For one model, we provide a worst-case bound on its LP relaxation. The heuristic and linear programming bounds grow at a rate of approximately the square root of the number of commodities; we also show that the heuristic worst-case performance ratios are tight for several models.

In a companion paper (Balakrishnan, Magnanti, and Mirchandani [1994]), we study a new class of multi-tier survivable network design problems incorporating both multiple facility types and differential node connectivities (i.e., requiring multiple edge-disjoint paths between select nodes), with potential applications to telecommunications. The general worst-case analysis presented in this paper applies to these telecommunications models as well. However, by exploiting the application's special problem structure, we obtain improved performance guarantees for certain versions of the survivable network design model (those with two-tiers, and requiring two-connectivity for certain critical nodes).

To conclude this section, we might note that the work reported in this paper is related to previous work in polyhedral combinatorics. Whenever the sets X and Y^k are discrete, and so can be represented as integer polyhedra, we can view our analysis as examining a problem concerning the coupling, through the linking constraints (1.3), of integer polyhedra. Therefore, this analysis is a special case of a more general situation: when we couple integer polyhedra in some general way, what can we say about the polyhedra that arise? Disjunctive programming (Balas [1979]) provides a general approach and set of tools for addressing these types of problems. More specifically, we might ask "when will the intersection of two integral polyhedra be an integral polyhedra?" Matroid intersection (Edmonds [1979]) provides the most noted such example. Two other examples are the intersection of forest and cover polyhedra (Gamble and Pulleyblank [1989]) and the intersection of tree and matching polyhedra (Hall and Magnanti [1992]). As another example, Barany, Edmonds, and Wolsey [1986] and Aghezzaf, Magnanti, and Wolsey [1992] have shown that the packing of certain polyhedra (to model rooted trees) produces an integer polyhedra. As we show in this paper, by combining integer polyhedra via forcing constraints as in the problem [P], we do not always create an integer polyhedron. How well does this coupled polyhedron approximate the convex hull of integer solutions to the overlay optimization problem? We are able to provide a partial answer to this question by bounding the degree of suboptimality of (i) the objective value of optimal LP solutions over the non-integral coupled polyhedron, and (ii) the objective value determined by heuristic solution procedures that solve problems over the constituent polyhedra.

2. Performance Analysis of the Overlay Optimization Problem

This section first describes a composite heuristic strategy for overlay optimization, and develops heuristic upper bounds and LP relaxation lower bounds on the optimal value of problem [P]. The upper-to-lower bound ratio characterizes both the worst-case performance of the composite heuristic as well as the maximum relative gap between the LP value and the optimal value of the problem. Our analysis relies on an important *feasible completion* assumption that we describe in Section 2.1.

To simplify the discussions, we consider the **single-overlay** problem with $K = 1$; Section 3 extends these results to the **multi-overlay** case in the context of the uncapacitated network design problem. For the single-overlay case, we omit the superscript k for the overlay variables and cost parameters. Let $\mathbf{c_j} = a_j + b_j$ denote the

total cost of activity j for all $j = 1, 2, ..., m$. For a given nonnegative cost vector $g = \{g_1, g_2, ..., g_m\}$, we define the following two subproblems:

Base subproblem [BP(g)]: $Z_B(g) = \min \{gx: x \in X\}$, and

Overlay subproblem [OP(g)]: $Z_O(g) = \min \{gy: y \in Y\}$.

Observe that, if we ignore the linking constraints (1.3), the overlay optimization problem [P] decomposes into two subproblems: BP(a) and OP(b).

To illustrate the solution method and understand the implications of our worst-case results, we will apply them to two special cases—hierarchical network design and two-level network design. The **hierarchical network design (HND)** problem (introduced by Current et al. [1986]) is defined over an undirected n-node graph G with two nodes designated as **primary nodes**. We can install either a primary or secondary facility on each edge j, incurring a **primary** cost c_j or a smaller **secondary** cost a_j ($c_j \geq a_j \geq 0$). The HND problem seeks a cost minimizing spanning tree that connects the two primary nodes via a primary path, i.e., a path containing only primary facilities. The HND problem's base subproblem is a minimum spanning tree (MST) problem, and its overlay subproblem requires finding the shortest path connecting the two primary nodes. The **two-level network design (TLND)** problem generalizes the HND problem by designating more than two nodes as primary nodes, i.e., we seek a minimum cost connected subgraph that interconnects all primary nodes via primary paths. The base subproblem is again the MST problem, but the TLND model's overlay subproblem is a Steiner tree problem with all the primary nodes as terminals.

2.1 Composite heuristic for overlay optimization

To approximately solve the overlay optimization problem, we consider a composite heuristic method that selects the better of two solutions generated by a Base Upgrading heuristic and an Overlay Completion heuristic. We next describe these two embedded heuristics.

Since the overlay optimization model assumes that $X \subseteq Y$, we can generate feasible solutions to [P] by finding feasible solutions $x \in X$ to the base subproblem, and setting $y = x$. When the solution x solves (approximately or optimally) the base subproblem BP(c), using total costs c_j, we refer to this method as the **Base Upgrading (BU) heuristic**. For the HND problem, the BU heuristic finds an MST of graph G using primary costs c_j, and installs primary facilities on all edges of this tree. In our subsequent analysis, we ignore the possibility of further reducing the cost of the BU heuristic solution by locally downgrading some primary facilities into secondary facilities whenever possible (for the HND problem, Balakrishnan et al. [1992] have shown that local improvement does not reduce the heuristic's worst-case performance ratio).

A complementary heuristic first generates a feasible solution $\hat{y}$ to the overlay subproblem, and then "completes" this overlay solution by solving the following **completion subproblem CP(a,$\hat{y}$)**:

$$Z_B'(a,\hat{y}) \;=\; \min\{ax\colon x \geq \hat{y},\, x \in X\}.$$

We refer to the particular implementation that generates $\hat{y}$ by solving the overlay subproblem OP(c), using total costs c_j, as the **Overlay Completion (OC) heuristic**. Since $x \geq \hat{y}$, the optimal value $Z_B'(a,\hat{y})$ of the completion problem must be greater than or equal to $a\hat{y}$. We refer to the difference $\delta(\hat{y}) = Z_B'(a,\hat{y}) - a\hat{y}$ as the **optimal completion cost**. For the HND problem, the solution to the overlay subproblem is the shortest path (using primary costs c_j) connecting the two primary nodes; to optimally complete this solution, we select every edge in this path, and sequentially add other edges in the order of increasing secondary costs to form a spanning tree. Note that the optimal completion cost for the HND problem must be less than or equal to the MST cost $Z_B(a)$ since any overlay solution has at least one optimal completion that installs secondary facilities only on MST edges. Similarly, for the TLND problem, the cost of completing any overlay subproblem solution (i.e., any Steiner tree) does not exceed $Z_B(a)$. Our subsequent analysis applies to problem classes that satisfy the following general *feasible completion property*:

> An overlay optimization problem is said to satisfy the **feasible completion property** if, for any feasible problem instance and a given overlay solution $\hat{y}$, the completion subproblem CP(a,$\hat{y}$) is feasible and has an optimal completion cost of no more than $\lambda\, Z_B(a)$ for some known finite constant λ.

We refer to the parameter λ as the **completion cost multiplier**; λ is 1 for the HND and TLND problems. Although our worst-case results extend to any finite value of λ, the remainder of this section assumes that $\lambda = 1$ for expositional convenience.

2.2 Upper and lower bounds on the optimal and LP values

2.2.1 Heuristic bounds

Let $Z^{\mathbf{Comp}}$ denote the cost of the composite heuristic solution. In general, the base and overlay subproblems might be difficult to solve to optimality (for example, the overlay subproblem for the TLND model is a Steiner tree problem). So, suppose we solve the base and overlay subproblems using methods with known worst-case performance guarantees of ρ_B and ρ_O, respectively. That is, the heuristics generate solutions costing no more than ρ_B and ρ_O times the optimal costs of the base and overlay problems. Let $\rho = \rho_O/\rho_B$. Then,

$$\begin{aligned} Z^{Comp} &\leq \min \{\rho_B Z_B(c), \rho_O Z_O(c) + \rho_B Z_B(a)\}, \\ &= \rho_B \min \{Z_B(c), \rho\, Z_O(c) + Z_B(a)\}. \end{aligned} \tag{2.1}$$

Implicitly, we assume here that a heuristic method with worst-case performance ratio ρ_B can also heuristically complete any overlay solution within a factor of ρ_B times the completion cost upper bound $Z_B(a)$. This assumption holds for all the problems we analyze. Notice that if we solve the base and overlay subproblems optimally, we obtain the following upper bound on the optimal objective value $\mathbf{Z}^*$ of the original overlay optimization model:

$$Z^* \leq \min \{Z_B(c), Z_O(c) + Z_B(a)\}. \tag{2.2}$$

We use the following two relaxations of problem [P] to generate a lower bound on Z^*:

(i) the **Base Relaxation** obtained by ignoring the base constraints (1.4): Since the base costs are nonnegative, x = y in an optimal solution to this relaxation, and so this relaxation reduces to problem OP(c), giving the lower bound $Z_O(c)$; and,

(ii) the **Linking Relaxation** obtained by eliminating the linking constraints (1.3): The formulation [P] then decomposes into two subproblems—the overlay subproblem OP(b) and the base subproblem BP(a)—giving the lower bound $Z_O(b) + Z_B(a)$.

Combining these bounds, we obtain

$$Z^* \geq \max \{Z_O(c), Z_O(b) + Z_B(a)\}. \tag{2.3}$$

2.2.2 Linear programming bounds

Suppose we can represent problem [P] as an integer (or mixed integer) program, i.e., we can express the implicit constraints $x \in X$ and $y \in Y$ by a set of linear inequalities, and require the vectors x and y to be integer-valued (our results also apply to overlay optimization problems containing continuous variables). Then, we can generate a lower bound on Z^* by solving formulation [P]'s linear programming relaxation, which we denote as **[LP]**. What is the relationship between the optimal objective value $\mathbf{Z^{LP}}$ of [LP] and Z^*, or between Z^{LP} and Z^{Comp}? To answer these questions, we first develop lower bounds on Z^{LP}.

Let $Z_B^{LP}(g)$ and $Z_O^{LP}(g)$ denote the optimal values of the LP relaxations of subproblems BP(g) and OP(g). Eliminating the linking constraints $y \le x$ from [LP], gives the lower bound

$$Z^{LP} \ \ge \ Z_O^{LP}(b) + Z_B^{LP}(a).$$

Also, if we remove the constraints $x \in X$ from [LP], setting $x = y$ is always optimal in the resulting problem (since $b \ge 0$), and so we obtain the alternate lower bound:

$$Z^{LP} \ \ge \ Z_O^{LP}(c).$$

Therefore,

$$Z^{LP} \ \ge \ \max \{Z_O^{LP}(c),\ Z_O^{LP}(b) + Z_B^{LP}(a)\}. \qquad (2.4)$$

Suppose we can bound the LP relaxation gaps for the base and overlay subproblems, i.e., for any cost vector g, $Z_B(g) \le \theta_B Z_B^{LP}(g)$ and $Z_O(g) \le \theta_O Z_O^{LP}(g)$ for some constants θ_B and θ_O that are both greater than or equal to 1; let $\theta = \theta_O/\theta_B$. Then, substituting for the LP values in (2.4) we obtain

$$Z^{LP} \ \ge \left(\frac{1}{\theta_O}\right) \max \{Z_O(c),\ Z_O(b) + \theta\, Z_B(a)\}. \qquad (2.5)$$

For the HND problem, $\theta_B = \theta_O = \theta = 1$ since the subproblems BP(g) and OP(g) are MST and shortest path problems, with known LP characterizations of their underlying polyhedra (see, for example, Nemhauser and Wolsey [1988] or Ahuja, Magnanti, and Orlin [1993]).

2.2.3 Worst-case performance ratios

The inequalities (2.1) and (2.3) provide the following upper bound on the ratio of the heuristic cost Z^{Comp} to the optimal value Z^* of the overlay optimization problem [P]:

$$\frac{Z^{Comp}}{Z^*} \ \le \ \rho_B \frac{\min \{Z_B(c),\ \rho Z_O(c)+Z_B(a)\}}{\max \{Z_O(c),\ Z_O(b)+Z_B(a)\}}\ . \qquad (2.6a)$$

Similarly, from inequalities (2.2) and (2.5), we have

$$\frac{Z^*}{Z^{LP}} \ \le \ \theta_O \frac{\min \{Z_B(c),\ Z_O(c)+Z_B(a)\}}{\max \{Z_O(c),\ Z_O(b)+\theta Z_B(a)\}}\ . \qquad (2.6b)$$

To determine the worst-case values of these ratios, we consider two cases: (i) problems for which total and base costs are **proportional**, i.e., the ratio of total to base costs has a constant value for all activities, and (ii) the general case with **unrelated** total to base costs.

2.3 Overlay optimization problems with proportional costs

Suppose all activities j have the same ratio of total to base costs, and let $\mathbf{r} = c_j/a_j \geq 1$ be this constant ratio. Then, $Z_O(c) = r\,Z_O(a)$, $Z_O(b) = (r-1)Z_O(a)$, and $Z_B(c) = rZ_B(a)$. Define $\mathbf{Z_O(a)/Z_B(a)} = s$; since $X \subseteq Y$, $s \leq 1$. Dividing the numerator and denominator in the right-hand side of inequality (2.6a) by $Z_B(a)$ (which we assume to be positive, since otherwise the BU solution is optimal), we obtain

$$\frac{Z^{Comp}}{Z^*} \leq \rho_B \frac{\min\{r, \rho rs+1\}}{\max\{rs, (r-1)s+1\}} . \qquad (2.7)$$

2.3.1 Worst-case performance of the composite heuristic

Let ω_{prop} be the worst-case performance ratio of the composite heuristic for the overlay optimization problem with proportional costs. Note that $rs \leq \{(r-1)s + 1\}$ since $s \leq 1$, and so the inequality (2.7) implies that

$$\omega_{prop} \leq \rho_B \frac{\min\{r, \rho rs+1\}}{\{(r-1)s+1\}} . \qquad (2.8)$$

For small values of s, the OC heuristic solution has a smaller upper bound (= $\rho rs+1$) than the BU heuristic solution, and vice versa. Since the value of s depends on the problem instance, we develop data-independent bounds by maximizing the right-hand side of (2.8) with respect to s and r. For this purpose, we consider two cases:

__Case 1__: $\rho r \geq (r-1)$

As s increases, the upper-to-lower bound ratios for the BU and OC heuristics decrease and increase, respectively. Since the composite heuristic selects the better of the two upper bounds, its worst-case ratio is maximum when $s^* = (r-1)/\rho r$. Substituting this value of s in (2.8), gives

$$\omega_{prop} \leq \frac{\rho_O\, r^2}{\{r^2 + (\rho-2)r + 1\}} . \qquad (2.9)$$

If $\rho \geq 2$, the function on the right-hand side of (2.9) is concave, since by dividing the numerator and denominator by $\rho_O r^2$, we can express it as the inverse of a positive convex function. If $\rho < 2$, then this function is pseudo-convex (see, for instance, Lasdon [1970]). We distinguish two subcases:

__Case 1a__. $\rho > 2$: the derivative of the right-hand side of inequality (2.9) with respect to r is positive; since the right-hand side approaches ρ_O as r approaches $+\infty$,

$$\omega_{prop} \leq \rho_O = \rho_B\,\rho \qquad (2.10)$$

Case 1b. $\rho \leq 2$: the derivative of the right-hand side is positive when $r < 2/(2-\rho)$ and is negative $r > 2/(2-\rho)$, and so the right-hand side achieves its maximum value when $r^* = 2/(2-\rho)$, i.e.,

$$\omega_{prop} \leq \rho_B \frac{4}{4-\rho}. \tag{2.11}$$

Case 2: $\rho r < (r-1)$

In this case (which applies only if $\rho < 1$), since the upper-to-lower bound ratios for both the BU and OC heuristics decrease with s, the composite heuristic's performance ratio is maximum when $s^* = 0$. Therefore,

$$\omega_{prop} \leq \rho_B. \tag{2.12}$$

Note that this bound is less than the bound in (2.11). So, inequalities (2.10) and (2.11) provide valid upper bounds on ω_{prop} for all values of r, establishing the following theorem.

Theorem 1:

For overlay optimization problems with completion cost multiplier $\lambda = 1$ and proportional costs, the performance ratio ω_{prop} of the composite heuristic is bounded from above as follows :

$$\begin{aligned} \omega_{prop} &\leq \rho_B \frac{4}{4-\rho} && \text{if } \rho \leq 2, \\ &\leq \rho_B\, \rho && \text{if } \rho > 2. \end{aligned}$$

For the **HND problem,** $\rho_O = \rho_B = \rho = 1$. So, Theorem 1 implies that the cost of the composite heuristic solution for proportional cost problems is at most 4/3 rds the optimal cost. For the TLND problem with proportional costs, a modified MST heuristic solves the overlay (Steiner network) subproblem with a worst-case ratio $\rho_O = 2$ (Goemans and Bertsimas [1993]). Since $\rho_B = 1$, Theorem 1 implies that the worst-case ratio of the composite heuristic does not exceed 2. Balakrishnan et al. [1992] have previously described worst-case examples for the HND and TLND problems showing that these bounds are tight. By using a more sophisticated heuristic for the overlay subproblem, we can improve the worst-case bound. For example, Berman and Ramaiyer's [1992] heuristic solves the Steiner network problem with a worst-case ratio $\rho_O = 16/9$, and so embedding this method in the composite procedure reduces the proportional cost TLND problem's worst-case bound to 1.8.

2.3.2 LP characterization ratio

We refer to the worst-case ratio of Z^* and Z^{LP} as the **LP characterization ratio.** Let $\nu_{\mathbf{prop}}$ represent this ratio for overlay optimization problems with proportional costs. Substituting $Z_O(a)/Z_B(a) = s$, $Z_O(b) = (r-1)\,Z_O(a)$, and so on in (2.6b), we obtain

$$\nu_{prop} \;\leq\; \theta_O \frac{\min\{r,\, rs+1\}}{(r-1)s+\theta} \;. \qquad (2.13)$$

To determine data-independent worst-case bounds on ν_{prop}, we again consider two cases:

<u>Case 1</u>: $r\theta \geq (r-1)$

The right-hand side of (2.13) is maximum when $s^* = (r-1)/r$. Therefore,

$$\nu_{prop} \;\leq\; \theta_O \frac{r^2}{(r-1)^2+\theta r} \;. \qquad (2.14)$$

<u>*Case 1a:*</u> $\theta > 2$. The right-hand side of (2.14) is a concave, increasing function of r, approaching θ_O as $r \rightarrow \infty$. Therefore,

$$\nu_{prop} \;\leq\; \theta_O \;=\; \theta_B\,\theta. \qquad (2.15)$$

<u>*Case 1b*</u>: $\theta \leq 2$. In this case, $r^* = 2/(2-\theta)$ maximizes the right-hand side of (2.14), giving the worst-case bound

$$\nu_{prop} \;\leq\; \theta_B \frac{4}{4-\theta} \;. \qquad (2.16)$$

<u>Case 2</u>: $r\theta < (r-1)$

In this case (which applies only when $\theta < 1$), $s^* = 0$ maximizes the right-hand side of (2.13), and so

$$\nu_{prop} \;\leq\; \theta_B \;. \qquad (2.17)$$

Because this bound is smaller than the bound in (2.16), inequalities (2.15) and (2.16) provide valid bounds for all values of r, establishing the following theorem:

Theorem 2:

For overlay optimization problems with completion cost multiplier $\lambda = 1$ and proportional costs, the LP characterization ratio ν_{prop} is bounded from above as follows:

$$\begin{aligned} \nu_{prop} &\leq \theta_B \frac{4}{4-\theta} && \text{if } \theta \leq 2, \\ &\leq \theta_B\,\theta && \text{if } \theta > 2. \end{aligned}$$

As special cases, these results provide what we believe are the first known bounds of 2 and 4/3 on the LP relaxations of the TLND and HND problems. Next, we describe TLND and HND examples for which the LP relaxations have nonzero gaps.

2.3.3 TLND and HND linear programming examples

Let us first consider the **TLND problem.** Since the base subproblem is the MST problem, $\theta_B = 1$. Since the LP characterization ratio for the cutset formulation of the Steiner network problem is 2 (Goemans and Bertsimas [1993]), $\theta_O = 2$. Therefore, Theorem 2 implies that $v_{prop} \leq 2$ for the TLND problem with proportional costs.

Figure 1(a) shows a worst-case example to prove that this bound of 2 is tight. Each of the q nodes on the rim of the circle is a primary node, and the node in the center is a secondary node. Every edge of the network has unit secondary cost. Figure 1(b) shows the LP solution to this problem; the cost of this solution is (rq/2 + q/2). The optimal solution, shown in Figure 1(c), costs r(q−1) + 1. Thus,

$$\frac{Z^*}{Z^{LP}} = \frac{r(q-1)+1}{r\,q/2 + q/2} = \frac{2 - 2/q + 2/rq}{1 + 1/r}$$

which approaches 2 as r and q approach infinity.

For the **HND problem**, since $\theta_B = \theta_O = 1$, Theorem 2 proves that the optimal HND cost is at most 4/3 rds the optimal LP value when the primary and secondary costs are proportional. We have not been able to construct an HND example that achieves this bound of 4/3. Figure 2(a) shows an example for which the optimal value is 8/7 ths of the LP relaxation value. This example has cost ratio r = 2; the values shown on the edges of the network in Figure 2(a) correspond to secondary costs. The primary nodes 1 and 2 can communicate via three paths: two 2-edge paths respectively containing nodes 3 and 4, and a q-edge path (q is a sufficiently large integer) that has a total (secondary) length of 3 units. Figures 2(b) shows the optimal LP solution; this solution has an objective function value of 7. We can interpret this LP solution as the convex combination of the two primary paths 1-3-2 and 1-4-2, and two (secondary) spanning trees—the tree T_1 containing edges (1,3), (1,4), and the q-edge path, and the tree T_2 containing edges (2,3), (2,4), and the q-edge path. The optimal HND solution, shown in Figure 2(c), has a cost of (8−3/q). Therefore, $Z^*/Z^{LP} \rightarrow 8/7$ as $q \rightarrow \infty$. We can marginally increase this optimal-to-LP ratio by changing the cost ratio r to $(1+ 1/\sqrt{2})$, and setting the total length of the q-edge path connecting nodes 1 and 2 to $(2+\sqrt{2})$. The ratio of the optimal value to the LP relaxation value then increases to $Z^*/Z^{LP} = \{5+2\sqrt{2}\}/\{4+2\sqrt{2}\} > 8/7$.

Several observations about this example lead us to conjecture that the LP bound of 4/3 is not tight for the HND problem. First, while the OC heuristic finds the optimal solution for this example, the actual cost of this solution is strictly less than the upper bound of $(rZ_O(a)+Z_B(a))$ that our analysis assumes; furthermore, the BU heuristic solution is not optimal. We can also show that if, in the optimal HND solution, the primary path visits all the nodes in the graph, then the LP value must equal the optimal value. In particular, if the BU heuristic finds the optimal solution, then the LP gap must be zero. On the other hand, to prove the tightness of the bound (2.16), we wish to construct an example for which both the OC and BU heuristics produce the optimal solution (since the BU and OC upper bounds are equal when $s^* = (r-1)/r$), the value of this solution equals our heuristic upper bound, and the optimal LP value equals the lower bound of $(r-1)Z_O(a) + Z_B(a)$. Simultaneously satisfying all these conditions appears to be difficult for the HND problem.

2.4 Overlay optimization problems with unrelated costs

2.4.1 Bounds on heuristic and LP performance

For arbitrary cost structures, we can readily compute a posteriori bounds on the worst-case ratio ω_{unrel} of the composite heuristic and the LP characterization ratio ν_{unrel}. For instance, inequality (2.6a) implies that

$$\omega_{unrel} \le \rho_B \frac{Z_B(c)}{Z_O(c)}. \tag{2.18}$$

For the HND problem, this bound implies that the ratio of the heuristic to optimal costs does not exceed the ratio of the MST cost to the shortest 1-to-2 path length (both using total costs as edge lengths). Since inequality (2.18) uses only the upper bound on the BU heuristic solution, it applies even when the overlay optimization model does not satisfy the feasible completion property.

To obtain an a priori bound, we consider just the OC heuristic. Inequality (2.6a) implies that

$$\omega_{unrel} \le \frac{\rho_O Z_O(c)+\rho_B Z_B(a)}{\max\{Z_O(c), Z_O(b)+Z_B(a)\}}.$$

If $Z_B(a) = 0$, this inequality implies that $\omega_{unrel} \le \rho_O$. Otherwise, we have:

$$\begin{aligned}\omega_{unrel} &\le \frac{\rho_O Z_O(c)}{Z_O(c)} + \frac{\rho_B Z_B(a)}{Z_O(b)+Z_B(a)} \\ &\le \rho_O + \rho_B \qquad \text{if } Z_B(a) > 0.\end{aligned} \tag{2.19a}$$

Similarly, since

$$\frac{Z^*}{Z^{LP}} \le \theta_O \frac{Z_O(c)+Z_B(a)}{\max\{Z_O(c), Z_O(b)+\theta Z_B(a)\}},$$

$$\nu_{unrel} \le \theta_O \{1+\frac{1}{\theta}\} \quad \text{if } Z_B(a) > 0. \qquad (2.19b)$$

These observations imply the following worst-case bounds for the unrelated costs case:

Theorem 3:

For overlay optimization problems with completion cost multiplier $\lambda = 1$ and unrelated costs, the worst-case performance ω_{unrel} of the OC heuristic and the LP characterization ratio ν_{unrel} are bounded from above as follows:

$$\begin{aligned} \omega_{unrel} &\le \rho_O + \rho_B && \text{if } Z_B(a) > 0, \text{ and} \\ &\le \rho_O && \text{if } Z_B(a) = 0; \qquad (2.20) \\ \nu_{unrel} &\le \theta_B + \theta_O && \text{if } Z_B(a) > 0, \text{ and} \\ &\le \theta_O && \text{if } Z_B(a) = 0. \qquad (2.21) \end{aligned}$$

These results indicate that both the heuristic and LP worst-case ratios deteriorate when we permit arbitrary (activity-dependent) total costs relative to base costs.

2.4.2 HND worst-case example

For the HND problem, Theorem 3 implies that installing primary facilities on the shortest 1-to-2 path, and completing the spanning tree with secondary facilities produces a solution that costs at most twice the optimal cost; furthermore, the optimal integer value is at most twice the LP lower bound. Balakrishnan et al. [1992] provide HND and TLND examples with unrelated costs to show that the worst-case bound (2.20) for the OC heuristic is tight. Let us now examine the LP gap.

Figure 3 shows a HND example for which the gap between the LP and optimal values is higher than the gap of 8/7 for the proportional costs case. This example has the same network topology as our previous HND example (Figure 2(a)), but differs in its cost structure. Each of the edges connecting the primary nodes 1 and 2 to the intermediate nodes 3 and 4 have unit primary and secondary costs; the q-edge path from node 1 to node 2 has a total primary cost of 1, and a total secondary cost of 0. The optimal LP and integer solutions have the same x and y values shown in Figures 2(b) and 2(c). Since the LP value is 2 and the optimal value is 3, for this example $Z^*/Z^{LP} = 3/2$. As before, we have not been able to develop an example to prove that the bound of 2 on the LP characterization ratio is tight.

The next section studies heuristic and linear programming worst-case performance for several versions of the uncapacitated, fixed-charge network design problem. This discussion generalizes the performance analysis methodology of Section 2 to a multi-overlay problem.

3. The Uncapacitated Network Design Problem

Given an undirected network G = (N,E), and a set of **K single-origin-destination commodities** indexed from k = 1 to K, the uncapacitated network design (UND) problem consists of selecting a subset of edges in E, and routing each commodity k from its **origin O(k)** to its **destination D(k)** along the chosen edges at minimum total cost. The cost function has two components: a nonnegative **fixed cost** $\mathbf{a_j}$ for each edge j, and a per unit nonnegative **routing cost** $\mathbf{b_j^k}$ to transport commodity k on edge j. The model does not incorporate flow capacity constraints, i.e., we can send unlimited flow on edge j if we install this edge. Therefore, we normalize the demand for each commodity to one unit. The UND model has applications in transportation, telecommunications, and production planning (e.g., Magnanti and Wong [1984]).

In general, if the pattern of required origin-to-destination flows is sparse and we do not impose any further restrictions on the topology of the network, the optimal design might contain more than one connected component. However, certain applications might explicitly require a connected design, i.e., although the volume of traffic between certain node pairs might be sporadic or relatively small, the designer might nevertheless wish to ensure that these nodes can communicate. Accordingly, we consider two variants of the UND model: a **connected** version in which we impose an explicit **full spanning constraint** (as constraint (1.4) in formulation [P]) requiring the design to be a connected subgraph spanning all the nodes, and an **unrestricted** version that permits multiple components. Since the worst-case analysis for the connected UND model is easier to explain, we first develop the results for this model (in Sections 3.1 to 3.4); we then discuss the ramifications (in Section 3.5) of removing the full spanning constraint. For each of these two models, we consider three different cost structures—uniform proportional costs, commodity-dependent proportional costs, and unrelated costs. To analyze the heuristic worst-case performance for these 6 problem variants, in Section 3.1 we propose two alternative implementations of the composite heuristic strategy for UND problems.

3.1 Overlay interpretations of the Connected UND model

We first present two alternative multi-overlay interpretations of the connected UND problem—a *simultaneous overlay* formulation and a *recursive overlay* formulation. These two interpretations give rise to two different implementations of the composite heuristic. For one cost structure (uniform proportional costs), the simultaneous composite heuristic is easier to analyze and provides tighter worst-case bounds than the recursive method (however, see our comments at the end of Section 3.3); this analysis is a natural extension of the developments in Section 2. The recursive composite heuristic has the advantage of permitting us to derive worst-case bounds for problems with commodity-dependent costs.

3.1.1 Simultaneous overlay formulation

In the simultaneous overlay formulation, the base subproblem selects the edges of a connected network, and we "simultaneously" overlay K origin-to-destination paths corresponding to each of the K commodities on the base design. In this interpretation, the base variable x_j of formulation [P] is 1 if we include edge j in the design, and 0 otherwise. the fixed cost a_j for edge j is the associated base cost, and the set X is the set of all connected subgraphs of G. The overlay decisions for each commodity k determine a route from node O(k) to node D(k), i.e., y_j^k is 1 if we route commodity k on edge j, and is 0 otherwise. Y^k is the set of all paths connecting node O(k) to D(k) in G, and the routing costs b_j^k serve as the overlay costs. The linking constraint (1.3) ensures that we route a commodity k on edge j only if we include this edge in the design. If we ignore this constraint, the simultaneous overlay formulation decomposes into:

(i) a base subproblem which is a **minimum spanning tree (MST)** problem using the fixed edge costs, and

(ii) K overlay subproblems, one corresponding to each commodity; for k = 1, 2, ..., K. the k^{th} overlay subproblem is a **shortest path** problem from node O(k) to node D(k) using commodity k's routing costs.

3.1.2 Recursive overlay formulation

Instead of simultaneously overlaying the flow paths for all K commodities, we can also view the K-commodity UND problem as the overlay of a single commodity, say commodity K, on a base solution that already routes the first (K–1) commodities. For L = 1, 2, ..., K, we refer to the UND subproblem containing only the first L commodities k =

1, 2, ..., L as the **L-commodity UND problem**. Thus, the base solution for the original K-commodity UND model is just a (K–1)-commodity UND solution, which in turn is an overlay of commodity (K–1) on a (K–2)-commodity UND solution, and so on. We call this interpretation of the UND problem as the recursive overlay model. Observe that this formulation extends our original definition of the overlay optimization problem: we now permit the base subproblem to contain variables that are unrelated to the overlay variables. For instance, at the L^{th} stage, the (L–1)-commodity base subproblem contains both the design variables x_j and the routing variables y_j^k for the first (L–1) commodities k = 1, 2, ..., L–1; however, the overlay variables y_j^L for commodity L are directly linked (via constraints (1.3)) only to the design variables x_j.

The connected UND problem satisfies the feasible completion property (see Section 2.1) with completion cost multiplier $\lambda = 1$, since appending any feasible base solution to the overlay solution, in either the simultaneous or recursive overlay formulation, produces a feasible UND solution.

3.1.3 Simultaneous and recursive versions of composite heuristic

The composite heuristic operates differently for the two formulations. Let us first describe the composite heuristic corresponding to the simultaneous overlay framework; we will refer to this method as the **simultaneous composite heuristic**. To construct the (simultaneous) BU solution, the heuristic first selects an MST, and routes each commodity k on the unique O(k)-to-D(k) path on this tree, for k = 1, 2,..., K. The (simultaneous) OC solution first routes each commodity k on its shortest O(k)-to-D(k) path (using the total costs $a_j + b_j^k$). If the union of these paths has more than one component or does not span all the nodes, then we add MST edges in increasing fixed cost sequence to create a connected, spanning network. The *simultaneous composite* solution is the better of the BU and OC solutions.

In contrast, the composite heuristic operates as follows in the recursive overlay framework; we will refer to this method as the **recursive composite heuristic**. At each stage L = 1, 2, ..., K, the algorithm selects the better of the (recursive) BU and OC solutions to the L-commodity UND problem as the L-commodity *composite* solution; the BU and OC heuristics at stage (L+1) build upon this composite heuristic solution. The following description summarizes the recursive composite procedure.

Recursive composite heuristic for Connected UND problems:

Initialization:

The 0-commodity composite design is the MST using the fixed costs a_j.

Iterative step:

For L = 1, 2, ..., K,

Step 1: Find the **L-commodity recursive BU solution**:

- Set the L-commodity BU design equal to the (L–1)-commodity composite design.
- For commodities k = 1,2,...,L–1, use the same routes as the (L–1)-commodity composite solution.
- Route commodity L on the shortest O(L)-to-D(L) path in the L-commodity BU design, using the routing costs b_j^L.

Step 2: Find the **L-commodity recursive OC solution:**

- Route commodity L on its shortest O(L)-to-D(L) path in G using total costs $(a_j+b_j^L)$.
- For commodities k = 1, 2, ..., L–1, use the same routes as the (L–1)-commodity composite solution
- Set the OC design equal to the union of all the edges in the L commodities' routes.

Step 2a: Satisfy the full spanning constraint

- Add to the OC design all other edges of the MST not already in the design.

Step 3: Find the **L-commodity recursive composite solution**:

- Select the better (lower total cost) of the L-commodity BU and OC solutions.

next L;

Note that in Step 2a we add all the MST edges even though some of these edges might create cycles in the existing design. Although this strategy is "suboptimal" (i.e., we can improve the cost by dropping redundant edges), it facilitates our subsequent worst-case analysis of the recursive heuristic. Moreover, the recursive heuristic's worst-case performance is the same whether or not we drop redundant edges, i.e., we later show an example for which the recursive method has the same heuristic-to-optimal ratio, equal to our worst-case bound, even if we drop redundant edges.

To summarize, the recursive composite method builds upon the composite solution to the (L–1)-commodity problem to construct the BU and OC solutions for the L-commodity problem, for L = 1, 2, ...K. We can also interpret the simultaneous composite heuristic as a K-stage procedure that successively adds commodities. Unlike the recursive method, however, the simultaneous method uses only the (L–1)-commodity BU (or OC) solution to build the BU (respectively, OC) solution to the L-commodity problem. That is, it retains the same heuristic (BU or OC) solution when adding each path, whereas the recursive strategy has the flexibility to alter this choice from step to step.

To help in navigating through the various worst-case results we will be considering, we briefly overview the remainder of this paper. Sections 3.2 to 3.4 deal with heuristic performance analysis for the connected UND problem. As before, we separately analyze the proportional costs case and the unrelated costs case. We further partition the proportional costs case into two subcases—**uniform proportional costs** or **commodity-dependent proportional costs**—corresponding to problems in which the ratio of fixed-to-routing costs is the same for all commodities and edges or varies by commodity but is the same for all the edges for a given commodity. Section 3.2 presents heuristic and LP worst-case results for problems with uniform proportional costs; the performance of the simultaneous composite heuristic is easier to analyze for this class of problems and directly extends the results of Section 2.3. In Section 3.3, we analyze the recursive composite heuristic's performance for the commodity-dependent proportional costs case. Section 3.4 develops worst-case results for the unrelated costs case. We provide worst-case examples in each section to show that our bounds are tight. Section 3.5 extends the results for all three cost structures–uniform proportional costs, commodity-dependent proportional costs, and unrelated costs–to the unrestricted UND model.

We note parenthetically that, for the proportional costs network design model (with or without the full spanning constraint), we can assume without loss of generality that the fixed and routing costs are triangular. This property rests upon the fact that, for the proportional costs model, the shortest path between any pair of nodes is the same using either fixed or routing costs. If the original costs do not satisfy the triangle inequality, we can solve the problem over an equivalent complete graph G' in which the fixed and routing costs of each edge {i,j} are the shortest path distances from node i to node j in G. To establish the validity of this claim, note that every feasible solution to the original problem is also feasible for the new graph G' and has the same or lower total cost since our network transformation does not increase any edge cost. Therefore, the optimal value of the new UND problem must be less than or equal to the optimal value of the original problem. Now consider an optimal solution to the new problem. If all the edges in this solution have the same costs as the original problem, then the solution is feasible and has the same total cost for the original problem. Otherwise, suppose the solution routes one or more commodities on an edge {i,j} that has a higher cost in the original graph G (or does not belong to G). Then, we construct an equivalent "original" solution by deleting every such edge {i,j} from the design, and instead rerouting all the commodities on the shortest path from node i to node j in G. Since this modification does not increase the total cost, the resulting solution must be optimal for the original problem.

3.2 Connected UND problems with uniform proportional costs: The simultaneous overlay bound

Let r denote the uniform proportionality constant, i.e., $r = (a_j+b_j^k)/a_j$ for all edges j and for all commodities k = 1, 2, ..., K. Consider the simultaneous overlay formulation, and its associated simultaneous composite heuristic. Let T(a) denote the cost of the MST using edge costs a_j; for connected UND problems, T(a) is the optimal value of the simultaneous overlay formulation's base subproblem. Let $S_{tot}(a)$ denote the sum of the shortest origin-to-destination path lengths for all K commodities using the fixed costs a_j; since $b_j^k = (r-1)a_j$, the sum of the optimal values of all K overlay subproblems in the multi-overlay formulation is $(r-1)S_{tot}(a)$. Therefore, relaxing the linking constraints for the optimal K-commodity UND provides a **lower bound** of $\{(r-1)S_{tot}(a)+T(a)\}$.

The simultaneous OC heuristic routes each commodity on its shortest path, possibly incurring the full cost (i.e., fixed + routing cost) on every edge of this path, and then completes the design by including additional edges in order to satisfy the full spanning constraint. Since these additional edges belong to the MST, the cost of the **OC heuristic solution** cannot exceed $\{rS_{tot}(a)+T(a)\}$. The simultaneous BU heuristic first installs edges of the MST, incurring a fixed cost of T(a), and then routes each commodity on this tree. Since the length of any origin-to-destination path on the MST cannot exceed T(a), the total routing cost for all K commodities cannot exceed K(r−1)T(a). Therefore, the cost of the **BU heuristic solution** is bounded from above by $\{K(r-1)T(a)+T(a)\}$. These observations give the following upper bound on the worst-case ratio ω_{unif} when we apply the simultaneous composite heuristic to the uniform, proportional costs connected UND problem:

$$\omega_{unif} \le \frac{\min\{T(a)(K(r-1)+1),\ rS_{tot}(a)+T(a)\}}{\{(r-1)S_{tot}(a)+T(a)\}},$$

$$= \frac{\min\{K(r-1)+1,\ rs_{tot}+1\}}{\{(r-1)s_{tot}+1\}}. \qquad (3.1)$$

In the last expression, $s_{tot} = S_{tot}(a)/T(a)$ (we assume $T(a) > 0$). To obtain a worst-case bound on ω_{unif}, we maximize the right-hand side of (3.1) by setting $s_{tot} = K(r-1)/r$, and $r = (1 + 1/\sqrt{K})$. Substituting these values in (3.1) gives the following theorem:

Theorem 4:

For the connected UND problem with uniform proportional costs, the simultaneous composite heuristic has a worst-case performance bound of:

$$\omega_{unif} \quad \leq \quad 1 + \frac{K}{2\sqrt{K}+1}. \tag{3.2}$$

Table 1 evaluates this worst-case bound for values of K ranging from 2 to 100.

Since both the overlay (shortest path) and the base (MST) subproblems have LP characterization ratios of 1, using arguments similar to those in Section 2.3, we can show that for connected UND problems with uniform proportional costs, the characterization ratio ν_{unif} of the simultaneous overlay formulation's LP relaxation has the same upper bound as (3.2), i.e.,

$$\nu_{unif} \quad \leq \quad 1 + \frac{K}{2\sqrt{K}+1}. \tag{3.3}$$

3.2.1 Worst-case example

Figure 4(a) shows a two-commodity connected UND example with uniform proportional costs for which the simultaneous composite heuristic achieves the bound of Theorem 4. The two commodities in this example flow from node 1 to nodes 2 and 3, respectively. Both commodities have the same total to fixed cost ratio of $r = 1 + 1/\sqrt{2}$. Figure 4(a) shows the fixed cost for each edge; here, $p = 1/(1+\sqrt{2})$. In the example, both the horizontal multi-node path (having a total fixed cost of p units), and the vertical multi-node path (with total fixed cost of $(1-p)$) have a sufficiently large number of equally spaced intermediate nodes, so that each segment has a small positive fixed cost δ.

The OC heuristic solution shown in Figure 4(b) has a total cost of $(2rp + 1) = \sqrt{2} + 1$. The BU heuristic solution (Figure 4(c)) incurs a fixed cost of 1, and a total flow cost of $2(r-1) = \sqrt{2}$; therefore, the total cost of the BU heuristic solution is $\sqrt{2}+1$. The optimal solution (Figure 4(d)) has a fixed cost of 1, and flow cost of $2p(r-1)$ giving a total cost of $Z^* = \frac{2\sqrt{2}+1}{\sqrt{2}+1}$. Therefore, Z^{Comp}/Z^* is $\{1+\frac{2}{2\sqrt{2}+1}\}$, proving that the bound (3.2) is tight.

We can also construct worst-case examples containing an arbitrary number of commodities, but the networks have a much more complex structure, and do not provide any additional intuition.

3.3 Connected UND problems with commodity-dependent proportional costs: The recursive overlay bound

When the total to fixed cost ratio varies by commodity, the worst-case ratio for the simultaneous heuristic depends on both the maximum and minimum cost ratios, r_{max} and r_{min}, instead of the single proportionality constant r (r_{max} and r_{min} replace r in the numerator and denominator, respectively, of inequality (3.1)). Therefore, for problems with commodity-dependent proportional costs, developing data-independent worst-case bounds for the simultaneous composite heuristic is difficult unless we make additional assumptions regarding r_{max} and r_{min}. However, as we show next, the performance analysis for the recursive composite heuristic does not require such assumptions.

For L = 1, 2,..., K, let ρ_L be the worst-case performance of the recursive composite heuristic for the L-commodity connected UND problem (containing the first L commodities, k = 1, 2,..., L) with commodity-dependent proportinal costs. The 1-commodity connected UND problem is the same as the HND problem, and so $\rho_1 = 4/3$. Since the L-commodity UND problem's recursive formulation contains the (L–1)-commodity problem as its base subproblem, we will first express ρ_L in terms of ρ_{L-1}. Using this expression and starting with $\rho_1 = 4/3$, we can recursively calculate numerical values of ρ_L for L = 2, 3, ..., K.

For k = 1, 2, ..., K, let r_k denote the total to fixed cost ratio for commodity k, i.e., $r_k = (a_j+b_j^k)/a_j$ for all edges j. Let $S_k(a)$ be the cost of the shortest origin-to-destination path for commodity k using the fixed costs a_j; the smallest possible routing cost for commodity k is, therefore, $(r_k-1)S_k(a)$. Assume we have indexed the commodities in decreasing order of their minimum routing cost $(r_k-1)S_k(a)$. Z_L is the (unknown) optimal value of the L-commodity UND problem.

Our analysis follows the same steps as before. We develop upper bounds for the (recursive) OC and BU heuristic solutions and a lower bound on the optimal value by relaxing the constraints that link the base and overlay problems. In the recursive setting, we have the following results.

- At stage L of our recursive composite heuristic procedure, we start with the composite solution for the (L–1)-commodity base subproblem; the total cost (= fixed cost + routing cost for the first (L–1) commodities) is no more than $\rho_{L-1}Z_{L-1}$. The **recursive OC heuristic** adds commodity L's origin-to-destination route (incurring the full cost $a_j+b_j^L$

on edges of this route) to the (L–1)-commodity recursive composite solution. Therefore, the L-commodity OC heuristic solution costs no more than $\{\mathbf{r_L S_L(a)} + \rho_{L-1}\mathbf{Z_{L-1}}\}$.

- Relaxing the linking constraints of the L-commodity recursive problem formulation gives a **lower bound** of $\{\mathbf{(r_L-1)S_L(a) + Z_{L-1}}\}$.

To express ρ_L in terms of ρ_{L-1}, we can use our single-overlay results from Section 2. Since the **recursive BU heuristic** upgrades the heuristic solution to the base problem, it has an upper bound of $\mathbf{r_L}\rho_{\mathbf{L-1}}\mathbf{Z_{L-1}}$. Dividing the minimum of the OC and BU upper bounds by the lower bound, we observe that this case corresponds to a special application of the results in Section 2 with $\rho_B = \rho_{L-1}$, $\rho_O = 1$, and so $\rho = 1/\rho_{L-1}$. Therefore, Theorem 1 provides us with the **basic bound**:

$$\rho_L \quad \leq \quad \frac{4\rho_{L-1}^2}{4\rho_{L-1} - 1}. \tag{3.4}$$

By analyzing the BU heuristic more carefully, we can develop a considerably stronger worst-case bound which we call the **enhanced bound**. Note that in obtaining the inequality (3.4), we used a bound of $r_L \rho_{L-1} Z_{L-1}$ on value of the BU heuristic. This bound assumes that, to route commodity L, we "upgrade" <u>all</u> the activities in the base solution including both the design arcs and the commodity routes for the first (L–1) commodities. To improve this bound, we would like to better estimate the incremental cost of routing commodity L on the (L–1) commodity base solution. We obtain this estimate by computing an upper bound on the total fixed cost of the shortest O(L)-to-D(L) path in the base design. Suppose we can show that the base design contains all the edges of the MST of graph G. Then, since the MST contains an O(L)-to-D(L) path, the route for commodity L chosen by the BU heuristic must have fixed cost less than or equal to the MST cost T(a), i.e., the overlay cost (= total routing cost for commodity L) incurred by the BU heuristic is at most $(r_L-1)T(a)$. Therefore, the **recursive BU heuristic** solution has an upper bound of $\{\rho_{\mathbf{L-1}}\mathbf{Z_{L-1} + (r_L-1)T(a)}\}$. We use induction to prove that the (L–1)-commodity composite solution must contain all the MST edges. For L = 1, this property holds since the MST solves the 0-commodity connected UND problem. Suppose the (L–2)-commodity composite design contains all the MST edges. Then the (L–1)-commodity BU design must also contain these edges since the BU heuristic builds upon the (L–2)-commodity composite solution. Also, at each stage, the recursive OC heuristic includes all the MST edges. Since the composite heuristic at stage (L–1) selects the better of the (L–1)-commodity BU and OC heuristic solutions, the (L–1)-commodity composite design must contain all the MST edges, establishing the desired property.

Let us now develop an upper bound for T(a) in terms of the shortest path length for commodity L and the optimal value Z_{L-1} of the base subproblem. Since Z_{L-1} represents the optimal value of the (L–1)-commodity UND problem, it must be greater than or equal to the lower bound $\{\sum_{k=1}^{L-1} (r_k-1)S_k(a) + T(a)\}$ obtained by ignoring the linking constraints in the simultaneous overlay formulation of the (L–1)-commodity UND problem. Since we have indexed the commodities k in decreasing order of their minimum routing costs $(r_k-1)S_k(a)$, replacing the first term in the lower bound with $(r_L-1)(L-1)S_L(a)$ gives

$$T(a) \quad \leq \quad Z_{L-1} - (r_L-1)(L-1)S_L(a).$$

Substituting for T(a) in the BU upper bound, we see that the cost of the recursive BU heuristic solution at stage L must not exceed

$$\{\rho_{L-1}Z_{L-1} + (r_L-1)[Z_{L-1} - (r_L-1)(L-1)S_L(a)]\}.$$

Normalizing all upper and lower bounds with respect to the optimal base value Z_{L-1}, dividing the minimum of the two heuristic upper bounds by the lower bound, and letting $s_L = S_L(a)/Z_{L-1}$, gives the worst-case performance bound

$$\rho_L \quad \leq \quad \frac{\min\{\rho_{L-1}+(r_L-1)[1-(r_L-1)(L-1)s_L],\ r_L s_L+\rho_{L-1}\}}{(r_L-1)s_L + 1}. \qquad (3.5)$$

To determine worst-case values of s_L and r_L, we consider two cases:

<u>Case 1</u>: $r_L \geq \rho_{L-1}(r_L-1)$, i.e., $r_L \leq \rho_{L-1}/(\rho_{L-1}-1)$.

In this case, the right-hand side of (3.6) achieves its maximum value when

$$s_L^* = \frac{(r_L-1)}{r_L+(L-1)(r_L-1)^2}. \qquad (3.6)$$

For this value of s_L^*, we maximize the right-hand side of (3.5) when

$$r_L^* \quad = \quad 1 + \frac{-(\rho_{L-1}-1) + \sqrt{(\rho_{L-1}-1)^2 + (\rho_{L-1}-1) + L}}{(\rho_{L-1}-1) + L}. \qquad (3.7)$$

We can verify that $r_L^* \leq \rho_{L-1}/(\rho_{L-1}-1)$, i.e., r_L^* satisfies the condition for case 1.

<u>Case 2:</u> $r_L > \rho_{L-1}/(\rho_{L-1}-1)$

In this case, $s_L^* = 0$ is the worst-case value of s_L, giving a worst-case bound of

$$\rho_L \quad \leq \quad \rho_{L-1}. \qquad (3.8)$$

The right-hand side of (3.5), using values of s_L^* and r_L^* from (3.6) and (3.7), is greater than ρ_{L-1}. Therefore, we get the following result.

Theorem 5:

For the connected L-commodity UND problem with commodity-dependent proportional costs, the recursive composite heuristic has a worst-case bound of:

$$\rho_L \quad \leq \quad \frac{r_L^* s_L^* + \rho_{L-1}}{(r_L^*-1)s_L^* \ + \ 1} \quad \text{for } L = 1, 2, ..., K, \qquad (3.9)$$

where s_L^* and r_L^* have the values specified in equations (3.6) and (3.7).

Starting with $\rho_1 = 4/3$ (for the HND problem), Table 1 recursively applies the **enhanced recursive overlay bound** (3.9) to compute ρ_K for various values of K from 2 to 100. The new recursive bound vastly improves upon the basic bound (3.4), providing worst-case ratios for the commodity-dependent proportional costs case that are very close to but larger than the simultaneous overlay bound for UND problems with uniform proportional costs.

Finally, we can specialize the recursive bound to the uniform proportional costs case by requiring that the r_L^* values must be the same, say, r^* for all L = 1, 2, ..., K when we maximize the right-hand side of inequality (3.5). This maximization exercise, sketched in Balakrishnan et al. [1993], is complicated, but produces a better recursive performance bound of $\{1+\frac{K}{2\sqrt{K}+1}\}$, corresponding to the worst-case value of $r^* = \{1+\frac{1}{\sqrt{K}}\}$, for connected UND problems with uniform proportional costs. Note that this bound is the same as the simultaneous heuristic's worst-case performance ratio (see Section 3.2).

The recursive composite heuristic achieves its worst-case bound for the uniform proportional costs example shown in Figure 4(a). In this figure, the (common) value of r is $1 + 1/\sqrt{2}$, and the parameter p is $1/(\sqrt{2}+1)$. Figures 5(a) and 5(b) show the BU and OC solutions at the first stage (L = 1) of the recursive procedure; both these solutions have a total cost of $\{\sqrt{2}+1\}/\sqrt{2}$. At the end of stage 1, the recursive composite heuristic chooses the BU solution (in the worst-case) as the 1-commodity composite solution. Figures 5(c) and 5(d) show the BU and OC solutions at the second stage. The 2-commodity BU solution, shown in Figure 5(c), costs $rp + r = (\sqrt{2}+1)$; the 2-commodity OC solution, shown in Figure 5(d), also costs $(2rp + 1) = (\sqrt{2}+1)$. The optimal solution (Figure 5(d)) costs $Z^* = \{2\sqrt{2}+1\}/\{\sqrt{2}+1\}$. Therefore, Z^{Comp}/Z^* is $\{1 + 2/(2\sqrt{2}+1)\}$, proving that the recursive composite heuristic's specialized bound for problems with uniform proportional costs is tight. This example also shows that the bound remains tight even if we drop redundant edges at each stage of the recursive procedure (for this example, we might drop the vertical edge incident to node 2 in Figure 5(b) and the vertical edge incident to node 3 in Figure 5(d)). As with the simultaneous overlay bound, we can also construct worst-case examples for arbitrary number of commodities.

3.4 Connected UND problems with unrelated costs

As the final problem variant for the connected UND model, we consider problems with unrelated costs, i.e., the ratio of total to fixed costs varies both by edge and commodity. We consider only the OC heuristic, and so our bound does not depend on whether we use the simultaneous or recursive formulations (i.e., recursively applying the single-overlay OC heuristic produces the same solution as the simultaneous OC heuristic).

If $S_k(c^k)$ denotes the shortest path length from origin O(k) to destination D(k) for commodity k using the total cost c_j^k as edge costs, T(a) is the cost of the MST using fixed costs a_j, and Z^* is the optimal value of the K-commodity connected UND problem, then

$$Z^* \geq S_k(c^k) \quad \text{for all } k = 1, 2, \ldots, K, \text{ and} \tag{3.10a}$$

$$Z^* \geq T(a). \tag{3.10b}$$

Furthermore, the cost Z^{OC} of the OC heuristic solution satisfies

$$Z^{OC} \leq \sum_{k=1}^{K} S_k(c^k) + T(a) \tag{3.11}$$

because the OC heuristic routes each commodity on its shortest path (incurring the full cost of this path, in the worst-case), and then adds MST edges to construct a connected design. Combining inequalities (3.10a), (3.10b), and (3.11) we obtain,

$$\omega_{unrel} = \frac{Z^{OC}}{Z^*} \leq K + 1. \tag{3.12}$$

Theorem 6:

For connected K-commodity UND problems with unrelated costs,

$$\omega_{unrel} \leq K + 1.$$

3.4.1 Worst-case example

Figure 6 contains a two-commodity UND example with unrelated costs for which the composite heuristic achieves the bound (3.12). This example is defined over the same network as our previous (proportional costs) example (Figure 4(a)); Figure 6 shows the fixed cost and routing cost for each commodity on every edge. The OC and optimal solutions have the same structure as Figures 4(b) and 4(d). If we use the costs shown in Figure 6, the OC solution has a total cost of 3, while the optimal solution incurs a fixed cost of 1, and no flow cost. Therefore, the ratio of heuristic to optimal value is 3 (= K+1). Again, we can extend this example to arbitrary number of commodities.

3.5 Worst-case analysis for the Unrestricted UND model

In Sections 3.1 to 3.4, we considered one version of the uncapacitated network design problem: one in which the design must be connected. We now examine worst-case results for the unrestricted network design problem. Since this model does not explicitly require the underlying design to span all the nodes, eliminating the connected UND model's full spanning constraint but imposing just the integrality requirements for the x variables (i.e., setting $X = \{x: x_j = 0 \text{ or } 1 \text{ for all edges } j\}$ in the constraint (1.4)) gives a valid formulation for the unrestricted UND problem. However, to obtain a tighter lower bound on the optimal LP relaxation value for our worst-case analysis, we will strengthen the base subproblem by adding certain valid design constraints to X.

Let S be the set of all transshipment nodes (i.e., nodes that do not serve as the origin or destination for any commodity) in G; we refer to these nodes as Steiner nodes. We partition the remaining nodes in N\S into M minimal node subsets N_m, for m = 1, 2, ..., M, each containing at least two nodes such that, for every commodity k, both its origin node O(k) and destination node D(k) belong to the same subset. Therefore, every feasible solution to the unrestricted UND problem must contain a component that spans all the nodes of N_m for m = 1, 2, ..., M, in order to permit O(k)-to-D(k) communication for every commodity k. If M = 1 then every feasible design must contain a Steiner tree with the nodes of N_1 as its terminals and some or all of the nodes in S serving as intermediate Steiner points. If M = 1 and $N_0 = \phi$, we say that the demand pattern is **spanning**. For instance, (n–1) commodities flowing from a single source to every other node of an n-node network define a spanning demand pattern.

To strengthen the unrestricted UND problem's (simultaneous or recursive) formulation we add, for all m = 1, 2, ..., M, a valid **connectedness constraint** specifying that every node of N_m must belong to a single component in the chosen network design. With these constraints, the unrestricted UND problem has the following **Steiner Forest** (SF) problem (also called the Generalized Steiner problem, Goemans and Williamson [1992]) as its base subproblem:

> **Steiner Forest problem:** Find the minimum cost forest of G that interconnects nodes in N_m for all m = 1, 2, ..., M. We permit the path connecting a pair of nodes $i,j \in N_m$ to optionally contain nodes from S and/or from other subsets $N_{m'}$, $m \neq m'$.

The SF problem reduces to the well-known Steiner network problem when M = 1. If the demand pattern is spanning then the SF problem becomes the MST problem.

Since the Steiner network problem is NP-hard, so is the SF problem. In fact, the problem is NP-hard even when S is empty (and M > 1) since we can formulate any given Steiner network problem SP as an SF problem with $S = \phi$. The equivalent SF problem has all the nodes and edges of SP. In addition, for each Steiner node i of SP, the SF problem contains an extra dummy node i', connected to node i by a zero cost edge. The dummy node i' has the same incident edges with the same cost as the edges incident to node i, i.e., for every edge {i,j} in the original problem SP, the SF network contains edge {i',j} with the same cost. This equivalent SF problem has a node subset $N_i = \{i,i'\}$ for each Steiner node i of SP, and an additional subset containing all the terminal nodes of SP. It is easy to show that the optimal solution to SP corresponds to an equal cost optimal solution to the SF problem and vice versa, proving that SF is NP-hard even when $S = \phi$.

Goemans and Williamson [1992] describe a labeling method to heuristically solve a general class of problems that includes the SF problem. The worst-case performance ratio of this heuristic is at most 2.

In the following discussions, we consider unrestricted UND problems with three cost structures—uniform proportional costs, commodity-dependent proportional costs, and unrelated costs. For the uniform proportional costs case, we analyze the worst-case performance of the simultaneous composite heuristic; for problems with commodity-dependent proportional costs, we apply the recursive method.

3.5.1 Simultaneous bound for Unrestricted UND problems with uniform proportional costs

The simultaneous composite heuristic for the unrestricted UND problem differs from its connected UND version in two ways: (i) since the base (SF) subproblem is computationally intractable (except in special cases when it reduces to the MST problem), we must incorporate a heuristic procedure to solve it; and (ii) the OC heuristic does not require an overlay complétion phase since the overlay subproblem's solution (i.e., the union of origin-to-destination paths for all commodities) itself produces a feasible design.

Let ρ_{SF} represent the worst-case performance ratio for the SF solution method, and let $Z_{SF}(a)$ denote the optimal value of the Steiner Forest problem using the fixed costs a_j as edge costs. As before, S(a) denotes the sum of the shortest origin-to-destination path lengths for all K commodities using a_j as edge costs, and r is the total to fixed cost ratio. Note that:

- the **simultaneous BU heuristic** selects the heuristic SF solution as the network design, and routes each of the K commodities on this design; consequently, this solution costs at most $\{\rho_{SF}Z_{SF}(a)+K(r-1)\rho_{SF}Z_{SF}(a)\}$;
- the **simultaneous OC heuristic** solution routes every commodity on its shortest origin-to-destination path, incurring the fixed and routing costs on each path. This solution has an upper bound of $rS(a)$; and,
- deleting the linking constraints (1.3) gives the **lower bound** of $\{(r-1)S(a)+Z_{SF}(a)\}$.

If we normalize all of the upper and lower bounds with respect to $Z_{SF}(a)$ and let $S = S(a)/Z_{SF}(a)$, the composite heuristic's worst-case performance ratio becomes:

$$\omega_{unif} \leq \frac{\min\{[K(r-1)+1]\rho_{SF},\ rS\}}{(r-1)S+1}. \tag{3.13}$$

The right-hand side of (3.13) achieves its maximum value when

$$S^* = \frac{\rho_{SF}\{K(r-1)+1\}}{r}. \tag{3.14}$$

Substituting this value of S in (3.13), and differentiating with respect to r gives the following worst-case value of r that maximizes the right-hand side of (3.13):

$$r^* = \frac{\rho_{SF}(K-1)}{(\rho_{SF}K-1)}\{1+\sqrt{\frac{1}{\rho_{SF}K}}\}. \tag{3.15}$$

Hence, we have:

Theorem 7:

The simultaneous composite heuristic for the unrestricted UND problem with uniform proportional costs has a worst-case performance ratio of

$$\omega_{unif} \leq \frac{r^*S^*}{(r^*-1)S^*+1},$$

where S^* and r^* have the values specified in equations (3.14) and (3.15).

If we solve the SF base subproblem optimally, then r* has the same value $(1+1/\sqrt{K})$ as it did for the connected UND problem (see Section 3.2), but S* is $\sqrt{K}$. In this case, the simultaneous composite heuristic has a worst-case ratio of

$$\omega_{unif} \leq \frac{\sqrt{K} + 1}{2}, \tag{3.16}$$

which is lower than the method's performance ratio of $\{1 + K/(2\sqrt{K}+1)\}$ for the connected UND problem (see inequality (3.2)). This observation has the following interesting implication. Recall that, when the demand pattern is spanning, the full spanning constraint is valid even for the unrestricted UND model, and so the connected UND model's simultaneous overlay bound (3.2) applies. However, bound (3.16) is superior to (3.2) because the bound (3.2) uses a higher value for the OC heuristic's upper bound that includes an additional cost to complete the overlay solution, whereas this completion step is unnecessary for spanning demand patterns.

Table 2 shows values of the simultaneous heuristic's performance ratio for the unrestricted UND problem with uniform proportional costs; we compute the bounds for both $\rho_{SF} = 1$ and $\rho_{SF} = 2$ (e.g., if we solve the overlay subproblem using Goemans and Williamson's heuristic) for selected values of K ranging from 2 to 100.

3.5.2 Recursive bound for Unrestricted UND problems with commodity-dependent proportional costs

In the recursive formulation of the unrestricted UND problem, we assume that, for all values of L from 0 to K, the L-commodity UND formulation contains the complete set of SF constraints corresponding to the original problem (i.e., although the first L commodities might have less stringent connectedness requirements, we retain the original SF constraints). With this assumption, the 1-commodity UND problem has the SF problem as its base subproblem. We discuss only the "enhanced" version of the recursive bound (i.e., the counterpart of the bound (3.9)) for the unrestricted UND problem. Let $S_k(a)$ denote the length of the shortest path from O(k) to D(k) using the fixed edge costs a_j. As in Section 3.3, we assume we have indexed the commodities k in order of decreasing routing costs $(r_k-1)S_k(a)$. Let Z_L and ρ_L denote the optimal value and heuristic worst-case ratio for the L-commodity problem, and let **HSF** be the heuristic solution to the SF problem; its cost does not exceed $\rho_{SF}Z_{SF}(a)$.

At stage 1 of the recursive composite heuristic procedure, the BU heuristic routes commodity 1 on HSF, while the OC heuristic adds the edges in HSF to commodity 1's shortest origin-to-destination path. The better of these two solutions is the composite 1-commodity solution. In subsequent stages L = 2, 3, ..., K,

(i) the **BU heuristic** routes commodity L on the shortest O(L)-to-D(L) path (using the flow costs b_j^L) in the (L–1)-commodity composite design. This design satisfies the SF constraints, and, therefore, contains at least one O(L)-to-D(L) path;

(ii) the **OC heuristic** first selects the shortest O(L)-to-D(L) route in G (using the total costs $a_j+b_j^L$), uses the same routes as the (L–1)-commodity composite solution for the first (L–1) commodities, and adds all the edges of HSF to complete the OC design; and,

(iii) we select the better of the BU and OC solutions as the L-commodity **recursive composite solution.**

Observe that, while the simultaneous OC heuristic for the unrestricted UND problem does not require an overlay completion step, the recursive OC heuristic must complete the solution at each stage (in step (ii) above) by including all the edges of the heuristic SF solution. This completion step is necessary in order to ensure that, in subsequent steps, the BU heuristic can build upon the composite solution (the BU heuristic assumes that the L-1 commodity composite solution, which could be either the BU or OC solution from previous steps, contains a route from O(L) to D(L)). Therefore, the recursive composite heuristic has the same underlying structure for both the connected and unrestricted UND problems except in the first stage (L = 0); the connected version starts with the MST at stage L = 0, while the "unrestricted" recursive procedure starts with the heuristic SF solution.

Since the composite (L–1)-commodity solution contains all the edges of HSF, the routing cost for commodity L in the BU heuristic solution cannot exceed $(r_L-1)\rho_{SF}Z_{SF}(a)$. Therefore, the total cost of the L-commodity **recursive BU heuristic** solution is at most $\{\rho_{L-1}Z_{L-1}+(r_L-1)\rho_{SF}Z_{SF}(a)\}$. Following the same arguments as in Section 3.3, we find that the **recursive OC heuristic** solution has the upper bound $\{r_L S_L(a)+\rho_{L-1}Z_{L-1}\}$, and the optimal value of the L-commodity UND problem must be at least $\{(r_L-1)S_L(a)+Z_{L-1}\}$.

Let us now express the optimal SF value $Z_{SF}(a)$ in terms of Z_{L-1}. Relaxing the linking constraints (1.3) in the simultaneous overlay formulation for the (L–1)-commodity UND problem gives the lower bound:

$$Z_{L-1} \;\geq\; \sum_{k=1}^{L-1} (r_k-1)S_k(a) + Z_{SF}(a). \qquad (3.17)$$

Since we have indexed the commodities in decreasing order of their minimum routing costs, $(r_k-1)S_k(a) \geq (r_L-1)S_L(a)$ for all $k = 1, 2, ..., L-1$. Therefore, from (3.17) we obtain

$$Z_{SF}(a) \;\leq\; \{Z_{L-1} - (r_L-1)(L-1)S_L(a)\}. \qquad (3.18)$$

Replacing $Z_{SF}(a)$ with its upper bound (3.18) in the BU heuristic's upper bound, and dividing the composite heuristic solution value (which is the minimum of the BU and OC heuristic solution values) by the L-commodity UND lower bound gives the following upper bound on the recursive composite heuristic's performance ratio ρ_L for the L-commodity UND problem:

$$\rho_L \;\leq\; \frac{\min\{\rho_{L-1}+(r_L-1)\rho_{SF}\{1-(r_L-1)(L-1)s_L\},\; r_L s_L+\rho_{L-1}\}}{(r_L-1)s_L + 1}. \qquad (3.19)$$

In this expression, $s_L = S_L(a)/Z_{L-1}$.

If $r_L < \rho_{L-1}(r_L-1)$, then $s_L = 0$ maximizes the right-hand side of (3.19), and hence $\rho_L \leq \rho_{L-1}$. Otherwise, if $r_L \geq \rho_{L-1}(r_L-1)$, substituting the following worst-case values of s_L and r_L in (3.19) gives a data-independent recursive upper bound for ρ_L in terms of ρ_{L-1}:

$$s_L^* \;=\; \frac{\rho_{SF}(r_L-1)}{\rho_{SF}(r_L-1)^2(L-1) + r_L}, \text{ and} \qquad (3.20)$$

$$r_L^* \;=\; \frac{\rho_{SF}L + \sqrt{(\rho_{L-1}-1)^2+(\rho_{L-1}-1)+\rho_{SF}L}}{(\rho_{L-1}-1) + \rho_{SF}L}. \qquad (3.21)$$

Theorem 8:

For the unrestricted L-commodity UND problem with commodity-dependent proportional costs, the recursive composite heuristic has a worst-case bound of

$$\rho_L \;\leq\; \frac{r_L^* s_L^* + \rho_{L-1}}{(r_L^*-1)s_L^* + 1} \quad \text{for } L = 1, 2, ..., K, \qquad (3.22)$$

where s_L^* and r_L^* have the values specified in equations (3.20) and (3.21).

Starting with $\rho_0 = \rho_{SF}$, we can compute a numerical value for the worst-case ratio ρ_K for the K-commodity unrestricted UND problem by recursively applying (3.22), and using values of s_L^* and r_L^* from (3.20) and (3.21), for $L = 1, ..., K$. Table 2 reports the worst-case ratios for selected values of K from 2 to 100, for both $\rho_{SF} = 1$ and 2. For problems with spanning demand patterns, the values corresponding to $\rho_{SF} = 1$ apply. Note that the bounds in this case are the same as the recursive bounds for connected UND problems shown in Table 1. This equivalence holds because, in the recursive framework, the OC heuristic for unrestricted UND problems does not differ from its connected counterpart, i.e., both versions apply the overlay completion step. Finally, the recursive bound for unrestricted UND problems with commodity-dependent costs is slightly larger than the simultaneous bound for the uniform proportional costs case; we previously observed the same phenomenon for the connected UND model.

3.5.3 Unrestricted UND problems with unrelated costs

If $S_k(c^k)$ denotes the length of the shortest origin-to-destination path for commodity k using the total costs c_j^k as edge lengths, then the OC heuristic has an upper bound of

$$Z^{OC} \quad \leq \quad \sum_{k=1}^{K} S_k(c^k),$$

while the optimal value Z^* of the unrestricted UND problem must be at least

$$Z^* \quad \geq \quad \text{Max } \{S_k(c^k): k = 1, 2, \ldots K\}.$$

Therefore, we have:

Theorem 9:

For unrestricted UND problems with unrelated costs, the OC heuristic's worst-case performance ratio is at most:

$$\omega_{unrel} \quad \leq \quad K. \tag{3.23}$$

Note that, for UND problems with spanning demand, this "unrestricted" OC heuristic provides a better bound than the tight bound of (K+1) that we developed for the connected UND problem.

In this section, we have shown how the principles underlying our heuristic and LP performance analysis of Section 2 apply to multi-overlay optimization problems. We proposed and analyzed two alternative overlay interpretations–a simultaneous multi-overlay formulation, and a recursive single-overlay representation–of the uncapacitated network design problem. Using composite heuristics based upon these two formulations, we developed heuristic worst-case performance guarantees for connected as well as unrestricted UND problems with uniform proportional costs, commodity-dependent proportional costs, and unrelated costs.

4. Concluding Remarks

Overlay optimization is a broad problem class, encompassing a variety of models that apply to problem contexts such as facility location and telecommunications planning. Our analysis in this paper began by developing bounds on a composite heuristic and an LP relaxation for the "single-overlay" optimization model. The bounds on the composite heuristic depend upon how accurately we can solve both the base and overlay subproblems. If we can solve them both optimally, then the composite heuristic is guaranteed to find a solution whose objective value is within 33% of the value of the optimal solution. As we lose accuracy in solving the base and overlay subproblems, the performance bound for the composite heuristic becomes worse. Similarly, if the linear programming relaxations of the base and overlay problems are exact (the LPs generate integer solutions), then the objective value of the linear programming relaxation of the overall overlay optimization problem (with the linking constraints) is within 33% of the optimal value of the problem, and as the linear programming relaxations of these subproblems becomes less accurate, then the bounds on the accuracy of the overall linear programming relaxation become worse.

We next explored multi-overlay extensions of the basic single-overlay model to characterize the worst-case performance for various versions of the multi-commodity, uncapacitated network design problem. In this discussion, we considered three different cost structures (proportional, uniformly proportional, and general) and two different solution strategies: simultaneous and recursive. Our results provide performance guarantees on these heuristics and, in one case, on a linear programming relaxation of the problem; the performance bounds on either the recursive or simultaneous composite heuristics depend upon the number of commodities in the network design problem (they grow as the square root of the number of commodities). We provided worst-case examples to prove that these bounds are tight.

Our general heuristic and linear programming worst-case results in Section 2 can apply directly to various types of overlay optimization problems such as the HND and TLND problems, and for these two problem classes the heuristic bounds are tight. We can use the general results to analyze other network configuration problems as well. For instance, consider a "**forest-on-tree**" problem, which generalizes the TLND problem in the following way. The solution to the TLND problem must connect every primary node to every other primary node via a primary path. Instead, suppose we are given Q (> 1) disjoint subsets of primary nodes. Every pair of nodes within each subset must communicate via a primary path, but we require only secondary paths (but also permit primary paths) connecting primary nodes belonging to different subsets. Each secondary node must connect to some primary node via a primary or secondary path. For this problem, the overlay subproblem is the Steiner Forest problem described in Section 3.5, and the base problem is a minimum spanning tree problem. The forest-on-tree problem satisfies the feasible completion property with $\lambda = 1$. Therefore, the results of Theorems 1 and 2 apply. If we use Goemans and Williamson's heuristic to solve the overlay (Steiner Forest) subproblem, then the composite heuristic procedure generates a solution that is at most twice as expensive as the optimal cost. The TLND problem is a special case of the forest-on-tree problem, and we know that for this problem the bound of 2 for the composite heuristic (but using a minimum spanning heuristic to solve the overlay subproblem) is tight. We conjecture that the forest-on-tree problem's worst-case bound is also tight even though we use a different heuristic for the overlay subproblem.

Even if the results of Section 2 do not apply directly or produce loose bounds, the general approach and principles underlying our performance analysis might itself prove to be worthwhile in studying other specific instances of the overlay optimization model. As we have seen in our discussion of the uncapacitated network design problems, if we examine problems with special structure or those that extend the overlay optimization model (for example, by adding additional variables to the base or overlay subproblems), the specifics of our analysis might be different, even though the overall approach is much the same.

In a companion paper, we address a new class of multi-tier, multi-connected (survivable) network design problems that arise in telecommunications planning. We apply the overlay optimization principles of this paper to different versions of two-tier, two-connected problems, exploiting their special structure to improve upon the generic worst-case bounds developed in Section 2. For instance, one problem variant that we study has a worst-case bound of only 1.25, lower than the 4/3 rds bound that we derived in this paper for general overlay optimization problems (assuming we solve both the overlay and base subproblems optimally).

Several questions, both theoretical and applied, arise as a result of this work. First, since the overlay optimization problem's heuristic and LP worst-case bounds depend upon the bounds for the corresponding base and overlay subproblems, strengthening the underlying subproblem formulations and developing better heuristics for these subproblems can improve the performance for the overlay optimization model. Second, for our analysis, we obtained lower bounds by ignoring the linking constraints (1.3). We can view this lower bound as the optimal value of the Lagrangian relaxation when we dualize the linking constraints using zero Lagrangian multipliers. Can we improve the worst-case performance ratios by considering certain special non-zero multiplier values that provide better, but analytically tractable, bounds?

A third and very alluring research direction to pursue is the extension of our approach to a more general decomposition framework. In this paper, we have considered only a "simple" class of forcing constraints $y_j^k \leq x_j$ for all j and for all k. To generalize this approach, we might consider a wider class of "complicating" constraints whose removal from the formulation decomposes the problem into a base subproblem, and one or more overlay problems. For instance, can we extend the analysis to problems with, say, "exclusivity" constraints of the form

$$y_j^{k1} + y_j^{k2} \leq x_j,$$

or more general "bundle" constraints and variable upper bounds? (In the network design context, the "exclusivity" constraint prevents commodities k1 and k2 from both flowing on the same edge j.) We note that the lower bound remains valid even for overlay models with these more general linking constraints since we obtain this bound by ignoring the linking constraints. However, our upper bounds depend on the structure of the linkage. In particular, for problems that have the simple forcing constraints (1.3) and satisfy the condition $X \subseteq Y^k$, the BU heuristic is valid, i.e., it produces a feasible solution to the overlay optimization problem by constructing a feasible base solution, and setting $y_j^k = x_j$ to satisfy the linking constraints (1.3) and the overlay constraints (1.2). This property might not hold for more general linking constraints unless we make appropriate feasibility assumptions.

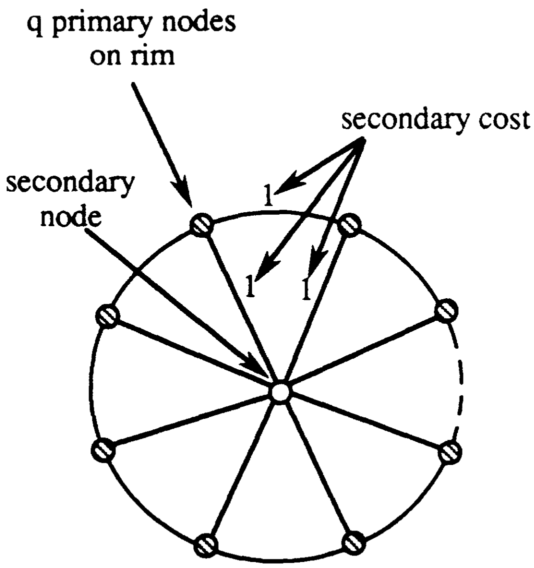

Figure 1(a): TLND example

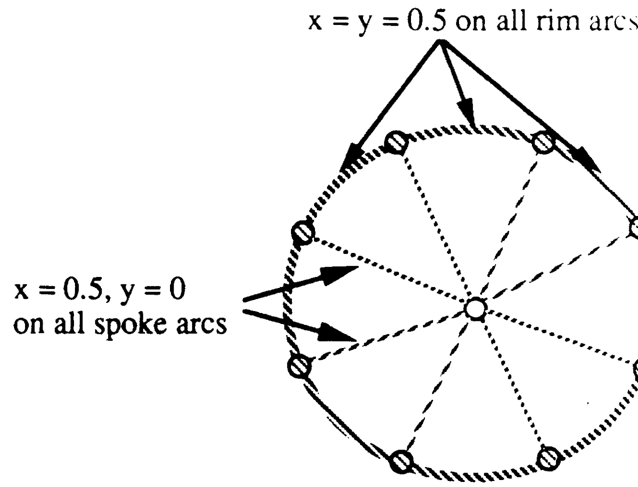

Figure 1(b): LP solution

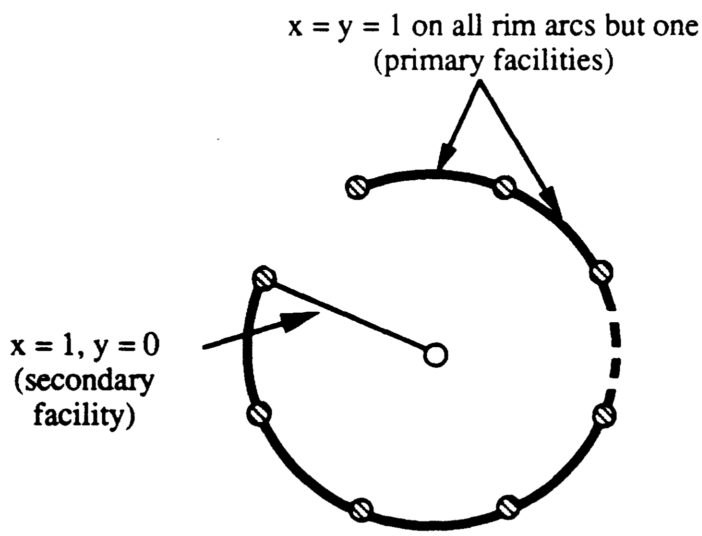

Figure 1(c): Optimal solution

Figure 1: LP worst-case example for TLND problem with proportional costs

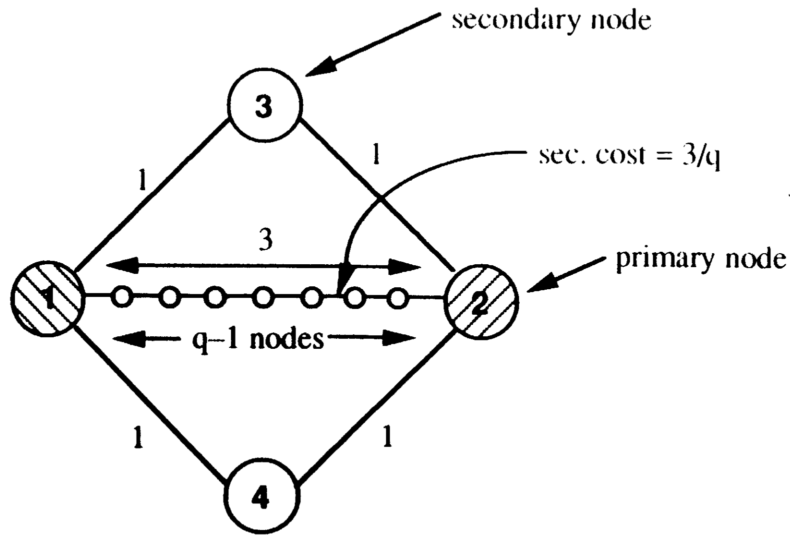

Figure 2(a): HND example:
cost ratio = 2; secondary costs shown on edges

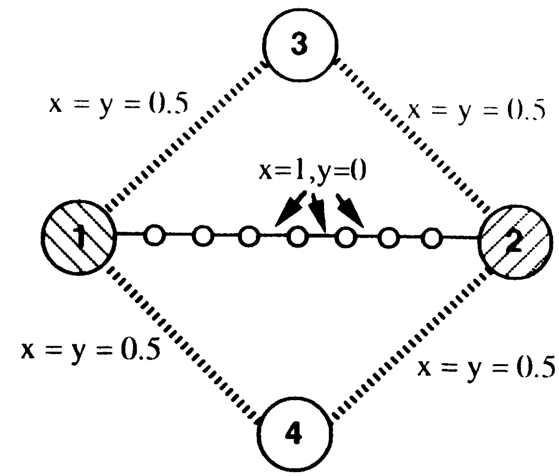

Figure 2(b): Optimal LP solution

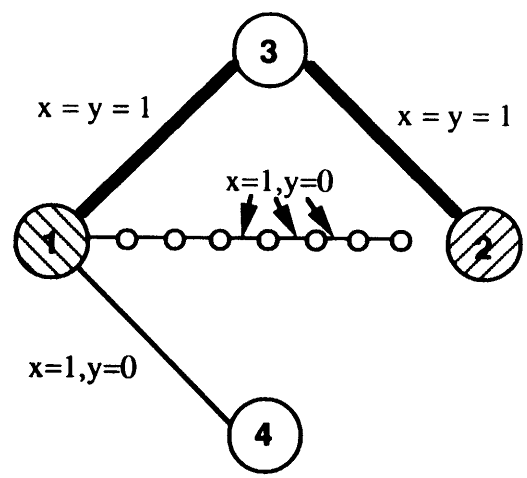

Figure 2(c): Optimal HND solution

Figure 2: LP performance for HND problem with proportional costs

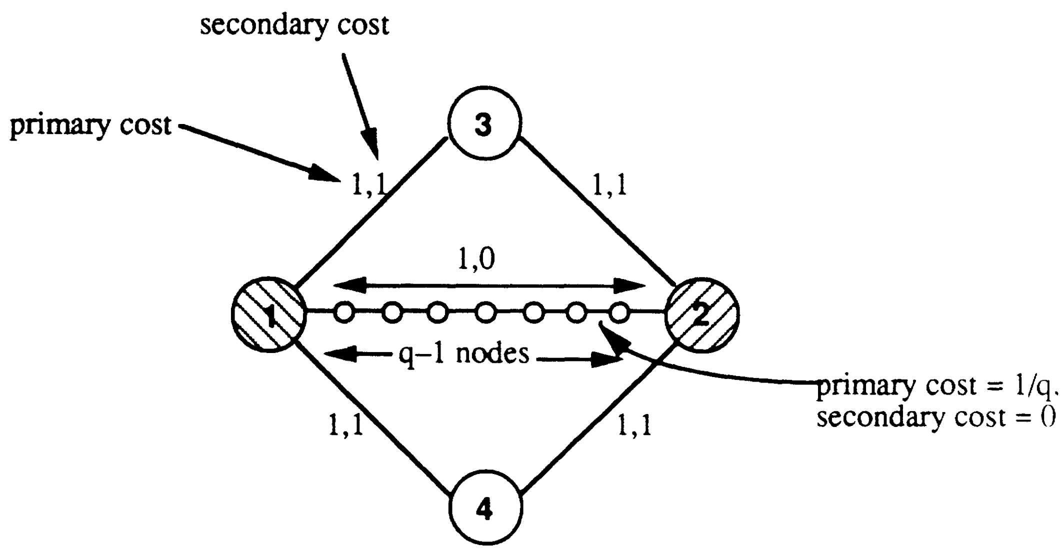

Figure 3: LP performance for HND problem with unrelated costs

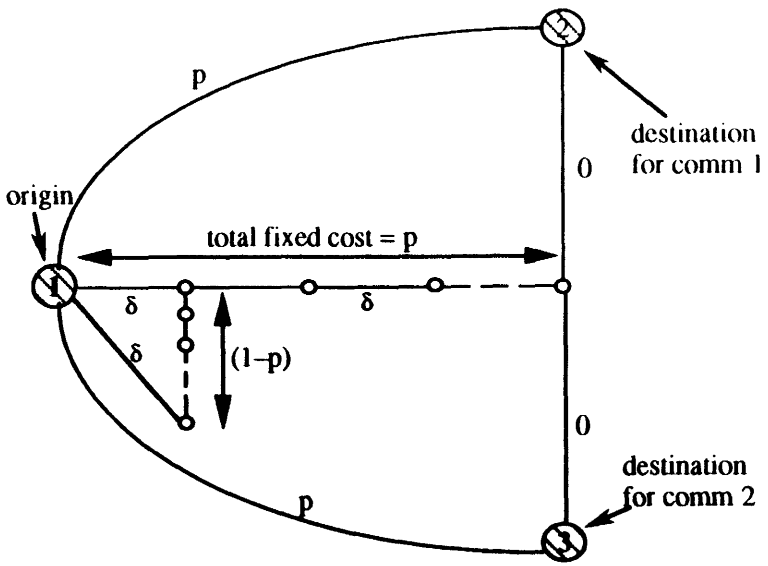

Figure 4(a): 2-commodity UND example

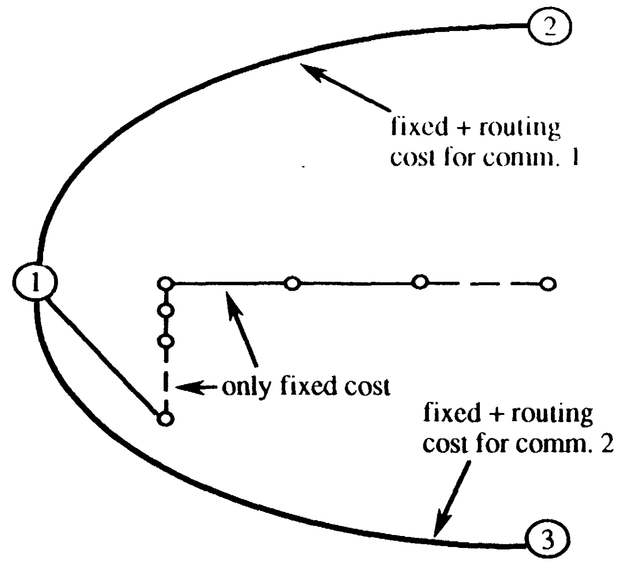

Figure 4(b): Simultaneous OC heuristic solution

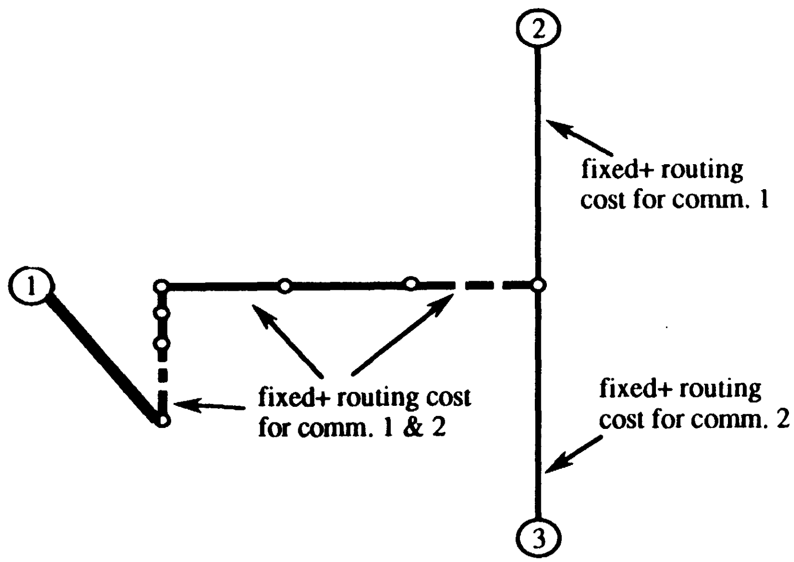

Figure 4(c): Simultaneous BU heuristic solution

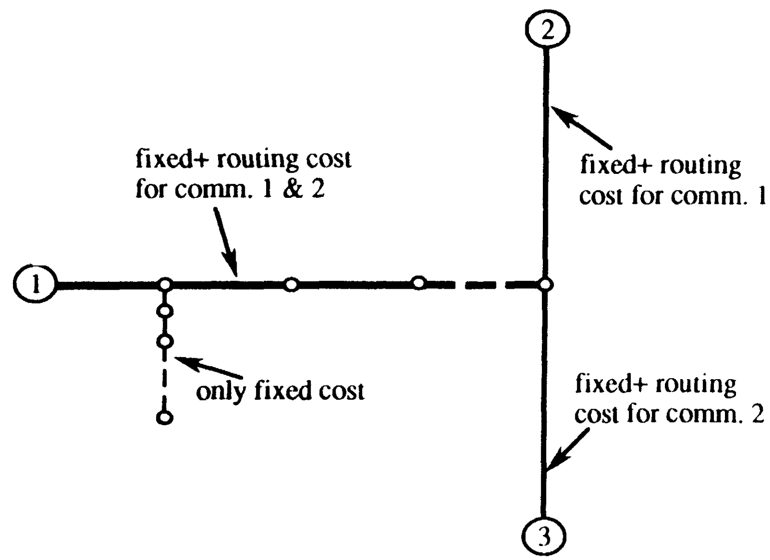

Figure 4(d): Optimal solution

Figure 4: Worst-case example for connected UND problem with uniform proportional costs using Simultaneous Composite heuristic

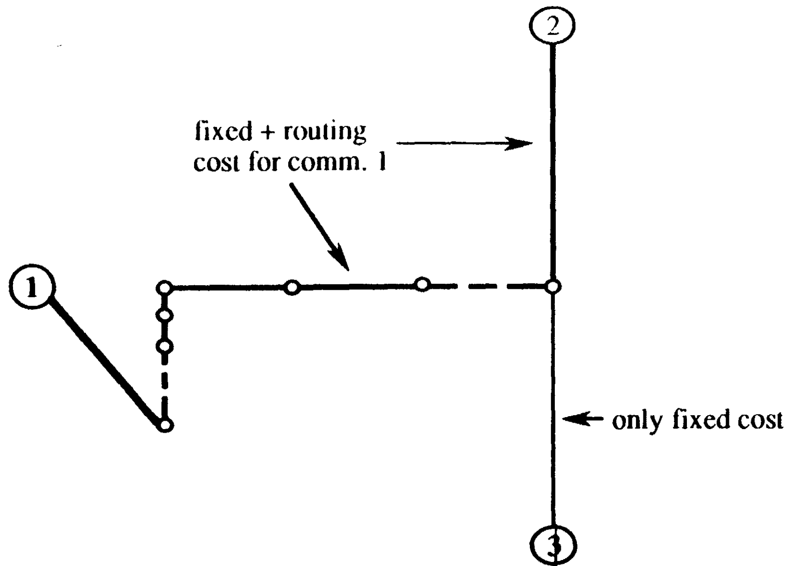

Figure 5(a): Recursive BU solution for 1-commodity problem

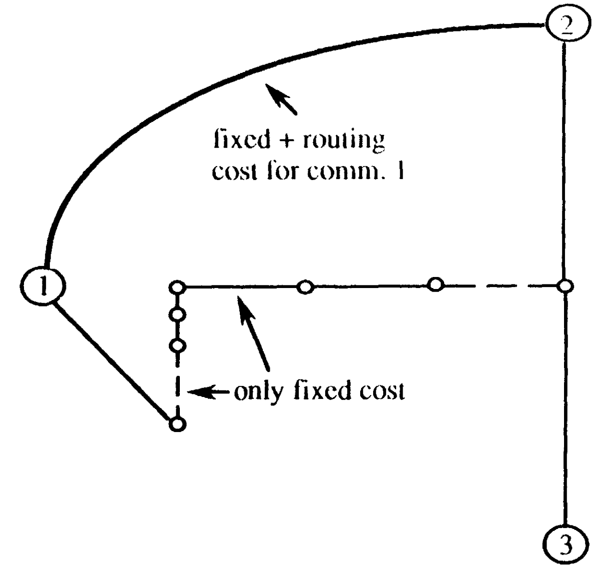

Figure 5(b): Recursive OC solution for 1-commodity problem

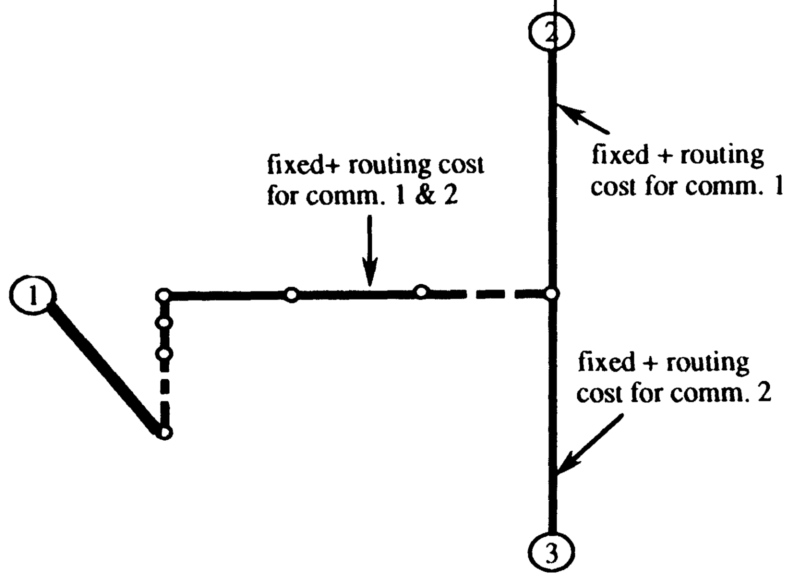

Figure 5(c): Recursive BU solution for 2-commodity problem

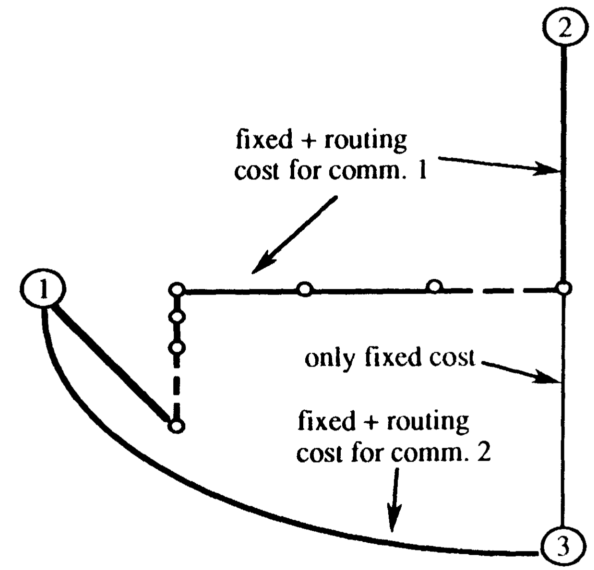

Figure 5(d): Recursive OC solution for 2-commodity problem

Figure 5: Worst-case example for connected UND problem with uniform proportional costs using Recursive Composite heuristic

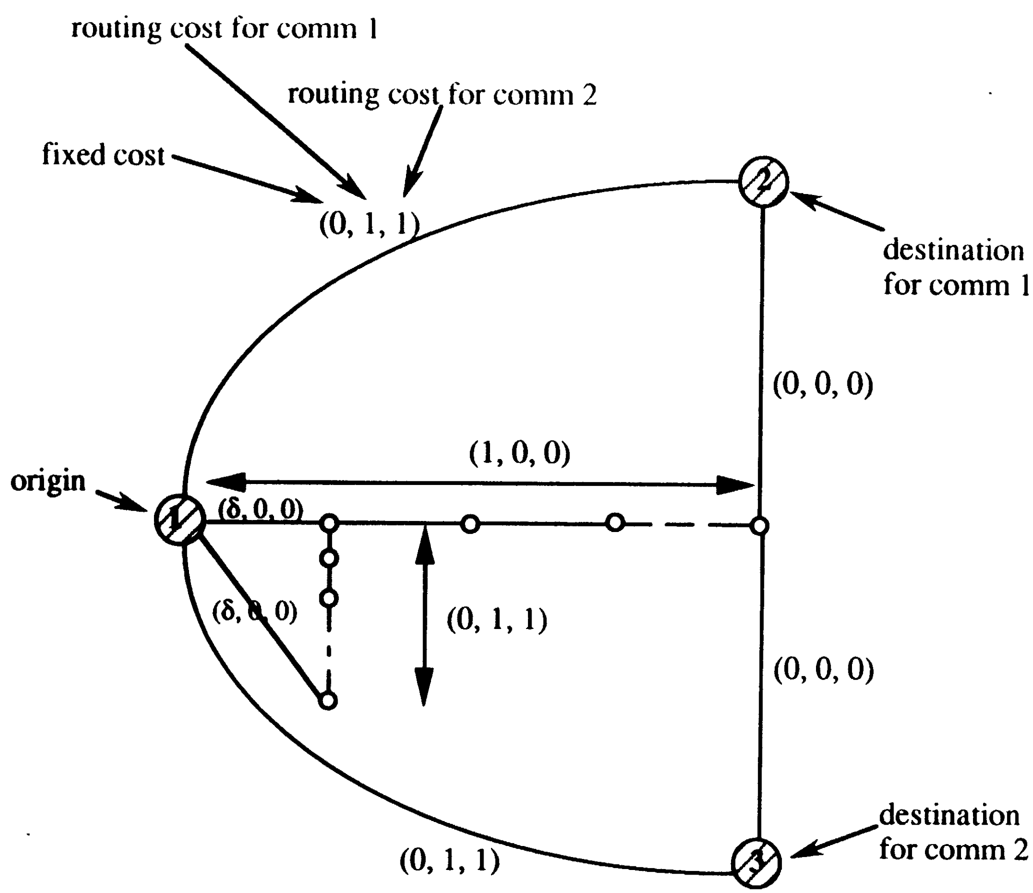

Figure 6: Worst-case example for Connected UND problem with Unrelated costs

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