Analysis of Vaidya's Volumetric Cutting Plane Algorithm

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Abstract

We analyze several aspects of Vaidya's volumetric cutting plane method for finding a point in a convex set $\mathcal{C} \subset \Re^n$. At each step of the algorithm we have a bounded polyhedron $\mathcal{P}$ that contains the convex set $\mathcal{C}$ and an interior point $x \in \mathcal{P}$. The polyhedron $\mathcal{P}$ undergoes either constraint additions or constraint deletions as we iterate through the algorithm with constraints that are added being provided by an oracle that furnishes a hyperplane that separates the interior point x from $\mathcal{C}$. The number of constraints are not allowed to grow indefinitely, but are deleted when they cease to have any significant effect on the system. Following the addition or deletion of a constraint, the algorithm takes a small number of Newton steps to re-optimize the volumetric barrier $\mathcal{V}(\cdot)$. The algorithm is terminated when either it is discovered that $x \in \mathcal{C}$, or $\mathcal{V}(\cdot)$ becomes large enough to demonstrate that the volume of $\mathcal{C}$ is smaller than a minimum allowed value indicating that $\mathcal{C}$ is empty.

Our theory follows that of Anstreicher that makes use of a quadratic convergence result for Newton's method applied to $\mathcal{V}(\cdot)$ that gives greater control over the proximity measures as well as allowing us to use the Hessian of the volumetric barrier $\mathcal{V}(\cdot)$ in the Newton steps that we take as opposed to the matrix that Vaidya uses that approximates the role played by the Hessian. We differ from Anstreicher's approach in that we seek to set the parameter τ that determines the placement of the separating hyperplane at its maximum value, thus bringing the separating hyperplane as close as possible to the test point. With this in mind, we arrive at a set of values for the algorithm's parameters; achieving an increase in the value of τ and also reducing the maximum number of constraints that are carried at the expense of taking additional Newton steps following both a constraint addition and deletion.

In the practical implementation stage we analyze a black box volumetric centering complexity model where we (i) remove all restrictions placed on τ, (ii) include a linesearch and (iii) we study the complexity under the assumption that the number of Newton steps taken will be $\mathcal{O}(1)$ in order to re-center after a constraint addition or deletion. Under (i) and (ii) we arrive at promising values for our parameters after runs of our algorithm on randomly generated instances of the convex set $\mathcal{C}$. This involves varying the value of τ, varying the number of bisections performed in the linesearch procedure and examining different dimensions of the problem to determine what combination of these parameters has the greatest influence on the efficiency of the algorithm.

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Chapter 1

Introduction

Let $\mathcal{C} \subset \Re^n$ be a convex set for which there is an oracle with the following property. For any $z \in \Re^n$, if $z \in \mathcal{C}$ then the oracle returns a 'Yes', otherwise the oracle returns a 'No' together with a vector $c \in \Re^n$ that acts as a separating hyperplane, ie $\mathcal{C} \subseteq \{x : c^T x \geq c^T z\}$. The feasibility problem that we consider is the problem of finding a point in the set $\mathcal{C}$ given an oracle for $\mathcal{C}$. We start off by making the assumption that $\mathcal{C}$ is contained in a ball of radius 2^L centered at the origin and that if $\mathcal{C}$ is non-empty then it contains a ball of radius 2^{-L}. Using these assumptions the volumetric cutting plane algorithm ensures that $\mathcal{C}$ will always be contained within a bounded polytope $\mathcal{P}$, represented by the constraint system $Ax \geq b$, and that at each step of the algorithm we will have an interior point $x \in \mathcal{P}$ that we will call our test point. Before calling the oracle to provide a separating hyperplane for the test point, it is verified that the constraints that define the polytope $\mathcal{P}$ all have some effect on the system, i.e., are not too far away from our test point, otherwise the algorithm removes one of these constraints in a constraint deletion operation. The volume of $\mathcal{P}$ is bounded by a function that decreases as a result of constraint additions and deletions as we iterate through the algorithm; progress being measured in terms of changes in the volumetric barrier function. During the course of the algorithm the description of $\mathcal{P}$ can become complicated as a result of many constraint additions; constraint deletions that will subsequently be performed cause $\mathcal{P}$ to be replaced by a simpler region that contains it and at the same time maintains the boundedness of the polytope. Such a replacement trades volume for computational efficiency, with the constraints to be deleted being those that cease to have any significant effect on the system.

At the heart of the algorithm is the *volumetric barrier* function ,$\mathcal{V}(\cdot)$, defined by

$$\mathcal{V}(x) = \tfrac{1}{2}\ln(\det(A^T S^{-2} A)) \tag{1.1}$$

where the matrix S is a diagonal matrix whose diagonal entries are the elements of the vector $s = Ax - b > 0$. The volumetric barrier function is originally due to Vaidya (1989), in which he presented his $\mathcal{O}(nL)$ iteration cutting plane algorithm for linear programming.

The test point that is used at each iteration is an approximation of the unique point w that minimizes the determinant of the Hessian of the logarithmic barrier for $\mathcal{P}$. Specifically, the logarithmic barrier is the function $-\sum_{i=1}^{m}\ln(a_i^T x - b_i)$ and its Hessian is given by $G(x) = A^T S^{-2} A$. Vaidya calls the point w that minimizes $\mathcal{V}(x) = \frac{1}{2}\ln(\det(G(x))$ over $\mathcal{P}$ the *volumetric center*. Other algorithms proposed by Sonnevend (1988), Goffin, Haurie and Vial (1992), and Ye (1992) used the analytic centers in the cutting plane framework, however the complexity of these algorithms are substantially inferior to that of Vaidyas's volumetric algorithm.

Owing to computational difficulties inherent in finding the volumetric center of a constraint system and making use of the strict convexity of the function $\mathcal{V}(\cdot)$, following a constraint addition or deletion we proceed to take a series of Newton steps starting at our test point and ending with a good approximation to the volumetric center of the new system thus formed. Here we follow the approach taken by Anstreicher (1994c) namely using what he refers to as 'true' Newton steps that employ the actual Hessian of $\mathcal{V}(\cdot)$ as opposed to Vaidya's damped 'Newton-like' steps that replace the Hessian of $\mathcal{V}(\cdot)$ with a matrix with promising properties that approximates the role played by the Hessian. This, together with a quadratic convergence result, from Anstreicher (1994b), for Newton's method applied to $\mathcal{V}(\cdot)$ in a sufficiently close vicinity of w provides greater control over the iterates proximity to the exact (but unknown) minimizer w. Anstreicher also succeeds in getting substantially better bounds on proximity measures than Vaidya does, and he accomplishes this through explicity working with infinity norms. Control over the number of planes defining $\mathcal{P}$ is maintained through constraint deletions in such a way so as to ensure that the number of defining hyperplanes does not grow beyond $\mathcal{O}(n)$. Finally, the volume of $\mathcal{P}$ decreases by a fixed constant factor (independent of the dimension n) at each iteration on the average, and the algorithm halts with a point in $\mathcal{C}$ or with the volume of the polytope $\mathcal{P}$ dropping below that of a ball of radius 2^{-L} in $\Re^n$ with the conclusion that $\mathcal{C}$ is empty, in $\mathcal{O}(nL)$ iterations.

From a theoretical perspective the volumetric algorithm has not been fully analyzed, this being mainly due to it being a novel interior point method particularly with the notion of the volumetric barrier that is not used elsewhere in the extensive interior point literature. Vaidya suggests an intuitive though crude answer to the question of why the volumetric center w is a good test point, that lies in the fact that the Dikin ellipse of the volumetric center has maximum volume among all Dikin ellipses within $\mathcal{P}$ and can be thought of as a local quadratic approximation to $\mathcal{P}$. Thus, he argues, that a plane through w while dividing its Dikin ellipse into two parts of equal volume has a good chance of dividing the polytope $\mathcal{P}$ into two parts of equal volume, and so if the process of cutting $\mathcal{P}$ through w is iterated the volume would be expected to decrease at a good rate. The algorithm fares well in relation to the ellipsoid algorithm because it makes more use of information as the cutting planes generated by the oracle are maintained for several steps and continue to directly influence the choice of the test point.

1.1 Overview

We start off with a convex set $\mathcal{C}$ that will always be contained within a bounded polytope $\mathcal{P}$ over which the algorithm maintains control through a series of constraint additions and deletions. The strict convexity of $\mathcal{V}(\cdot)$ and the boundedness of $\mathcal{P}$ ensures

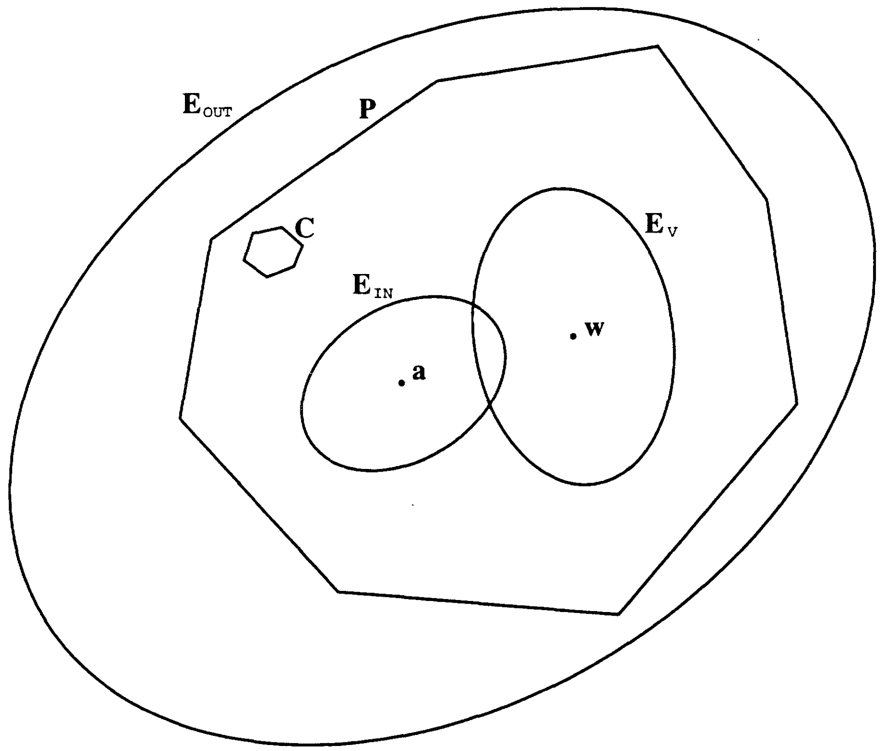

Figure 1-1: Underlying geometric representation

that the volumetric center w and the analytic center (denoted by a in Figure 1-1) will always exit. The various centers and associated Dikin ellipses are depicted in Figure 1-1 together with the convex set $\mathcal{C}$ and the bounding polytope $\mathcal{P}$. Now, for a symmetric positive definite matrix A, we let $E(A, x, r)$ denote the ellipsoid given by

$$E(A, x, r) \;=\; \{y \,:\, (y-x)^T A (y-x) \leq r^2\}$$

By Proposition A.1.1 (see Appendix) we have that $\mathcal{P} \subseteq E_{OUT} = \{x \mid (x-a)^T G(a)(x-a) \leq m^2\}$, where $G(\cdot)$ is the Hessian of the logarithmic barrier function associated with $\mathcal{P}$, i.e., on expanding the Dikin ellipse represented by $E_{IN} = \{x \mid (x-a)^T G(a)(x-a) \leq 1\} \subset \mathcal{P}$ by a factor of m we manage to contain the polytope $\mathcal{P}$. The point $w \in \mathcal{P}$ is unique in that amongst all Dikin ellipses associated with points in $\mathcal{P}$, the Dikin ellipse associated with w, E_V, has maximum volume. This is easily seen from the definition of the point w that minimizes $\det(A^T S^{-2} A)$ over $\mathcal{P}$. Thus, as evidenced in

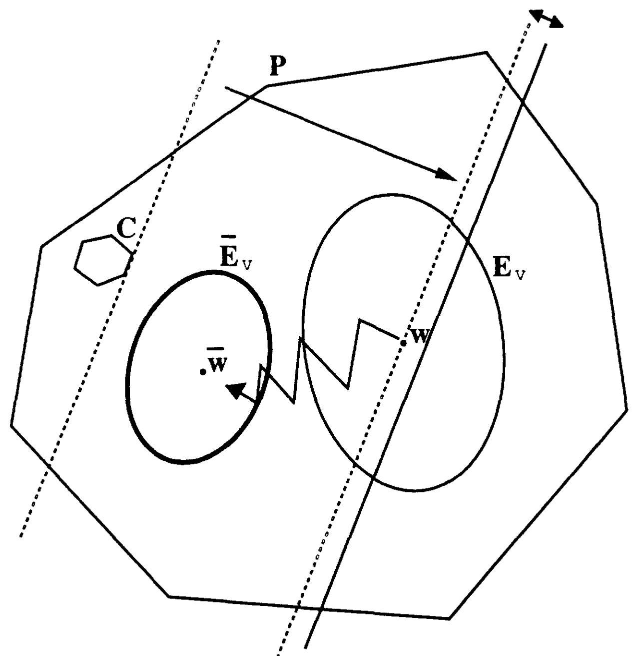

Figure 1-2: Adding a constraint and moving closer to the new w

Figure 1-1, we have that,

$$Vol\,E_{IN} \leq Vol\,E_V \leq Vol\mathcal{P} \leq Vol\,E_{OUT} \tag{1.2}$$

Now bounding the volume of the polytope $\mathcal{P}$,

$$\begin{aligned} Vol\,E_{OUT} &= Vol\,E(\frac{1}{m^2}G, a, 1) \\ &= S_n.\det\left[\frac{1}{m^2}G(a)\right]^{-1/2} \\ &\leq S_n.m^n\det\left[G(a)\right]^{-1/2} \\ &\leq S_n.m^n\det\left[G(w)\right]^{-1/2} \leq (2m)^n e^{-\mathcal{V}(w)} \end{aligned} \tag{1.3}$$

where S_n is the volume of the unit ball in $\Re^n$. It follows from (1.2) and (1.3) that $Vol\mathcal{P} \leq (2m)^n e^{-\mathcal{V}(w)}$.

The number of planes defining $\mathcal{P}$ is not allowed to increase indefinitely, but is kept in check through the use of a parameter that decreases with constraint additions. This parameter which is denoted by $\sigma_{\min}$ and is the smallest diagonal element of

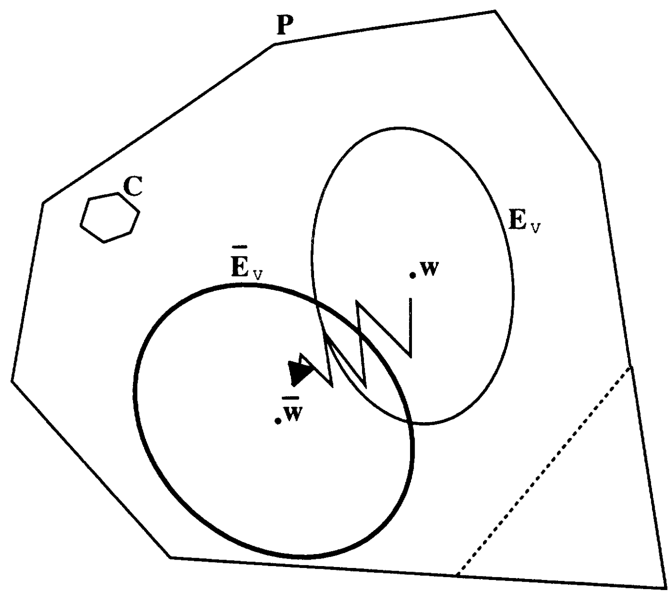

Figure 1-3: Deleting a constraint and moving closer to the new w

the projection matrix P associated with the polytope $\mathcal{P}$ (see Appendix A.3 for some properties of projection matrices). Thus, the criterion employed for dropping a plane a_i is if $\min_{1\leq i\leq m} \sigma_i(z) < \epsilon$ where z is our test point and ϵ is set beforehand. An important result, $\sum_{i=1}^{m} \sigma_i(x) = n$ (see A.4.1 in the Appendix), means that the number of planes that define $\mathcal{P}$ never exceed n/ϵ which implies that $m = \mathcal{O}(n)$. Thus if our algorithm can guarantee the long term increase of $\mathcal{V}(\cdot)$ we can succeed in driving the volume of $\mathcal{P}$ down to zero and in so doing we can be assured of termination. So our approach will be to drive the volume of E_V to zero and this directly causes $\mathcal{V}(\cdot)$ to increase indefinitely and in turn will cause the volume of $\mathcal{P}$ to fall.

The type of computation performed during an iteration is either one based on a constraint addition or a constraint deletion depending on the value of

$$t = \min_{1\leq i\leq m} \{\sigma_i(z)\},$$

where z is the current test point.

If $t \geq \epsilon$ then we proceed to add a plane to the polytope $\mathcal{P}$, as shown in Figure 1-2, with the separating hyperplane that the oracle returns being used as the constraint that is to be added to give the new system. If the convex set $\mathcal{C}$ that we use is itself a polytope, the separating hyperplane can simply be taken to be the first constraint of this set that our test point violates. This new hyperplane is 'backed off' from the test point allowing Newton steps to be taken which are complemented by line searches to move closer towards the new volumetric center.

If $t < \epsilon$ then we delete the constraint that corresponds to the minimum σ_i from the constraint system that defines $\mathcal{P}$, (see Figure 1-3). Since the volumetric center w shifts as a result of a plane removal we again take Newton steps complemented by line searches to move closer towards the new volumetric center.

1.2 Notation, assumptions and preliminaries

If x, s, or σ is a vector in $\Re^n$ then X, S, or Σ refers to the $n \times n$ diagonal matrix with diagonal entries corresponding to the components of x, s, or σ. Let e be the vector of ones, $e = (1, \ldots, 1)$. If $x \in \Re^n$ then $\|x\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$ refers to the p-norm of x, where $1 \leq p \leq \infty$; thus we have,

$\|x\|_1 = |x_1| + \cdots + |x_n|$;

$\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$ (Euclidean norm which we also denote by $\|x\|$) ;

$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$ (modulus of the largest component of x) ;

Next, for any positive semi-definite matrix B we use the notation $\|\xi\|_B$ to denote the proximity measure $\sqrt{\xi^T B \xi}$ and in order to compare the positive definiteness of matrices the following notation is used $A \succ (\succeq) B \iff A - B$ is positive (semi) definite, with $\prec$, $\preceq$ defined analogously. For the matrix M we define $|M| = \sqrt{\max \lambda_i(M^T M)}$, i.e., the square root of the largest eigenvalue of the matrix $M^T M$.

The Schur, or Hadamard, product of two matrices which we denote by $A \circ B$ is defined by multiplying corresponding entries in the respective matrices, ie. $(A \circ B)_{ij} = A_{ij} \times B_{ij}$ and we will denote the Schur product of a matrix A with itself by $A^{(2)}$.

For $x \in int(\mathcal{P})$ let $\Sigma(x, r)$ be the region

$$\Sigma(x,r) = \{y : \forall i,\ 1 \le i \le m,\ \ 1 - r \le \frac{a_i^T y - b_i}{a_i^T x - b_i} \le 1 + r\} \tag{1.4}$$

and note that if $r \le 1$ then $\Sigma(x, r) \subseteq \mathcal{P}$.

The volumetric barrier function $\mathcal{V}(x)$ is defined by $\mathcal{V}(x) = \frac{1}{2}\ln(\det(A^T S^{-2} A))$, where $s(x) = Ax - b > 0$, and A is an $m \times n$ matrix with linearly independent columns. Here, n refers to the dimension of the underlying space and m is the number of constraints of the system $Ax \ge b$. For a given $s > 0$, the projection onto the range of $S^{-1}A$ (see Appendix A.3 for some properties of projection matrices) can then be written as

$$P(s) = S^{-1}A(A^T S^{-2} A)^{-1} A^T S^{-1} \tag{1.5}$$

For $s > 0$ we then define the vector $\sigma(s)$ to be the vector of the diagonal entries of $P(s)$, ie. $\sigma_i = P_{ii}(s)$, $i = 1, \ldots, m$. From the definition of the projection matrix P we then have that

$$\sigma_i = \frac{1}{s_i^2} a_i^T (A^T S^{-2} A)^{-1} a_i \tag{1.6}$$

The gradient and Hessian of $\mathcal{V}(\cdot)$ at x (see Appendix A.4) are then given by

$$\begin{aligned} g &= g(x) = \nabla \mathcal{V}(x)^T = -A^T S^{-1} \sigma \\ H &= H(x) = \nabla^2 \mathcal{V}(x) = A^T S^{-1}(3\Sigma - 2P^{(2)}) S^{-1} A \end{aligned} \tag{1.7}$$

Letting $Q = Q(x) = A^T S^{-2} \Sigma A$, then $Q(x)$ is a good approximation to $H(x)$, in that (see Appendix A.5)

$$Q(x) \preceq H(x) \preceq 3Q(x) \tag{1.8}$$

There is a quantity that plays an important role in maintaining control over the proximity of the iterates in the algorithm. It is defined differently in Vaidya (1989) than in Anstreicher (1994c). Vaidya denotes this quantity by $\mu(x)$ and defines it as the largest number λ satisfying the condition that $Q(x) \succeq \lambda G(x)$ and later bounds $\mu(x)$ by $1/4m$. However, Anstreicher defines this quantity explicitly as $\mu(x) = (2\sqrt{\sigma_{\min}} - \sigma_{\min})^{-1/2}$ after obtaining the bound shown in Lemma 2.1 in the next chapter. The role that $\mu(x)$ plays will become apparent in the subsequent chapters.

Let $p = p(x) = -H^{-1}g$ denote the Newton direction for $\mathcal{V}(\cdot)$ at x. The new point after taking a Newton step is denoted by means of the bar ($\bar{}$) notation, ie. $\bar{x} = x + p$, $\bar{s} = s(\bar{x})$, $\bar{\sigma} = \sigma(\bar{s})$, $\bar{\mu} = \mu(\bar{x})$, $\bar{g} = g(\bar{x})$, $\bar{p} = p(\bar{x})$, $\bar{H} = H(\bar{x})$, $\bar{Q} = Q(\bar{x})$.
To represent the constraint system after a constraint addition or a constraint deletion has occured we use the tilde ($\tilde{}$) notation, e.g. $\tilde{s} = \tilde{s}(x) = \tilde{A}x - \tilde{b}$, $\tilde{Q}(x) = \tilde{A}^T\tilde{S}^{-2}\tilde{\Sigma}\tilde{A}$, $\tilde{\mathcal{V}}(x) = \frac{1}{2}\ln(\det(\tilde{A}^T\tilde{S}^{-2}\tilde{A}))$, etc., to denote quantities which depend on the current point x, but are defined using the new constraint system $[\tilde{A}, \tilde{b}]$. On a constraint addition the system $[A, b]$ will be augmented to obtain the new sytem $[\tilde{A}, \tilde{b}]$ such that

$$\tilde{A} = \begin{pmatrix} A \\ a_{m+1}^T \end{pmatrix}, \qquad \tilde{b} = \begin{pmatrix} b \\ b_{m+1} \end{pmatrix} \tag{1.9}$$

and on constraint deletions (assuming for simplicity that the mth constraint is the one to be deleted) the new reduced system is of the form $[\tilde{A}, \tilde{b}]$, where

$$A = \begin{pmatrix} \tilde{A} \\ a_m^T \end{pmatrix}, \qquad b = \begin{pmatrix} \tilde{b} \\ b_m \end{pmatrix} \tag{1.10}$$

Finally, as we progress through the algorithm we denote the sequence of iterates by x^k, where $k \geq 0$ is the current iteration. Thus, we are naturally led to use the following abbreviated nomenclature: $s^k = s(x^k)$, $\sigma^k = \sigma(x^k)$, $\mu^k = \mu(x^k)$, $g^k = g(x^k)$, $H^k = H(x^k)$, $Q^k = Q(x^k)$. Also, at each iteration the bounded polytope that contains $\mathcal{C}$ is denoted by $\mathcal{P}^k$ and is of the form

$$\mathcal{P}^k = \{x \in \Re^n \mid A^k x \geq b^k\}$$

where A^k is an $m_k \times n$ matrix with independent columns, and $b^k \in \Re^{m_k}$. Whenever we refer to the set $\mathcal{P}^k$, we are implicitly refering to the algebraic representation given by the constraint system $[A^k, b^k]$, and the volumetric barrier associated with $\mathcal{P}^k$ is the function $\mathcal{V}^k(x) = \frac{1}{2}\ln(\det(A^{k^T} S^{-2} A^k))$, where $s = A^k x - b^k$.

Chapter 2

The volumetric barrier

In this chapter we will collect together a number of properties of the volumetric barrier function $\mathcal{V}(\cdot)$ which will be used in subsequent analysis. Many of the results that will be established and the approach that will be taken relies on Anstreicher's (1994b) quadratic convergence result for Newton's method applied to $\mathcal{V}(\cdot)$ for points sufficiently close to the volumetric center w (see Appendix B).

We are now in a position to analyze some of the properties of the volumetric barrier and the proximity measure $\|\cdot\|_H$. Let us begin by presenting the following lemma, Anstreicher (1994c), that provides a better bound on the $\|\cdot\|_Q$ measure than Vaidya (1989) achieves by explicitly working with the infinity norm $\|S^{-1}A\xi\|_\infty$.

Lemma 2.1 Let x have $s = s(x) > 0$, and let $\sigma = \sigma(s)$. Then $\forall \xi \in \Re^n$,

$$\xi^T Q \xi \geq (2\sqrt{\sigma_{\min}} - \sigma_{\min})\|S^{-1}A\xi\|_\infty^2$$

Proof : Applying the same technique as in the proof of Theorem A.5.2 (see Appendix) and using the same change of variables, proving the lemma is equivalent to proving that

$$\bar{\xi}^T U^T \Sigma U \bar{\xi} \geq (2\sqrt{\sigma_{\min}} - \sigma_{\min})\,\|U\bar{\xi}\|_\infty^2 \tag{2.1}$$

Proceeding as in Theorem A.5.2 but replacing (A.14) by the following relaxation of the problem

$$\begin{aligned} \min \quad & \|u_1\|^2 + \sigma_{\min} \sum_{i=2}^{m} (u_i^T \bar{\xi})^2 \\ \text{s.t.} \quad & \sum_{i=2}^{m} (u_i^T \bar{\xi})^2 = \|\bar{\xi}\|^2 - 1 \end{aligned} \tag{2.2}$$

the solution value of which is obviously $\|u_1\|^2 + \sigma_{\min}(\|\bar{\xi}\|^2 - 1)$. Since $1 = |u_1^T \bar{\xi}| \leq \|u_1\| \, \|\bar{\xi}\| \Rightarrow \|u_1\|^2 \geq 1/\|\bar{\xi}\|^2$. Letting $\theta = \|\bar{\xi}\|^2 \geq 1$, the solution value in (2.2) is therefore no lower than the solution value in the minimization problem

$$\min_{\theta \geq 1} \left\{ \frac{1}{\theta} + \sigma_{\min}(\theta - 1) \right\} \tag{2.3}$$

A straightforward calculation shows that the solution in (2.3) is $\theta = 1/\sqrt{\sigma_{\min}}$, with objective value $2\sqrt{\sigma_{\min}} - \sigma_{\min}$, proving (2.1) and the lemma. □

Let us define $\mu = \mu(x) = (2\sqrt{\sigma_{\min}} - \sigma_{\min})^{-1/2}$. Then Lemma 2.1 and (1.8) imply that

$$\delta = \|S^{-1}Ap\|_\infty \leq \mu \|p\|_Q \leq \mu \|p\|_H \tag{2.4}$$

There are two quantities that the algorithm will need to maintain explicit control over, namely the measure $\|p\|_H$ and the quantity $\mu\|p\|_H$ and they will later be used to argue that following a constraint addition or deletion only a small number of Newton steps suffice to return the current iterate to a suitable proximity of w. The results of the following lemma follow from the quadratic convergence result, namely Theorem B.7 (see Appendix) and (2.4). It establishes quadratic convergence properties written entirely in terms of the measures we seek control over, in addition to a relationship between Q and $\bar{Q}$ that will be needed in the proof of Theorem 2.4.

Lemma 2.2 Let x have $s = s(x) > 0$. Assume that $\mu\|p\|_H \leq .014$, and let $\bar{x} = x + p$. Then

i) $\|\bar{p}\|_{\bar{H}} \leq 21.6\mu\|p\|_H^2$,

ii) $\bar{\mu}||\bar{p}||_{\bar{H}} \leq 21.6\mu^2||p||_H^2$,

iii) $Q \preceq \exp(6.02\mu||p||_H)\bar{Q}$

Proof : Using Theorem B.7, (2.4) and the fact that $||p||_Q \leq ||p||_H$, we get that

$$||\bar{p}||_{\bar{H}} \leq \frac{19\mu(1+\mu||p||_H)^2}{(1-\mu||p||_H)^6}\,||p||_H^2 < 21.27\mu||p||_H^2 \tag{2.5}$$

where the last inequality uses the assumption that $\mu||p||_H \leq .014 \Rightarrow$ i) holds true. Next, since $\delta \leq 1$ $\bar{x} \in \Sigma(x,\delta) \subset \Sigma(x,\mu||p||_H)$ and it follows from Proposition B.3 that

$$\frac{(1-\delta)^2}{(1+\delta)^2} \leq \frac{\bar{\sigma}_i}{\sigma_i} \leq \frac{(1+\delta)^2}{(1-\delta)^2} \tag{2.6}$$

and this gives us that $\bar{\sigma}_{\min} \geq \sigma_{\min}(1-\delta)^2/(1+\delta)^2$. Since the function $2\sqrt{\gamma}-\gamma$ is monotone increasing for $\gamma \in [0,1]$, it follows that

$$2\sqrt{\bar{\sigma}_{\min}} - \bar{\sigma}_{\min} \geq 2\sqrt{\sigma_{\min}}\left(\frac{1-\delta}{1+\delta}\right) - \sigma_{\min}\left(\frac{1-\delta}{1+\delta}\right)^2 \geq (2\sqrt{\sigma_{\min}} - \sigma_{\min})\left(\frac{1-\delta}{1+\delta}\right) \tag{2.7}$$

and therefore

$$\bar{\mu} = \mu(\bar{x}) = (2\sqrt{\bar{\sigma}_{\min}} - \bar{\sigma}_{\min})^{-1/2} \leq \mu\left(\frac{1+\delta}{1-\delta}\right)^{1/2} \leq \mu\left(\frac{1+\mu||p||_H}{1-\mu||p||_H}\right)^{1/2} \tag{2.8}$$

where the last inequality uses (2.4). Substituting $\mu||p||_H \leq .014$ into (2.8), and combining the resulting bound with (2.5), proves part ii). Next from combining Proposition B.3 and (B.1) we get that

$$\xi^T Q \xi \leq \frac{(1+\delta)^4}{(1-\delta)^2}\,\xi^T \bar{Q} \xi$$

$\forall \xi \in \Re^n$, and therefore

$$\ln\left(\frac{\xi^T Q \xi}{\xi^T \bar{Q} \xi}\right) \leq 4\ln(1+\delta) - 2\ln(1-\delta) \tag{2.9}$$

But for $0 \leq \delta < 1$, $\ln(1+\delta) \leq \delta$, and $\ln(1-\delta) \geq -\delta - .5\delta^2/(1-\delta) = -\delta[1+.5\delta/(1-\delta)]$.

Combining these facts with (2.9), and using $\mu||p||_H \leq .014$, we obtain

$$\ln\left(\frac{\xi^T Q \xi}{\xi^T \bar{Q} \xi}\right) \leq 4\mu||p||_H + 2\mu||p||_H\left(1 + \frac{.014}{2(1-.014)}\right) < 6.02\mu||p||_H$$

proving iii). □

Lemma 2.3 Let $\mathcal{P} = \{x \mid Ax \geq b\}$, where the columns of A are independent, and assume that the interior of $\mathcal{P}$ is nonempty. Then $\mathcal{P}$ is bounded $\Leftrightarrow$ $\mathcal{V}(\cdot)$ attains its minimum at a unique point w of $\mathcal{P}$.

Proof : If $\mathcal{P}$ is bounded then $\mathcal{V}(\cdot)$ clearly attains its minimum over $\mathcal{P}$ at a unique interior point, since $\mathcal{V}(\cdot)$ is strictly convex in the interior of $\mathcal{P}$, and $\mathcal{V}(x) \to \infty$ as x approaches a boundary point of $\mathcal{P}$. Now assume that $\mathcal{P}$ is not bounded, hence $\mathcal{P}$ must contain a ray, say r, such that if $x \in \mathcal{P}$ then $x' = x + \theta r \in \mathcal{P}$ for all $\theta \geq 0$. Letting $s(\theta) = A(x+\theta w) - b = s + \theta A w > 0$ (since $Aw \geq 0$, $Aw \neq 0$) and using the fact that $A^T S(\theta)^{-2} A = \sum_i \frac{a_i a_i^T}{(s_i + \theta a_i^T w)^2}$ on letting $\theta \to \infty$, we see that the matrix $A^T S(\theta)^{-2} A$ tends to become more and more like that of a Null matrix $\Rightarrow$ $\det(A^T S(\theta)^{-2} A) \to 0 \Rightarrow \mathcal{V}(x') \to -\infty$ and therefore no minimizer w can exist. □

Theorem 2.4 Let x have $s = s(x) > 0$, and assume that $\mu||p||_H \leq .014$. Then $\mathcal{V}(\cdot)$ has a unique minimizer $w \in int(\mathcal{P})$, and $\mathcal{V}(x) - \mathcal{V}(w) \leq 1.11||p||_H^2$.

Proof : Consider an infinite sequence of Newton steps initiated at $x^0 = x$, $x^{k+1} = x^k + p^k$ for $k \geq 0$. Applying Lemma 2.2,

$$\mu^1||p^1||_{H^1} \leq 21.6(\mu^0||p^0||_{H^0})^2 \leq 21.6(.014)^2 < .014$$

and by induction it follows that $\mu^k||p^k||_{H^k} \leq .014$ for all $k \geq 0$. Lemma 2.4 then implies

$$||p^{k+1}||_{H^{k+1}} \leq 21.6\mu^k||p^k||_{H^k}^2 \leq 21.6(.014)||p^k||_{H^k} < .31||p^k||_{H^k} \tag{2.10}$$

forall $k \geq 0$, and therfore $||p^k||_{H^k} \to 0$. Also, since $\mathcal{V}(\cdot)$ is strictly convex, the subgradient inequality implies that

$$\mathcal{V}(x^{k+1}) \geq \mathcal{V}(x^k) + g^{k^T} p^k = \mathcal{V}(x^k) - \|p^k\|^2_{H^k} \tag{2.11}$$

If it is the case that $x^k \rightarrow w \in int(\mathcal{P})$. Then $H(w)$ being positive definite, (2.10) implies that $g(w) = 0$, and therefore w is the unique minimizer of $\mathcal{V}(\cdot)$. Moreover, using (2.10) and (2.11) we have

$$\begin{aligned} \mathcal{V}(w) &= \mathcal{V}(x^0) + \sum_{k=0}^{\infty}[\mathcal{V}(x^{k+1}) - \mathcal{V}(x^k)] \\ &\geq \mathcal{V}(x^0) - \sum_{k=0}^{\infty} \|p^k\|^2_{H^k} \\ &\geq \mathcal{V}(x^0) - \sum_{k=0}^{\infty} (.097)^k \|p^0\|^2_{H^0} \\ &\geq \mathcal{V}(x^0) - 1.11\|p^0\|^2_{H^0} \end{aligned} \tag{2.12}$$

as claimed in the lemma. To complete the proof we must prove that the sequence $\{x^k\}$ converges to a point $w \in int(\mathcal{P})$ and to bound the sequence we will first prove that

$$Q = Q^0 \preceq 1.14\, Q^k \tag{2.13}$$

for all $k \geq 0$. Now, since

$$\frac{\xi^T Q^0 \xi}{\xi^T Q^k \xi} = \prod_{j=0}^{k-1} \frac{\xi^T Q^j \xi}{\xi^T Q^{j+1} \xi} \tag{2.14}$$

and part iii) of Lemma 2.2 implies that

$$\ln\left(\frac{\xi^T Q^0 \xi}{\xi^T Q^k \xi}\right) = \sum_{j=0}^{k-1} \frac{\xi^T Q^j \xi}{\xi^T Q^{j+1} \xi} \leq \sum_{j=0}^{k-1} 6.02 \mu^j \|p^j\|_{H^j} \tag{2.15}$$

and through repeated application of part ii) of Lemma 2.2 we get

$$\mu^j \|p^j\|_{H^j} \leq (21.6)^{2^j - 1} (\mu^0 \|p^0\|_{H^0})^{2^j}, \quad j \geq 0 \tag{2.16}$$

Substituting (2.16) into (2.15), using $\mu^0\|p^0\|_{H^0} \leq .014$, gives us

$$\begin{aligned}
\ln\left(\frac{\xi^T Q^0 \xi}{\xi^T Q^k \xi}\right) &\leq 6.02 \sum_{j=0}^{k-1} (21.6)^{2^j-1} (\mu^0 \|p^0\|_{H^0})^{2^j} \\
&\leq \frac{6.02}{21.06} \sum_{j=0}^{\infty} (21.6\mu^0 \|p^0\|_{H^0})^{2^j} \\
&< .28 \sum_{j=0}^{\infty} (.31)^{j+1} \\
&= \frac{.28(.31)}{1-.31} < .13 \qquad (2.17)
\end{aligned}$$

Exponentiating (2.17) proves (2.13). Using (2.13), (1.8) and (2.10) we then have that

$$\begin{aligned}
\|x^k - x^0\|_Q &\leq \sum_{j=0}^{k-1} \|x^{j+1} - x_j\|_Q \\
&\leq \sqrt{1.14} \sum_{j=0}^{k-1} \|p^j\|_{H^j} \\
&\leq \sqrt{1.14} \sum_{j=0}^{\infty} (.31)^j \|p^0\|_{H^0} \\
&< 1.55 \|p^0\|_{H^0} \qquad (2.18)
\end{aligned}$$

But from Lemma 2.1, $\|S^{-1}A\xi\|_\infty \leq \mu\|\xi\|_Q \; \forall \xi$. Letting $\xi = x^k - x^0$, (2.18) implies that

$$\|S^{-1}A(x^k - x^0)\|_\infty \leq 1.55\mu\|p^0\|_{H^0} \leq .022$$

and therefore

$$.978 \leq \frac{s_i^k}{s_i^0} \leq 1.022, \quad k \geq 0 \qquad (2.19)$$

From (2.18) we get that $\|x^k\|_Q < \|x^0\|_Q + 1.55\|p^0\|_{H^0}$ and since $Q = R^T D R \succeq \lambda_{\min} I \Rightarrow \|x^k\|^2 \leq \frac{1}{\lambda_{\min}} \|x^k\|_Q^2$, we get that the entire sequence $\{x^k\}$ is bounded and from (2.19) the sequence lies in the interior of $\mathcal{P}$. By the Bolzano Weirstrass theorem there exists at least one accumulation point and it is an interior point of $\mathcal{P}$. Then $\|p^k\|_{H^k} \to 0$, from (2.10) implies that $g(w) = 0$ at any accumulation point w of $\{x^k\}$. But there can only be one such point, the minimizer w of $\mathcal{V}(\cdot)$, and therefore $x^k \to w$ as claimed. $\square$

Chapter 3

The algorithm and its complexity

In this chapter we will present the cutting plane algorithm. The original version of this algorithm was first developed in Vaidya (1989), later went through some changes in Anstreicher (1994c). These changes were mainly improvements in the definitions of some key parameters and the use of the Hessian of $\mathcal{V}(\cdot)$ in the computation of the Newton steps, the effect of which was a dramatic reduction in both the number of Newton steps required for termination and the maximum number of constraints used to define $\mathcal{P}$. We further introduce a linesearch into the algorithm following each Newton step that brings in an additional parameter, namely the number of steps taken in the Bisection Method [1] that we have used in performing the linesearch and that we denote by $\mathcal{K}$. The performance of the algorithm and the efficacy of the linesearch is measured by the total number of inversions carried out until termination. The values assigned to our parameters was done in such a way so as to minimize the number of inversions carried out by the algorithm and this is discussed in Chapter 5.

3.1 The volumetric cutting plane algorithm

The Bisection Method [1] is used with termination after $\mathcal{K}$ bisections. Given the current point x and the Newton direction p the problem

$$\min_{0<\alpha<\alpha_{\max}} f(\alpha) = \mathcal{V}(x') = \mathcal{V}(x + \alpha p)$$

where α is the step-length and x' will be the next current point, has a unique solution since the function $\mathcal{V}(\cdot)$ is strictly convex. The quantity $\alpha_{\max}$ is the value that will take x' to the boundary of the polytope $\mathcal{P}$ and is computed using the min-ratio test, namely

$$\min_{1\leq i\leq m} \quad -\frac{s_i}{a_i^T p}$$
$$\text{s.t.} \quad a_i^T p < 0$$

The Bisection Method [1] uses a gradient function and a closed step-length range. In our case a simple differentiation with the aid of the chain rule yields

$$f'(\alpha) \;=\; -\sigma^T S(\alpha)^{-1} A p$$

We now present the pseudo-code of the cutting plane algorithm with a linesearch following every Newton step.

Step 0. Given x^0, $\mathcal{P}^0 = \{x | A^0 x \geq b^0\}$, $0 < \epsilon < 1$, $\gamma_1 > 0$, $\gamma_2 > 0$, L, $\mathcal{K}$ and $\mathcal{V}^k_{\max}$. Go to Step 1.

Step 1. If $\mathcal{V}^k(x^k) \geq \mathcal{V}^k_{\max}$, then STOP. Else go to Step 2.

Step 2. If $\sigma^k_{\min} \geq \epsilon$, go to Step 3. Else go to Step 4.

Step 3. (Constraint Addition) Call the oracle to see if $x^k \in \mathcal{C}$. If so, STOP. Otherwise the oracle returns a vector $a^k \in \Re^n$ such that ${a^k}^T x > {a^k}^T x^k \;\forall x \in \mathcal{C}$. Let $[A^{k+1}, b^{k+1}]$ be an augmented constraint system having $a^{k+1}_{m_{k+1}} = a^k$, $b^{k+1}_{m_{k+1}} < {a^k}^T x^k$. Go to Step 5.

Step 4. (Constraint Deletion) Suppose that $\sigma^k_j = \sigma^k_{\min} < \epsilon$. Let $[A^{k+1}, b^{k+1}]$ be the reduced system obtained by removing the jth contraint. Go to Step 5.

Step 5. (Newton Direction) Let $\bar{x}^0 = x^k$. Compute the Newton direction and take a sequence of steps with the opitmal step-length α of the form $\bar{x}^{j+1} = \bar{x}^j + \alpha^j \bar{p}^j$, where $\bar{p}^j = p(\bar{x}^j)$, $j \geq 0$ (go to Step 6 at each iteration) until $\|\bar{p}^J\|_{\bar{H}^J} \leq \gamma_1$,

$\bar{\mu}^J \|\bar{p}^J\|_{\bar{H}^J} \leq \gamma_2$, where $\bar{H}^j = H(\bar{x}^j)$, $\bar{\mu}^j = \mu(\bar{x}^j)$. Let $x^{k+1} = \bar{x}^J$, set $k = k+1$, and go to Step 1.

Step 6. (Linesearch) Initialize $\alpha_{\min}$ to 0 and $\alpha_{\max}$ to $\min_{1 \leq i \leq m} -s_i/a_i^T p$ s.t. $a_i^T p < 0$. Then, For $i = 1$ to $\mathcal{K}$ Do: $\alpha = (\alpha_{\min} + \alpha_{\max})/2$. If $f'(\alpha) < 0$ then $\alpha_{\min} = \alpha$. Else $\alpha_{\max}$ is set to α, End. Go back to Step 5.

In Step 3 the value of $b_{m_k+1}^{k+1}$ that corresponds to the placement of the new constraint is not arbitrary, but will be prescribed precisely in terms of a parameter $\tau > 0$ in Chapter 4. Also, throughout the algorithm the iterates x^k will have $\|p^k\|_{H^k} \leq \gamma_1$, $\mu^k \|p^k\|_{H^k} \leq \gamma_2$ for all k.

3.2 Initialization

In Step 0, the initial system is taken to be

$$\mathcal{P}^0 = \{x \in \Re^n \,|\, x_j \geq -2^L, j = 1, \cdots, n \,, e^T x \leq n 2^L\}$$

Note that $\mathcal{P}^0$ then contains a sphere of radius 2^L, centered at the origin. What is the volumetric center of $\mathcal{P}^0$? It is the point x such that $A^T S^{-1} \sigma = 0$. To simplify matters in calculating this point it will suffice to consider the scaled system $\tilde{\mathcal{P}}^0$ given by

$$\tilde{\mathcal{P}}^0 = \{x \in \Re^n \,|\, x_j \geq -1, j = 1, \cdots, n \,, e^T x \leq n\}$$

since if $x \in \mathcal{P}^0$ then $x/2^L \in \tilde{\mathcal{P}}^0$, and so the volumetric center of $\mathcal{P}^0$ will simply be 2^L times the volumetric center of $\tilde{\mathcal{P}}^0$. Proceeding therefore with

$$A^0 = \begin{pmatrix} I \\ -e^T \end{pmatrix}, \qquad b^0 = \begin{pmatrix} -e \\ -n \end{pmatrix} \tag{3.1}$$

and

$$s_i = \begin{cases} x_i + 1 & 1 \le i \le n \\ -e^T x + n & i = n+1 \end{cases} \tag{3.2}$$

we get that

$$A^T S^{-1}\sigma = \left(\begin{array}{ccc|c} \frac{1}{x_1+1} & & 0 & \frac{1}{e^Tx-n} \\ & \ddots & & \vdots \\ 0 & & \frac{1}{x_n+1} & \frac{1}{e^Tx-n} \end{array}\right) \sigma = \begin{pmatrix} \frac{\sigma_1}{x_1+1} + \frac{\sigma_{n+1}}{e^Tx-n} \\ \vdots \\ \frac{\sigma_n}{x_n+1} + \frac{\sigma_{n+1}}{e^Tx-n} \end{pmatrix} = 0 \tag{3.3}$$

and so our point x that will be our volumetric center must have the following property

$$\sigma_j = -\frac{(x_j+1)\sigma_{n+1}}{e^Tx - n} \tag{3.4}$$

Now, using (1.6), (3.1) and (3.2) we have

$$\sigma_j = \left[\frac{(A^TS^{-2}A)^{-1}}{(x_j+1)^2}\right]_{jj}, \quad j = 1, \cdots, n \quad , \text{ and } , \; \sigma_{n+1} = \frac{e^T(A^TS^{-2}A)^{-1}e}{(x_j+1)^2}$$

and together with

$$(A^TS^{-2}A)^{-1} = \begin{pmatrix} (x_1+1)^2 & & 0 \\ & \ddots & \\ 0 & & (x_n+1)^2 \end{pmatrix} - \frac{ww^T}{(e^Tx-n)^2 + \Sigma_i(x_i+1)^2} \tag{3.5}$$

where $w^T = ((x_1+1)^2, \cdots, (x_n+1)^2)$, we get that

$$\sigma_j = \frac{(x_j+1)^2[\Sigma_i(x_i+1)^2 + (\Sigma_i x_i - n)^2 - (x_j+1)^2]}{\Sigma_i(x_i+1)^2 + (\Sigma_i x_i - n)^2}, \quad j = 1, \cdots, n \tag{3.6}$$

and

$$\sigma_{n+1} = \frac{(\Sigma_i x_i - n)^2 \Sigma_i (x_i+1)^2}{\Sigma_i(x_i+1)^2 + (\Sigma_i x_i - n)^2} \tag{3.7}$$

from (3.6) and (3.7) we find that (3.4) is satisfied if $\Sigma_i x_i + x_j = n-1, j = 1, \cdots, n$, ie. for $Dx = (n-1)e$, where $D = I + ee^T$, a straightforward calculation using the

Sherman-Morrison-Woodbury formula A.2.1 gives us that $x_i = \frac{n-1}{n+1}$, $i = 1, \cdots, n$. This is a strictly interior point of $\mathcal{P}^0$ and is therefore the unique minimizer, i.e., the volumetric center. It is also interesting to note that the analytic center for $\mathcal{P}^0$ happens to coincide with the volumetric center for this case.

It can easily be shown through induction that the determinant of a matrix D of the form $D = cI + dee^T$, where I is the identity matrix is given by $\det(D) = c^{n-1}(nd + c)$. In computing $\mathcal{V}^0(x^0)$, it can be seen from (3.1) and (3.2) that the matrix $G^0(x^0)$ will be of this form and a straightforward computation gives us that $\det(G^0) = \left(\frac{n+1}{n2^{L+1}}\right)^{2n}(n+1)$ and so the value of $\mathcal{V}^0(\cdot)$ at x^0 is

$$\mathcal{V}^0(x^0) \; = \; -\ln(2)n(L+1) + nln(1 + 1/n) + \tfrac{1}{2}ln(n+1) \; > \; -.7n(L+1) \qquad (3.8)$$

3.3 Termination

The value of $\mathcal{V}^k_{\max} > 0$ which depends on m_k and that is set in Lemma 3.3.1 is such that $\mathcal{V}^k(x^k) \geq \mathcal{V}^k_{\max}$ implies that $Vol(\mathcal{P}^k)$, and hence also $\mathcal{C} \subset \mathcal{P}^k$, is less than that of an n dimensional ball of radius 2^{-L}. However, since from the outset we have assumed that if $\mathcal{C}$ is non-empty then it must contain a ball of radius 2^{-L}, this result would mean that the convex set $\mathcal{C}$ is empty. Note that by construction, from Step 5 of the algorithm, all of the iterates will satisfy $||p^k||_{H^k} \leq \gamma_1$, $\mu^k||p^k||_{H^k} \leq \gamma_2$.

Lemma 3.3.1 Assume that the iterates of the volumetric cutting plane algorithm satisfy $||p^k||_{H^k} \leq .014$ forall k. Then on setting $\mathcal{V}^k_{\max} = .7nL + n\ln(m_k)$ termination in Step 1 establishes that $Vol(\mathcal{C})$ is less than that of an n dimensional sphere of radius 2^{-L}

Proof : By Lemma 2.3 and Theorem 2.4, $\mathcal{P}^k$ is bounded for each k, and therefore the analytic and volumetric centers of $\mathcal{P}^k$ both exit. From (1.2) and (1.3) we have that

$$Vol(\mathcal{P}^k) \leq S_n m_k^n e^{-\mathcal{V}^k(w^k)} \tag{3.9}$$

and since $\mathcal{C} \subset \mathcal{P}^k \; \forall k \geq 0$ to show that $Vol(\mathcal{C})$ is less than that of an n dimensional ball of radius 2^{-L} it suffices from (3.9) to have

$$S_n m_k^n e^{-\mathcal{V}^k(w^k)} < S_n 2^{-nL}$$

and on taking logarithms this is equivalent to

$$\mathcal{V}^k(w^k) > nL\ln(2) + n\ln(m_k) \tag{3.10}$$

But Theorem 2.4 and $\|p^k\|_{H^k} \leq .014$ imply that

$$\mathcal{V}^k(w^k) \geq \mathcal{V}^k(x^k) - .00022$$

and so (3.10) is satisfied if $\mathcal{V}^k(x^k) \geq .7nL + n\ln(m_k)$. □

3.4 Complexity

Assuming for a fixed $\epsilon > 0$, and $\gamma_2 \leq .014$ the algorithm achieves

$$\mathcal{V}^{k+1}(x^{k+1}) \geq \mathcal{V}^k(x^k) + \Delta\mathcal{V}^+ \tag{3.11}$$

on steps where a constraint is added, where $\Delta\mathcal{V}^+ > 0$, while on steps where a constraint is deleted it achieves

$$\mathcal{V}^{k+1}(x^{k+1}) \geq \mathcal{V}^k(x^k) - \Delta\mathcal{V}^- \tag{3.12}$$

where $\Delta\mathcal{V}^- > 0$. From the boundedness property of $\mathcal{P}^k$ and noting that $\mathcal{P}^0$ is defined by the least number of constraints needed to bound a polytope in $\Re^n$ it is apparent that at any point in our algorithm we will have that the number of constraint additions that have occurred will be greater than the number of constraint deletions. Thus, if we can guarantee that $\Delta\mathcal{V} = \Delta\mathcal{V}^+ - \Delta\mathcal{V}^- > 0$, then as the next theorem shows our algorithm will terminate in $\mathcal{O}(nL)$ iterations.

Theorem 3.4.1 Assume that the iterates of the volumetric cutting plane algorithm, using $\epsilon > 0$, $\gamma_2 \leq .014$, satisfy (3.11) and (3.12) on iterations where a constraint is added or deleted, respectively. Assume further that $\Delta\mathcal{V} = \Delta\mathcal{V}^+ - \Delta\mathcal{V}^-$ is $\Omega(1)$ and positive, and that the number of Newton steps in Step 5 of the algorithm is $\mathcal{O}(1)$. Then using $\mathcal{V}^k_{\max}$ as in Lemma 3.3.1, the algorithm terminates in $\mathcal{O}(nL)$ iterations, using a total of $\mathcal{O}(nLT + n^4L)$ operations, where T is the cost of a call to the separation oracle.

Proof : The number of constraint additions being greater than the number of constraint deletions, together with (3.11) and (3.12), implies that

$$\mathcal{V}^k(x^k) \geq \mathcal{V}^0(x^0) + k(\Delta\mathcal{V}^+ - \Delta\mathcal{V}^-)/2 = \mathcal{V}^0(x^0) + k\Delta\mathcal{V}/2 \tag{3.13}$$

Next, the fact that on steps where a constraint is added we always have $\sigma^k_{\min} \geq \epsilon$, and $m_k\sigma^k_{\min} \leq e^T\sigma_k = n$ for all k, implies that $m_k \leq (1/\epsilon)n + 1 \leq (1+1/\epsilon)n$ for all k. Using this fact with (3.8) and (3.13), we see that $\mathcal{V}^k(x^k) \geq \mathcal{V}^k_{max}$ certainly occurs if

$$-.7n(L+1) + k\Delta\mathcal{V}/2 \geq .7nL + n\ln(1+1/\epsilon) + n\ln(n)$$

and therefore the algorithm must terminate for

$$k \geq \frac{2n(1.4L + \ln(n) + \ln(1+1/\epsilon) + .7)}{\Delta\mathcal{V}} = \mathcal{O}(nL) \tag{3.14}$$

Finally, noting that $m_k \leq n(1+1/\epsilon)$, we have that the work per iteration for the algorithm using standard linear algebra is $\mathcal{O}(n^3)$ and as a result the total complexity of the algorithm is $\mathcal{O}(nLT + n^4L)$ operations, where T is the cost of a call to the oracle. □

Chapter 4

Adding and deleting constraints

Results will be proved that characterize the effects of constraint additions and deletions, that occur in Steps 3 and 4 of our algorithm, on $\mathcal{V}(\cdot)$, σ, and $||p||_H$, respectively. We will see that from the observations following Lemma 4.1.1 and Lemma 4.2.1 we will have that $\Delta\mathcal{V} = \Delta\mathcal{V}^+ - \Delta\mathcal{V}^- = \ln(1+\tau)^{1/2} - \ln(1-\epsilon)^{-1/2}$, where the quantity τ is yet to be defined. We will leave the analysis of the Newton steps and linesearches that occur immediately after a constraint addition or deletion to the following chapter.

4.1 Constraint additions

We now consider in detail the effect of adding a constraint in Step 3 of the algorithm. Dropping all dependence on the iteration k to reduce the burden of notation, and with our new system defined by (1.9) with the assumption that $a_{m+1}^T x > b_{m+1}$, so that $s_{m+1} = a_{m+1}^T x - b_{m+1} > 0$ we let

$$\tau = s_{m+1}^{-2} a_{m+1}^T (A^T S^{-2} A)^{-1} a_{m+1} \tag{4.1}$$

This definition of τ has an interesting geometric interpretation, for if we consider

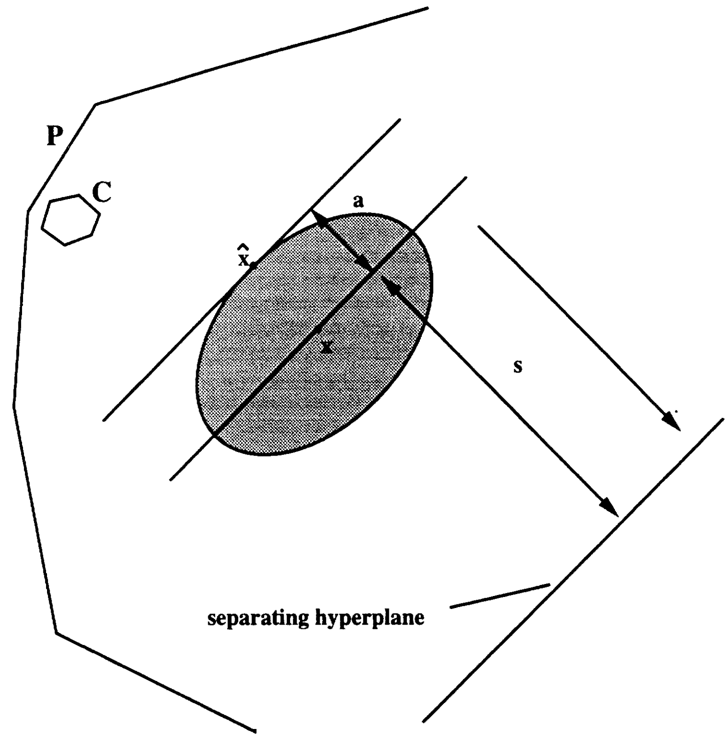

Figure 4-1: Setting the value of τ

the following program:

$$\text{Max } a_{m+1}^T(x-\bar{x})$$
$$\text{s.t. } (x-\bar{x})^T A^T S^{-2} A(x-\bar{x}) \leq 1$$

we have that the solution occurs at $\hat{x}$ (see Figure 4-1) with maximum objective function value given by

$$\alpha = \sqrt{a_{m+1}^T (A^T S^{-2} A)^{-1} a_{m+1}}$$

and so $\tau = (\alpha/s_{m+1})^2$

From a geometric perspective $\tau = (a/s)^2$ is the ratio of the distances a and s squared as can be seen in Figure 4-1, and so decreasing τ has the effect of pushing the separating hyperplane ever further away. It is advantageous to have the separating hyperplane as close to the test point as possible, as this will result in the greatest decrease in the volume of the polytope $\mathcal{P}$ on the next iteration and hence the greatest increase in the function $\mathcal{V}(\cdot)$ and that is what we hope to achieve. Thus, it will be attempted to set τ at a maximum value in such a way as to still be able to satisfy the assumptions of our theorems in Chapter 2 that prove convergence of the algorithm. The quantity τ being set beforehand in this way, will necessitate computation of the value s_{m+1} during each iteration in order to satisfy (4.1).

We will now prove three results that demonstrate the effect of a constraint addition on $\mathcal{V}(\cdot)$, σ, and $||p||_H$, respectively.

Lemma 4.1.1 Suppose that a constraint (a_{m+1}^T, b_{m+1}) is added, and τ is given as in (4.1). Then $\tilde{\mathcal{V}}(x) = \mathcal{V}(x) + 1/2\ln(1+\tau)$

Proof By definition,

$$\begin{aligned}
\tilde{\mathcal{V}}(x) &= \tfrac{1}{2}\ln\Big[\det(\tilde{A}^T\tilde{S}^{-2}\tilde{A})\Big] \\
&= \tfrac{1}{2}\ln\Big[\det(A^TS^{-2}A + s_{m+1}^{-2}a_{m+1}a_{m+1}^T)\Big] \\
&= \tfrac{1}{2}\ln\Big[\det\Big((A^TS^{-2}A)(I + s_{m+1}^{-2}(A^TS^{-2}A)^{-1}a_{m+1}a_{m+1}^T)\Big)\Big] \\
&= \mathcal{V}(x) + \tfrac{1}{2}\ln\Big[\det(I + s_{m+1}^{-2}(A^TS^{-2}A)^{-1}a_{m+1}a_{m+1}^T)\Big]
\end{aligned}$$

The lemma then follows from the definition of τ, and the fact that $\det(I + uv^T) = 1 + u^Tv$. □

Note that from the above lemma we have that following a constraint addition $\tilde{\mathcal{V}}(\tilde{w}) - \mathcal{V}(\tilde{w}) \geq \ln(1+\tau)^{1/2}$ and thus in Theorem 3.4.1 $\Delta\mathcal{V}^+$ will be represented by $\ln(1+\tau)^{1/2}$.

Lemma 4.1.2 Suppose that a constraint (a_{m+1}^T, b_{m+1}) is added, and τ is given as in (4.1). Then $\tilde{\sigma}_{m+1} = \tau/(1+\tau)$, and $\sigma_i \geq \tilde{\sigma}_i \geq \sigma_i/(1+\tau)$, $i = 1, \ldots, m$.

Proof We have that $\tilde{A}^T\tilde{S}^{-2}\tilde{A} = A^TS^{-2}A + s_{m+1}^{-2}a_{m+1}a_{m+1}^T$, so the Sherman - Morrison-Woodbury formula A.2.1 obtains

$$(\tilde{A}^T\tilde{S}^{-2}\tilde{A})^{-1} = (A^TS^{-2}A)^{-1} - \frac{s_{m+1}^{-2}(A^TS^{-2}A)^{-1}a_{m+1}a_{m+1}^T(A^TS^{-2}A)^{-1}}{1+\tau} \tag{4.2}$$

Now $\tilde{\sigma}_i = s_i^{-2} a_i^T (\tilde{A}^T \tilde{S}^{-2} \tilde{A})^{-1} a_i$, so from (4.2) we immediately obtain,

$$\tilde{\sigma}_i = \sigma_i - \frac{s_i^{-2} s_{m+1}^{-2} (a_i^T (A^T S^{-2} A)^{-1} a_{m+1})^2}{1+\tau}, \quad i = 1, \dots, m \tag{4.3}$$

Note that (4.3) implies that $\sigma_i \geq \tilde{\sigma}_i$, $i = 1, \dots, m$. Applying Proposition A.2.6

$$\begin{aligned} |s_i^{-1} s_{m+1}^{-1} a_i^T (A^T S^{-2} A)^{-1} a_{m+1}| &\leq \|s_i^{-1} a_i\|_{(A^T S^{-2} A)^{-1}} \|s_{m+1}^{-1} a_{m+1}\|_{(A^T S^{-2} A)^{-1}} \\ &= \sqrt{\sigma_i \tau} \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) then obtains $\tilde{\sigma}_i \geq \sigma_i \tau/(1+\tau)$, $i = 1, \dots, m$, which is exactly the bound of the lemma. Finally, from (4.2) we have

$$\tilde{\sigma}_{m+1} = s_{m+1}^{-2} a_{m+1}^T (\tilde{A}^T \tilde{S}^{-2} \tilde{A})^{-1} a_{m+1} = \frac{\tau}{1+\tau}. \quad \square$$

Theorem 4.1.3 Suppose that a constraint (a_{m+1}^T, b_{m+1}) is added, $\sigma_{min} \geq \epsilon > 0$, and τ is given as in (4.1). Then

$$\|\tilde{p}\|_{\tilde{H}} \leq \sqrt{1+\tau} \left(\sqrt{3} \|p\|_H + \frac{\tau(1+\sqrt{\tau/\epsilon})}{1+\tau} \right).$$

Proof Using Lemma 4.1.2, we have

$$\tilde{Q} = \sum_{i=1}^{m+1} \frac{\tilde{\sigma}_i}{s_i^2} a_i a_i^T \succeq \frac{1}{1+\tau} \sum_{i=1}^{m} \frac{\sigma_i}{s_i^2} a_i a_i^T = \frac{1}{1+\tau} Q$$

and therefore $\tilde{Q}^{-1} \preceq (1+\tau) Q^{-1}$, by Claim B.2. As a result,

$$\|\tilde{p}\|_{\tilde{H}} = \|\tilde{g}\|_{\tilde{H}^{-1}} \leq \|\tilde{g}\|_{\tilde{Q}^{-1}} \leq \sqrt{1+\tau} \|\tilde{g}\|_{Q^{-1}} = \sqrt{1+\tau} \|\tilde{A}^T \tilde{S}^{-1} \tilde{\sigma}\|_{Q^{-1}} \tag{4.5}$$

where the first inequality uses (1.8) and Claim B.2. We also have that

$$\tilde{A}^T \tilde{S}^{-1} \tilde{\sigma} = \sum_{i=1}^{m+1} \frac{\tilde{\sigma}_i}{s_i} a_i = \sum_{i=1}^{m} \frac{\sigma_i}{s_i} a_i + \sum_{i=1}^{m} \frac{\tilde{\sigma}_i - \sigma_i}{s_i} a_i + \frac{\tilde{\sigma}_{m+1}}{s_{m+1}} a_{m+1} \tag{4.6}$$

Since $g = A^T S^{-1}\sigma$, combining (4.5) and (4.6), and using the triangle inequality, obtains

$$\|\tilde{p}\|_{\tilde{H}} \le \sqrt{1+\tau}\left(\|g\|_{Q^{-1}} + \|\sum_{i=1}^{m} \frac{\tilde{\sigma}_i - \sigma_i}{s_i} a_i\|_{Q^{-1}} + \|\frac{\tilde{\sigma}_{m+1}}{s_{m+1}} a_{m+1}\|_{Q^{-1}}\right) \tag{4.7}$$

Next, from (4.3) we have

$$\|\sum_{i=1}^{m} \frac{\tilde{\sigma}_i - \sigma_i}{s_i} a_i\|_{Q^{-1}}^2 = d^T \Sigma^{1/2} S^{-1} A (A^T S^{-2} \Sigma A)^{-1} A^T S^{-1} \Sigma^{1/2} d \le \|d\|^2 \tag{4.8}$$

where

$$d_i = \frac{s_i^{-2} s_{m+1}^{-2} (a_i^T (A^T S^{-2} A)^{-1} a_{m+1})^2}{\sigma_i^{1/2}(1+\tau)}, \quad i = 1, \dots, m$$

and the last inequality follows from the properties of projection matrices (see Appendix A.3).

Using the bound from (4.4), we have $|d_i| \le \tau \sigma_i^{1/2}/(1+\tau)$, $i = 1, \dots, m$, so

$$\begin{aligned}
\sum_{i=1}^{m} d_i^2 &\le \sum_{i=1}^{m} \frac{s_i^{-2} s_{m+1}^{-2} a_{m+1}^T (A^T S^{-2} A)^{-1} a_i a_i^T (A^T S^{-2} A)^{-1} a_{m+1}}{\sigma_i^{1/2}(1+\tau)} \frac{\tau \sigma_i^{1/2}}{1+\tau} \\
&= \frac{\tau}{(1+\tau)^2} s_{m+1}^{-2} a_{m+1}^T (A^T S^{-2} A)^{-1} a_{m+1} \\
&= \frac{\tau^2}{(1+\tau)^2}
\end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9) then obtains

$$\|\sum_{i=1}^{m} \frac{\tilde{\sigma}_i - \sigma_i}{s_i} a_i\|_{Q^{-1}} \le \frac{\tau}{1+\tau} \tag{4.10}$$

The fact that $\sigma_{\min} \ge \epsilon \Rightarrow Q = A^T S^{-2} \Sigma A \succeq \epsilon A^T S^{-2} A$, giving us that $Q^{-1} \preceq (1/\epsilon)(A^T S^{-2} A)^{-1}$, and as a result,

$$\|\frac{\tilde{\sigma}_{m+1}}{s_{m+1}} a_{m+1}\|_{Q^{-1}}^2 \le \epsilon^{-1} \tilde{\sigma}_{m+1}^2 s_{m+1}^{-2} a_{m+1}^T (A^T S^{-2} A)^{-1} a_{m+1} = \frac{\tau}{\epsilon} \frac{\tau^2}{(1+\tau)^2} \tag{4.11}$$

where the last inequality uses $\tilde{\sigma}_{m+1} = \tau/(1+\tau)$, from Lemma 4.1.1. Finally, using

$Q^{-1} \preceq 3H^{-1}$ from (1.8) and Claim B.2 we get

$$\|g\|_{Q^{-1}} \leq \sqrt{3}\|g\|_{H^{-1}} = \sqrt{3}\|p\|_H \tag{4.12}$$

The proof is completed by combining (4.7), (4.10), (4.11) and (4.12). □

4.2 Constraint deletions

We now consider the effect of deleting a constraint, as occurs in Step 4 of the algorithm. We again drop all dependence on the iteration k to simplify notation and simply consider the system given by (1.10), where once again for simplicity we assume without loss of generality that the mth constraint is the one to be deleted. Assuming that the columns of A are linearly independent then linear independence of the columns of $\tilde{A}$ is a consequence of $\sigma_m \leq \epsilon < 1$, as will be seen from (4.13), where for σ_m in that range we get that $(\tilde{A}^T\tilde{S}^{-2}\tilde{A})^{-1}$ is positive definite $\Rightarrow \tilde{A}^T\tilde{S}^{-2}\tilde{A}$ is positive definite and thus the columns of $\tilde{A}$ must be independent. This is an important observation as the proof of the boundedness of $\mathcal{P}^k$ deduced from Lemma 2.3 and Theorem 2.4, requires that the columns of A^k be linearly independent for all k.

We now proceed to establish the three results (as in the case for constraint additions) to show the effect of a constraint deletion on $\mathcal{V}(\cdot)$, σ, and $\|p\|_H$, respectively. For the latter, we give a result in terms of σ_{min}, and not $\epsilon > \sigma_{\min}$, for reasons that will become clear in the next chapter.

Lemma 4.2.1 Suppose that the constraint (a_m^T, b_m) is deleted, where $\sigma_m \leq \epsilon$. Then $\tilde{\mathcal{V}}(x) \geq \mathcal{V}(x) + 1/2\ln(1-\epsilon)$.

Proof By definition,

$$\begin{aligned}\tilde{\mathcal{V}}(x) &= \tfrac{1}{2}\ln\Big[\det(\tilde{A}^T\tilde{S}^{-2}\tilde{A})\Big] \\ &= \tfrac{1}{2}\ln\Big[\det(A^TS^{-2}A - s_m^{-2}a_ma_m^T)\Big]\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\ln\Big[\det\Big((A^TS^{-2}A)(I - s_m^{-2}(A^TS^{-2}A)^{-1}a_m a_m^T)\Big)\Big] \\
&= \mathcal{V}(x) + \tfrac{1}{2}\ln\Big[\det(I - s_m^{-2}(A^TS^{-2}A)^{-1}a_m a_m^T)\Big]
\end{aligned}
$$

The lemma then follows from $\sigma_m \leq \epsilon$, and the fact that $\det(I - uv^T) = 1 - u^Tv$. □

It is worth noting that from Lemma 4.2.1 we can establish that $0 < \mathcal{V}(\tilde{w}) - \tilde{\mathcal{V}}(\tilde{w}) = \ln(1-\sigma_m)^{-1/2} \leq \ln(1-\epsilon)^{-1/2}$ and thus in Theorem 3.4.1 $\Delta\mathcal{V}^-$ will be represented by $\ln(1-\epsilon)^{-1/2}$.

Lemma 4.2.2 Suppose that the constraint (a_m^T, b_m) is deleted, where $\sigma_m \leq \epsilon$. Then $\sigma_i \leq \tilde{\sigma}_i \leq \sigma_i/(1-\epsilon)$, $i = 1, \ldots, m-1$.

Proof We have that $\tilde{A}^T\tilde{S}^{-2}\tilde{A} = A^TS^{-2}A - s_m^{-2}a_m a_m^T$, so the Sherman - Morrison-Woodbury formula A.2.1 obtains

$$
(\tilde{A}^T\tilde{S}^{-2}\tilde{A})^{-1} = (A^TS^{-2}A)^{-1} + \frac{s_m^{-2}(A^TS^{-2}A)^{-1}a_m a_m^T(A^TS^{-2}A)^{-1}}{1-\sigma_m} \tag{4.13}
$$

Now $\tilde{\sigma}_i = s_i^{-2}a_i^T(\tilde{A}^T\tilde{S}^{-2}\tilde{A})^{-1}a_i$, so from (4.13) we immediately obtain,

$$
\tilde{\sigma}_i = \sigma_i + \frac{s_i^{-2}s_m^{-2}(a_i^T(A^TS^{-2}A)^{-1}a_m)^2}{1-\sigma_m}, \quad i = 1, \ldots, m-1 \tag{4.14}
$$

Note that (4.14) implies that $\sigma_i \leq \tilde{\sigma}_i$, $i = 1, \ldots, m-1$. Applying Proposition A.2.6 as in (4.4), then obtains

$$
|s_i^{-1}s_m^{-1}a_i^T(A^TS^{-2}A)^{-1}a_m| \leq \sqrt{\sigma_i\sigma_m} \tag{4.15}
$$

Combining (4.14) and (4.15) and using $\sigma_m \leq \epsilon$, then obtains $\tilde{\sigma}_i \leq \sigma_i + \sigma_i\epsilon/(1-\epsilon)$, $i = 1, \ldots, m-1$, which is exactly the bound of the lemma. □

Theorem 4.2.3 Suppose that the constraint (a_m^T, b_m) is deleted, where $\sigma_m = \sigma_{\min}$. Then

$$
\|\tilde{p}\|_{\tilde{H}} \leq \frac{1}{\sqrt{1-\sigma_{\min}}}\left(\sqrt{3}\|p\|_H + \frac{2\sigma_{\min}}{1-\sigma_{\min}}\right). \tag{4.16}
$$

Proof Using Lemma 4.2.2, we have

$$\tilde{Q} = \sum_{i=1}^{m-1} \frac{\tilde{\sigma}_i}{s_i^2} a_i a_i^T \succeq \sum_{i=1}^{m} \frac{\sigma_i}{s_i^2} a_i a_i^T - \frac{\sigma_m}{s_m^2} a_m a_m^T = Q - \frac{\sigma_m}{s_m^2} a_m a_m^T$$

Using Claim B.2, and the Sherman-Morrison-Woodbury formula A.2.1, we then have

$$\tilde{Q}^{-1} \preceq \left(Q - \frac{\sigma_m}{s_m^2} a_m a_m^T\right)^{-1} = Q^{-1} + \frac{\sigma_m s_m^{-2} Q^{-1} a_m a_m^T Q^{-1}}{1 - \sigma_m s_m^{-2} a_m^T Q^{-1} a_m} \tag{4.17}$$

Since we know that $Q = A^T S^{-2} A \Sigma A \succeq \sigma_{\min} A^T S^{-2} A$, Claim B.2 implies that $Q^{-1} \preceq (1/\sigma_{\min})(A^T S^{-2} A)^{-1}$, and therefore

$$s_m^{-2} a_m^T Q^{-1} a_m \le \frac{1}{\sigma_{\min}} s_m^{-2} a_m^T (A^T S^{-2} A)^{-1} a_m = \frac{\sigma_m}{\sigma_{\min}} = 1 \tag{4.18}$$

Combining (4.17) and (4.18), and using $\sigma_m = \sigma_{\min}$, then produces

$$\tilde{Q}^{-1} \preceq Q^{-1} + \frac{\sigma_{\min}}{1 - \sigma_{\min}} s_m^{-2} Q^{-1} a_m a_m^T Q^{-1} \tag{4.19}$$

and therefore

$$\|\tilde{p}\|_{\tilde{H}}^2 = \|\tilde{g}\|_{\tilde{H}^{-1}}^2 \le \|\tilde{g}\|_{\tilde{Q}^{-1}}^2 \le \|\tilde{g}\|_{Q^{-1}}^2 + \frac{\sigma_{\min}}{1 - \sigma_{\min}} \left(\frac{\tilde{g}^T Q^{-1} a_m}{s_m}\right)^2 \tag{4.20}$$

where the first inequality uses (1.8) and Claim B.2. Next from Proposition A.2.6 we have

$$|s_m^{-1} \tilde{g}^T Q^{-1} a_m| \le \|\tilde{g}\|_{Q^{-1}} \|s_m^{-1} a_m\|_{Q^{-1}} \le \|\tilde{g}\|_{Q^{-1}} \tag{4.21}$$

where the last inequality uses (4.18). Combining (4.20) and (4.21) then obtains

$$\|\tilde{p}\|_{\tilde{H}}^2 \le \|\tilde{g}\|_{Q^{-1}}^2 + \frac{\sigma_{\min}}{1 - \sigma_{\min}} \|\tilde{g}\|_{Q^{-1}}^2 = \frac{1}{1 - \sigma_{\min}} \|\tilde{g}\|_{Q^{-1}}^2 \tag{4.22}$$

Now

$$\tilde{g} = \tilde{A}^T \tilde{S}^{-1} \tilde{\sigma} = \sum_{i=1}^{m-1} \frac{\tilde{\sigma}_i}{s_i} a_i = \sum_{i=1}^{m} \frac{\sigma_i}{s_i} a_i + \sum_{i=1}^{m-1} \frac{\tilde{\sigma}_i - \sigma_i}{s_i} a_i - \frac{\sigma_m}{s_m} a_m$$

so (4.22) and the triangle inequality imply that

$$\|\tilde{p}\|_{\tilde{H}} \le \frac{1}{\sqrt{1-\sigma_{\min}}} \left(\|g\|_{Q^{-1}} + \|\sum_{i=1}^{m-1} \frac{\tilde{\sigma}_i - \sigma_i}{s_i} a_i\|_{Q^{-1}} + \|\frac{\sigma_m}{s_m} a_m\|_{Q^{-1}} \right) \tag{4.23}$$

Next, from (4.14) we have

$$\|\sum_{i=1}^{m-1} \frac{\tilde{\sigma}_i - \sigma_i}{s_i} a_i\|_{Q^{-1}}^2 = d^T \Sigma^{1/2} S^{-1} A (A^T S^{-2} \Sigma A)^{-1} A^T S^{-1} \Sigma^{1/2} d \le \|d\|^2 \tag{4.24}$$

where

$$d_i = \frac{s_i^{-2} s_m^{-2} (a_i^T (A^T S^{-2} A)^{-1} a_m)^2}{\sigma_i^{1/2} (1-\sigma_m)}, \quad i = 1, \ldots, m-1$$

and $d_m = 0$, with the last inequality following from the properties of projection matrices (see Appendix A.3).

Using the bound from (4.15), and the fact that $\sigma_m = \sigma_{\min}$, and also noting that $|d_i| \le \sigma_i^{1/2} \sigma_{\min}/(1-\sigma_{\min})$, $i = 1, \ldots, m-1$, we have

$$\begin{aligned}
\sum_{i=1}^{m} d_i^2 &\le \sum_{i=1}^{m-1} \frac{s_i^{-2} s_m^{-2} a_m^T (A^T S^{-2} A)^{-1} a_i a_i^T (A^T S^{-2} A)^{-1} a_m}{\sigma_i^{1/2} (1-\sigma_{\min})} \frac{\sigma_i^{1/2} \sigma_{\min}}{1-\sigma_{\min}} \\
&\le \sum_{i=1}^{m} \frac{s_i^{-2} s_m^{-2} a_m^T (A^T S^{-2} A)^{-1} a_i a_i^T (A^T S^{-2} A)^{-1} a_m}{\sigma_i^{1/2} (1-\sigma_{\min})} \frac{\sigma_i^{1/2} \sigma_{\min}}{1-\sigma_{min}} \\
&= \frac{\sigma_{\min}}{(1-\sigma_{\min})^2} s_m^{-2} a_m^T (A^T S^{-2} A)^{-1} a_m \\
&= \frac{\sigma_{\min}^2}{(1-\sigma_{\min})^2}
\end{aligned} \tag{4.25}$$

Combining (4.24) and (4.25) then obtains

$$\|\sum_{i=1}^{m-1} \frac{\tilde{\sigma}_i - \sigma_i}{s_i} a_i\|_{Q^{-1}} \le \frac{\sigma_{\min}}{1-\sigma_{\min}} \tag{4.26}$$

Finally, using (4.18),

$$\|\frac{\sigma_m}{s_m}a_m\|_{Q^{-1}} = \sigma_m\|s_m^{-1}a_m\|_{Q^{-1}} \leq \sigma_m = \sigma_{\min} \leq \frac{\sigma_{\min}}{1-\sigma_{\min}} \tag{4.27}$$

Combining (4.23), (4.26), (4.27) and (4.12) completes the proof. □

Chapter 5

Analysis

In seeking to find the maximum number of Newton steps that would guarantee the next iterate satisfies the proximity conditions we will use the more general results

$$\|\bar{p}\|_{\bar{H}} \leq \frac{19(1+\mu\|p\|_H)^2}{(1-\mu\|p\|_H)^6}\mu\|p\|_H^2 \tag{5.1}$$

and

$$\bar{\mu}\|\bar{p}\|_{\bar{H}} \leq \frac{19(1+\mu\|p\|_H)^2.5}{1-\mu\|p\|_H)^6.5}(\mu\|p\|_H)^2 \tag{5.2}$$

which follows from the analysis used in the proofs of parts i) and ii) of Lemma 2.2.

There are four parameters that will play a role in the analysis, namely τ, ϵ, γ_1 and γ_2. Intuition tells us that it is wise to set τ large; however, we are restricted by the bounds in Theorem 4.1.3 and Theorem 4.2.3 on the proximity measure $\|\tilde{p}\|_{\tilde{H}}$ that must be maintained for all the iterates in a run of the algorithm. These bounds play an important role in establishing that the number of Newton steps that are needed to recover the proximity conditions after a constraint addition or deletion is $\mathcal{O}(1)$ and this is required by the convergence Theorem 3.4.1. In order to set τ at an optimum value and still satisfy the bounds the following parameter settings were used: τ was set to .0062, $\epsilon = .0049$, $\gamma_1 = .000006$ and $\gamma_2 = .0001$. With these settings, the maximum number of Newton steps on a constraint addition was shown to be 7, while the maximum number of Newton steps on a constraint deletion was shown to be 4. These results are established in the following two theorems.

Theorem 5.1 Let x be a point with $s = s(x) > 0$. Assume that $\gamma_1 \leq .000006$, and $\gamma_2 \leq .0001$. With $\sigma_{\min} \geq \epsilon = .00475$ and with τ set to $.0062$ suppose that a constraint (a_{m+1}^T, b_{m+1}) is added, and let $[\tilde{A}, \tilde{b}]$ be the augmented constraint system. Let $\bar{x}$ be obtained by taking 7 Newton steps for $\tilde{\mathcal{V}}(\cdot)$ starting at x. Then

i) $||\bar{p}||_{\bar{H}} \leq .000006$,

ii) $\bar{\mu}||\bar{p}||_{\bar{H}} \leq .0001$,

iii) $\tilde{\mathcal{V}}(\bar{x}) \geq \mathcal{V}(x) + .0025438$

Proof : Since $\sigma_{\min} \geq \epsilon$, and $\tau > \epsilon$, Lemma 4.1.2 implies that $\tilde{\sigma}_{\min} \geq \epsilon/(1+\tau) \geq .00472$ and therefore

$$\tilde{\mu} = (2\sqrt{\tilde{\sigma}_{\min}} - \tilde{\sigma}_{\min})^{-1/2} \leq (2\sqrt{.00472} - .00472)^{-1/2} < 2.745 \tag{5.3}$$

Also, Theorem 4.1.3, with $\tau = .0062$, gives

$$||\tilde{p}||_{\tilde{H}} \leq \sqrt{1.0062}\left(\sqrt{3}(.000006) + \frac{.0062(1+\sqrt{1.31})}{1.0062}\right) < .01325 \tag{5.4}$$

Combining (5.3) and (5.4) , $\tilde{\mu}||\tilde{p}||_{\tilde{H}} \leq 2.745(.01325) < .0364$. Using the same notation as in Step 5 of the algorithm, repeatedly applying (5.1) and (5.2) then obtains

$$\begin{aligned}
||\bar{p}^1||_{\bar{H}^1} &< .012290, & \bar{\mu}^1||\bar{p}^1||_{\bar{H}^1} &< .034989 \\
||\bar{p}^2||_{\bar{H}^2} &< .010837, & \bar{\mu}^2||\bar{p}^2||_{\bar{H}^2} &< .031952 \\
||\bar{p}^3||_{\bar{H}^3} &< .008514, & \bar{\mu}^3||\bar{p}^3||_{\bar{H}^3} &< .025916 \\
||\bar{p}^4||_{\bar{H}^4} &< .005165, & \bar{\mu}^4||\bar{p}^4||_{\bar{H}^4} &< .016137 \\
||\bar{p}^5||_{\bar{H}^5} &< .001803, & \bar{\mu}^5||\bar{p}^5||_{\bar{H}^5} &< .005724 \\
||\bar{p}^6||_{\bar{H}^6} &< .000205, & \bar{\mu}^6||\bar{p}^6||_{\bar{H}^6} &< .000655 \\
||\bar{p}^7||_{\bar{H}^7} &< .000003, & \bar{\mu}^7||\bar{p}^7||_{\bar{H}^7} &< .000008
\end{aligned}$$

proving parts i) and ii). The proof of part iii) follows from repeated application of the convexity property of $\tilde{\mathcal{V}}(\cdot)$, namely if $\bar{x} = x + \tilde{p}$ then $\tilde{\mathcal{V}}(\bar{x}) \geq \tilde{\mathcal{V}}(x) + \tilde{g}^T\tilde{p} = \tilde{\mathcal{V}}(x) - ||\tilde{p}||^2_{\tilde{H}}$. Using this convexity property and Lemma 4.1.1 we have

$$\begin{aligned}
\tilde{\mathcal{V}}(\bar{x}) &\geq \mathcal{V}(x) + \tfrac{1}{2}\ln(1+\tau) - \sum_{j=0}^{6} ||\tilde{p}^j||^2_{\tilde{H}^j} \\
&= \mathcal{V}(x) + .5\ln(1.0062) - (.01325^2 + .01229^2 + \cdots + .00021^2) \\
&= \mathcal{V}(x) + .0025438
\end{aligned}$$

proving part iii) and the theorem. □

We now consider the case where a constraint is deleted.

Theorem 5.2 Let x be a point with $s = s(x) > 0$. Assume that $\gamma_1 \leq .000006$, $\gamma_2 \leq .0001$ and that $\sigma_m = \sigma_{\min} \leq \epsilon = .00475$. Suppose that the constraint (a_m^T, b_m) is deleted, and let $[\tilde{A}, \tilde{b}]$ be the reduced constraint system. Let $\bar{x}$ be obtained by taking 4 Newton steps for $\tilde{\mathcal{V}}(\cdot)$ starting at x. Then

i) $||\bar{p}||_{\bar{H}} \leq .000006$,

ii) $\bar{\mu}||\bar{p}||_{\bar{H}} \leq .0001$,

iii) $\tilde{\mathcal{V}}(\bar{x}) \geq \mathcal{V}(x) - .0025125$

Proof : By Theorem 4.2.3, using $\sigma_m \leq \epsilon$,

$$||\tilde{p}||_{\tilde{H}} \leq \frac{1}{\sqrt{.99525}}\left(\sqrt{3}(.000006) + \frac{.0095}{.99525}\right) < .009579$$

Also, by Lemma 4.2.2, $\tilde{\sigma}_{\min} \geq \sigma_{\min}$, and therefore $\tilde{\mu} \leq \mu$. Applying Theorem 4.2.3 again, using $\sigma_{\min} < \epsilon$, and $\mu||p||_H \leq .0001$, then obtains

$$\begin{aligned}
\tilde{\mu}||\tilde{p}||_{\tilde{H}} &\leq \mu||\tilde{p}||_{\tilde{H}} \\
&\leq \frac{1}{\sqrt{1-\epsilon}}\left(\sqrt{3}\mu||p||_H + \frac{2\sigma_{\min}\mu}{1-\epsilon}\right) \\
&\leq \frac{1}{\sqrt{.99525}}\left(.000173 + \frac{2\sigma_{\min}\mu}{.99525}\right) \\
&\leq .0001736 + 2.014\sigma_{\min}\mu \qquad (5.5)
\end{aligned}$$

But $\sigma_{\min} \leq \epsilon = .00475$, so $\sigma_{\min} \leq \sqrt{\epsilon}\sqrt{\sigma_{\min}} = .0689\sqrt{\sigma_{\min}}$ and therefore

$$\mu = (2\sqrt{\sigma_{\min}} - \sigma_{\min})^{-1/2} \leq (1.931\sqrt{\sigma_{\min}})^{-1/2} \leq .7197\sigma_{\min}^{-1/4} \tag{5.6}$$

Combining (5.5) and (5.6) and using $\sigma_{\min} \leq \epsilon = .00475$ we then have that

$$\tilde{\mu}\|\tilde{p}\|_{\tilde{H}} \leq .0001736 + 2.014(.7197)(.00475)^{3/4} < .00264$$

Using the same notation as in Step 5 of the algorithm, repeatedly applying (5.1) and (5.2) then obtains

$$\begin{aligned}
\|\bar{p}^1\|_{\bar{H}^1} &< .005943\,, & \bar{\mu}^1\|\bar{p}^1\|_{\bar{H}^1} &< .016819 \\
\|\bar{p}^2\|_{\bar{H}^2} &< .002174\,, & \bar{\mu}^2\|\bar{p}^2\|_{\bar{H}^2} &< .006257 \\
\|\bar{p}^3\|_{\bar{H}^3} &< .000272\,, & \bar{\mu}^3\|\bar{p}^3\|_{\bar{H}^3} &< .000787 \\
\|\bar{p}^4\|_{\bar{H}^4} &< .000004\,, & \bar{\mu}^4\|\bar{p}^4\|_{\bar{H}^4} &< .000012
\end{aligned}$$

proving parts i) and ii). To prove part iii) we again repeatedly use the convexity property of $\mathcal{V}(\cdot)$ and Lemma 4.2.1 to give us

$$\begin{aligned}
\tilde{\mathcal{V}}(\bar{x}) &\geq \mathcal{V}(x) + \tfrac{1}{2}\ln(1-\epsilon) - \sum_{j=0}^{3}\|\bar{p}^j\|_{\bar{H}^j}^2 \\
&\geq \mathcal{V}(x) + .5\ln(.99525) - (.009579^2 + 0.005943^2 + \cdots + .000004^2) \\
&\geq \mathcal{V}(x) - .00251
\end{aligned}$$

proving part iii) and the theorem. □

From Theorem 5.1 and Theorem 5.2 we see that $\Delta\mathcal{V} = \Delta\mathcal{V}^+ - \Delta\mathcal{V}^- = .0025438 - .0025125 = .000031 > 0$. It is not possible to increase τ much further and still satisfy the proximity conditions. Insignificant increases in τ beyond this value merely increases the number of Newton steps that will be needed after a constraint addition or deletion and also results in a huge decrease in $\Delta\mathcal{V}$. Further increases in τ merely results in $\Delta\mathcal{V}$ becoming negative and thus violating the assumptions of Theorem 3.4.1.

5.1 Comparison with Anstreicher's and Vaidya's constants

In terms of specification of the algorithm, this algorithm differs from Anstreicher's algorithm in that we have included a linesearch prior to every Newton step and have used a different set of parameters. As for Vaidya's algorithm it takes Newton steps based on directions $d = -Q^{-1}g$ and uses a proximity measure based on $\mathcal{V}(x) - \mathcal{V}(w)$, where w is the true minimizer of $\mathcal{V}(\cdot)$. By contrast the fundamental proximity measure used here is $||p||_H$, but explicit control over the measure $\mu||p||_H$ is also necessary. Anstreicher's quadratic convergence result gives much sharper control over the proximity measures, using Newton steps, than Vaidya has over his measure $\mathcal{V}(x) - \mathcal{V}(w)$ and this means that τ and ϵ can be increased on steps with constraint addition and deletion while still returning the proximity measures to their prescribed values using a very small number of Newton steps. This is obviously very desirable from a practical perspective and is what motivated us to find how large we can increase τ and still be able to establish the same complexity result.

Now, a larger ϵ means that we will carry fewer constraints (the maximum number of constraints carried being $n/\epsilon+1$), and in practice a larger setting of τ translates into an immediate larger value for $\Delta\mathcal{V}$ that will lead to fewer iterations of the algorithm. In his analysis Vaidya uses $\epsilon = 10^{-7}$ and his $\Delta\mathcal{V}$ is about 1.325×10^{-7}. Furthermore, on a step where a constraint is added Vaidya's algorithm takes 2197 Newton-like steps (based on the matrix Q), while on a step where a constraint is deleted his algorithm takes 1493 Newton-like steps. Anstreicher sets $\tau = .0035$ and $\epsilon = .0025$, in one of the instances that he considers, resulting in $\Delta\mathcal{V} = .00033$ with a total of 3 Newton steps taken on an a constraint addition and 2 on a constraint deletion. In our attempt to set τ at a maximum value, while still satisfying the requirements of the convergence Theorem 3.4.1, we get that τ can be increased to .0062 (an increase of more that 77% over Anstreicher's setting of τ) with $\epsilon = .0049$, $\gamma_1 = .000006$ and $\gamma_2 = .0001$ and this gives $\Delta\mathcal{V} = .000031$ with 7 Newton steps taken on a constraint addition and 4 on a constraint deletion. The restrictions imposed by the proximity measures and the bounds that ensure that the number of Newton steps will be $\mathcal{O}(1)$, causing the value of $\Delta\mathcal{V}$ to in fact decrease by about a factor of 9. Any further increase in τ using this procedure would result in a negative $\Delta\mathcal{V}$.

Anstreicher thus reduces the number of constaints that are carried by Vaidya by a factor of 2.5×10^4 while increasing $\Delta\mathcal{V}$ by a factor of about 2490 ($\approx .00033/(1.325 \times 10^{-7})$) more than Vaidya. Also, Vaidya requires a factor of 738 ($= (2197 + 1493)/5$) more Newton steps following a pair consisting of a constraint addition and deletion and since by (3.14) the maximum number of iterations of the algorithm is inversely proportional to $\Delta\mathcal{V}$, Anstreicher's analysis succeeds in reducing the total number of Newton steps required by the algorithm by a factor of about 1.8 million ($\approx 2490 \times 738$) over that of Vaidya's. With our modifications we have an improvement by about a factor of 2 ($\approx .0049/.0025$) over Anstreicher and 5×10^4 over Vaidya in the maximum number of constraints that are carried. As remarked earlier our $\Delta\mathcal{V}$ has decreased by about a factor of 9 over that of Anstreicher's result, but is still greater by about a factor of 230 over Vaidya's result. Vaidya requires about 335 more Newton steps than us following a pair consisting of a constraint addition and deletion, whereas we exceed the number of Newton steps that Anstreicher takes by 6. We decrease the total number of Newton steps required by the algorithm by a factor of about 77,000 over Vaidya but Anstreicher reduces our total number of Newton steps by a factor of about 23.

Our algorithm therefore further reduces the maximum number of constraints that will be carried, at a cost of a decrease in the value of $\Delta\mathcal{V}$. These results have been obtained from our attempt to make the algorithm more efficient while implementing it in practice (through increasing τ) and while still trying to satisfy the theory that establishes that the number of Newton steps at each iteration will be $\mathcal{O}(1)$. In the following section we consider the case where we can allow τ to increase indefinitely under the 'black box' assumption that the number of Newton steps that will be taken at each iteration will be $\mathcal{O}(1)$.

5.2 Analysis using a black box volumetric centering complexity model (BBVC)

In this section we consider a Black Box Volumetric Centering complexity scenario (BBVC) where we remove all restrictions placed on the parameter τ, (i.e. we can have it set at any value greater than zero) and make the assumption that the number of Newton steps taken will be $\mathcal{O}(1)$ in order to re-center after a constraint addition or deletion. Under this assumption it is easy to see that we can satisfy the requirements of Theorem 3.4.1 that establishes that termination will be achieved in $\mathcal{O}(nL)$ steps. By Lemma 4.1.1 and Lemma 4.2.1 we have that

$$\Delta\mathcal{V} \geq \tfrac{1}{2}\ln(1+\tau) + \tfrac{1}{2}\ln(1-\epsilon) \;=\; \tfrac{1}{2}\ln[(1+\tau)(1-\epsilon)]$$

and thus $\Delta\mathcal{V}$ will be $\Omega(1)$ and positive if $\epsilon < \tau/(1+\tau)$ and this will always be the case in our analysis. Thus, we define the BBVC as our volumetric cutting plane algorithm together with linesearch and our complexity assumption for larger τ. The computer code representing the BBVC (see Appendix C) was used to draw conclusions about what parameter settings would be best to enable the algorithm to perform at an optimal level in practice

We proceeded to analyze the BBVC during runs of our algorithm on randomly generated instances of the convex set $\mathcal{C}$ and try and arrive at some promising values for both the parameter τ that determines how far from the test point our separating hyperplane would be placed and the parameter $\mathcal{K}$ that specifies the number of bisections the Bisection Method [1] would perform while doing a linesearch. The cases that we considered were instances of our problem using dimensions (2×6), (5×15) and (10×30). The parameter τ and the parameter $\mathcal{K}$ were allowed to vary within ranges that most influenced the efficiency of the algorithm. As the bulk of computation is in matrix inversion it is reasonable to use the total number of matrix inversions that have been carried out in a run of the model as a yardstick by which to measure efficiency. It can easily be seen that the calculation of the Hessian

$\mathcal{K}$	τ .1	.5	1	2	5	10	15	20	30	50	80
8	530	405	440	340	410	392	350	420	327	330	325
9	613	376	404	315*	393	346	324*	338	341	352	345
10	504	408	429	336	426	393	342	369	404	354	444

Table 5.1: Average number of matrix inversions required for 2×6 instance on 5 problems

$\mathcal{K}$	τ .1	.5	1	2	5	10	15	20	30	50	80
8	5070	2123	1710	1533	1473	1263	1266	1330	1310	1376	1396
9	3945	2273	1727	1426	1294	1224	1191*	1155*	1235	1345	1345
10	3920	2348	1796	1480	1392	1308	1228	1232	1344	1428	1404

Table 5.2: Average number of matrix inversions required for 5×15 instance on 5 problems

involves two such matrix inversions whereas the calculation of the gradient involves only one matrix inversion. Hence, it can be easily verified that a single Newton step requires two matrix inversions while every step in the Bisection Method requires one inversion. Three separate problem sizes were considered and each entry in the tables shown below represent averages of several runs of the program for each pair of τ and $\mathcal{K}$ with promising results indicated by asterisks.

Looking at the results across the dimensions considered, it appeared that a promising value for τ would be 15 whereas a good value for $\mathcal{K}$ would be 9 for smaller dimension problems and 10 for larger ones. In Table 5.1 for the 2×6 instance taking $\tau = 15$ and $\mathcal{K} = 9$ the algorithm took a maximum of 3 Newton steps at any iteration and the total number of Newton steps taken ranged from 24 to 35. In Table 5.2 for the 5×15 instance, with $\tau = 15$ and $\mathcal{K} = 9$ the algorithm took a maximum of 4 Newton steps at any iteration and the total number of Newton steps taken ranged from 90 to 120. Finally in the 10×30 instance with τ again taken to be 15 and $\mathcal{K}$ taken to be

* indicates favourable results

$\mathcal{K}$	τ									
	.5	1	2	5	10	15	20	30	50	80
9	6163	4264	3527	3157	3069	2944*	3021	2929*	3080	3138
10	5980	4308	3520	3232	3072	3004	2968*	2940*	3120	3152
11	7921	4563	3802	3601	3094	3121	3282	3139	3484	3427

Table 5.3: Average number of matrix inversions required for 10×30 instance on 3 problems

9 we got that the algorithm took a maximum of 5 Newton steps at any iteration and the total number of Newton steps taken ranged from 260 to 273.

Another important observation that can be seen from the tables is that for larger values of τ the performance drops (as is the case for smaller values of τ). The reason behind this is that if the separating hyperplane is placed too close to the test point then it hampers the progress of the Newton steps that are taken starting at the test point. It seems that for the best performance the hyperplane must be backed off a short distance from the test point before starting the Newton steps. Anstreicher (1994c) refers to this as a fundamental limitation in that the constraint cannot be placed through the current point and this is clearly shown by our results.

Appendix A

Proofs of some theorems

A.1 The analytic center

The analytic center of a polytope $\mathcal{P} = \{x|Ax \geq b\}$ is the point that maximizes the logarithmic barrier function $f(x) = -\sum_{i=1}^{m}(\ln a_i^T x - b_i)$ over $\mathcal{P}$, ie. it is the solution of the following program

$$\begin{aligned} \max \quad & -\sum_{i=1}^{m} \ln(a_i^T x - b_i) \\ \text{s.t.} \quad & Ax - s = b\,,\ s > 0 \end{aligned} \tag{A.1}$$

We have that $\nabla f(x) = -A^T S^{-1} e$ and so for $\bar{x}$ and $\bar{s}$ to solve (A.1) then it must be the case that $\nabla f(\bar{x}) = -A^T \bar{S}^{-1} e = 0$, else we could find a descent direction $d = -\nabla f(\bar{x})$.

Proposition A.1.1 $\mathcal{P} = \{x \mid Ax \geq b\} \subset E_{OUT} = \{x \mid (x - \bar{x})^T G(\bar{x})(x - \bar{x}) \leq m^2\}$

Proof We first observe that $\forall\, x, s$ with $Ax - s = b$, $s \geq 0$

$$e^T \bar{S}^{-1} s = e^T \bar{S}^{-1}(Ax - b) = -e^T \bar{S}^{-1} b = -e^T \bar{S}^{-1}(A\bar{x} - \bar{s}) = e^T \bar{S}^{-1} \bar{s} = m \tag{A.2}$$

Next, using (A.2) we have that for $x \in \mathcal{P}$

$$\begin{aligned}
\|\bar{S}^{-1}A(x-\bar{x})\|^2 &= \sum_{i=1}^{m}\left(\frac{a_j^T}{\bar{s}_j}(x-\bar{x})\right)^2 = \sum_{i=1}^{m}\left(\frac{a_j^T x - b_j - \bar{s}_j}{\bar{s}_j}\right)^2 = \sum_{i=1}^{m}\left(\frac{s_j}{\bar{s}_j}-1\right)^2 \\
&= \sum_{i=1}^{m}\left(\frac{s_j}{\bar{s}_j}\right)^2 - 2\sum_{i=1}^{m}\frac{s_j}{\bar{s}_j} + m \le \left(\sum_{i=1}^{m}\frac{s_j}{\bar{s}_j}\right)^2 = m^2
\end{aligned}$$

$\Rightarrow x \in E_{OUT} = \{x \mid (x-\bar{x})^T A^T \bar{S}^{-2} A(x-\bar{x}) \le m^2\}$. □

A.2 Some properties of matrices

Proposition A.2.1 [Sherman–Morrison–Woodbury formula]

$$\left(A + vw^T\right)^{-1} = A^{-1} - \frac{A^{-1}vw^TA^{-1}}{1+w^TA^{-1}v}$$

Proof

$$\begin{aligned}
\left(A^{-1} - \frac{A^{-1}vw^TA^{-1}}{1+w^TA^{-1}v}\right)\left(A+vw^T\right) &= I + A^{-1}vw^T - \frac{A^{-1}vw^TA^{-1}A + A^{-1}vw^TA^{-1}vw^T}{1+w^TA^{-1}v} \\
&= I + A^{-1}vw^T - \frac{A^{-1}vw^T(1+w^TA^{-1}v)}{1+w^TA^{-1}v} = I. \quad \square
\end{aligned}$$

Corollary

$$\left(A + vv^T\right)^{-1} = A^{-1} - \frac{A^{-1}vv^TA^{-1}}{1+v^TA^{-1}v}. \quad \square$$

Proposition A.2.2 If $P \succeq 0$, then $P \circ (aa^T) \succeq 0$

Proof Let $M = P \circ (aa^T)$, then $M_{ij} = P_{ij}a_ia_j$ and $\xi^TM\xi = \sum_i\sum_j \xi_i M_{ij}\xi_j = \sum_i\sum_j \xi_i a_i P_{ij} a_j \xi_j = v^TPv \succeq 0$, where $v_k = a_k\xi_k$. □

Proposition A.2.3 If $P \succeq 0$ and $Q \succeq 0$, then $P \circ Q \succeq 0$

Proof Let $Q = RR^T$ and let $M^k = R^k(R^k)^T$, where R^k is the kth column of R. Then $M^k_{ij} = R_{ik}R_{jk}$ and $\left(\sum_k M^k\right)_{ij} = \sum_k R_{ik}R_{jk} = \sum_k R_{ik}(R^T)_{kj}$. Also we have that, $Q_{ij} = \sum_k R_{ik}(R^T)_{kj} = \sum_k (M^k)_{ij}$, ie. $Q = \sum_k M^k$. Thus, $P \circ Q = \sum_k P \circ M^k = \sum_k P \circ (R^k(R^k)^T) \succeq 0$, since by Proposition A.2.2 $P \circ (R^k(R^k)^T) \succeq 0 \, \forall k$. □

Proposition A.2.4 If A and B are symmetric positive semi-definite matrices, and $A \preceq B$, then $A^{(2)} \preceq B^{(2)}$.

Proof We have that $B - A \succeq 0$ and $B + A \succeq 0$ and so by Proposition A.2.3 $(B + A) \circ (B - A) \succeq 0$, ie. $B^{(2)} - A^{(2)} \succeq 0$. □

Theorem A.2.5 [Gershgorin Circle Theorem] The eigenvalues of a symmetric matrix W are contained in the union of the intervals $W_{ii} \pm \sum_{j \neq i} |w_{ij}|$, $i = 1, \ldots, m$.

Proof Take any eigenvalue λ and let x be the corresponding eigenvector. Choose i such that $|x_i| \geq |x_j| \; \forall j$. Now, $Wx = \lambda x \;\Rightarrow\; W_{ii}x_i + \sum_{j \neq i} W_{ij}x_j = \lambda x_i$, and it follows that $(W_{ii} - \lambda) = \sum_{j \neq i} W_{ij}\left(\frac{-x_j}{x_i}\right) \;\Rightarrow\; -\sum_{j \neq i} |W_{ij}| \leq (W_{ii} - \lambda) \leq \sum_{j \neq i} |W_{ij}|$, and so we have that $W_{ii} - \sum_{j \neq i} |W_{ij}| \leq \lambda \leq W_{ii} + \sum_{j \neq i} |W_{ij}|$. Thus, $\lambda_k \in \bigcup_i \left(W_{ii} \pm \sum_{j \neq i} |w_{ij}|\right) \; \forall k$. □

Proposition A.2.6 If B is a symmetric positive definite matrix, then

$$|\xi_1^T B \xi_2| \leq \|\xi_1\|_B \, \|\xi_2\|_B$$

Proof Letting $B = M^T M$, we get that

$$|\xi_1^T B \xi_2| = |(M\xi_1)^T (M\xi_2)| \leq |M\xi_1| \, |M\xi_2| = \|\xi_1\|_B \, \|\xi_2\|_B. \quad \square$$

Proposition A.2.7 $\|Mx\| \leq |M| \|x\|$

Proof $\|Mx\| = \sqrt{x^T M^T M x} = \sqrt{x^T Q^T D Q x}$, (since $M^T M$ is symmetric and can be written as $Q^T D Q$ where the columns of the matrix Q are orthogonal and the diagonal

matrix D contains the eigenvalues of M^TM),
$= \sqrt{\sum_i D_{ii}(Qx)_i^2} \leq \sqrt{\sum_i |M|^2(Qx)_i^2} = |M|\sqrt{x^TQ^TQx} = |M|\sqrt{x^Tx} = |M|\|x\|$. □

Proposition A.2.8 M is symmetric $\Rightarrow |M| = \max |\lambda_i(M)|$

Proof Self-evident. If λ, x are eigenvalue, vector of M then $Mx = \lambda x \Rightarrow M^TM = \lambda M^Tx = \lambda Mx = \lambda^2 x$, ie. λ^2, x are eigenvalue, vector of M^TM and the result follows.

□

Proposition A.2.9 Let A and B be $n \times n$ symmetric matrices such that $|\xi^TA\xi| \leq \xi^TB\xi \quad \forall\, \xi \in \Re^n$ and suppose that the matrix B is positive definite. Then $|\xi^{1^T}A\xi^2| \leq \|\xi^1\|_B \, \|\xi^2\|_B \quad \forall\, \xi^1, \xi^2 \in \Re^n$.

Proof Let $\bar{A} = B^{-1/2^T}AB^{-1/2}$; where B is written as $B^{1/2^T}B^{1/2}$ as it is positive definite. Then $\forall\, \xi \in \Re^n$, $|\xi^T\bar{A}\xi| = |\xi^TB^{-1/2^T}AB^{-1/2}\xi| \leq \xi^TB^{-1/2^T}BB^{-1/2}\xi = \xi^TI\xi \Rightarrow -I \preceq \bar{A} \preceq I$. Now $\bar{A}$ is symmetric and thus it can be written as R^TDR where R is the matrix of orthonormal eigenvectors of $\bar{A}$ and D is the diagonal matrix of eigenvalues of $\bar{A}$. Thus, $-I \preceq R^TDR \preceq I \Rightarrow \forall\, \eta \in \Re^n$, $-\eta^TI\eta \leq (\eta^TR)R^TDR(R^T\eta) = \eta^TD\eta \leq \eta^TI\eta$ and so $\lambda_i(\bar{A}) \in [-1, 1]$ giving that $|\bar{A}\,| \leq 1$.
Finally,

$$x^TAy = x^TB^{1/2^T}B^{-1/2^T}AB^{-1/2}B^{1/2}y = x^TB^{1/2^T}\bar{A}B^{1/2}y \leq \|B^{1/2}x\| \, \|\bar{A}B^{1/2}y\|$$

$$\leq \|B^{1/2}x\| \, |\bar{A}\,| \, \|B^{1/2}y\| \leq \|B^{1/2}x\| \, \|B^{1/2}y\| = \|x\|_B \, \|y\|_B. \quad \square$$

A.3 Projection matrices

A matrix P is a projection matrix if the following two properties hold

1. $P^T = P$

2. $PP = P$

Proposition A.3.1 For any projection matrix P the following holds true:

1. $I - P$ is a projection matrix
2. P is positive semi-definite
3. $||Px|| \leq ||x||$

Proof

1. $(I-P)^T = I - P^T = I - P$, and $(I-P)(I-P) = I - 2P + PP = I - 2P + P = I - P$. $\square$
2. $x^T P x = x^T P P x = x^T P^T P x = ||Px||^2 \geq 0$. $\square$
3. $||x||^2 = ||Px + (I-P)x||^2 = ||Px||^2 + ||(I-P)x||^2 + 2x^T P(I-P)x = ||Px||^2 + ||(I-P)x||^2$ and since $P(I-P)$ is just the null matrix it is evident that, $||x||^2 \geq ||Px||^2 \Rightarrow ||x|| \geq ||Px||$. $\square$

A.4 Properties of the volumetric barrier function $\mathcal{V}(\cdot)$

Lemma A.4.1 Fix $s > 0$, and let $\sigma = \sigma(s)$, then $0 < \sigma_i \leq 1$, $i = 1, \ldots, m$, and $\sum_{i=1}^m \sigma_i = n$.

Proof Let $P = P(s)$, and let e_i denote the vector with a 1 in the ith component, and all other components equal to zero. Then $\sigma_i = e_i^T P e_i = ||Pe_i||^2 \leq ||e_i||^2 = 1$, establishing that $0 \leq \sigma_i \leq 1$. Also, note that as $s > 0$, $(A^T S^{-2} A)$ is positive definite and so $(A^T S^{-2} A)^{-1}$ is also positive definite $\Rightarrow \sigma_i = \dfrac{1}{s_i^2} a_i^T (A^T S^{-2} A)^{-1} a_i > 0$.

That $\sum_{i=1}^m \sigma_i = n$ follows from the fact that P has n eigenvalues equal to 1, and $m - n$ eigenvalues equal to 0. This is seen as follows.

First for any $m \times m$ symmetric matrix A, $\sum_{i=1}^m A_{ii} = \sum_{i=1}^m \lambda_i(A)$

(To see this, let $A = R^T DR$, where R^T is an orthonormal matrix whose columns contain the eigenvectors of A and D is the diagonal matrix consisting of the eigenvalues $\lambda_i(A)$ then $A_{ii} = \sum_{j=1}^m \lambda_j R_{ji}^2 \Rightarrow \sum_{i=1}^m A_{ii} = \lambda_1[R_{11}^2 + \cdots + R_{1m}^2] + \cdots + \lambda_m[R_{m1}^2 + \cdots + R_{mm}^2] = \sum_{i=1}^m \lambda_i$, since $\|r_i\| = 1$)

Secondly, the dimension of $\mathcal{N}ull(A^T S^{-1})$ is $m - n$, since only n out of m columns of A^T are linearly independent and if $x \in \mathcal{N}ull(A^T S^{-1})$ then $Px = 0x$; and the dimension of $\mathcal{R}ange(S^{-1}A) = n$, since the columns of A are linearly independent and if $x \in \mathcal{R}ange(S^{-1}A)$ then $Px = x$. Thus, A has $m - n$ eigenvalues equal to 0 and n eigenvalues equal to 1. □

Lemma A.4.2 Let $u, v \in \Re^n$. Then $\det(I - uv^T) = 1 - u^T v$.

Proof This follows from the fact that the matrix $(I - uv^T)$ has $n - 1$ eigenvalues equal to 1 with corresponding eigenvectors spanning $\mathcal{R}ange(S^{-1}A)$, and one eigenvalue equal to $1 - u^T v$ with corresponding eigenvector u; and that for any $n \times n$ matrix A, $\det(A) = \prod_{i=1}^n \lambda_i(A)$

(To see this, we have that $(I - uv^T)x = x \Leftrightarrow uv^T x = 0$ and $n - 1$ eigenvectors span the space $\mathcal{Z} = \{x \in \Re^n | \, v^T x = 0\}$. Also, we have that $(I - uv^T)u = u - uv^T u = u(1 - v^T u) \Rightarrow u$ is an eigenvector and $(1 - u^T v)$ its eigenvalue.

Finally, for any $n \times n$ matrix A let R be the matrix whose columns contain the eigenvectors of A and let D be the diagonal matrix of corresponding eigenvalues $\lambda_i(A)$ of A. Then $AR = RD \Rightarrow \det(A)\det(R) = \det(R)\det(D) \Rightarrow \det(A) = \prod_{i=1}^n \lambda_i(A)$.) □

Lemma A.4.3 Let x have $s = s(x) > 0$, and let $\sigma = \sigma(s)$. Then $\nabla \mathcal{V}(x)^T = -A^T S^{-1} \sigma$.

Proof Consider the function $v(\cdot) : \Re^m \to \Re$ given by $v(s) = \frac{1}{2} \ln(\det(A^T S^{-2} A))$. Then

$$\frac{\partial v(s)}{\partial s_j} = \lim_{\lambda \to 0} \frac{v(s + \lambda e_j) - v(s)}{\lambda} \tag{A.3}$$

However, $v(s + \lambda e_j) = \frac{1}{2} \ln(\det(A^T (S + \lambda E_j)^{-2} A))$, where $E_j = \mathrm{diag}(e_j)$. Letting

a_j^T denote the jth row of A, we get

$$A^T(S+\lambda E_j)^{-2}A = A^TS^{-2}A + ((s_j+\lambda)^{-2} - s_j^{-2})a_ja_j^T$$

(this is because $A^T(S+\lambda E_j)^{-2}A = A^T(S^{-2} + ((s_j+\lambda)^{-2} - s_j^{-2})E_j)A = A^TS^{-2}A + ((s_j+\lambda)^{-2} - s_j^{-2})A^TE_jA$, and $A^TE_jA = a_ja_j^T$.)

$$\begin{aligned} &= A^TS^{-2}A - \frac{\lambda^2+2s_j\lambda}{(s_j+\lambda)^2s_j^2}a_ja_j^T \qquad \text{(A.4)} \\ &= A^TS^{-2}A\left(I - \frac{\lambda^2+2s_j\lambda}{(s_j+\lambda)^2s_j^2}(A^TS^{-2}A)^{-1}a_ja_j^T\right) \end{aligned}$$

Now $P(s) = S^{-1}A(A^TS^{-2}A)^{-1}A^TS^{-1}$, and so $\sigma_j = \frac{1}{s_j^2}a_j^T(A^TS^{-2}A)^{-1}a_j$, giving us that $v(s+\lambda e_j) = \frac{1}{2}\ln\left[\det(A^TS^{-2}A)\det\left(I - \frac{\lambda^2+2s_j\lambda}{(s_j+\lambda)^2s_j^2}a_ja_j^T\right)\right]$. On applying lemma (A.4.2) with $u = \frac{\lambda^2+2s_j\lambda}{(s_j+\lambda)^2s_j^2}(A^TS^{-2}A)^{-1}a_ja_j^T$ and $v = a_j$ we get that

$$v(s+\lambda e_j) = v(s) + \frac{1}{2}\ln\left(1 - \frac{\lambda^2+2s_j\lambda}{(s_j+\lambda)^2}\sigma_j\right) \qquad \text{(A.5)}$$

Substituting (A.5) into (A.3), we have

$$\frac{\partial v(s)}{\partial s_j} = \frac{1}{2}\lim_{\lambda\to 0}\frac{1}{\lambda}\ln\left(1 - \frac{\lambda^2+2s_j\lambda}{(s_j+\lambda)^2}\sigma_j\right) = \frac{1}{2}\frac{d}{d\lambda}\bigg|_{\lambda=0}\ln\left((1-\sigma_j) + \frac{s_j^2\sigma_j}{(s_j+\lambda)^2}\right) \qquad \text{(A.6)}$$

where the last equality follows from,

(i) $1 - \frac{\lambda^2+2s_j\lambda}{(s_j+\lambda)^2}\sigma_j = \frac{(s_j+\lambda)^2-\lambda^2\sigma_j-2s_j\lambda\sigma_j-s_j^2\sigma_j+s_j^2\sigma_j}{(s_j+\lambda)^2} = \frac{(s_j+\lambda)^2(1-\sigma_j)+s_j^2\sigma_j}{(s_j+\lambda)^2}$
$= (1-\sigma_j) + \frac{s_j^2\sigma_j}{(s_j+\lambda)^2}$

(ii) if we define the function f by $f(\lambda) = \frac{1}{2}\ln\left((1-\sigma_j) + \frac{s_j^2\sigma_j}{(s_j+\lambda)^2}\right)$, then $f(0) = 0$ and $\lim_{\lambda\to 0}\frac{f(\lambda)-f(0)}{\lambda} = \frac{d}{d\lambda}\Big|_{\lambda=0}f(\lambda)$

A straightforward computation then gives,

$$\frac{d}{d\lambda}\ln\left((1-\sigma_j) + \frac{s_j^2\sigma_j}{(s_j+\lambda)^2}\right) = \frac{-2s_j^2\sigma_j}{(s_j+\lambda)[(s_j+\lambda)^2 - \lambda\sigma_j(2s_j+\lambda)]} \qquad \text{(A.7)}$$

and on putting $\lambda = 0$ we get that $\frac{\partial v(s)}{\partial s_j} = -\frac{\sigma_j}{s_j} \Rightarrow \nabla v(s) = -\sigma^T S^{-1}$

And since $s = Ax - b$, the chain rule gives that $\nabla_x V(s(x)) = \nabla_s V(s(x)) \nabla_x s(x) = \sigma^T S^{-1} A$

(we have that v is a function of $s_1, \ldots, s_m$ and that each s_i is a function of $x_1, \ldots, x_n$ then by the chain rule we have that,

$$\frac{\partial v}{\partial x_j} = \frac{\partial v}{\partial s_1}\frac{\partial s_1}{\partial x_j} + \cdots + \frac{\partial v}{\partial s_m}\frac{\partial s_m}{\partial x_j}$$

$$= \left(\frac{\partial v}{\partial s_1}, \ldots, \frac{\partial v}{\partial s_m} \right) \begin{pmatrix} \frac{\partial s_1}{\partial x_j} \\ \vdots \\ \frac{\partial s_m}{\partial x_j} \end{pmatrix} = -\sigma^T S^{-1} A_j \text{ , where } A_j \text{ is the } j\text{th column of } A$$

$$\Rightarrow \nabla_x V(s(x)) = -\sigma^T S^{-1} A) . \quad \square$$

Lemma A.4.4 Let x have $s = s(x) > 0$ and let $\sigma = \sigma(s), P = P(s)$. Then, $\nabla^2 \mathcal{V}(x) = A^T S^{-1}(3\Sigma - 2P^{(2)})S^{-1}A$, where $\Sigma = \text{diag}(\sigma)$.

Proof Let $g(s) = \nabla v(s)^T = -S^{-1}\sigma(s)$. Then,

$$g_i(s) = -\frac{\sigma_i(s)}{s_i} = -\frac{\nu_i(s)}{s_i^3}, \; i = 1, \cdots, m \tag{A.8}$$

where $\nu_i(s) = a_i^T(A^T S^{-2} A)^{-1} a_i = s_i^2 \sigma_i(s)$. We will compute,

$$\frac{\partial \nu_i(s)}{\partial s_j} = \lim_{\lambda \to 0} \frac{a_i^T a_i - a_i^T (A^T S^{-2} A)^{-1} a_i}{\lambda} \tag{A.9}$$

First, using (A.4) and the Sherman-Morrison formula we obtain,

$$\begin{aligned}(A^T(S + \lambda E_j)^{-2} A)^{-1} &= (A^T S^{-2} A)^{-1} + \left(\frac{\lambda^2 + 2\lambda s_j}{(s_j + \lambda)^2 s_j^2} \right) \frac{(A^T S^{-2} A)^{-1} a_j a_j^T (A^T S^{-2} A)^{-1}}{1 - \sigma_j \lambda (2s_j + \lambda)/(s_j + \lambda)^2} \\ &= (A^T S^{-2} A)^{-1} + \frac{\lambda(2s_j + \lambda)}{s_j^2[(s_j + \lambda)^2 - \sigma_j \lambda(2s_j + \lambda)]} w_j w_j^T \end{aligned} \tag{A.10}$$

where $w_j = (A^T S^{-2} A)^{-1} a_j$. Substituting (A.10) into (A.9) and noting that

$$p_{ij} = \frac{1}{s_i s_j} a_i^T (A^T S^{-2} A)^{-1} a_j = \frac{a_i^T w_j}{s_i s_j}$$

we obtain

$$\frac{\partial \nu_i(s)}{\partial s_j} = \lim_{\lambda \to 0} \frac{1}{\lambda} \frac{\lambda (2s_j + \lambda) p_{ij}^2 s_i^2}{(s_j + \lambda)^2 - \sigma_j \lambda (2s_j + \lambda)} = \frac{2p_{ij}^2 s_i^2}{s_j} \tag{A.11}$$

From (A.8) and (A.11) we then have

$$\frac{\partial g_i(s)}{\partial s_j} = \begin{cases} -2p_{ij}^2/(s_i s_j) & \text{if } j \neq i \\ (3\sigma_i - 2p_{ii}^2)/s_i^2 & \text{otherwise} \end{cases}$$

which is exactly $\nabla^2 v(s) = S^{-1}(3\Sigma - 2P^{(2)})S^{-1}$. The formula for $\nabla^2 \mathcal{V}(x)$ follows from the relationship $s(x) = Ax - b$ and the chain rule

(we have that v is a function of $s_1, \ldots, s_m$ and that each s_i is a function of $x_1, \ldots, x_n$ then by the chain rule we have that,

$$\frac{\partial v}{\partial x_j} = \frac{\partial v}{\partial s_1}\frac{\partial s_1}{\partial x_j} + \cdots + \frac{\partial v}{\partial s_m}\frac{\partial s_m}{\partial x_j}$$

applying the chain rule again gives,

$$\begin{aligned} \frac{\partial^2 v}{\partial x_i \partial x_j} &= \left(\frac{\partial^2 v}{\partial x_i \partial s_1}\right)\frac{\partial s_1}{\partial x_j} + \cdots + \left(\frac{\partial^2 v}{\partial x_i \partial s_m}\right)\frac{\partial s_m}{\partial x_j} \quad , \quad \text{since } \frac{\partial^2 s_i}{\partial x_i \partial x_j} = 0 \\ &= \left[\left(\frac{\partial^2 v}{\partial s_1 \partial s_1}\right)\frac{\partial s_1}{\partial x_i} + \cdots + \left(\frac{\partial^2 v}{\partial s_m \partial s_1}\right)\frac{\partial s_m}{\partial x_i} \; , \; \ldots \; , \; \left(\frac{\partial^2 v}{\partial s_1 \partial s_m}\right)\frac{\partial s_1}{\partial x_i} + \cdots + \right. \\ &\qquad \left. \left(\frac{\partial^2 v}{\partial s_m \partial s_m}\right)\frac{\partial s_m}{\partial x_i}\right]^T [A_j] \\ &= [A_i]^T \left[\nabla^2 v(s)\right][A_j] \Rightarrow \nabla^2 \mathcal{V}(x) = A^T S^{-1}(3\Sigma - 2P^{(2)})S^{-1}A) \; . \quad \square \end{aligned}$$

A.5 Properties of the matrix $Q(x)$

Theorem A.5.1 Let x have $s = s(x) > 0$ and let $\sigma = \sigma(s)$. Define $Q(x) = A^T S^{-2} \Sigma A$. Then $\forall \xi \in \Re^n$, $\xi^T Q(x) \xi \leq \xi^T \nabla^2 \mathcal{V}(x) \xi \leq 3\, \xi^T Q(x) \xi$.

Proof Recalling that $\nabla^2 \mathcal{V}(x) = A^T S^{-1}(3\Sigma - 2\,P^{(2)})S^{-1}A$ and noting that $3\Sigma - 2\,P^{(2)} = \Sigma + 2\,(\Sigma - P^{(2)})$, then the relation $\Sigma \preceq 3\Sigma - 2\,P^{(2)} \preceq 3\Sigma$ and hence the proof will follow if we can show that the two matrices $(\Sigma - P^{(2)})$ and $P^{(2)}$ are positive semi-definite.

First we have that the matrix $P \succeq 0$ since $A^T S^{-2} A \succeq 0$ and by Theorem A.2.3 it follows that $P^{(2)} = P \circ P \succeq 0$. Secondly, using the properties of a projection matrix, namely $PP^T = PP = P$ we get that $\sigma_i = p_{ii} = \sum_{j=1}^m p_{ij}^2 = \sum_{j=1}^m (P^{(2)})_{ij}$, ie. $\sigma_i - \sigma_i^2 = \sum_{j \neq i}(P^{(2)})_{ij} \Rightarrow (\Sigma - P^{(2)})$ is diagonally dominant and so by the Gershgorin Circle Theorem A.2.5 all the eigenvalues are ≥ 0, ie. the matrix is positive semi-definite. $\square$

Theorem A.5.2 Let x have $s = s(x) > 0$, and let $\sigma = \sigma(s)$. Then for every $0 \leq \rho \leq 1$, and $\xi \in \Re^n$, $\xi^T A^T S^{-2}(\Sigma + \rho I)A\xi \geq \left[2\sqrt{\rho(m-1)+1}/(1+\sqrt{m})\right] \|S^{-1}A\xi\|_\infty^2$

Proof Let $\bar{A} = S^{-1}A$. Since the columns of $\bar{A}$ are linearly independent, we can write $\bar{A} = UR$, where U is an $m \times n$ matrix with orthonormal columns, and R is a nonsingular $n \times n$ matrix. Using the change of variables $\bar{\xi} = R\xi$, and noting that $\|U\bar{\xi}\| = \sqrt{\xi^T U^T U \xi} = \|\xi\|\ \forall \bar{\xi} \in \Re^n$, proving the theorem is equivalent to proving that

$$\rho\|\bar{\xi}\|^2 + \bar{\xi}^T U^T \Sigma U \bar{\xi}^T \geq \frac{2\sqrt{\rho(m-1)+1}}{1+\sqrt{m}} \|U\bar{\xi}\|_\infty^2 \tag{A.12}$$

$\forall \bar{\xi} \in \Re^n$. Pick a $\bar{\xi} \in \Re^n$ with $\|U\bar{\xi}\|_\infty = 1$. Note that the projection matrix P simplifies to UU^T, since R is nonsingular, and therefore $\sigma_i = p_{ii} = u_i^T u_i = \|u_i\|^2$, where u_i^T denotes the ith row of U. Taking note that $\bar{\xi}^T U^T \Sigma U \bar{\xi}^T = \sum_{i=1}^m \|u_i\|^2 (u_i^T \bar{\xi})^2$ and that $\|U\bar{\xi}\| = \sum_{i=1}^m (u_i^T \bar{\xi})^2$, a natural minimization problem to consider towards proving (A.12) is

$$\begin{aligned} \min \quad & \rho\|\bar{\xi}\|^2 + \sum_{i=1}^m \|u_i\|^2 (u_i^T \bar{\xi})^2 \\ \text{s.t.} \quad & \sum_{i=1}^m \|u_i\|^2 = n \\ & \sum_{i=1}^m (u_i^T \bar{\xi})^2 = \|\bar{\xi}\|^2 \end{aligned} \tag{A.13}$$

Since $||U\bar{\xi}||_\infty = 1$ there is a component i with $|u_i^T\bar{\xi}| = 1$, so assume WLOG that $|u_1^T\bar{\xi}| = 1$. Notice also that $|u_1^T\bar{\xi}| \le ||u_1||\,||\bar{\xi}||$, so $||u_1|| \ge 1/||\bar{\xi}||$. A relaxation of (A.13) is then the problem

$$\begin{aligned} \min \quad & \rho||\bar{\xi}||^2 + (1/||\bar{\xi}||^2) + \sum_{i=2}^{m} ||u_i||^2 (u_i^T\bar{\xi})^2 \\ \text{s.t.} \quad & \sum_{i=2}^{m} (u_i^T\bar{\xi})^2 = ||\bar{\xi}||^2 - 1 \end{aligned} \tag{A.14}$$

To obtain a lower bound on the solution value for (A.14), first consider the solution value of the problem

$$\begin{aligned} \min \quad & \sum_{i=2}^{m} ||u_i||^2 (u_i^T\bar{\xi})^2 \\ \text{s.t.} \quad & \sum_{i=2}^{m} (u_i^T\bar{\xi})^2 = ||\bar{\xi}||^2 - 1 \end{aligned} \tag{A.15}$$

as a function of $||\bar{\xi}||$. Let $\theta = ||\bar{\xi}||^2 \ge 1$. Using the fact that $(u_i^T\bar{\xi})^2 \le ||u_i||^2||\bar{\xi}||^2 = \theta||u_i||^2$ we then have that $||u_i||^2 \ge (u_i^T\bar{\xi})^2/\theta\,\forall i \ge 2$. Defining $v_i = (u_i^T\bar{\xi})^2$, the minimum value in (A.15), with $||\bar{\xi}||^2 = \theta$, is no less than the solution value in the minimization problem

$$\begin{aligned} \min \quad & (1/\theta) \sum_{i=2}^{m} v_i^2 \\ \text{s.t.} \quad & \sum_{i=2}^{m} v_i = \theta - 1 \end{aligned} \tag{A.16}$$

But the solution of (A.16) has $v_i = (\theta - 1)/(m-1)\,\forall i \ge 2$, and the solution value is

$$\frac{m-1}{\theta}\left(\frac{\theta-1}{m-1}\right)^2 = \frac{(\theta-1)^2}{\theta(m-1)} \tag{A.17}$$

Using (A.17), a lower bound on the minimum value for (A.14) is

$$\min_{\theta \ge 1}\left\{\rho\theta + \frac{1}{\theta} + \frac{(\theta-1)^2}{\theta(m-1)}\right\} = \left(\frac{1}{m-1}\right)\min_{\theta \ge 1}\left\{\frac{m}{\theta} + [\rho(m-1)+1]\theta - 2\right\} \tag{A.18}$$

A straightforward differentiation shows that the minimizing θ in (A.18) is

$$\theta = \sqrt{\frac{m}{\rho(m-1)+1}} \tag{A.19}$$

and the solution value for (A.18) is then

$$\begin{aligned}\frac{2}{m-1}\left(\sqrt{m[\rho(m-1)+1]}-1\right) &\geq \frac{2}{m-1}(\sqrt{m}-1)\sqrt{\rho(m-1)+1} \\ &= \frac{2\sqrt{\rho(m-1)+1}}{1+\sqrt{m}}\end{aligned}$$

and we have just shown that

$$\rho\|\bar{\xi}\|^2 + \bar{\xi}^T U^T \Sigma U \bar{\xi}^T \geq \frac{2\sqrt{\rho(m-1)+1}}{1+\sqrt{m}}$$

$\forall\bar{\xi}$ with $\|U\bar{\xi}\|_\infty = 1$, proving (A.12) and the theorem. □

Corollary Let x have $s = s(x) > 0$, and let $\sigma = \sigma(s)$. Then for every $\xi \in \Re^n$ $\xi^T Q\xi \geq 2/(1+\sqrt{m})\ \|S^{-1}A\xi\|_\infty^2$

Proof Follows on setting ρ to 0. □

Appendix B

Quadratic convergence result

We will show the quadratic convergence result for Newton's method applied to the volumetric function $\mathcal{V}(\cdot)$ for points in a close enough vicinity of the volumetric center w. We will first start by establishing two claims.

Claim B.1 Let B be an $n \times n$ symmetric positive definite matrix. Then

$$\max_{y \in E(B,x,r)} \{(w^T(y-x))^2\} = r^2 w^T B^{-1} w$$

where $w \in \Re^n$ is an arbitrary fixed vector.

Proof Since we are maximizing the square of a linear function over a convex set the optimal value will be on the boundary. Let $\tilde{x}$ be the point that gives the maximum value, then from the Karush-Kuhn-Tucker conditions we have that

$$\begin{aligned} -2w + 2vB(\tilde{x} - x) &= 0 \\ (\tilde{x} - x)^T B(\tilde{x} - x) &= r^2 \end{aligned}$$

which gives us that $\tilde{x} = \frac{rB^{-1}w}{\sqrt{w^T B^{-1} w}} + x$, and the maximum value $(w^T(\tilde{x} - x))^2 = r^2 w^T B^{-1} w$. □

Claim B.2 Let $\theta > 0$, and let B_1, B_2 be $n \times n$ positive definite matrices. Then

$$B_1 \succeq \theta B_2 \;\Rightarrow\; B_1^{-1} \preceq \frac{1}{\theta} B_2^{-1}$$

Proof Suppose that $\forall \xi \in \Re^n$, $\xi^T B_1 \xi \geq \theta \xi^T B_2 \xi$. Then $\xi^T B_1 \xi \leq 1 \Rightarrow \xi^T B_2 \xi \leq \frac{1}{\theta}$. Thus,

$$E(B_1, 0, 1) \subseteq E(B_2, 0, \frac{1}{\sqrt{\theta}})$$

and hence for $w \in \Re^n$,

$$\max_{\xi \in E(B_1,0,1)} \{(w^T\xi)^2\} \;\leq\; \max_{\xi \in E(B_2,0,\frac{1}{\sqrt{\theta}})} \{(w^T\xi)^2\}$$

So by Claim 1,

$$w^T B_1^{-1} w \;\leq\; \frac{1}{\theta} w^T B_2^{-1} w$$

and the proof is complete. □

Let $p \in \Re^n$ have $\delta = \|S^{-1}Ap\|_\infty < 1$, and let $\bar{x} = x + p$, where $x \in int(\mathcal{P})$. Let $\bar{s} = s(\bar{x})$, $\bar{P} = P(\bar{s})$, $\bar{\sigma} = \sigma(\bar{s})$, $\bar{H} = \nabla \mathcal{V}(\bar{x})$.

Proposition B.3 Let $\bar{x} = x + p$, where $\delta = \|S^{-1}Ap\|_\infty < 1$, then $\bar{x} \in \Sigma(x, \delta) \subset int(\mathcal{P})$ and

$$\frac{(1-\delta)^2}{(1+\delta)^2} \leq \frac{\bar{\sigma}_i}{\sigma_i} \leq \frac{(1+\delta)^2}{(1-\delta)^2}$$

Proof By the definition of the infinity norm we have that $-\|S^{-1}Ap\|_\infty \leq \frac{a_i^T p}{s_i} \leq \|S^{-1}Ap\|_\infty \Rightarrow (1-\delta)s_i \leq a_i^T p + s_i \leq s_i(1+\delta)$. Now $a_i^T p + s_i = \bar{s}_i \Rightarrow \bar{x} \in \Sigma(x, \delta) \subset int(\mathcal{P})$, since $\delta < 1$.

From the definition of $G(x)$ we have

$$\xi^T G(\bar{x})\xi = \sum_{i=1}^{m} \frac{(a_i^T\xi)^2}{(a_i^T\bar{x} - b_i)^2} = \sum_{i=1}^{m} \left(\frac{(a_i^T x - b_i)}{(a_i^T\bar{x} - b_i)}\right)^2 \frac{(a_i^T\xi)^2}{(a_i^T x - b_i)^2}$$

Hence,

$$\min_{1\le i\le m}\left\{ \left(\frac{(a_i^T x - b_i)}{(a_i^T\bar{x} - b_i)}\right)^2 \right\} \xi^T G(x)\xi \le \xi^T G(\bar{x})\xi \le \max_{1\le i\le m}\left\{ \left(\frac{(a_i^T x - b_i)}{(a_i^T\bar{x} - b_i)}\right)^2 \right\} \xi^T G(x)\xi$$

and since $\bar{x} \in \Sigma(x,\delta)$ we get that

$$\frac{\xi^T G(x)\xi}{(1+\delta)^2} \le \xi^T G(\bar{x})\xi \le \frac{\xi^T G(x)\xi}{(1-\delta)^2} \tag{B.1}$$

applying Claim B.2 we get that

$$(1+\delta)^2 a_i^T G(x)^{-1} a_i \ge a_i^T G(\bar{x})^{-1} a_i \ge (1-\delta)^2 a_i^T G(\bar{x})^{-1} a_i$$

Then noting that $\sigma_i(\bar{x}) = \frac{a_i^T G(\bar{x})^{-1} a_i}{(a_i^T\bar{x} - b_i)^2}$ it follows that for $1 \le i \le m$,

$$(1-\delta)^2 (a_i^T x - b_i)^2 \sigma_i(x) \le (a_i^T\bar{x} - b_i)^2 \sigma_i(\bar{x}) \le (1+\delta)^2 (a_i^T x - b_i)^2 \sigma_i(x)$$

giving that

$$\frac{(1-\delta)^2}{(1+\delta)^2}\sigma_i(x) \le \sigma_i(\bar{x}) \le \frac{(1+\delta)^2}{(1-\delta)^2}\sigma_i(x)$$

and proving the lemma. □

Inorder to arrive at some quadratic convergence result for $\|\cdot\|_H$ the magnitude of $H(x) - H(\bar{x})$ must be bounded. There are clearly two components in this difference, namely one involving $\Sigma - \bar{\Sigma}$ and the other $P^{(2)} - \bar{P}^{(2)}$, and these are both bounded in the next two lemmas. To facilitate let $R = R(x) = A^T S^{-1} P^{(2)} S^{-1} A$, and $\bar{R} = R(\bar{x}) = A^T \bar{S}^{-1} \bar{P}^{(2)} \bar{S}^{-1} A$, so that $H = 3Q - 2R$, and $\bar{H} = 3\bar{Q} - 2\bar{R}$.

Lemma B.4 Let $\bar{x} = x + p$, where $\delta = \|S^{-1}Ap\|_\infty < 1$. Then $\forall \xi \in \Re^n$,

$$|\xi^T(\bar{Q} - Q)\xi| \leq \frac{6\delta}{(1-\delta)^4}\xi^T Q \xi$$

Proof From Proposition B.3 we immediately obtain

$$\frac{(1-\delta)^2}{(1+\delta)^4}\xi^T Q\xi \leq \xi^T \bar{Q}\xi \leq \frac{(1+\delta)^2}{(1-\delta)^4}\xi^T Q\xi \ \ \forall \xi \tag{B.2}$$

Subtracting $\xi^T Q \xi$ throughout in (B.2), and noting that

$$1 - \frac{(1-\delta)^2}{(1+\delta)^4} \leq \frac{(1+\delta)^2}{(1-\delta)^4} - 1 \ \text{ for } \ 0 \leq \delta < 1$$

we then have that

$$|\xi^T(\bar{Q} - Q)\xi| \leq \left(\frac{(1+\delta)^2}{(1-\delta)^4} - 1\right)\xi^T Q \xi \tag{B.3}$$

But $(1+\delta)^2 - (1-\delta)^4 = \delta(3-\delta)(2-\delta+\delta^2)$, so (B.3) can be written as

$$|\xi^T(\bar{Q} - Q)\xi| \leq \frac{\delta(3-\delta)(2-\delta+\delta^2)}{(1-\delta)^4}\xi^T Q \xi \tag{B.4}$$

and $(3-\delta)(2-\delta+\delta^2) \leq 6$ for $0 \leq \delta < 1$. □

Lemma B.5 Let $\bar{x} = x + p$, where $\delta = \|S^{-1}Ap\|_\infty < 1$. Then $\forall \xi \in \Re^n$,

$$|\xi^T(\bar{R} - R)\xi| \leq \frac{16\delta}{(1-\delta)^6}\xi^T Q \xi$$

Proof Let $U = A(A^T S^{-2} A)^{-1} A^T$, and $\bar{U} = A(A^T \bar{S}^{-2} A)^{-1} A^T$. Then $P^{(2)} = S^{-2} U^{(2)} S^{-2}$, and $\bar{P}^{(2)} = \bar{S}^{-2}\bar{U}^{(2)}\bar{S}^{-2}$. Applying Claim B.2 and Lemma B.3, we obtain

$$\begin{gathered} (1-\delta)^2 U \preceq \bar{U} \preceq (1+\delta)^2 U \\ (1-\delta)^4 U^{(2)} \preceq \bar{U}^{(2)} \preceq (1+\delta)^4 U^{(2)} \end{gathered} \tag{B.5}$$

we begin by obtaining an upper bound for $\xi^T(\bar{R} - R)\xi$. Letting $z_i = s_i/\bar{s}_i$, and

$Z = \text{diag}(z)$, (B.5) implies that

$$\begin{aligned}
\xi^T(\bar{R} - R)\xi &= \xi^T A^T \left(\bar{S}^{-3}\bar{U}^{(2)}\bar{S}^{-3} - S^{-1}P^{(2)}S^{-1}\right) A\xi \\
&\leq \xi^T A^T \left((1+\delta)^4 \bar{S}^{-3}U^{(2)}\bar{S}^{-3} - S^{-1}P^{(2)}S^{-1}\right) A\xi \\
&= \xi^T A^T S^{-1}\Sigma^{1/2}\Sigma^{-1/2}\left((1+\delta)^4 Z^3 P^{(2)} Z^3 - P^{(2)}\right)\Sigma^{-1/2}\Sigma^{1/2}S^{-1}A\xi \\
&\leq \lambda_{\max}\, \xi^T Q \xi \qquad \text{(B.6)}
\end{aligned}$$

where $\lambda_{\max}$ is the maximum eigenvalue of the matrix

$$\Sigma^{-1/2}\left((1+\delta)^4 Z^3 P^{(2)} Z^3 - P^{(2)}\right)\Sigma^{-1/2}$$

By similarity, $\lambda_{\max}$ is also the maximum eigenvalue of the matrix

$$W = \Sigma^{-1}\left((1+\delta)^4 Z^3 P^{(2)} Z^3 - P^{(2)}\right)$$

Also by the Gershgorin Circle Theorem A.2.5, the eigenvalues of W are contained in the union of the intervals $w_{ii} \pm \sum_{j\neq i} |w_{ij}|$, $i = 1, \ldots, m$ and since $P_{ii}^{(2)} = \sigma_i^2$ we have

$$w_{ii} = \left((1+\delta)^4 z_i^6 - 1\right)\sigma_i \leq \left(\frac{(1+\delta)^4}{(1-\delta)^6} - 1\right)\sigma_i \qquad \text{(B.7)}$$

where the inequality follows from Proposition B.3. Moreover,

$$\begin{aligned}
\sum_{j\neq i} |w_{ij}| &= \frac{1}{\sigma_i}\sum_{j\neq i} |(1+\delta)^4 p_{ij}^2 z_i^3 z_j^3 - p_{ij}^2| \\
&\leq \frac{1}{\sigma_i}\left(\frac{(1+\delta)^4}{(1-\delta)^6} - 1\right)\sum_{j\neq i} p_{ij}^2 \\
&= \left(\frac{(1+\delta)^4}{(1-\delta)^6} - 1\right)(1-\sigma_i) \qquad \text{(B.8)}
\end{aligned}$$

and the last step follows since $\sum_{j\neq i} p_{ij}^2 = \sigma_i - \sigma_i^2$. Combining (B.7) and (B.8) we have that $\lambda_{\max} \leq [(1+\delta)^4/(1-\delta)^6] - 1$, and therefore by (B.6)

$$\xi^T(\bar{R}-R)\xi \le \left(\frac{(1+\delta)^4}{(1-\delta)^6}-1\right)\xi^T Q\xi \tag{B.9}$$

Similarly from Proposition B.3, we get the following condition

$$w_{ii} = \left((1+\delta)^4 z_i^6 - 1\right)\sigma_i \ge \left(\frac{1}{(1+\delta)^2}-1\right)\sigma_i \tag{B.10}$$

and combining this with (B.8) we get that

$$\lambda_{\min} \ge \left[\left(\frac{1}{(1+\delta)^2}-1\right)\sigma_i - \left(\frac{(1+\delta)^4}{(1-\delta)^6}-1\right)(1-\sigma_i)\right] \tag{B.11}$$

It follows that $\xi^T(R-\bar{R})\xi \le -\lambda_{\min}\xi^T Q\xi$ and on noting that $1-1/(1+\delta)^2$ $\le [(1+\delta)^4/(1-\delta)^6]-1 \ \forall\delta\in[0,1]$ we get that

$$\xi^T(R-\bar{R})\xi \le \left(\frac{(1+\delta)^4}{(1-\delta)^6}-1\right)\xi^T Q\xi \tag{B.12}$$

But $(1+\delta)^4-(1-\delta)^6 = \delta(5-2\delta+\delta^2)(2-\delta+4\delta^2-\delta^3)$, so (B.9) and (B.12) together imply

$$\xi^T(\bar{R}-R)\xi \le \frac{\delta(5-2\delta+\delta^2)(2-\delta+4\delta^2-\delta^3)}{(1-\delta)^6}\,\xi^T Q\xi \tag{B.13}$$

Finally, the maximum of the polynomial $(5-2\delta+\delta^2)(2-\delta+4\delta^2-\delta^3)$ for $0\le\delta\le 1$ occurs at $\delta=1$, with value 16. □

Theorem B.6 Let $\bar{x}=x+p$, where $\delta=\|S^{-1}Ap\|_\infty<1$. Then $\forall\xi\in\Re^n$,

$$|\xi^T(\bar{H}-H)\xi| \le \frac{38\delta}{(1-\delta)^6}\,\xi^T Q\xi$$

Proof By definition $H=3Q-2R$ and $\bar{H}=3\bar{Q}-2\bar{R}$. Then $|\xi^T(\bar{H}-H)\xi|$

$\leq 3|\xi^T(\bar{Q}-Q)\xi| + 2|\xi^T(\bar{R}-R)\xi|$. From (B.4) and (B.13) we then have

$$
\begin{aligned}
|\xi^T(\bar{H}-H)\xi| &\leq \left(\frac{3\delta(3-\delta)(2-\delta+\delta^2)}{(1-\delta)^4} + \frac{2\delta(5-2\delta+\delta^2)(2-\delta+4\delta^2-\delta^3)}{(1-\delta)^6}\right)\xi^T Q\xi \\
&= \delta\left(\frac{3(3-\delta)(2-\delta+\delta^2)(1-\delta)^2+2(5-2\delta+\delta^2)(2-\delta+4\delta^2-\delta^3)}{(1-\delta)^6}\right)\xi^T Q\xi
\end{aligned}
$$

and the polynomial $3(3-\delta)(2-\delta+\delta^2)(1-\delta)^2+2(5-2\delta+\delta^2)(2-\delta+4\delta^2-\delta^3) = 38-69\delta+108\delta^2-70\delta^3+30\delta^4-5\delta^5$ is maximized for $0 \leq \delta \leq 1$ at $\delta = 0$. □

Now that we have obtained this bound on H the quadratic convergence result for Newton's method applied to $\mathcal{V}(\cdot)$ is established in the following theorem.

Theorem B.7 Let $\bar{x} = x + p$, where $p = -H^{-1}g$ and $\delta = \|S^{-1}Ap\|_\infty < 1$. Then

$$
\|\bar{p}\|_{\bar{H}} \leq \frac{19\delta(1+\delta)^2}{(1-\delta)^6}\|p\|_Q
$$

Proof For $0 \leq \alpha \leq 1$, and $\xi \in \Re^n$, define $h(\alpha,\xi) = g(x+\alpha p)^T\xi - (1-\alpha)g(x)^T\xi$. Then for any ξ, $h(0,\xi) = 0$, $h(1,\xi) = g(x+p)^T\xi$, and the chain rule gives us that

$$
\frac{d}{d\alpha}h(\alpha,\xi) = p^T H(x+\alpha p)\xi + g^T\xi = p^T(H(x+\alpha p)-H)\xi
$$

where the last equality follows from $p = -H^{-1}g$. Now for $0 \leq \alpha \leq 1$, $\tilde{x} = x + \alpha p \in \Sigma(x, \alpha\delta)$ and on appealing to Theorem B.6 and Proposition A.2.9 with symmetric matrix $A = (\tilde{H} - H)$, and symmetric positive definite matrix $B = \frac{38\alpha\delta}{(1-\alpha\delta)^6}Q$ we get that

$$
\left|\frac{d}{d\alpha}h(\alpha,\xi)\right| \leq \frac{38\alpha\delta}{(1-\alpha\delta)^6}\|p\|_Q\,\|\xi\|_Q \quad \text{for } 0 \leq \alpha \leq 1
$$

integrating both sides,

$$
|h(1,\xi)| = |g(x+p)^T\xi| \leq 38\|p\|_Q\|\xi\|_Q\int_0^1 \frac{\alpha\delta}{(1-\alpha\delta)^6}\,d\alpha \tag{B.14}
$$

Note that $\alpha\delta/(1-\alpha\delta)^6 = (1-\alpha\delta)^{-6} - (1-\alpha\delta)^{-5}$. A straightforward integration then obtains

$$
\begin{aligned}
\int_0^1 \frac{\alpha\delta}{(1-\alpha\delta)^6}\, d\alpha &= \frac{1}{20\delta}\left(\frac{(1-\delta)^5+5\delta-1}{(1-\delta)^5}\right) \\
&= \frac{\delta}{20}\left(\frac{10-10\delta+5\delta^2-\delta^3}{(1-\delta)^5}\right) \\
&\le \frac{\delta}{2(1-\delta)^5} \qquad \text{(B.15)}
\end{aligned}
$$

for $0 \le \delta \le 1$. Combining (B.14) and (B.15),

$$
|g(x+p)^T\xi| \le \frac{19\delta}{(1-\delta)^5}\, \|\xi\|_Q \, \|p\|_Q \quad \forall \xi \in \Re^n \qquad \text{(B.16)}
$$

Now let $\xi = -H(x+p)^{-1}g(x+p) = \bar{p}$, so that $|g(x+p)^T\xi| = \bar{p}^T\bar{H}\bar{p} = \|\bar{p}\|_{\bar{H}}^2$. Then (B.16) is exactly

$$
\begin{aligned}
\|\bar{p}\|_{\bar{H}}^2 &\le \frac{19\delta}{(1-\delta)^5}\, \|\bar{p}\|_Q \, \|p\|_Q \\
&\le \frac{19\delta(1+\delta)^2}{(1-\delta)^6}\, \|\bar{p}\|_{\bar{H}} \, \|p\|_Q \qquad \text{(B.17)}
\end{aligned}
$$

where the second inequality follows from (1.8) and (B.2). □

Theorem B.7 establishes the quadratic convergence for the $\|\cdot\|_H$ measure associated with $\mathcal{V}(\cdot)$, this is seen as follows. Since from Theorem A.5.2 with ρ set equal to zero we have

$$
\frac{\|\xi\|_Q^2}{\|S^{-1}A\xi\|_\infty^2} \ge \frac{2}{1+\sqrt{m}} \ge \frac{1}{\sqrt{m}} \quad \forall \xi \in \Re^n \qquad \text{(B.18)}
$$

and this immediately implies that

$$
\delta = \|S^{-1}Ap\|_\infty \le m^{1/4}\, \|p\|_Q \le m^{1/4}\, \|p\|_H \qquad \text{(B.19)}
$$

Finally, combining (B.19) with Theorem B.7, using $||p||_Q \leq ||p||_H$, gives

$$||\bar{p}||_{\bar{H}} \leq \frac{19m^{1/4}(1+m^{1/4}||p||_H)^2}{(1-m^{1/4}||p||_H)^6}\,||p||_H^2 \tag{B.20}$$

and it is straightforward to verify that (B.20) implies $||\bar{p}||_{\bar{H}} \leq ||p||_H$ for roughly $||p||_H \leq .038m^{-1/4}$.

Appendix C

Computer code

The algorithm was coded and implemented in Matlab. The main program starts by prompting the user for his/her choice of parameter values and dimensions for the problem. Based on these it then maintains control over the iterations and determines when the algorithm will terminate. The main program also calls the linesearch subroutine and two procedures, one that updates the vectors and matrices after the current test point has shifted and the other that furnishes a separating hyperplane for use during the following iteration of the algorithm. The computer code is now presented.

C.1 The main program

```
clear;
L = 6;
k = input ('Enter the number of lsearch steps:');
tau = input ('Enter the value of tau:');
eps = input ('Enter the value of epsilon:');
gamma1 = input ('Enter the value of gamma1:');
gamma2 = input ('Enter the value of gamma2:');
q = input ('Generate a new matrix, or use old one? [1 = Yes, 0 = No] :');
```

```
if q == 1
  n = input ('Enter dimension n of C:');
  m = input ('Enter dimension m of C:');
  C = normrnd(0,1,m,n);
  d = - abs(normrnd(0,1,m,1));
  save temp C d m n;
else
  load temp;
end

n_rows = n+1;
V_max = 0.7 × n × L + n × log(n_rows);
A = eye(n);
A(n+1,:) = −ones(1,n);
b = −(2^L) × ones(n,1);
b(n+1) = −n × (2^L);
x = 2^L× ((n-1)/(n+1)) × ones(n, 1);
addcounter = 0;
deccounter = 0;
newtmax = 0;
newtmin = 10;
counter = 0;
totnewt = 0;
update;
V = 0.5 × log(det(B));

while (V_max > V)
        counter = counter + 1;
        if (sigm_min >= eps)
          [flag, new_row] = oracle(C,d,x);
          if (flag == 1)
```

```
    break;
  end;
  A = [A; new_row'];
  b1 = new_row' × x − ((new_row' × inv(B) × new_row)/tau)^0.5;
  b = [b; b1];
  n_rows = n_rows + 1;
  addcounter = addcounter+1;
else
  ind = find(sigm == sigm_min)
  if (length(ind) > 1)
    ind = ind(1);
  end;
  if (ind == 1)
    A = A(2:n_rows,:);
    b = b(2:n_rows);
  elseif (ind == n_rows)
    A = A(1:n_rows-1,:);
    b = b(1:n_rows-1);
  else
    A = [A(1:ind−1,:); A(ind+1:n_rows,:)];
    b = [b(1:ind−1); b(ind+1:n_rows)];
  end;
  n_rows = n_rows − 1;
  deccounter = deccounter +1;
end
update;
t = 0;
while (crit > gamma1) | (m × crit > gamma2)
      t = t+1;
      lambda = lsearch(A,p,s,n_rows,k)
```

```
            x = x + lambda × p;
            update;
        end;
        totnewt = totnewt + t;
        if ( newtmax < t)
          newtmax = t;
        end;
        if (newtmin > t)
          newtmin = t;
        end;
        V = 0.5 × log(det(B))
        V_max = 0.7 × n × L + n × log(n_rows)
end;

counter
addcounter
deccounter
avgnewt = totnewt/(counter−1)
totnewt
newtmax
newtmin
inversions = totnewt × 2 + totnewt × k
```

C.2 The oracle procedure

This is the procedure that checks to see whether the current test point lies within the convex set (the result being indicated by the state a returned flag is in) and if not, it provides the separating hyperplane.

```
function [flag, y] = oracle(C,d,x)
```

```
ind = (d > C × x);
flag = 0;
z = find(ind == 1);

if z == [];
  flag = 1;
  y = x;
else
  y = C(z(1), :)';
end;
```

C.3 The update procedure

This procedure merely updates all matrices, vectors and associated value after our test point has shifted, both during the linesearch subroutine and after a Newton step has occured.

```
s = A × x − b;
S = diag(s);
S_inv = inv(S);
AS_inv = A' × S_inv;
B = AS_inv × AS_inv';
P = AS_inv' × inv(B) × AS_inv;
sigm = diag(P);
sigm_min = min(sigm);
g = −AS_inv × sigm;
H = AS_inv × (3 × diag(sigm) − 2 × P.^2) × AS_inv';
p = −inv(H) × g;
m = (2 × sigm_min^.5 − sigm_min)^−.5;
```

```
crit = (p' × H × p)^.5;
```

C.4 The linesearch procedure

The Bisection Method is used as our linesearch algorithm and is presented here. It is called prior to every Newton step that is taken.

```
function flambda = lsearch(A,p,s,n_rows,k)
z = 1;

for w = 1:n_rows,
    if A(w,:) × p < 0
        alph(z) = − s(w) / (A(w,:) × p);
        z = z+1;
    end;
end;

maxalph = min(alph)
a = 0;
b = maxalph;
tlambda = (a+b)/2

for i = 1:k,
    temp = s + tlambda × A × p;
    TS = diag(temp);
    TS_inv = inv(TS);
    TAS_inv = A' × TS_inv;
    TB = TAS_inv × TAS_inv';
    TP = TAS_inv' × inv(TB) × TAS_inv;
    Tsigm = diag(TP);
```

```
    Tg = −Tsigm' × TS_inv × A × p;
    if Tg > 0
      b = tlambda;
    else
      a = tlambda;
    end;
    tlambda = (a+b)/2
end;

    flambda = tlambda
    end;
```

References

G. Sonnevend (1988), "New algorithms in convex programming based on a notion of 'center' (for systems of analytic inequalities) and on rational extrapolation," in *Trends in Mathematical Optimization*, K.H. Hoffmann et al., editors, *International Series of Numerical Mathematics* 84, 311-327.

J. Goffin, A. Haurie, and J. Vial (1992), "Decomposition and nondifferentiable optimization with the projection algorithm," *Management Sciences* 38, 284-302.

K.M. Anstreicher (1994a), "Large step volumetric potential reduction algorithms for linear programming," Department of Management Seciences, University of Iowa (Iowa City, IA).

K.M. Anstreicher (1994b), "Volumetric path following algorithms for linear programming," Department of Management Seciences, University of Iowa (Iowa City, IA).

K.M. Anstreicher (1994c), "On Vaidya's volumetric cutting plane method for convex programming," Department of Management Seciences, University of Iowa (Iowa City, IA).

Mokhtar S. Bazaraa, Hanif D. Sherali, and C.M. Shetty. *Nonlinear Programming*, second edition, 1993.

P.M Vaidya (1989), "A new algorithm for minimizing convex functions over convex sets," AT&T Bell Laboratories, Murray Hill, NJ.