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Modeling Travel Times in Dynamic Transportation Networks; A Fluid Dynamics Approach

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Abstract

In this paper, we take a fluid dynamics approach to determine the travel time in traversing a network's link. We propose a general model for travel time functions that utilizes fluid dynamics laws for compressible flow to capture a variety of flow patterns such as the formation and dissipation of queues, drivers' response to upstream congestion or decongestion and drivers' reaction time. We examine two variants of the model, in the case of separable velocity functions, which gives rise to two families of travel time functions for the problem; a polynomial and an exponential family. We analyze these travel time functions and examine several special cases. Our investigation also extends to the case of non-separable velocity functions starting with an analysis of the interaction between two links, and then extending it to the general case of acyclic networks.

1 Introduction

In recent years, traffic congestion has rapidly grown in transportation networks and has become an acute problem. In fact, it is estimated that the presence of congestion costs around $100 billion each year to Americans alone in the form of lost productivity (see Barnhart *et al.* (1998)). Therefore, it is critical to investigate and understand its nature and address questions of the type: *how are traffic patterns formed?* and *how can traffic congestion be alleviated?* Answering these questions and designing accurate traffic flow models is important for the development of efficient control strategies.

The way flows circulate in traffic networks, the way queues form and disappear, and the way spillback and shock wave phenomena occur, are striking evidence that traffic flows are similar to gas and water flows. It is therefore normal to use physical laws of fluid dynamics for compressible flow to model traffic flow patterns.

Lighthill and Whitham (1955), and Richards (1956) introduced the first continuum approximation of traffic flows using kinematic wave theory (see Haberman (1977) for a detailed analysis). The dynamic nature of these models gave them instant credibility. Indeed, with the increase of urban and highway congestion, the variations of flow with time are too important to be neglected. Dynamic traffic flow modeling captured the focus of most researchers interested in theoretical or applied research in the transportation area. A variety of dynamic traffic flow models have been proposed in the literature that can be classified in two major categories: microscopic models and macroscopic models.

Microscopic models, or car-following models, have the ability to describe, at a level of detail, the network geometry, the traffic flow and its kinematics and the traffic control logic. Such models enable simulated tests of traffic flow control strategies, and help design safety procedures by better understanding the driver's behavior. These models consist of difference equations expressing the acceleration of a vehicle as a function of the behavior of downstream vehicles. Reuschell (1950) proposed the first car-following model. Pipes (1953) and Herman *et al.* (1959) extended this model. Gerlough and Huber (1975), Bekey, Burnham and Seo (1977), and Papageorgiou (1983) and (1998), Papageorgiou, Blosseville and Haj-Salem (1989) and references therein provide an extensive analysis of these models.

On the other hand, analytical models usually possess mathematical properties that are useful in understanding the properties of a model and in designing solution algorithms to solve instances of this problem. Developing a good understanding of such phenomena is important since they arise not only in transportation systems but also in manufacturing and communication systems. In an attempt to improve modeling accuracy, the model of Lighthill and Whitham (1955) was extended by Payne (1971) and Whitham (1974). These models are widely applied in practice. However, these models contradict the anisotropic property of traffic flow (see Daganzo (1995) for more details) and faced criticism from Daganzo (1995), Papageorgiou, Blosseville and Haj-Salem (1990), and Heidemann (1999).

The purpose of this paper is to determine the travel time of a driver in traversing a network's link. Practitioners in the transportation area have been using several families of travel time functions. Akcelik (1988) proposed a polynomial-type travel time function for links at signalized intersections. The BPR function (Bureau of Public Roads (1964)), that is used to estimate travel times at priority intersections, is also a polynomial function. Finally, Meneguzzer *et al.* (1990) proposed an exponential travel time function for all-way-stop intersections. Our goal is to lay the theoretical foundations for using these polynomial and exponential families of travel time functions in practice. While most analytical models in traffic modeling assume an a priori knowledge of drivers' travel time functions, in this paper, travel time is part of the model and comes as an output. To determine the travel time, we examine and further extend the analytical model proposed by Perakis (1997). This model provides a macroscopic fluid dynamics approach to the dynamic network equilibrium problem.

The main contributions of this paper are:

- We propose two models to estimate travel times as functions of departure flow rates: the Polynomial Travel Time (PTT) Model and the Exponential Travel Time (ETT) Model.
- We propose a general framework for the analysis of the PTT Model. This framework allows us to reduce the analysis of the model to solving a single ordinary differential equation.
- Based on piecewise linear and piecewise quadratic approximations of the flow rates, we propose several classes of travel time functions for the separable PTT and ETT models. We further establish a connection between these travel time functions.
- We extend the analysis of the PTT Model to non-separable velocity functions in the case of acyclic networks.

The paper is organized as follows. In Section 2, we provide some useful notations and introduce the reader to the hydrodynamic theory of traffic flow on a single stretch of road. We also review the analytical model proposed by Perakis (1997). To make the problem more tractable, we propose, in Section 3, two approximations of Perakis' model: the Polynomial Travel Time Model (PTT Model) and the Exponential Travel Time Model (ETT Model). In Section 4, we examine the case of separable velocity functions. We show how the PTT Model reduces, in this case, to the analysis of one single ordinary differential equation. Based on an approximation of departure flows by piecewise linear and piecewise quadratic functions, we propose several classes of travel time functions for the problem that rely on a variety of assumptions. We analyze the relationship between these travel time functions and show that the assumptions we impose are indeed reasonable. We further show that the analysis of the ETT Model is more complex, and propose another class of travel time functions for the piecewise linear approximation of departure flows. We then summarize our results and establish a relationship between the travel time functions obtained by the two approximation models. In Section 5, we extend our investigation to non-separable velocity functions. Starting with the case of a network with two links that interact, we show how the analysis of the PTT Model in Section 4 extends to the general case of acyclic networks. Finally, we conclude in Section 6 by providing future steps for the study of this model.

2 The Hydrodynamic Theory of Traffic Flow and a General Travel Time Model

In Subsection 2.1, we summarize the notation that we use throughout the paper. In Subsection 2.2, we consider a single link network and introduce the hydrodynamic theory of traffic flow developed by Lighthill and Whitham (1955). In Subsection 2.3, we establish a relationship between path and link flows. In Subsection 2.4, we review a general model for travel time functions proposed by Perakis (1997).

2.1 Notation

The physical traffic network is represented by a directed network $G = (N, I)$, where N is the set of nodes and I is the set of directed links. Index w denotes an Origin-Destination (O-D) in the set W of origin destination pairs. Index P denotes the set of paths and index P_w denotes the set of paths between O-D w.

Origin-Destination variables:

W	:	number of O-D pairs in the network;
n_w	:	number of paths on O-D pair w;
$d_w(t)$	:	demand rate function on O-D pair w;

Path variables:

$\|P\|$	:	number of paths in the network;
x_p	:	position on path p;
L_p	:	length of path p;
$F_p(x_p, t)$	:	flow rate at time t on path p at position x_p;
$F(0, t)$	:	vector of departure path flow rates;
$T_p(L_p, t)$	:	path travel time function on path p starting the trip at time t;
$u_p(x_p, t)$	:	traffic speed on path p at position x_p at time t;
$k_p(x_p, t)$	:	traffic density on path p at position x_p at time t;
u_p^{max}	:	maximum traffic speed on path p;
k_p^{max}	:	maximum traffic density on path p;

Link variables:

$\|I\|$	:	number of directed links in the network;
x_i	:	position on link i;
L_i	:	length of link i;
$f_i(x_i, t)$	:	flow rate at time t on link i at position x_i;
$f(0, t)$	:	vector of departure link flow rates;
$T_i(L_i, t)$	:	link travel time function on link i starting the trip at time t;
$u_i(x_i, t)$	:	traffic speed on link i at position x_i at time t;
$k_i(x_i, t)$	:	traffic density on link i at position x_i at time t;
u_i^{max}	:	maximum traffic speed on link i;
k_i^{max}	:	maximum traffic density on link i;

Link-path flow variables:

ip	:	a link-path pair;
i^-p	:	predecessor of link i on path p;
δ_{ip}	=	1 if link i belongs to path p, and 0 otherwise;
L_{ip}	:	length from the origin of path p until the beginning of link i;
$T_{ip}(L_{ip}, t)$	:	partial path travel time function from the origin of path p until the beginning of link i starting the trip at time t;

Time variables:

$[0, T]$: Time period.

2.2 Hydrodynamic Theory of Traffic Flow on a Single Stretch of Road

In this subsection, we describe the laws of fluid dynamics for compressible flow in a single stretch of road. Lighthill and Whitham (1955) introduced these laws. See Haberman (1977) for a more detailed analysis.

Let us consider a link of length L. We denote by $\tau = \tau(x,t)$ the travel time to reach position x when departing at time t. The three fundamental traffic variables of fluid dynamics are:

- the flow rate function $f(x, t+\tau)$ that measures, in vehicles per unit of time, that is, the flow rate that crosses point x at time $t+\tau$,

- the density function $k(x, t+\tau)$ that measures, in vehicles per mile, that is, the density rate at point x at time $t+\tau$, and

- the velocity function $u(x, t+\tau)$ that measures, in miles per unit of time, that is, the instantaneous speed at point x at time $t+\tau$.

Two relationships connect these three variables.

$$f(x, t+\tau) = k(x, t+\tau).u(x, t+\tau), \qquad \forall x, \tau. \tag{1}$$

Assuming that there are no exits in this stretch of road between the entrance position $x = 0$ and the exit position $x = L$, the second relationship expresses a conservation of vehicles in this stretch:

$$\frac{\partial f(x, t+\tau)}{\partial x} + \frac{\partial k(x, t+\tau)}{\partial \tau} = 0. \tag{2}$$

If we knew the velocity $u(.)$, then conservation law (2) and equation (1) would allow us to obtain the flow rate $f(.)$ and as a result the density $k(.)$. Nevertheless the velocity is a consequence of the drivers' behavior. In the mid-1950's Lighthill and Whitham (1955) and independently Richards (1956), proposed the additional assumption that the velocity at any point depends only on the density. In mathematical terms:

$$u = \widehat{u}(k). \tag{3}$$

The function $\widehat{u}$ is empirically measured and is an input to the model.

Several models have been proposed in the literature for the velocity function $\widehat{u}(.)$. Mahmassani and Hernan (1984) proposed a linear model:

$$\widehat{u}(k) = u^{max}(1 - \frac{k}{k^{max}}), \tag{4}$$

where they assume that:

- the free flow speed is the maximum speed: $\widehat{u}(0) = u^{max}$,

- at maximum density, the speed is zero: $\widehat{u}(k^{max}) = 0$.

From equations (1) and (3), we obtain:

$$f(x,t+\tau) \quad = \quad k(x,t+\tau).\widehat{u}(k(x,t+\tau)) \tag{5}$$

In the case of the linear model of Mahmassani and Hernan, $f(x,t+\tau) = u^{max}.k(x,t+\tau)(1 - \frac{k(x,t+\tau)}{k^{max}})$. More generally, there exists a function $g(.)$ such that

$$f(x,t+\tau) \quad = \quad g(k(x,t+\tau)). \tag{6}$$

If $g(.)$ is an invertible function, then:

$$k(x,t+\tau) = g^{-1}(f(x,t+\tau)). \tag{7}$$

If we further assume that $g(.)$ is differentiable, using the above expression in the conservation law (2), we derive:

$$\frac{\partial f(x,t+\tau)}{\partial x} + \frac{\partial g^{-1}(f(x,t+\tau))}{\partial f} . \frac{\partial f(x,t+\tau)}{\partial \tau} = 0. \tag{8}$$

Equation (8) is a partial differential equation that can be solved using our knowledge of the boundary term $f(0,t)$ corresponding to the entrance flow rate in the stretch of road.

Once we solve this partial differential equation in $f(.)$, we use equation (7) to obtain the density function $k(.)$ and subsequently equation (3) to obtain the velocity. Using the velocity field equation

$$\frac{dx}{d\tau} = u(x,t+\tau), \tag{9}$$

we derive the travel time function $\tau = \tau(x,t)$ using as an initial condition the fact that $\tau(0,t) = 0$.

In the case of a network of multiple links, we will call the velocity function $u_i(.)$ of link i separable if it only depends on the density function k_i. In Section 4 we will consider the separable case. In Section 5, we will examine the more complex case of non-separable velocity functions.

2.3 Relationship between path and link variables

After determining travel time functions on the network's links, we need to determine the travel times to traverse the network's paths. Determining path travel times becomes complicated due to the dynamic nature of traffic. Two approaches have been proposed in the literature to address this problem. The first approach assumes that travelers consider only the current travel time information in the network. That is, travelers compute their path travel time at time t as the sum of all the link travel times along their route, based on the current information available to the travelers at time t. For example, up-to-the-minute radio broadcasts could be a source of such information. This type of travel time function is called instantaneous travel time (see for example Boyce, Ran and Leblanc (1995)). The second approach assumes that travelers consider predicted or estimates of travel times. That is, the travel time to traverse a path is the summation of the link travel times that the traveler experiences when he/she reaches each link along the path (see for example Friesz *et al.* (1993)). Traveler information systems could provide, for example, such information.

In this paper, we follow the second approach. To illustrate this, let us first consider a network with one path p and two links 1 and 2. We have:

$$T_p(L_p, t) = T_1(L_1, t) + T_2(L_2, t + T_1(L_1, t)).$$

Since $L_{2p} = L_1$ and $T_{2p}(L_{2p}, t) = T_1(L_1, t)$, it follows that:

$$T_p(L_p, t) = T_1(L_1, t) + T_2(L_2, t + T_{2p}(L_{2p}, t)).$$

Similarly, if we add a third link 3 to path p, it follows that:

$$\begin{aligned} T_p(L_p, t) &= T_1(L_1, t) + T_2(L_2, t + T_1(L_1, t)) + T_3(L_3, t + T_1(L_1, t) + T_2(L_2, t + T_1(L_1, t)) \\ &= T_1(L_1, t) + T_2(L_2, t + T_{2p}(L_{2p}, t)) + T_3(L_3, t + T_{3p}(L_{3p}, t)). \end{aligned}$$

The above formulas easily extend to the general case as follows:

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}.$$

2.4 A General Travel Time Model

Perakis (1997) proposed a model for computing travel times and linked this model to the user-equilibrium problem.

Perakis' model relies on the following two assumptions:
A1 Links in the network have no exits. Therefore, the conservation equation (2) of Subsection 2.3 holds.
A2 The velocity function u_i on link i can be expressed as a function $\widehat{u}_i$ that depends only on the vector of density functions and hence equation (3) of Subsection 2.3 holds.

Below, we introduce a family of velocity functions that verifies Assumption (A2). We consider the general case of non-separable velocity functions. We model link interactions by considering that the velocity of link i, at position x_i and at time t, can be expressed as a function $\widehat{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i + \sum_{j \in B(i)} \alpha_{ij}(x_i) R_{ij}(\overline{x}_j, t - \Delta_{ij})$, where b_i is a constant; $\alpha_{ij}(x_i)$ is the density correlation function between link i and link j and depends on the position x_i on link i; R_{ij} is a function of k_j and ∇k_j; $\overline{x}_j$ is a fixed position of a detector of density on link j; Δ_{ij} is a propagation time between link j and link i; and $B(i)$ denotes a set of links neighboring link i. In Sections 3 and 4, we consider separable velocity functions (e.g. $\alpha_{ij}(.) = 0$). In Section 5, we consider the more general case of non-separable velocity functions for acyclic networks.

Below, we provide the general model:

Model 1
For all $t \in [0, T]$, we have:

$$T_p(L_p, t) = \textstyle\sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \quad \text{for all } p \in P, \tag{10}$$

$$f_i(x_i, t) = \textstyle\sum_{p \in P} F_p(x_i, t)\delta_{ip}, \quad \text{for all } i \in I, \tag{11}$$

$$u_i = \widehat{u}_i(k), \quad \text{for all } i \in I, \tag{12}$$

$$
\begin{aligned}
&f_i(x_i,t) = k_i(x_i,t)u_i(x_i,t), && \text{for all } i \in I, && (13)\\
&\tfrac{\partial f_i(x_i,T_i)}{\partial x_i} + \tfrac{\partial k_i(x_i,T_i)}{\partial T_i} = 0, && \text{for all } i \in I, && (14)\\
&\tfrac{dT_i(x_i,t)}{dx_i} = \tfrac{1}{u_i}, && \text{for all } i \in I, && (15)\\
&T_i(0,t) = 0, && \text{for all } i \in I. && (16)
\end{aligned}
$$

Given path flow rate functions $F_p(0,t)$, $p \in P$, and a density velocity relationship $u_i = \widehat{u}_i(k)$, $i \in I$, the Dynamic Travel Time Problem is the problem of determining $F_p(x_p,t)$ $p \in P$, $f_i(x_i,t)$ $i \in I$, $u_i(x_i,t)$ $i \in I$, $k_i(x_i,t)$ $i \in I$ and $T_i(x_i,t)$ $i \in I$, as functions of x_p $p \in P$, x_i $i \in I$ and $t \in [0,T]$.

3 Two Approximations of the General Travel Time Model

Model 1 is hard to analyze in its current form. For this reason, in this section, we consider two simplified models of Model 1 for the case of separable velocity functions (where $\alpha_{ij}(.) = 0$). This will give rise to the Separable Polynomial Travel Time Model (Separable PTT Model) and the Separable Exponential Travel Time Model (Separable ETT Model) that we formulate in this section. The analysis of these two models is the focus of Section 4.

In addition to Assumptions (A1) and (A2) introduced in Subsection 2.4, we further assume that:
A3 $\widehat{u}_i(.)$ is a separable and linear function of the density k_i. That is, $\widehat{u}_i(k_i) = u_i^{max} - b_i(u_i^{max})^2 k_i$, where $b_i = -\frac{1}{(u_i^{max})^2}\frac{d\widehat{u}_i(0)}{dk_i}$.
A4 The term $\frac{1}{(u_i^{max})^2} << 1$.
A5 For all t, the link flow rate $f_i(0, t+\tau_i)$ can be approximated by $h_i^t(\tau_i)$, which is a continuously differentiable function of τ_i.

Remarks:

- Note that Assumption (A3) is a particular case of the density-velocity relationship introduced in Subsection 2.4 where we consider that $\alpha_{ij}(.) = 0$.
- Our analysis in Section 4 will rely on the separability Assumption (A3). However, in Section 5, we relax this assumption and consider the non-separable case as formulated in Subsection 2.4.
- As an example, consider an arc i with speed limit of 40 miles per hour. Then $(u_{max}^i)^2 = 1,600$. This example demonstrates that Assumption (A4) is reasonable.

Our goal is to solve Model 1 and propose specific travel time functions. To achieve this, the first step is to eliminate some of the variables involved in the model. We eliminate the density variables by expressing them as functions of the flow rates.

Lemma 1 *Under Assumption (A3), the link density as a function of the link flow rate function can be expressed as:*

$$k_i = \frac{1}{2b_i u_i^{max}}(1 - (1 - 4b_i f_i)^{\frac{1}{2}}). \tag{17}$$

Proof: Using Assumption (A3), equation (13) can be expressed as a second degree polynomial in terms of the density k_i. Solving in terms of k_i gives rise to $k_i = \frac{1}{2b_i u_i^{max}}(1 - (1 - 4b_i f_i)^{\frac{1}{2}})$.

□

3.1 Separable Polynomial Travel Time Model

In this subsection, we consider an approximation of the density flow rate relationship (17). This approximation enables us to express conservation law (14) in terms of the link flow rates only. We present a formulation of the Separable Polynomial Travel Time Model and provide a necessary and sufficient condition for existence of a solution.

3.1.1 Preliminary result

Lemma 2 *Under Assumptions (A3)-(A4), the link density as a function of the link flow rate function can be expressed as:*

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}. \tag{18}$$

Proof: Assumption (A4) implies that all the terms of order higher than or equal to 3 in the Taylor expansion of equation (17) are negligible. That is,

$$1 - (1 - \epsilon)^{\frac{1}{2}} = \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + O(\epsilon^3) \approx \frac{\epsilon}{2} + \frac{\epsilon^2}{8}. \tag{19}$$

This gives rise to equation (18).

□

Using the above result, the following theorem provides a first-order partial differential equation satisfied by the link flow rate functions.

Theorem 1 *Under Assumptions (A3)-(A4) and equation (18), the link flow rate functions f_i are solutions of the first-order partial differential equation:*

$$\frac{\partial f_i}{\partial t} + \frac{u_i^{max}}{1 + 2b_i f_i} \frac{\partial f_i}{\partial x_i} = 0. \tag{20}$$

Assumption (A5) provides a boundary condition.

Proof: Replacing the value of k_i from equation (18) in the conservation equation (14), it follows that:

$$\frac{\partial f_i}{\partial x_i} + \frac{1 + 2b_i f_i}{u_i^{max}} \frac{\partial f_i}{\partial t} = 0. \tag{21}$$

The result of the theorem easily follows.

□

This new conservation law (20) is the basis of the PTT Model that we present below and analyze in Sections 4 and 5.

3.1.2 Model Formulation

Theorem 1 gives rise to the following formulation:

PTT Model

For all $t \in [0, T]$:

$$\frac{\partial f_i}{\partial t} + \frac{u_i^{max}}{1+2b_i f_i}\frac{\partial f_i}{\partial x_i} = 0, \quad \text{for all } i \in I, \tag{22}$$

$$f_i(0, t + T_i) = h_i^t(T_i), \quad \text{for all } i \in I, \tag{23}$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}, \quad \text{for all } i \in I, \tag{24}$$

$$u_i = \frac{f_i}{k_i}, \quad \text{for all } i \in I, \tag{25}$$

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1}{u_i}, \quad \text{for all } i \in I, \tag{26}$$

$$T_i(0, t) = 0, \quad \text{for all } i \in I, \tag{27}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \quad \text{for all } p \in P. \tag{28}$$

Equation (22) is a first-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model. Moreover, equation (23) provides the boundary condition for this partial differential equation.

If we assume that equations (22) and (23) possess a continuously differentiable solution f_i, then, equations (24) and (25) determine the density function k_i and the velocity function u_i. The ordinary differential equation (26), under boundary condition (27), determines travel times on the network's links. Finally, path travel times follow from equation (28). Therefore, if we assume that equations (22) and (23) possess a continuously differentiable solution f_i, the PTT Model, as formulated by equations (22)-(28), also possesses a solution.

Remark: Note that equations (24) and (25) simplify the travel time differential equation (26) into

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1 + b_i f_i}{u_i^{max}}. \tag{29}$$

3.1.3 Existence of Solution to the PTT Model

The following theorem provides an existence result for a continuously differentiable solution of the PTT Model as formulated above.

Theorem 2 *(Perakis (1997)) The PTT Model as formulated in equations (25)-(31) possesses a solution if and only if the first derivative of the link flow rate function $h_i^t(T_i)$ satisfies the following boundedness condition:*

$$\frac{dh_i^t(T_i)}{dT_i} > -\frac{u_i^{max}}{2b_i L_i}. \tag{30}$$

3.2 Separable Exponential Travel Time Model

In this subsection, we take a different approach. We use the exact expression of the density flow rate relationship (17) to derive a conservation law in terms of the link flow rates. We then approximate this equation to obtain a first-order partial differential equation in terms of the link flow rates. We present a formulation of the Separable Exponential Travel Time Model and provide a necessary and sufficient condition for existence of a solution.

3.2.1 Preliminary result

Theorem 3 *Under Assumption (A3), the link flow rate functions f_i are solutions of the partial differential equation:*

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 4b_i f_i)^{\frac{1}{2}} \frac{\partial f_i}{\partial x_i} = 0. \tag{31}$$

Furthermore, under Assumption (A4), the link flow rate functions f_i are solutions of the first-order partial differential equation:

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 2b_i f_i) \frac{\partial f_i}{\partial x_i} = 0. \tag{32}$$

Assumption (A5) provides a boundary condition.

Proof: Under Assumption (A3), equation (17) holds. Differentiating equation (17) with respect to t gives rise to $\frac{\partial k_i}{\partial t} = \frac{\frac{\partial f_i}{\partial t}}{u_i^{max}(1-4b_i f_i)^{\frac{1}{2}}}$.

Replacing the above value of $\frac{\partial k_i}{\partial t}$ in conservation equation (14), we obtain $\frac{\partial f_i}{\partial t} + u_i^{max}(1-4b_i f_i)^{\frac{1}{2}} \frac{\partial f_i}{\partial x_i} = 0$.

Assumption (A4) implies that all the terms of order higher than or equal to 2 in the Taylor expansion of the above equation are negligible. That is,

$$(1 - \epsilon)^{\frac{1}{2}} = 1 - \frac{\epsilon}{2} + O(\epsilon^2) \approx 1 - \frac{\epsilon}{2}. \tag{33}$$

Using this observation and Assumption (A6), we obtain that

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 2b_i f_i) \frac{\partial f_i}{\partial x_i} = 0.$$

□

This new conservation law (32) is the basis of the ETT Model that we present below and analyze in Sections 4 and 5.

3.2.2 Model Formulation

Theorem 3 gives rise to the following formulation:

ETT Model

For all $t \in [0, T]$:

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 2b_i f_i)\frac{\partial f_i}{\partial x_i} = 0, \quad \text{for all } i \in I, \tag{34}$$

$$f_i(0, t + T_i) = h_i^t(T_i), \quad \text{for all } i \in I, \tag{35}$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}, \quad \text{for all } i \in I, \tag{36}$$

$$u_i = \frac{f_i}{k_i}, \quad \text{for all } i \in I, \tag{37}$$

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1}{u_i}, \quad \text{for all } i \in I, \tag{38}$$

$$T_i(0, t) = 0, \quad \text{for all } i \in I, \tag{39}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \quad \text{for all } p \in P. \tag{40}$$

Equation (34) is a first-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model. Moreover, equation (35) provides the boundary condition for this equation.

Assuming that equations (34) and (35) possess a continuously differentiable solution f_i, equations (36) and (37) determine the density k_i and the velocity u_i. The ordinary differential equation (38) under its boundary condition (39) determines travel times on the network's links. Finally, path travel times follow from equation (40). Therefore, if we assume that equations (34) and (35) possess a continuously differentiable solution f_i, the ETT Model, as formulated by equations (34)-(40), also possesses a solution.

Remark: Replacing equations (36) and (37) in equation (38), leads to the same equation as for the PTT Model, that is

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1 + b_i f_i}{u_i^{max}}. \tag{41}$$

3.2.3 Existence of Solution to the ETT Model

The following theorem provides an existence result for a continuously differentiable solution of ETT Model as formulated above.

Theorem 4 *The ETT Model as formulated in equations (34)-(35) possesses a solution if and only if the first derivative of the link flow rate function $h_i^t(T_i)$ satisfies the following boundedness condition:*

$$\frac{dh_i^t(T_i)}{dT_i} > -\frac{u_i^{max}}{2b_i L_i}(1 - 2b_i h_i^t(T_i))^2. \tag{42}$$

Proof: The proof is similar to the one for Theorem 2. We include the details in the Appendix.

4 Analysis of Separable Velocity Functions

In this section, we study the PTT and the ETT Models in further details.

In particular, in Subsection 4.1, we extensively analyze the PTT Model for piecewise linear and piecewise quadratic functions $h_i^t(T_i)$ (see Assumption (A7)). We show how Model 1 reduces in this case to the analysis of a single ordinary differential equation. We provide families of travel time functions.

In Subsection 4.2, we analyze the ETT Model by approximating the initial flow rate with piecewise linear functions $h_i^t(T_i)$. Moreover, we show why the analysis of the ETT Model is more complex than the one of the PTT Model. Finally, we propose a family of travel time functions. In Subsection 4.3, we summarize our results and show how the families of travel time functions we propose in Subsections 4.1 and 4.2 relate.

4.1 Separable PTT Model

In this subsection, we analyze the PTT Model for piecewise linear and piecewise quadratic approximations of departure flow rates. We provide families of travel time functions under a variety of assumptions.

4.1.1 A General Framework for the Analysis of the PTT Model

The purpose of this subsection is to provide a general framework for the analysis of the PTT Model that reduces the problem to solving a single ordinary differential equation.

Applying this general framework to piecewise linear departure link flow rate functions will result in an easy derivation of link travel times. Furthermore, applying this framework to piecewise quadratic departure link flow rate functions will provide us with a closed form solution of link travel time functions.

As a first step towards establishing the main result of this subsection, we introduce the classical method of characteristics in fluid dynamics. Haberman (1977) provides a detailed analysis of this method. Along the characteristic line that passes through $(x_i, t + T_i)$ with slope $\frac{1+2b_i f_i}{u_i^{max}}$, the solution $f_i(x_i, t + T_i)$ of equation (22) remains constant. If $(0, t + s_i(x_i, t + T_i))$ denotes the point at which the characteristic line intersects the time axis, we have

$$f_i(x_i, t + T_i) = h_i^t(s_i(x_i, t + T_i)). \tag{43}$$

Perakis (1997) establishes that

$$s_i(x_i, t + T_i) = \frac{T_i u_i^{max} - x_i - 2b_i x_i h_i^t(s_i(x_i, t + T_i))}{u_i^{max}}. \tag{44}$$

We introduce two new variables $m_i(.)$ and $g_i(.)$ defined by $m_i(s_i) = \frac{b_i}{u_i^{max}}(h_i^t(s_i) - A_i)$ and $g_i(x_i, t) = T_i(x_i, t) - \frac{1+b_i A_i}{u_i^{max}} x_i$.

Theorem 5 *(General framework) The PTT Model reduces to solving the following ordinary differential equation:*

$$\frac{ds_i}{dx_i} = \frac{-m_i(s_i) - \frac{b_i A_i}{u_i^{max}}}{1 + 2x_i m_i'(s_i)}, \tag{45}$$

with $s_i(0) = 0$ *as an initial condition. The link flow rate functions and the link travel time functions follow from:*

$$f_i(x_i,t) = h_i^t(s_i) \tag{46}$$

$$T_i(x_i,t) = s_i + \frac{x_i + 2b_i x_i h_i^t(s_i)}{u_i^{max}}. \tag{47}$$

Proof: Introducing g_i, m_i and A_i in equations (44) and (29), we derive the following two relations:

$$s_i = g_i - 2x_i m_i(s_i) - \frac{b_i A_i}{u_i^{max}} x_i, \tag{48}$$

$$\frac{dg_i}{dx_i} = m_i(s_i), \tag{49}$$

with $s_i(0) = g_i(0) = m_i(0) = 0$.

From equations (48) and (49), it follows that

$$\begin{aligned}\frac{ds_i}{dx_i} &= \frac{dg_i}{dx_i} - 2m_i(s_i) - 2x_i\frac{ds_i}{dx_i}m_i'(s_i) - \frac{b_i A_i}{u_i^{max}} \\ &= -m_i(s_i) - 2x_i\frac{ds_i}{dx_i}m_i'(s_i) - \frac{b_i A_i}{u_i^{max}}.\end{aligned}$$

Hence, $\frac{ds_i}{dx_i}(1 + 2x_i m_i'(s_i)) = -m_i(s_i) - \frac{b_i A_i}{u_i^{max}}$. Then the results of the theorem follow.

□

4.1.2 Piecewise Linear Departure Link Flow Rate Functions

In this subsection, we apply the general framework to simplify the analysis of the piecewise linear approximation of departure flow rates.

We assume that during a time period $[t, t+\Delta]$, travelers make the approximation that the departure link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. That is,

$$f_i(0, t+T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i. \tag{50}$$

Over the time period $[0, T]$, this results into a piecewise linear approximation of link departure flow rates as shown in Figure 1.

Remark:
Note that equation (30) is a necessary and sufficient condition for the existence of solution of the PTT Model. In this case, the condition becomes:

$$B_i(t) > -\frac{u_i^{max}}{2b_i L_i}. \tag{51}$$

We call the system of equations (22)-(28) and (50) the Linear PTT Model. Next, we provide a closed form solution for the Linear PTT Model.

Theorem 6 *If (51) holds, then:*
(i) The Linear PTT Model possesses a solution,
(ii) The link flow rate functions $f_i(x_i, t+T_i)$ are continuously differentiable,

$$f_i(x_i, t+T_i) = \frac{B_i(t)u_i^{max}T_i - B_i(t)x_i + A_i(t)u_i^{max}}{u_i^{max} + 2b_iB_i(t)x_i}, \tag{52}$$

(iii) The link travel time functions $T_i(x_i, t)$ are given by:

$$T_i(x_i, t) = \frac{x_i}{u_i^{max}} + \frac{A_i(t)}{B_i(t)}((1 + \frac{2b_iB_i(t)x_i}{u_i^{max}})^{\frac{1}{2}} - 1). \tag{53}$$

Proof: Since $h_i^t(s_i) = A_i + B_is_i$, it follows that $m_i(s_i) = \frac{b_iB_i}{u_i^{max}}s_i$. Replacing in equation (45), we obtain

$$\frac{ds_i}{dx_i} = \frac{-\frac{b_iB_i}{u_i^{max}}s_i - \frac{b_iA_i}{u_i^{max}}}{1 + 2x_i\frac{b_iB_i}{u_i^{max}}}.$$

The above equation can be written as the following separable equation:

$$\frac{ds_i}{s_i + \frac{A_i}{B_i}} = -\frac{\frac{b_iB_i}{u_i^{max}}dx}{1 + 2x_i\frac{b_iB_i}{u_i^{max}}}. \tag{54}$$

Integrating both parts (see Bender and Orszag (1978)) gives rise to $\frac{s_i + \frac{A_i}{B_i}}{\frac{A_i}{B_i}} = \frac{1}{(1+2\frac{b_iB_i}{u_i^{max}}x_i)^{\frac{1}{2}}}$. Therefore it follows that

$$s_i = \frac{A_i}{B_i}(\frac{1}{(1 + 2\frac{b_iB_i}{u_i^{max}}x_i)^{\frac{1}{2}}} - 1). \tag{55}$$

Using equations (46) and (47), the results of the theorem follow.

□

Corollary 1 *Assume that*

$$|B_i(t)| << \frac{u_i^{max}}{2b_iL_i}. \tag{56}$$

Then:
(i) The Linear PTT Model possesses a solution,
(ii) The link travel time functions $T_i(x_i, t)$ simplifies as follows:

$$T_i(x_i, t) = \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}}x_i^2]. \tag{57}$$

Proof:
(i) Note that equation (56) implies that equation (51) holds. From Theorem 2, part (i) follows.
(ii) Equation (56) justifies why a second order Taylor expansion of equation (53) is reasonable. This leads to equation (57).

□

Example

To illustrate our results, we consider a network of four links connecting one O/D pair as shown in Figure 2.

The total length of each of the four links is $L_1 = 4$ miles, $L_2 = 5$ miles, $L_3 = 6$ miles and $L_4 = 7.5$ miles, respectively. The speed limit on each link is $u_1^{max} = 40$ miles/hr, $u_2^{max} = 25$ miles/hr, $u_3^{max} = 25$ miles/hr and $u_4^{max} = 30$ miles/hr, respectively. Finally, the maximum density on each link is $k_1^{max} = 200$ cars per mile, $k_2^{max} = 160$ cars per mile, $k_3^{max} = 192$ cars per mile and $k_4^{max} = 250$ cars per mile, respectively.

We illustrate our results using the example of Figure 2. We consider various choices for $A_i(t)$ and $B_i(t)$.

1) The traveler estimates his/her travel time on link i by assuming that the departure link flow rate $f_i(0, t + T_i) = f_i(0, t)$, that is the flow rate remains constant during the time period $[t, t + \Delta]$. Then, $A_i(t) = f_i(0, t)$ and $B_i(t) = 0$.
2) The traveler assumes that the departure link flow rate is equal to the average of the departure link flow rate over a previous time interval of length h, that is, $f_i(0, t + T_i) = \frac{1}{h}\int_{t-h}^{t} f_i(0, w)dw$. Then, $A_i(t) = \frac{1}{h}\int_{t-h}^{t} f_i(0, w)dw$ and $B_i(t) = 0$.
3) The traveler uses information prior to t as in 2). The traveler considers the departure link flow rate on link i to be

$$f_i(0, t + T_i) = f_i(0, t) + \frac{1}{h}[f_i(0, t) - f_i(0, t - h)]T_i.$$

For this choice, $A_i(t) = f_i(0, t)$ and $B_i(t) = \frac{1}{h}[f_i(0, t) - f_i(0, t - h)]$.
4) The traveler takes into account the first order information of the departure link flow rate function

$$f_i(0, t + T_i) = f_i(0, t) + \frac{df_i(0, t)}{dt}T_i.$$

For this choice, $A_i(t) = f_i(0, t)$ and $B_i(t) = \frac{df_i(0,t)}{dt}$.

Using the first two choices 1) and 2), Corollary 1 gives rise to the following travel times

$$T_1(L_1, t) = \frac{1}{10000}[\frac{1}{8}A_1(t) + 1000],$$
$$T_2(L_2, t) = \frac{1}{10000}[\frac{1}{2}A_2(t) + 2000],$$
$$T_3(L_3, t) = \frac{1}{10000}[\frac{1}{2}A_3(t) + 2400],$$
$$T_4(L_4, t) = \frac{1}{10000}[\frac{1}{3}A_4(t) + 2500],$$

Using the latter two choices 3) and 4), the travel times become

$$T_1(L_1, t) = \frac{1}{10000}[\frac{1}{8}A_1(t) + 1000 - \frac{A_1(t)B_1(t)}{1280000}],$$
$$T_2(L_2, t) = \frac{1}{10000}[\frac{1}{2}A_2(t) + 2000 - \frac{A_2(t)B_2(t)}{80000}],$$
$$T_3(L_3, t) = \frac{1}{10000}[\frac{1}{2}A_3(t) + 2400 - \frac{A_3(t)B_3(t)}{80000}],$$
$$T_4(L_4, t) = \frac{1}{10000}[\frac{1}{3}A_4(t) + 2500 - \frac{A_4(t)B_4(t)}{180000}].$$

4.1.3 Piecewise Quadratic Departure Link Flow Rate Functions

In this subsection, we assume that during a time period $[t, t+\Delta]$, travelers make the approximation that the departure link flow rate for subsequent times $t+T_i$ is quadratic in terms of the travel time T_i. That is,

$$f_i(0, t+T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i + C_i(t)(T_i)^2. \tag{58}$$

Over the time period $[0, T]$, this results into a piecewise quadratic approximation of link departure flow rates as shown in Figure 3.

Note that equation (30), which is a necessary and sufficient condition for existence of a solution, becomes in this case:

$$B_i(t) + 2C_i(t)(t+\Delta) > -\frac{u_i^{max}}{2b_iL_i}. \tag{59}$$

We call the system of equations (22)-(28) and (58) the Quadratic PTT Model. Next, we provide a closed form solution to the Quadratic PTT Model. Note that when the quadratic term is neglected (i.e. $C_i = 0$), we capture the previously studied case of piecewise linear departure link flow rate functions.

Let $\alpha_1 = \frac{b_iB_i(t)}{u_i^{max}}$, $\alpha_2 = \frac{b_iA_i(t)}{u_i^{max}}$ and $\alpha_3 = \frac{4b_iC_i(t)}{u_i^{max}}$.

Theorem 7 *Assume that*

$$|B_i(t) + 2C_i(t)(t+\Delta)| << \frac{u_i^{max}}{2b_iL_i}. \tag{60}$$

Then, the following holds
(i) The Quadratic PTT Model possesses a solution.
(ii) The link characteristic line functions s_i are continuously differentiable and are given by

$$s_i(x_i, t) = \frac{\alpha_2}{\alpha_1} e^{-\alpha_1 x_i + (2\alpha_1^2 + \alpha_2\alpha_3)\frac{x_i^2}{2}} \int_0^{x_i} e^{\alpha_1 t - (2\alpha_1^2+\alpha_2\alpha_3)\frac{t^2}{2}} (-\alpha_1 + 2\alpha_1^2 t)dt. \tag{61}$$

(iii) The third degree Taylor expansion of the link characteristic line functions s_i becomes

$$s_i(x_i, t) = -\frac{\alpha_2}{\alpha_1}(\alpha_1 x_i - \frac{3\alpha_1^2}{2}x_i^2 + (7\alpha_1^3 + 2\alpha_1\alpha_2\alpha_3)\frac{x_i^3}{6}). \tag{62}$$

(iv) The third degree Taylor expansion of the link travel time functions $T_i(x_i, t)$ becomes

$$\begin{aligned} T_i(x_i, t) &= \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}}x_i^2 + (\frac{2A_i(t)^2C_i(t)(b_i)^3}{3(u_i^{max})^2} \\ &- \frac{7A_i(t)B_i(t)^2(b_i)^3}{6(u_i^{max})^2} + \frac{3B_i(t)^2C_i(t)(b_i)^3}{(u_i^{max})^2})x_i^3]. \end{aligned} \tag{63}$$

Proof: The analysis involved in this proof is very tedious. For the sake of simplicity and brevity, we only include the most important steps of the analysis in the Appendix.

Example

We illustrate our results using the example of Figure 2. We consider two additional choices for $A_i(t)$, $B_i(t)$ and $C_i(t)$.

5) The traveler considers second-order information prior to t. That is, the departure link flow rate on link i to be

$$f_i(0, t+T_i) = f_i(0,t) + \frac{df_i(0,t)}{dt}T_i + \frac{1}{2h}[\frac{df_i(0,t)}{dt} - \frac{df_i(0,t-h)}{dt}]T_i^2.$$

For this choice, $A_i(t) = f_i(0,t)$, $B_i(t) = \frac{df_i(0,t)}{dt}$ and $C_i(t) = \frac{1}{2h}[\frac{df_i(0,t)}{dt} - \frac{df_i(0,t-h)}{dt}]$.
6) The traveler considers second-order information of the departure link flow rate function

$$f_i(0, t+T_i) = f_i(0,t) + \frac{df_i(0,t)}{dt}T_i + \frac{1}{2}\frac{d^2 f_i(0,t)}{dt^2}T_i^2.$$

For this choice, $A_i(t) = f_i(0,t)$, $B_i(t) = \frac{df_i(0,t)}{dt}$ and $C_i(t) = \frac{1}{2}\frac{d^2 f_i(0,t)}{dt^2}$.

Using choices 5) and 6), Theorem 7 gives rise to

$$T_1(L_1,t) = \frac{1}{10000}[\frac{1}{8}A_1(t) + 1000 - \frac{A_1(t)B_1(t)}{1280000} + \frac{\frac{2A_1^2(t)C_1(t)}{3} - \frac{7A_1(t)B_1^2(t)}{6} + 3B_1^2(t)C_1(t)}{1280000000}],$$

$$T_2(L_2,t) = \frac{1}{10000}[\frac{1}{2}A_2(t) + 2000 - \frac{A_2(t)B_2(t)}{80000} + \frac{\frac{2A_2^2(t)C_2(t)}{3} - \frac{7A_2(t)B_2^2(t)}{6} + 3B_2^2(t)C_2(t)}{32000000}],$$

$$T_3(L_3,t) = \frac{1}{10000}[\frac{1}{2}A_3(t) + 2400 - \frac{A_3(t)B_3(t)}{80000} + \frac{\frac{2A_3^2(t)C_3(t)}{3} - \frac{7A_3(t)B_3^2(t)}{6} + 3B_3^2(t)C_3(t)}{32000000}],$$

$$T_4(L_4,t) = \frac{1}{10000}[\frac{1}{3}A_4(t) + 2500 - \frac{A_4(t)B_4(t)}{180000} + \frac{\frac{2A_4^2(t)C_4(t)}{3} - \frac{7A_4(t)B_4^2(t)}{6} + 3B_4^2(t)C_4(t)}{90000000}].$$

Equation (63) provides us with a general family of travel time functions. In Subsection 4.3, we will discuss the relationship between this family of travel time functions and the one obtained by the Linear PTT Model.

4.2 Separable ETT Model

In this subsection, we study the ETT Model. We show that the analysis of the ETT Model is more complex than the PTT Model, and propose a different class of travel time functions for piecewise linear approximations of departure flow rates.

4.2.1 Piecewise Linear Departure Link Flow Rate Functions

In this subsection, we assume that during a time period $[t, t+\Delta]$, travelers make the approximation that the departure link flow rate for subsequent times $t+T_i$ is linear in terms of the travel time T_i (see Figure 1 in Subsection 4.1.2). That is,

$$f_i(0, t+T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i. \tag{64}$$

Note that equation (42), which is a necessary and sufficient condition for the existence of a solution, becomes in this case:

$$B_i(t) > -\frac{u_i^{max}}{2b_iL_i}(1 - 2b_iA_i(t) - 2b_iB_i(t)(t + \Delta))^2. \tag{65}$$

We call the system of equations (34)-(40) and (64) the Linear ETT Model. Next, we provide a closed form solution of the Linear ETT Model.

To make our notation more tractable, we introduce variables $\theta_1 = \frac{b_iB_i(t)}{1-2b_iA_i(t)}$, $\theta_2 = \frac{b_iB_i(t)}{u_i^{max}}$ and $\theta_3 = \frac{1+b_iA_i(t)}{u_i^{max}} - \frac{1}{u_i^{max}(1-2b_iA_i(t))}$.

Theorem 8 *Assume that*

$$|B_i(t)| << \frac{u_i^{max}}{2b_iL_i}(1 - 2b_iA_i(t) - 2b_iB_i(t)(t + \Delta))^2. \tag{66}$$

The following holds,
(i) The Linear ETT Model possesses a solution,
(ii) The link characteristic line functions s_i are continuously differentiable and can be expressed as a function of the link travel travel time functions, that is,

$$s_i(x_i, t) = \frac{T_iu_i^{max}(1 - 2b_iA_i(t)) - x_i}{u_i^{max}(1 - 2b_iA_i(t) + 2b_iB_i(t)T_i}, \tag{67}$$

(iii) The link travel time functions $T_i(x_i, t)$ are given by

$$T_i(x_i, t) = \theta_3(\frac{e^{\theta_2x_i} - 1}{\theta_2}) + \frac{\theta_1x_i}{\theta_2(u_i^{max})^2}, \tag{68}$$

(iv) If condition (56) holds, the link travel time functions $T_i(x_i, t)$ are

$$T_i(x_i, t) = \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}}x_i^2]. \tag{69}$$

Proof: The analysis involved in this proof is quite tedious. For the sake of brevity, we only include the key steps of the analysis in the Appendix.

Example

Let us illustrate our results using the example in Figure 2. We consider for $A_i(t)$ and $B_i(t)$ the four choices introduced in Subsection 4.1.4.

Using the first two choices 1) and 2), equation (69) gives rise to

$$T_1(L_1, t) = \frac{1}{10000}[\frac{1}{8}A_1(t) + 1000],$$
$$T_2(L_2, t) = \frac{1}{10000}[\frac{1}{2}A_2(t) + 2000],$$
$$T_3(L_3, t) = \frac{1}{10000}[\frac{1}{2}A_3(t) + 2400],$$
$$T_4(L_4, t) = \frac{1}{10000}[\frac{1}{3}A_4(t) + 2500],$$

while for the latter two choices 3) and 4), it follows that

$$T_1(L_1,t) = \frac{1}{10000}[\frac{1}{8}A_1(t) + 1000 - \frac{A_1(t)B_1(t)}{1280000}],$$
$$T_2(L_2,t) = \frac{1}{10000}[\frac{1}{2}A_2(t) + 2000 - \frac{A_2(t)B_2(t)}{80000}],$$
$$T_3(L_3,t) = \frac{1}{10000}[\frac{1}{2}A_3(t) + 2400 - \frac{A_3(t)B_3(t)}{80000}],$$
$$T_4(L_4,t) = \frac{1}{10000}[\frac{1}{3}A_4(t) + 2500 - \frac{A_4(t)B_4(t)}{180000}].$$

It is not a coincidence that the above results match exactly the results we obtained in Subsection 4.1.4. Subsection 4.3 will further clarify this similarity.

Equation (68) is an exponential family of travel time functions. In the following subsection, we analyze the relationship between the exponential family of travel time functions from this subsection and the one we obtained through the Linear PTT Model and the Quadratic PTT Model.

4.3 Summary and Models Comparison

In summary, we have so far derived two families of travel time functions. The Linear PTT Model which leads to the polynomial family of travel time functions

$$T_i(x_i,t) = \frac{x_i}{u_i^{max}} + \frac{A_i(t)}{B_i(t)}((1 + \frac{2b_iB_i(t)x_i}{u_i^{max}})^{\frac{1}{2}} - 1), \tag{70}$$

and the Linear ETT Model which leads to the exponential family of travel time functions

$$T_i(x_i,t) = \theta_3(\frac{e^{\theta_2 x_i} - 1}{\theta_2}) + \frac{\theta_1 x_i}{\theta_2(u_i^{max})^2}, \tag{71}$$

where, θ_i, $i \in \{1,2,3\}$ defined in Subsection 4.2.2.

It is very important to note that equations (70) and (71) coincide when $|B_i(t)| << \frac{u_i^{max}}{2b_iL_i}$ holds. That is, they possess the same second order Taylor expansion

$$T_i(x_i,t) = \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}}x_i^2]. \tag{72}$$

This relationship shows that the assumptions made for both the Linear PTT Model and the Linear ETT Model are indeed reasonable.

Furthermore, the Quadratic PTT Model gives rise to a more complicated expression of link travel time functions. The third degree Taylor expansion leads to

$$\begin{aligned} T_i(x_i,t) &= \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}}x_i^2 + (\frac{2A_i(t)^2C_i(t)(b_i)^3}{3(u_i^{max})^2} \\ &- \frac{7A_i(t)B_i(t)^2(b_i)^3}{6(u_i^{max})^2} + \frac{3B_i(t)^2C_i(t)(b_i)^3}{(u_i^{max})^2})x_i^3]. \end{aligned} \tag{73}$$

We observe that

- If the quadratic term is neglected (i.e. $C_i = 0$), then a second order approximation of equation (73) leads to equation (72) and, as one would expect, we fall in the case of the Linear PTT Model. Hence, it appears that the assumptions made for the Quadratic PTT Model are also reasonable.

- If the constant term is neglected (i.e. $A_i = 0$), equation (73) provides us with a non-zero third order degree term.

This concludes our analysis of the separable case. In the following section, we study the non-separable case of this problem.

5 A Non-Separable Model

In this section, we generalize the Polynomial Travel Time Model (PTT Model) to the case of non-separable velocity functions. We show how the results obtained for the separable case extend to the non-separable case. The proofs are similar to the ones of Subsection 4.1.

In order to ease the transition to the non-separable PTT Model, we first consider in Subsection 5.1 the case of a two-link network. In Subsection 5.2, we extend our results to the more general case of acyclic networks.

5.1 Two Links Interaction

In this subsection, we consider the case of two links: link 1 and link 2, as shown in Figure 4. We consider the case of non-separable velocity functions. We model the two-link network interaction by considering that the velocity of link 2, at position x_2 and at time t, can be expressed as in Subsection 2.4 by

$$u_2 = \widehat{u}(k_2(x_2,t)) = u_2^{max} - b_2(u_2^{max})^2 k_2(x_2,t) + \alpha_{21}(x_2) R_{21}(\overline{x}_1, t-\Delta_{21}), \tag{74}$$

where $\alpha_{21}(x_2)$ is the density correlation function between link 2 and link 1 and depends on the position x_2 on link 2; R_{21} is a function of k_1 and ∇k_1; $\overline{x}_1$ is a fixed position of a detector of density on link 1; and Δ_{21} is a propagation time between link 1 and link 2.

For the sake of simplicity, let us consider $R_{21}(.) = k_1(.)$. Moreover, for the sake of simplifying notation, we introduce the term $J_{21} = \frac{\alpha_{21}(x_2)}{u_2^{max}} k_1(\overline{x}_1, t-\Delta_{21})$.

Lemma 2, from Subsection 3.1.1, that relates the density on a link to the link flow rate, extends in this case as well. Using a similar proof, we derive

$$k_2 = \frac{f_2}{u_2^{max}(1+J_{21})} + \frac{b_2(f_2 u_2^{max})^2}{(u_2^{max}(1+J_{21}))^3}. \tag{75}$$

Furthermore, through similar arguments as in Subsection 4.1, we show that the general framework of Theorem 5 (see equation (45)) leads to the conservation law $\frac{\partial f_2}{\partial x_2} + \frac{1}{u_2^{max}}(\frac{1}{1+J_{21}} + \frac{2b_2 f_2}{(1+J_{21})^3})\frac{\partial f_2}{\partial T_2} = 0$.

Therefore, the Non-Separable PTT Model for link 2 becomes:

Non-Separable Polynomial Travel Time Model

For all $t \in [0,T]$,

$$\frac{\partial f_2}{\partial x_2} + \frac{1}{u_2^{max}}\left(\frac{1}{1+J_{21}} + \frac{2b_2 f_2}{(1+J_{21})^3}\right)\frac{\partial f_2}{\partial T_2} = 0,$$
$$f_2(0, t+T_2) = h_2^t(T_2),$$
$$k_2 = \frac{f_2}{u_2^{max}(1+J_{21})} + \frac{b_2(f_2 u_2^{max})^2}{(u_2^{max}(1+J_{21}))^3},$$
$$u_2 = \frac{f_2}{k_2},$$
$$\frac{dT_2(x_2,t)}{dx_2} = \frac{1}{u_2},$$
$$T_2(0,t) = 0.$$

Similarly to the separable case, we can express the link flow rate function $f_2(x_2, t+T_2)$ as

$$f_2(x_2, t+T_2) = \frac{B_2(t)u_2^{max}T_2 - \frac{B_2(t)x_2}{1+J_{21}} + A_2(t)u_2^{max}}{u_2^{max} + \frac{2b_2B_2(t)x_2}{(1+J_{21})^3}}, \tag{76}$$

and consider a linear ordinary differential equation for determining link travel times

$$\frac{dT_2}{dx_2} - \frac{b_2B_2(t)}{(1+J_{21})^3u_2^{max} + 2b_2x_2B_2(t)}T_2 - \frac{\frac{b_2x_2B_2(t)}{1+J_{21}} + b_2A_2(t)u_2^{max} + (1+J_{21})^2u_2^{max}}{u_2^{max}((1+J_{21})^3u_2^{max} + 2b_2x_2B_2(t))} = 0. \tag{77}$$

The complexity of equation (77) depends on the complexity of the density correlation function $\alpha_{21}(x_2)$ expressed through the term J_{21}. Notice that we can establish similar results as in Subsection 4.1 if $\alpha_{21}(x_2)$ is a constant. However, deriving analytical closed form solutions is more complex if $\alpha_{21}(x_2)$ is linear in x_2 and too difficult in other cases. If $\alpha_{21}(x_2)$ is neither constant nor linear, numerical methods seem to be the only approach to solve differential equation (77) and determine travel times.

We are now ready to extend our results to acyclic networks.

5.2 Acyclic Networks

In this subsection, we consider an acyclic network (see Figure 5). The acyclicity assumption will enable us to extend the results of Subsection 4.1 to the case of non-separable velocity functions. We model link interactions by considering that the velocity of link i, at position x_i and at time t, can be expressed as in Subsection 2.4 by

$$\widehat{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2k_i + \sum_{j\in B(i)} \alpha_{ij}(x_i)R_{ij}(\overline{x}_j, t-\Delta_{ij}), \tag{78}$$

where $B(i)$ is the set of predecessors of link i.

A predecessor of a link i is any link that comes before i on a path. It does not restrict to only the immediate parent of a link. Note that since we consider the case of acyclic networks, we can talk of predecessors of a link as shown in Figure 5. Note as well that the results we will establish for the case, where we consider the set of predecessors, also apply to the case where we consider the set of successors instead.

For the sake of simplicity, let us consider $R_{ij}(.) = k_j(.)$. Moreover, for the sake of simplifying notation, we introduce $J_i = 1 + \sum_{j\in B(i)} \frac{\alpha_{ij}(x_i)}{u_i^{max}}k_j(\overline{x}_j, t-\Delta_{ij})$.

Therefore, the Non-Separable PTT Model becomes:

Non-Separable Polynomial Travel Time Model
For all $t \in [0, T]$,

$$\frac{\partial f_i}{\partial x_i} + \frac{1}{u_i^{max}}\left(\frac{1}{J_i} + \frac{2b_i f_i}{J_i^3}\right)\frac{\partial f_i}{\partial T_i} = 0, \quad \text{for all } i \in I, \tag{79}$$

$$f_i(0, t + T_i) = h_i^t(T_i), \quad \text{for all } i \in I, \tag{80}$$

$$k_i = \frac{f_i}{u_i^{max} J_i} + \frac{b_i f_i^2}{u_i^{max} J_i^3}, \quad \text{for all } i \in I, \tag{81}$$

$$u_i = \frac{f_i}{k_i}, \quad \text{for all } i \in I, \tag{82}$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \quad \text{for all } i \in I, \tag{83}$$

$$T_i(0, t) = 0, \quad \text{for all } i \in I, \tag{84}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \quad \text{for all } p \in P. \tag{85}$$

Next, we show how the general framework (see Subsection 4.1.3) for analyzing the Separable PTT Model, extends to the non-separable model.

5.2.1 General Framework for Constant Density Correlation Functions

In this subsection, we assume that the density correlation function $\alpha_{ij}(x_i)$ between link i and link j is a constant function of x_i. In this case, J_i is also a constant function of x_i.

As in Subsection 4.1.1, we introduce two new variables $m_i(.)$ and $g_i(.)$ defined by $m_i(s_i) = \frac{b_i}{u_i^{max}}(h_i^t(s_i) - A_i)$ and $g_i(x_i, t) = T_i(x_i, t) - \frac{1 + b_i A_i}{u_i^{max}} x_i$.

Theorem 9 *(A General Framework for Constant Density Correlation Functions)*
If the density correlation functions are constant, the Non-Separable PTT Model reduces to solving the ordinary differential equation:

$$\frac{ds_i}{dx_i} = \frac{-m_i(s_i) - \frac{b_i A_i}{u_i^{max}}}{J_i^3 + 2x_i m_i'(s_i)}, \tag{86}$$

with $s_i(0) = 0$ as an initial condition. The link flow rate functions and the link travel time functions follow from:

$$f_i(x_i, t) = h_i^t(s_i) \tag{87}$$

$$T_i(x_i, t) = s_i + x_i \frac{\frac{1}{J_i} + \frac{2b_i h_i^t(s_i)}{J_i^3}}{u_i^{max}}. \tag{88}$$

Proof: See Appendix.

We now consider that during a time period $[t, t + \Delta]$, travelers make the approximation that the link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. That is, $f_i(0, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i$.

The following theorem is an extension of Theorem 6 to the non-separable case with constant density correlation functions.

Theorem 10 *If* $B_i > -\frac{u_i^{max} J_i^3}{2b_i L_i}$ *holds, then:*
(i) The Non-Separable Linearized PTT Model possesses a solution,
(ii) The link flow rate functions $f_i(x_i, t+T_i)$ *are continuously differentiable, and we have:*

$$f_i(x_i, t+T_i) = \frac{B_i(t) u_i^{max} T_i - \frac{B_i(t) x_i}{J_i} + A_i(t) u_i^{max}}{u_i^{max} + \frac{2b_i B_i(t) x_i}{J_i^3}},$$

(iii) The link travel time functions $T_i(x_i, t)$ *are given by:*

$$T_i(x_i, t) = \frac{x_i}{u_i^{max} J_i} + \frac{A_i(t)}{B_i(t)}((1 + \frac{2b_i B_i(t) x_i}{u_i^{max} J_i^3})^{\frac{1}{2}} - 1).$$

Proof: The proof is fairly similar to the one of Theorem 6. We include it in the Appendix.

Note that when the density correlation functions are set to zero, we have $J_i = 1$. The results of Theorem 10 then reduce to the results of Theorem 6 introduced in Subsection 4.1.2.

5.2.2 Linear Density Correlation Functions

In this subsection, we consider the more complex case of linear density correlation functions. That is, for every link $j \in B(i)$, $\alpha_{ij}(x_i) = a_{ij} + b_{ij} x_i$. In addition to the acyclicity assumption we imposed on the network, we further assume that the influence of neighboring links has only a first order effect. This translates into $\sum_{j \in B(i)} \frac{\alpha_{ij}(x_i)}{u_i^{max}} k_j(\overline{x_j}, t - \Delta_{ij}) << 1$. Therefore, we can make the following first-order approximation

$$J_i^3 \approx 1 + 3 \sum_{j \in B(i)} \frac{\alpha_{ij}(x_i)}{u_i^{max}} k_j(\overline{x_j}, t - \Delta_{ij}). \tag{89}$$

For every integer n, let $\theta_{in} = n \sum_{j \in B(i)} b_{ij} k_j$ and $\gamma_{in} = u_i^{max} + n \sum_{j \in B(i)} a_{ij} k_j$. The following result provides a linear ordinary differential equation satisfied by link travel time functions T_i for the case of linear density correlation functions.

Theorem 11 *If* $B_i > -\frac{u_i^{max} J_i^3}{2b_i L_i}$ *holds, then:*
(i) The Linear PTT Model possesses a solution,
(ii) The link travel time functions $T_i(x_i, t)$ *satisfy*

$$\frac{dT_i}{dx_i} - \frac{b_i B_i(t)}{\gamma_{i3} + (\theta_{i3} + 2b_i B_i(t)) x_i} T_i - \frac{\frac{b_i x_i B_i(t)}{J_i} + b_i A_i(t) u_i^{max} + u_i^{max} J_i^2}{u_i^{max}(u_i^{max} J_i^3 + 2b_i x_i B_i(t))} = 0, \tag{90}$$

Proof: Equation (90) is an easy extension of equation (77). We can obtain a complicated closed form solution of equation (90). Its derivation is too complicated and involves several integration by parts that we do not include for the sake of simplicity.

□

6 Conclusions

In this paper, we took a fluid dynamics approach to determine the delay (travel time) of a traveler in traversing a network's link. We extended a model proposed by Perakis (1997) by considering two approximation models: the Polynomial (PTT) and the Exponential (ETT) Travel Time models. We proposed a general framework for the analysis of the PTT Model. This framework allowed us to reduce the analysis of the model to solving a single ordinary differential equation. In the case of separable velocity functions, we proposed families of travel time functions for the problem that rely on piecewise linear and piecewise quadratic approximations of departure flow rates. We further established a connection between these travel time functions. Our analysis of the ETT Model applied to the case of separable velocity functions while the analysis of the PTT Model applied to both separable and non-separable velocity functions. The latter applied to the case of acyclic networks.

Continuing this work, we intend to investigate the extension of our results in the case of non-separable velocity functions as they apply to non-acyclic networks. We also intend to extend our models to incorporate second-order effects such as reaction of drivers to upstream and downstream congestion as well as second-order link interaction effects. We plan to examine other fluid dynamics models. For example, we can consider a different model for relating speed and density. Moreover, we will investigate alternate approaches including queuing models. We wish to connect these models with the dynamic user-equilibrium problem. We plan to investigate the solution to this problem and propose algorithms for computing the solution to our models. We also intend to perform a numerical study for realistic networks using the models and the analysis that we already performed in order to show how a numerical solution approach compares to an analytical one.

7 Appendix

Proof of Theorem 4:
This result relies on the classical method of characteristics in fluid dynamics. Along the characteristic line that passes through $(x_i, t+T_i)$ with slope $\frac{1}{(1-2b_i f_i)u_i^{max}}$, the solution $f_i(x_i, t+T_i)$ of equation (34) remains constant. Let $(0, t+s_i(x_i, t+T_i))$ be the point at which the characteristic line intersects the time axis. We know that:

$$f_i(x_i, t+T_i) = h_i^t(s_i(x_i, t+T_i)). \tag{91}$$

Using the slope information, we obtain that

$$\frac{1}{(1-2b_i f_i)u_i^{max}} = \frac{t+T_i-(t+s_i(x_i,t+T_i))}{x_i-0} = \frac{T_i-s_i(x_i,t+T_i))}{x_i}.$$

It follows that

$$s_i(x_i, t+T_i) = T_i - \frac{x_i}{(1-2b_i h_i^t(s_i(x_i,t+T_i)))u_i^{max}}. \tag{92}$$

Differentiating the above equation with respect to T_i, and rearranging terms, leads to the following expression of $\frac{\partial s_i(x_i,t+T_i)}{\partial T_i}$,

$$\frac{\partial s_i(x_i,t+T_i)}{\partial T_i} = \frac{1}{2b_i x_i \frac{dh_i^t(s_i(x_i,t+T_i))}{ds_i} u_i^{max}(1-2b_i h_i^t(T_i))^2+1}.$$

Using the method of characteristics, equations (34) and (35) possess a continuously differentiable solution f_i if and only if $\frac{\partial s_i(x_i,t+T_i)}{\partial T_i} > 0$. This gives rise to equation (42).

Therefore, we conclude that the ETT Model possesses a solution if and only if the derivative of $h_i^t(T_i)$ satisfies the boundedness condition

$$\frac{dh_i^t(T_i)}{dT_i} > -\frac{u_i^{max}}{2b_iL_i}2b_iL_i(1-2b_ih_i^t(T_i))^2.$$

□

Proof of Theorem 6:

(i) Note that condition (60) implies that condition (59) holds. Hence, the result of Theorem 3 applies. Therefore, the Quadratic PTT Model possesses a solution.

(ii) Equation (45) can be rewritten as

$$\frac{ds_i}{dx_i} = \frac{-\frac{b_i}{u_i^{max}}(B_is_i + C_is_i^2) - \frac{b_iA_i}{u_i^{max}}}{1 + 2x_i\frac{b_i}{u_i^{max}}(B_i + 2C_is_i)}.$$

Condition (60) allows us to consider a first order Taylor expansion of the denominator in the above equation. Introducing α_i, $i \in \{1, 2, 3\}$ as defined above, using a Taylor expansion and rearranging terms leads to the following linear ordinary differential equation:

$$\frac{ds_i}{dx_i} - (-\alpha_1 + (2\alpha_1^2 + \alpha_2\alpha_3)x_i)s_i = -\alpha_2(1 - 2\alpha_1x_i), \tag{93}$$

with $s_i(0) = 0$ as an initial condition.

The integrating term $I(x_i)$ of this equation (see Bender and Orszag (1978) for more details) can be written as

$$I(x_i) \quad = \quad e^{\int_0^{x_i}(\alpha_1+(2\alpha_1^2+\alpha_2\alpha_3)t)dt} = e^{\alpha_1x_i-(2\alpha_1^2+\alpha_2\alpha_3)\frac{x_i^2}{2}}.$$

Equation (61) then follows.

(iii) Let $N(x_i)$ denote the following function:

$$N(x_i) = e^{-\alpha_1x_i+(2\alpha_1^2+\alpha_2\alpha_3)\frac{x_i^2}{2}}\int_0^{x_i} e^{\alpha_1t-(2\alpha_1^2+\alpha_2\alpha_3)\frac{t^2}{2}}(-\alpha_1 + 2\alpha_1^2t)dt. \tag{94}$$

Tedious analysis leads to $N(0) = 0$, $N^{(1)}(0) = \alpha_1$, $N^{(2)}(0) = -3\alpha_1^2$ and $N^{(3)}(0) = 7\alpha_1^3 + 2\alpha_1^2 + \alpha_2\alpha_3$. From

$$N(x_i) = N(0) + N'(0)x_i + N^{(2)}(0)\frac{x_i^2}{2} + N^{(3)}(0)\frac{x_i^3}{6} + o(x_i^3), \tag{95}$$

equation (62) follows.

(iii) Equations (62) and (47), with α_i, $i \in \{1, 2, 3\}$, lead to equation (63).

□

Proof of Theorem 8:

(i) Note that equation (66) implies that equation (65) holds. Hence, the result of Theorem 7 applies. This implies that the Linear ETT Model possesses a solution.

(ii) Using equation (64), equation (92) can be rewritten as

$$s_i(x_i, t+T_i) = T_i - \frac{x_i}{u_i^{max}(1 - 2b_iA_i(t) - 2b_iB_i(t)s_i(x_i, t+T_i))}. \tag{96}$$

Equation (96) leads to a second degree polynomial in terms of s_i. Equation (66) justifies a first order Taylor expansion of the solution to this polynomial. This Taylor expansion leads to equation (67).

(iii) Using equations (64) and (67), we derive the following expression for the link flow rate functions f_i

$$f_i(x_i, t+T_i) = \frac{B_i(t)T_iu_i^{max} - B_i(t)x_i + A_i(t)u_i^{max}(1 - 2b_iA_i(t))}{u_i^{max}(1 - 2b_iA_i(t) + 2b_iB_i(t)T_i} \tag{97}$$

Replacing in equation (41) the link flow rate functions we found in equation (97) gives rise to the linear ordinary differential equation

$$\frac{dT_i}{dx_i} - (\frac{\theta_1(1 - 2b_iA_i(t))}{u_i^{max}})T_i = \frac{1 + b_iA_i(t) - \frac{\theta_1 x_i}{u_i^{max}}}{u_i^{max}}, \tag{98}$$

with $T_i(0, t) = 0$ as a boundary condition.

The integrating term $I(x_i)$ of this equation (see Bender and Orszag (1978) for more details) can be written as

$$I(x_i) = e^{-\frac{\theta_1(1-2b_iA_i(t))}{u_i^{max}}x_i}.$$

Using the boundary condition $T_i(0, t) = 0$, it follows that

$$T_i(x_i, t) = \frac{1}{I(x_i)} \int_0^{x_i} I(w) \frac{1 + b_iA_i(t) - \frac{\theta_1 w}{u_i^{max}}}{u_i^{max}} dw. \tag{99}$$

The calculation of the integral in equation (99), using an integration by parts, leads to equation (68).

(iv) To make the link travel time function T_i more tractable, we assume that condition (56) holds. Condition (56) allows us to perform a second order Taylor expansion of equation (68), which leads to a simpler form,

$$T_i(x_i, t) = (\theta_3 + \frac{\theta_1}{\theta_2(u_i^{max})^2})x_i + \theta_2\theta_3\frac{x_i^2}{2}.$$

Our definition of θ_i, $i \in \{1, 2, 3\}$ leads to equation (69).

□

Proof of Theorem 9:

In the non-separable case, equations (44) and (29) become

$$s_i(x_i, t+T_i) = \frac{T_iu_i^{max} - \frac{x_i}{J_i} - \frac{2b_ix_ih_i^t(s_i(x_i,t+T_i))}{J_i^3}}{u_i^{max}},$$

$$\text{and,} \quad \frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i} = \frac{\frac{1}{J_i} + \frac{b_if_i}{J_i^3}}{u_i^{max}}.$$

After introducing g_i, m_i and A_i, we derive the following two relations:

$$s_i = g_i - \frac{2x_i m_i(s_i)}{J_i^3} - \frac{b_i A_i}{u_i^{max} J_i^3} x_i, \tag{100}$$

$$\frac{dg_i}{dx_i} = \frac{m_i(s_i)}{J_i^3}, \tag{101}$$

with $s_i(0) = g_i(0) = m_i(0) = 0$.

From equations (100) and (101), it follows that

$$\begin{aligned}\frac{ds_i}{dx_i} &= \frac{dg_i}{dx_i} - \frac{2m_i(s_i)}{J_i^3} - \frac{2x_i \frac{ds_i}{dx_i} m_i'(s_i)}{J_i^3} - \frac{b_i A_i}{u_i^{max} J_i^3} \\ &= -\frac{m_i(s_i)}{J_i^3} - \frac{2x_i}{J_i^3}\frac{ds_i}{dx_i} m_i'(s_i) - \frac{b_i A_i}{u_i^{max} J_i^3}.\end{aligned}$$

Hence, $\frac{ds_i}{dx_i}(1 + \frac{2x_i m_i'(s_i)}{J_i^3}) = -\frac{m_i(s_i)}{J_i^3} - \frac{b_i A_i}{u_i^{max} J_i^3}$. The results of the theorem follow.

□

Proof of Theorem 10:
Since $h_i^t(s_i) = A_i + B_i s_i$, it follows that $m_i(s_i) = \frac{b_i B_i}{u_i^{max}} s_i$. Replacing in equation (86), we obtain

$$\frac{ds_i}{dx_i} = \frac{-\frac{b_i B_i}{u_i^{max}} s_i - \frac{b_i A_i}{u_i^{max}}}{J_i^3 + 2x_i \frac{b_i B_i}{u_i^{max}}}.$$

The above equation can be written as the following separable equation:

$$\frac{ds_i}{s_i + \frac{A_i}{B_i}} = -\frac{\frac{b_i B_i}{u_i^{max}} dx}{J_i^3 + 2x_i \frac{b_i B_i}{u_i^{max}}}. \tag{102}$$

Integrating both parts gives rise to $\frac{s_i + \frac{A_i}{B_i}}{\frac{A_i}{B_i}} = \frac{1}{(1+2\frac{b_i B_i}{u_i^{max} J_i^3} x_i)^{\frac{1}{2}}}$. Therefore it follows that

$$s_i = \frac{A_i}{B_i}\left(\frac{1}{(1 + 2\frac{b_i B_i}{u_i^{max} J_i^3} x_i)^{\frac{1}{2}}} - 1\right). \tag{103}$$

Using equations (87) and (88), we easily derive the results of the theorem.

□

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Figure 1: A Possible Profile of Approximated Departure Flow Rates

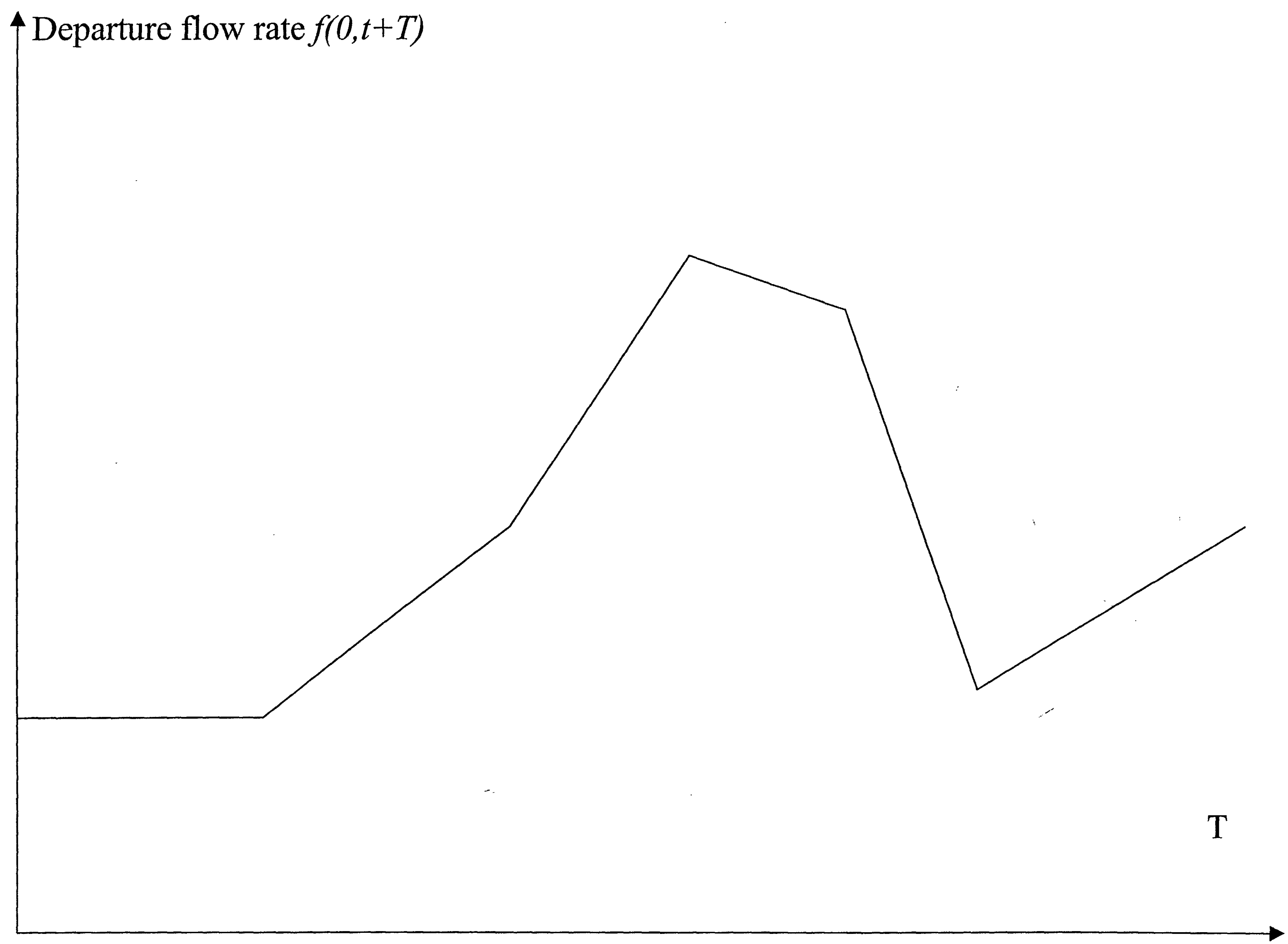

Departure flow rate f(0,t+T)
T

Figure 2: Example

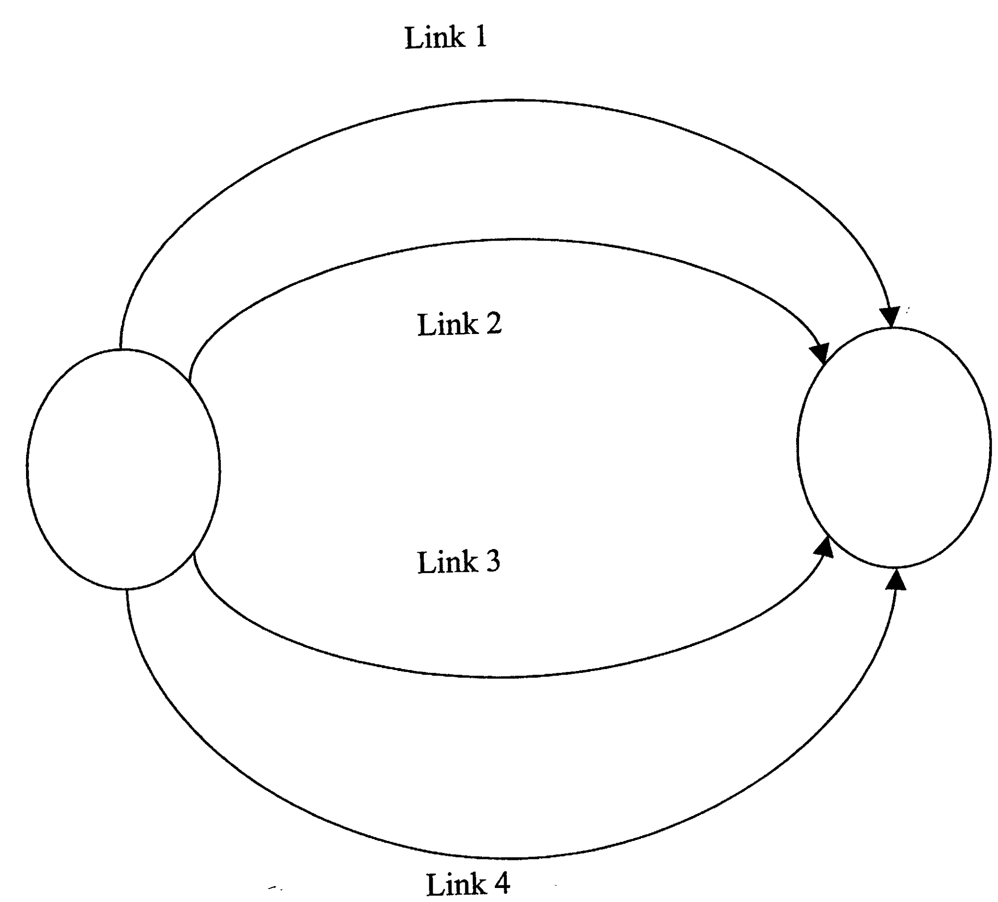

Link 1
Link 2
Link 3
Link 4

Figure 3: A Possible Profile of Approximated Departure Flow Rates

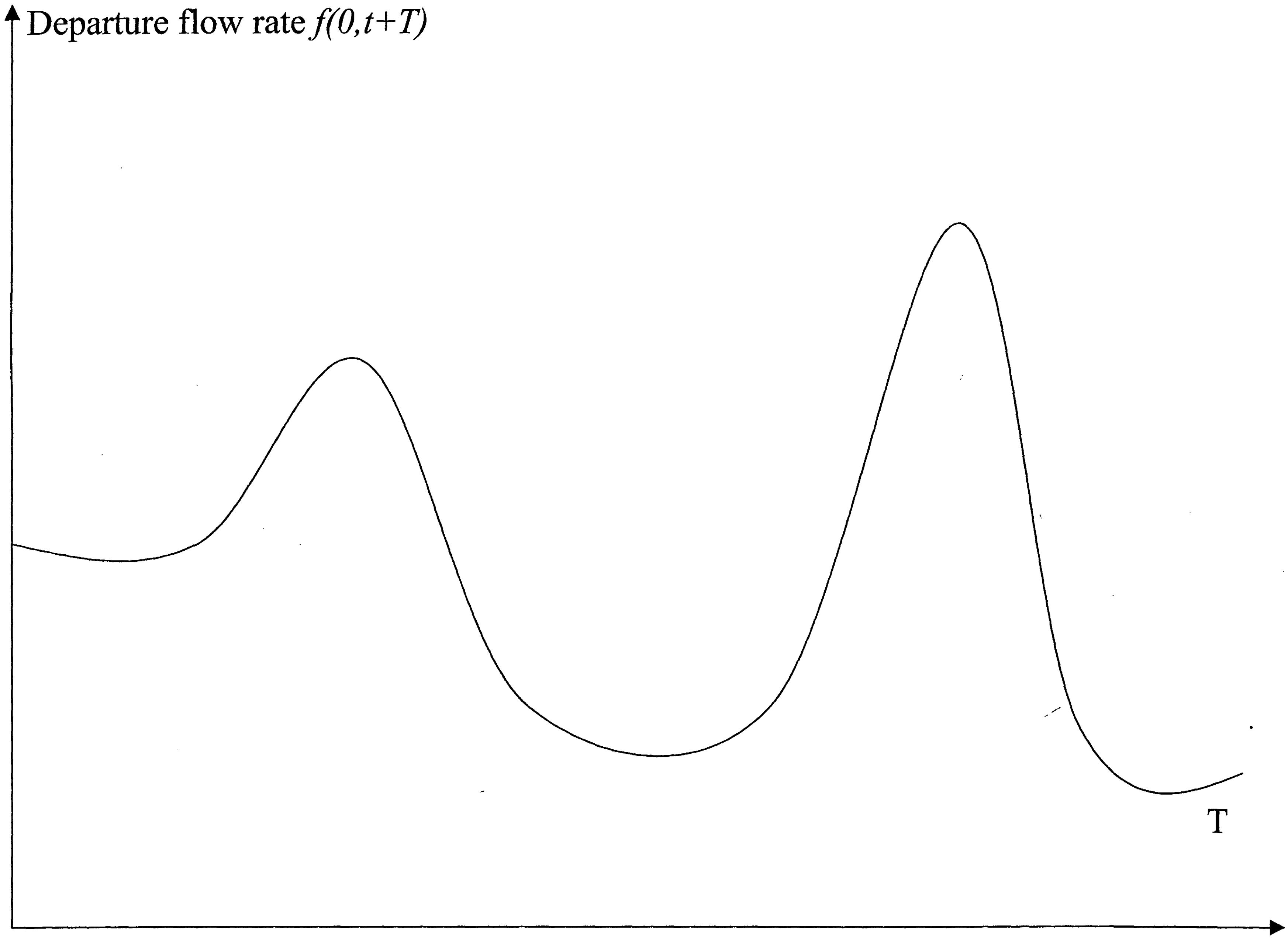

Departure flow rate f(0,t+T)
T

Figure 4: A Network with Two Links

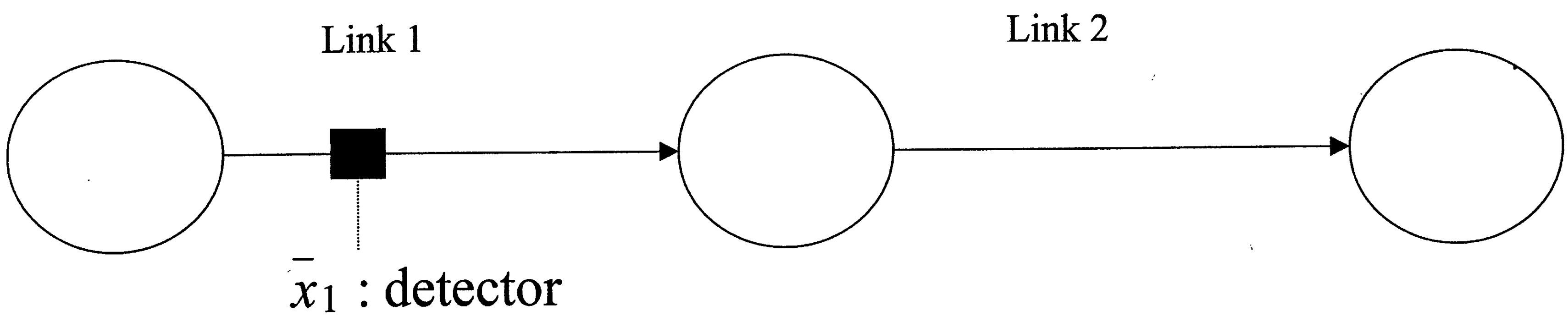
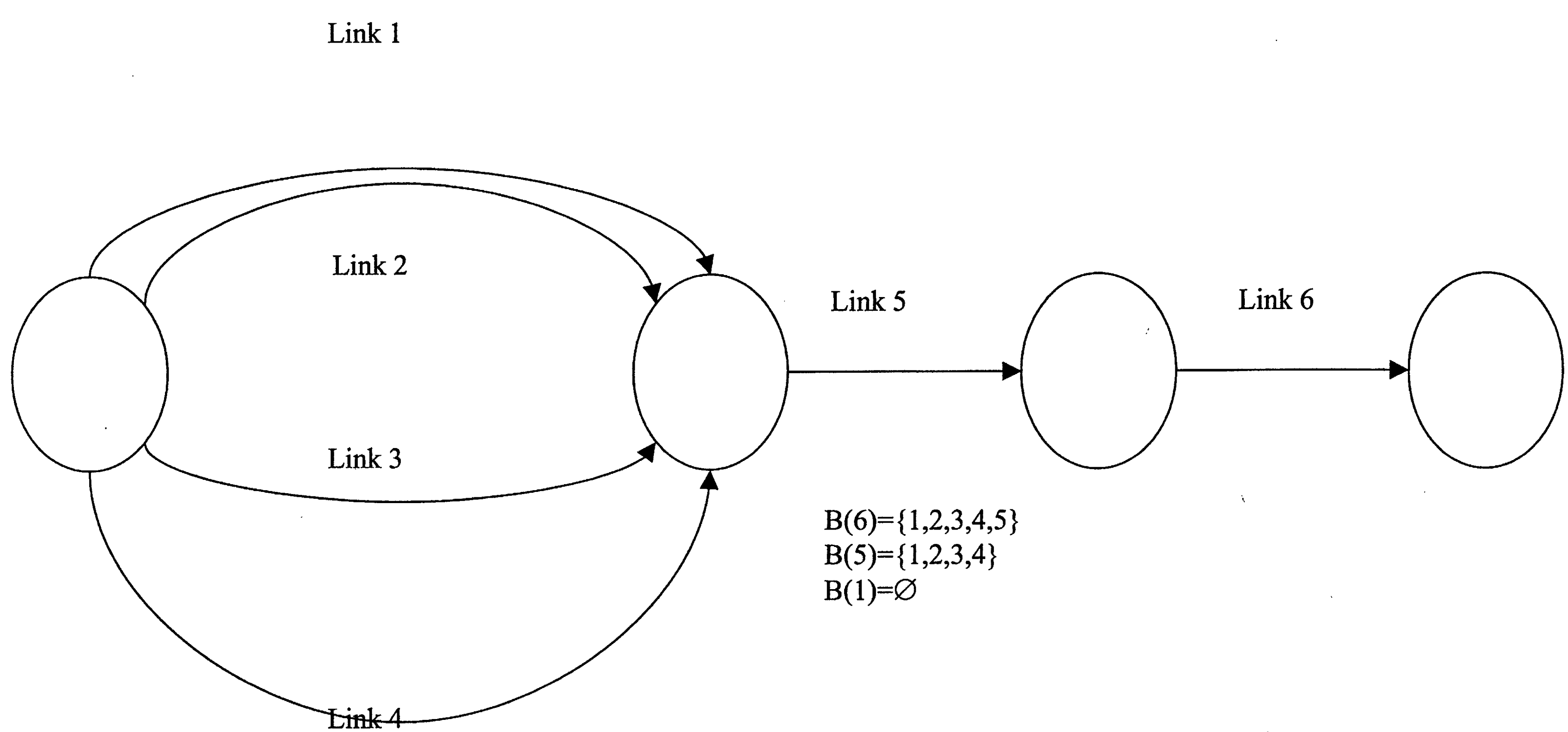

Link 1
Link 2
$\bar{x}_1$: detector

Figure 5: Set of Predecessors


Link 1
Link 2
Link 3
Link 4
Link 5
Link 6
B(6)={1,2,3,4,5}
B(5)={1,2,3,4}
B(1)=∅