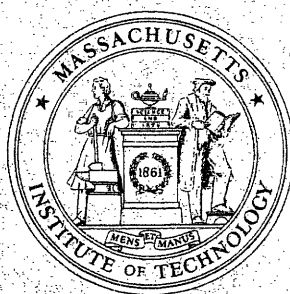


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**MASSACHUSETTS INSTITUTE
OF TECHNOLOGY**

**Convergent Duality for the Traveling
Salesman Problem**

by

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CONVERGENT DUALITY FOR THE TRAVELING SALESMAN PROBLEM

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Abstract

A constructive method is presented for optimizing exactly the Traveling Salesman Problem as a sequence of shortest route problems. The method combines group theoretic and Lagrangean relaxation constructions.

Key Words: Traveling Salesman Problem, Lagrangean relaxation, shortest route problem, generalized linear programming, group theory.

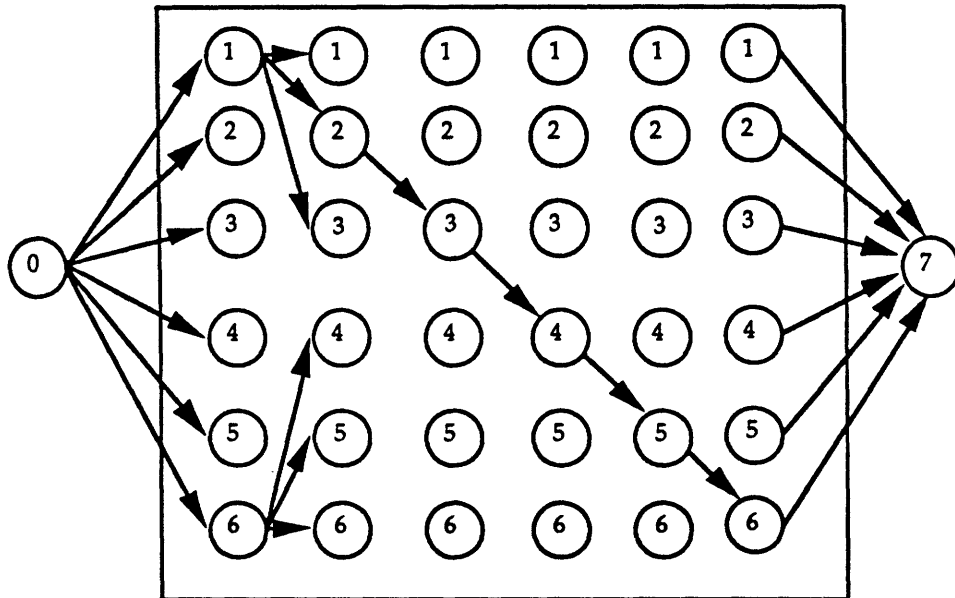
1. Introduction

In this paper, we present a constructive method for optimizing exactly the traveling salesman problem (TSP) by solving a sequence of shortest route problems. The method is a synthesis of two computational methods: (1) Lagrangean relaxation methods for formulating the TSP as a shortest route problem with side constraints (Picard and Queyranne [6], Houch et al [4], Fox et al [3]); and (2) group theoretic methods for resolving duality gaps in integer programs (Bell and Shapiro [1]). The two constructive procedures merge seamlessly because the group optimization problems required to resolve duality gaps are themselves shortest route problems.

The plan of this paper is as follows. Section 2 contains a statement of the TSP as a shortest route problem with side constraints and the ordinary relaxation of this problem. In Section 3, we discuss how the ordinary Lagrangean can be iteratively strengthened until some shortest route calculation yields a TSP tour, which by construction must be an optimal tour. A numerical example is presented in Section 4. A few concluding remarks are given in Section 5.

2. Statement of the Model and the Initial Dual

Consider a network with nodes $0, 1, 2, \dots, N$, directed arcs (i, j) for all i and $j \neq i$, and associated arc lengths c_{ij} . The arc lengths need not be symmetric; that is, we allow $c_{ij} \neq c_{ji}$. We define a new node with the label $N+1$ which is located identically with node 0 but corresponds to terminating there. Thus, the arc costs $c_{i,N+1} = c_{i0}$ for all i . The TSP can be expressed as the problem of sending one unit of flow from node 0 to node $N+1$ so that the distance traveled is minimized (this is a shortest route problem), but with the additional constraints that each city must be visited exactly once. Shortest route solutions that satisfy the additional constraints are called tours. The construction is illustrated in Figure 1 (without all possible arcs) for a 7 node problem.



Shortest Route Representation of TSP

Figure 1

Letting x_{ijt} ($i \neq j$) denote zero-one decision variables, the model can be stated as

Traveling Salesman Problem

$$V = \min \sum_{j=1}^N c_{0j} x_{0j0} + \sum_{t=1}^{N-1} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N c_{ij} x_{ijt} + \sum_{i=1}^N c_{i,N+1} x_{i,N+1,N} \quad (1)$$

Subject to

$$\sum_{j=1}^N x_{0j} = 1 \quad (2)$$

$$\left. \begin{aligned} -x_{0i} + \sum_{\substack{j=1 \\ j \neq i}}^N x_{ij} &= 0 && \text{for } i=1, \dots, N \\ -\sum_{\substack{k=1 \\ k \neq i}}^N x_{ki, t-1} + \sum_{\substack{j=1 \\ j \neq i}}^N x_{ijt} &= 0 && \text{for } i=1, \dots, N \\ &&& t=2, \dots, N-1 \\ -\sum_{\substack{k=1 \\ k \neq i}}^N x_{ki, N-1} + x_{i, N+1, N} &= 0 && \text{for } i=1, \dots, N \end{aligned} \right\} \quad (3)$$

$$\sum_{i=1}^N x_{i, N+1, N} = 1 \quad (4)$$

$$x_{0j} + \sum_{t=1}^{N-1} \sum_{\substack{i=1 \\ i \neq j}}^N x_{ijt} = 1 \quad \text{for } j=1, \dots, N \quad (5)$$

$$x_{ijt} = 0 \text{ or } 1 \quad (6)$$

The constraints (2) – (4) describe the shortest route problem. The constraints (5) require that each node be visited exactly once.

The constraints (5) are the ones that make the model combinatorially complex. The traditional Lagrangean relaxation approach in such a circumstance is to dualize on these constraints. Letting π_j denote the dual variable on row j in (5), and letting π denote the N -vector of these variables, the result is

$$L^0(\pi) = \sum_{j=1}^N \pi_j + \min \sum_{j=1}^N (c_{0j} - \pi_j) x_{0j0} + \sum_{t=1}^{N-1} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N (c_{ij} - \pi_j) x_{ijt} + \sum_{i=1}^N c_{i,N+1} x_{i,N+1,N} \quad (7)$$

subject to: (2), (3), (4), (6)

The Lagrangean calculation (7) is a shortest route problem in which the arc lengths c_j are adjusted to $c_{ij} - \pi_j$ to reflect the relative economy or diseconomy of visiting node j .

Let $x_{ijt}(\pi)$ denote an optimal solution to (6). It is easy to show that this solution is optimal in TSP if

$$x_{0j0}(\pi) + \sum_{t=1}^{N-1} \sum_{\substack{i=1 \\ i \neq j}}^N x_{ijt}(\pi) = 1 \quad \text{for } j=1, \dots, N \quad (8)$$

Thus, if the Lagrangean L^0 , or any of the strengthened Lagrangeans derived in the next section, ever yields an optimal shortest route path that is a TSP tour, such a tour is optimal. To try to select a $\pi \in \mathbb{R}^N$ that will cause this fortuitous outcome, we are led to consider the dual problem

$$\begin{aligned} D^0 &= \max L^0(\pi) && \text{TSPD(0)} \\ &\text{s.t. } \pi \in \mathbb{R}^N && (9) \end{aligned}$$

Since $L^0(\pi) \leq V$ for all $\pi \in \mathbb{R}^N$, we have $D^0 \leq V$. It is also easy to show that if π produces an optimal solution to TSP, π is optimal in the dual TSP (9); moreover, $D^0 = V$. Thus, the indicated constructive approach is first to compute π^* that is optimal in TSPD(0), and then seek $x_{ijt}(\pi^*)$ that satisfies the degree constraints (8).

The deficiency of the Lagrangean approach described thus far is that we cannot guarantee the existence of any π such that $x_{ijt}(\pi)$ satisfies (8). This deficiency is equivalent to the condition that $D^0 < V$, in which case we say there is a duality gap. The main purpose of this paper is to present a constructive method for

resolving duality gaps, thereby allowing the Lagrangean shortest route calculations to continue until a solution satisfying (8) is computed.

The point of departure for the new constructive method is the generalized linear programming algorithm (see [5] or [7]) for solving (9). Let

$$x_r = (x_{ijtr}) \quad \text{for } r=1, \dots, R_0$$

represent an enumeration of the x_{ijtr} satisfying (2), (3), (4), (6). These correspond to the set of feasible paths in the shortest route network. Let

$$C_r = \sum_{j=1}^N c_{0j} x_{0j0r} + \sum_{t=1}^{N-1} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N c_{ij} x_{ijtr} + \sum_{i=1}^N c_{i,N+1} x_{i,N+1,N,r} \quad (10)$$

and

$$d_{jr} = x_{0j0r} + \sum_{t=1}^{N-1} \sum_{\substack{i=1 \\ i \neq j}}^N x_{ijtr} \quad \text{for } j=1, \dots, N \quad (11)$$

It is well known that TSPD(0) is equivalent to (see [5])

MLP (0)

$$\begin{aligned} D^0 &= \min \sum_{r=1}^{R_0} C_r \Omega_r \\ \text{s.t. } &\sum_{r=1}^{R_0} d_{jr} \Omega_r = 1 \quad \text{for } j=1, \dots, N \\ &\sum_{r=1}^{R_0} \Omega_r = 1 \\ &\Omega_r \geq 0 \end{aligned} \quad (12)$$

This master LP has an enormous number of columns. Starting with a small subset, new columns can be generated iteratively by computing $L^0(\pi)$ where π is the vector

of optimal LP shadow prices on the degree rows in (12). This computation will either establish that π is optimal in TSPD(0), or uncover a new column to add to the Master (12). Generalized linear programming will compute an optimal π after generating a finite number of new columns since there are only a finite number of paths in the shortest route network. Finally, we note that MLP (0) is equivalent to the LP model that results if we relax the zero-one integrality restriction (6) and allow the x_{ijt} to take on continuous values between 0 and 1. In particular, any feasible solution to MLP (0) corresponds

$$x_{ijt} = \sum_{r=1}^{R_0} x_{ijtr} \Omega_r$$

which in turn is a feasible solution to the ordinary LP relaxation.

Our interest in this paper is how to proceed when optimizing MLP(0) and therefore TSPD(0) fails to find an optimal solution to TSP. This is the subject of the following section.

3. Iterative Dual Model Construction

We begin with a brief discussion, which is merely algebraic and non-constructive, about how the Lagrangean $L^0(\pi)$ can be strengthened if it fails to yield an optimal solution to TSP. The remainder of this section is devoted to constructive procedures for selecting the algebraic groups from which the stronger Lagrangeans are derived. The reader is referred to Shapiro [7] for background.

The degree constraints (5) are equations defined over Z^N , the infinite abelian group of integer N-vectors under ordinary addition. These equations can be aggregated or relaxed as follows. Choose any finite abelian group G and an homomorphism ψ mapping Z^N onto G. Let

$$\varepsilon_j \equiv \psi(e_j) \quad \text{for } j=1, \dots, N$$

where e_j is the j-th unit vector in Z^N and " \equiv " denotes equality in G. The aggregation of (5) is achieved by multiplying each equation by ε_j and summing. The result is

$$\sum_{j=1}^N \varepsilon_j x_{0j0} + \sum_{j=1}^N \sum_{t=1}^{N-1} \sum_{\substack{i=1 \\ i \neq j}}^N \varepsilon_j x_{ijt} \equiv \beta \tag{13}$$

where

$$\beta \equiv \sum_{j=1}^N \varepsilon_j .$$

It is straightforward to demonstrate that any zero-one solution x_{ijt} that satisfies (5) satisfies (13)

The new Lagrangean is

$$L^\Psi(\pi) = \sum_{j=1}^N \pi_j + \min \sum_{j=1}^N (c_{0j} - \pi_j) x_{0j0} + \sum_{t=1}^{N-1} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N (c_{ij} - \pi_j) x_{ijt} + \sum_{i=1}^N c_{i,N+1} x_{i,N+1,N} \quad (14)$$

subject to: (2), (3), (4), (13), (6)

In particular, the Lagrangean has been strengthened in the sense that $L^\Psi(\pi) \geq L^0(\pi)$ for any $\pi \in \mathbb{R}^N$ because we have added the constraint (13). The Lagrangean L^Ψ is still a shortest route problem, but over an expanded network. Formally, the nodes in the network comprise the starting and ending nodes 0 and $N+1$ as before, plus nodes labelled i, t, α for $i=1, \dots, N$; $t=1, \dots, N$; and for all $\alpha \in G$. The arcs in this network are:

- | | | | |
|------|--|------------------------------|--|
| (i) | $(0, (j, 1, \varepsilon_j))$ | with length $c_{0j} - \pi_j$ | for $j=1, \dots, N$ |
| (ii) | $((i, t, \alpha), (j, t+1, \alpha + \varepsilon_j))$ | with length $c_{ij} - \pi_j$ | for $i=1, \dots, N$
$j=1, \dots, N$ and $j \neq i$
$t=1, \dots, N-1$ |

and

- | | | | |
|-------|------------------------|-------------------------|---------------------|
| (iii) | $((i, N, \beta), N+1)$ | with length $c_{i,N+1}$ | for $i=1, \dots, N$ |
|-------|------------------------|-------------------------|---------------------|

Note that the terminal node $N+1$ can be reached only from nodes (i, N, β) for some i . An example of the expanded network is given in Figure 3.

We turn now to constructive procedures that rationalize the choice of G and Ψ . The construction begins with the assumption that the ordinary Lagrangean relaxation $L^0(\pi)$ and associated dual problem $TSPD(0)$, which we optimize by generalized linear programming and the simplex method, fails to find an optimal solution to the original TSP. The construction is based on the observation that

MLP(0) with the additional constraint that the variables $\Omega_r = 0$ or 1 solves TSP. With this constraint and all feasible columns included, the model would select a shortest route path that visits nodes $1,2,\dots,N$ exactly once before returning to node 0 (that is, node $N+1$).

Thus, viewing MLP(0) as the linear programming relaxation of the integer program that we wish to solve, we apply general purpose integer programming methods based on number (abelian group) theory to the relaxation if it fails to produce an optimal solution to TSP. The following theorem characterizes the possible outcomes of the dual analysis of the previous section.

Theorem 1: Suppose MLP(0) is optimized using the simplex method. Let Ω_r^* for some $r=1,\dots,R$ denote an optimal solution, where $R \leq R_0$ is the number of columns generated by generalized linear programming in solving TSPD(0).

Case (a): If $\Omega_s^* = 1$ for some s , (x_{ijts}) is an optimal TSP tour.

Case (b): If more than one Ω_r^* is positive, say $\Omega_r^* > 0$ for $r = 1,2,\dots,K$ with $K \geq 2$, (x_{ijtr}) is not a feasible TSP tour for $r = 1,2,\dots,K$.

Proof: The proof is the same as Lemma 2 of [1].

The ordinary Lagrangean relaxation and analysis of the previous section fails to solve TSP under case (b) of Theorem 1. This probably means that $D^0 < V$ and we need to somehow strengthen the Lagrangean to close up the gap. It may be, however, that $D^0 = V$ but we have not yet identified a TSP tour with length D^0 via the linear program MLP(0) and the implied column generation procedure. The constructive procedure we are about to present will in any event resolve the difficulty.

Suppose Case (b) of Theorem 1 obtains. The method proceeds by considering the $(N+1) \times K$ matrix

$$Q = \left(\begin{array}{c} \left[\begin{array}{c} d_{11} \\ \vdots \\ d_{N1} \\ 1 \end{array} \right] \quad \left[\begin{array}{c} d_{12} \\ \vdots \\ d_{N2} \\ 1 \end{array} \right] \quad \left[\begin{array}{c} d_{1K} \\ \vdots \\ d_{NK} \\ 1 \end{array} \right] \end{array} \right)$$

This integer matrix induces a unique abelian group G_1 and a homomorphism ψ_1 mapping Z^{N+1} , the infinite abelian group of integer $(N+1)$ vectors under ordinary addition, onto G_1 . The mapping can be characterized by computing

$$\psi_1(e_j) \equiv \varepsilon_j \quad \text{for } j=1, \dots, N+1$$

since for any $a \in Z^{N+1}$

$$\psi_1(a) \equiv \sum_{j=1}^{N+1} a_j \varepsilon_j$$

The mapping has the property that

$$\sum_{j=1}^N d_{jr} \varepsilon_j + \varepsilon_{N+1} \equiv 0 \quad \text{for } j=1, \dots, K$$

whereas

$$\sum_{j=1}^{N+1} \varepsilon_j + \varepsilon_{N+1} \neq 0$$

since the computed optimal solution for $MLP(0)$ is not integer. Letting

$$\beta \equiv \sum_{j=1}^N \varepsilon_j, \text{ we have}$$

$$\sum_{j=1}^N d_{jr} \varepsilon_j \neq \beta \quad \text{for } j=1, \dots, K \quad (15)$$

The group G_1 and the mapping can be computed by using the Smith reduction procedure (see [7]) to diagonalize Q .

We use ψ_1 in strengthening the Lagrangean to the form (14), thereby creating a stronger dual problem. By (15), the new Lagrangean is strictly stronger than L^0 in the sense that each shortest route path (x_{ijtr}) corresponding to an infeasible tour that was given a positive weight $\Omega_r^* > 0$ in $MLP(0)$ is infeasible in the Lagrangean L^{ψ_1} .

There is no guarantee, however, that the dual problem TSPD(1) analogous to (9) defined for L^{Ψ^1} will yield an optimal TSP tour. In particular, we may find that case (b) of Theorem 1 obtains for the Master linear program MLP(1) that solves TSPD(1). In such an event, we repeat the construction just outlined to derive a group G^2 , a homomorphism Ψ^2 , and a still stronger Lagrangean L^2 . Equation (13) is now written over the abelian group $G^1 \oplus G^2$. If necessary, the procedure can be applied an arbitrary number of times. Since we eliminate at least two shortest route paths each time we strengthen the Lagrangean and since the number of such paths is finite, our constructive procedure must terminate finitely with a Lagrangean, a vector $\pi \in \mathbb{R}^N$, and a solution $x_{ijt}(\pi)$ such that (8) holds; that is, such that $x_{ijt}(\pi)$ is an optimal tour. Thus, we have proved the following.

Theorem 2: The constructive dual procedure for the TSP will, in a finite number of increasingly strong dual model constructions, produce a shortest route network and a vector of dual variables $\pi \in \mathbb{R}^N$ such that the calculated shortest route path (Lagrangean optimization) will be an optimal TSP tour.

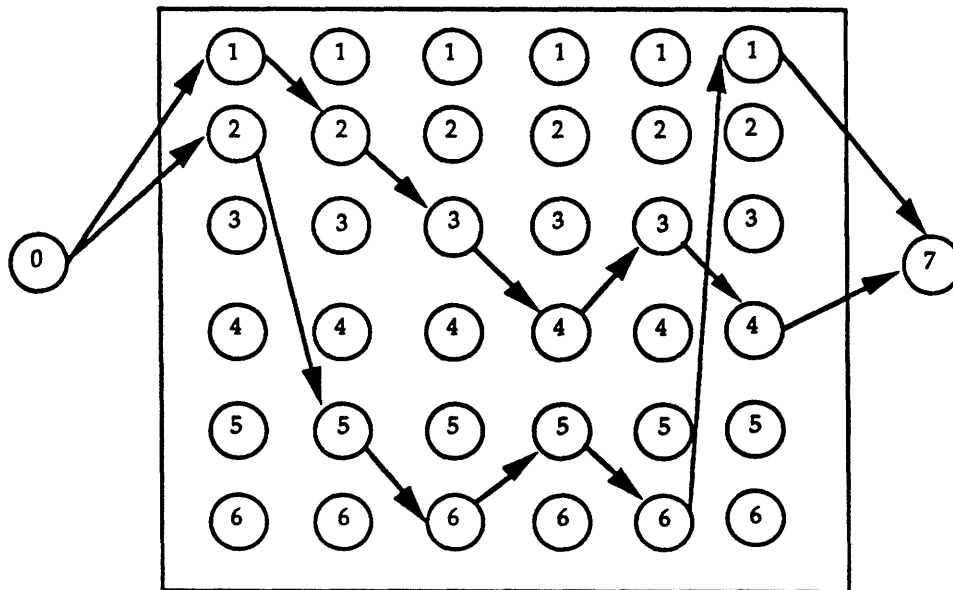
4. Numerical Example

We illustrate the procedure for a 7 city TSP with the distance matrix shown in Table 1.

		To City					
		1	2	3	4	5	6
From City	0	52	80	120	71	218	178
	1		57	116	90	188	139
	2			61	61	121	101
	3				58	132	115
	4					176	154
	5						62

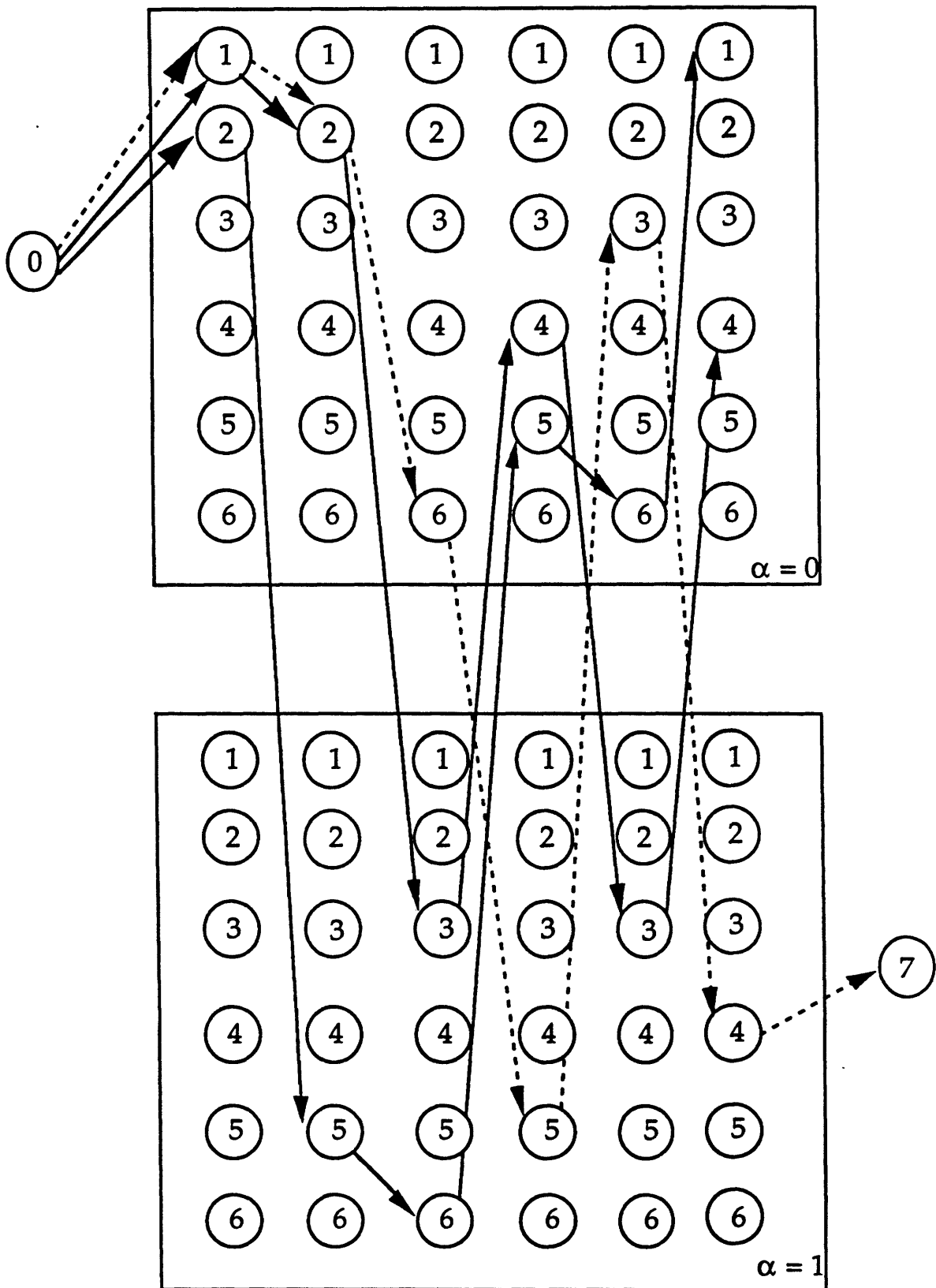
The initial dual problem TSPD(0) produces two optimal shortest routes paths, as depicted in Figure 2, that correspond to optimal columns in MLP(0), each with weight equal to 1/2. These columns induce the abelian group $G_1 = \{0, 1\}$ and the homomorphism ψ_1 with the property $\psi_1(e_1) \equiv \psi_1(e_2) \equiv \psi_1(e_6) \equiv 0$ and $\psi_1(e_3) \equiv \psi_1(e_4) \equiv \psi_1(e_5) \equiv 1$.

The extended shortest route network using G_1 and ψ_1 is shown in Figure 3. The solid lines represent the shortest route tours of Figure 2 except for the final transition to node 7 which is not allowed since these paths are drawn to nodes at $t = 6$ on the replicate $\alpha = 0$. By contrast, the optimal TSP tour $(0, 1, 2, 6, 5, 3, 4, 7)$ is depicted with a dotted line. The transition to node 7 from the replicate $\alpha = 1$ is allowed according to our choice of G_1 and ψ_1 .



Shortest Route Paths Producing a Duality Gap

Figure 2



Extended Shortest Route Representation of TSP

Figure 3

5. Conclusions

The constructive method presented in this paper computes an optimal TSP tour by solving a finite sequence of shortest route problems. Small computational experiments performed to date with the method are insufficient for judging the ultimate effectiveness of the method. The experiments do indicate, however, an interesting array of constructive ambiguities whose study might reveal new insights into the TSP. For example, the underlying group structures change radically if the degree constraints (8) are written as

$$x_{0j0}(\pi) + \sum_{t=1}^{N-1} \sum_{\substack{i=1 \\ i \neq j}}^N x_{ijt}(\pi) \geq 1 \quad \text{for } j=1, \dots, N$$

These inequalities will, of course, be satisfied as equalities by any TSP tour.

Another ambiguity is the selection of columns from the linear programming basis in the generalized linear programming Master from which to derive the group and associated homomorphism. The TSP has been optimized exactly if any column, even one with zero weight (that is, a degenerate column), corresponds to a TSP tour. Otherwise, all columns in the basis with weight equal to zero can be included in the matrix form from which the group and the homomorphism are derived since they are not tours and might as well be eliminated in the strengthened Lagrangean.

The method presented here was motivated in part by column generation applications involving the TSP. One such application to vehicle routing and scheduling is given in [2]. As we have shown, integer programming column generation involving TSP's can be made theoretically exact by our method. The practical use of the method awaits further experimentation.

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