Analyzing Multi-Objective Linear Programs by Lagrange Multipliers

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Introduction

A new method for multi-objective optimization of linear programs based on Lagrange multiplier methods is developed. The method resembles, but is distinct from, objective function weighting and gaol programming methods. A subgradient optimization algorithm for selecting the multipliers is presented and analyzed. The method is illustrated by its application to a model for determining the weekly re-distribution of railroad cars from excess supply areas to excess demand areas.

Key Words: Programming (Multiple Criteria; Subgradient)

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Introduction

In this paper, we present a new approach to analyzing multi-objective linear programs based on Lagrangean techniques. The approach resembles classical methods for using non-negative weights to combine multiple objectives into a single objective function. The weights in our construction, however, are Lagrange multipliers whose selection is determined iteratively by reference to targeted values for the objectives. Thus, the Lagrangean approach also resembles goal programming due to the central role played by the target values (goals) in determining the values of the multipliers. The reader is referred to Steuer [1986] for a review of weighting and goal programming methods in multi-objective optimization.

The plan of this paper is as follows. In the next section, we formulate the multi-objective linear programming model as an Existence Problem. We then demonstrate how to convert the Existence Problem to an optimization problem by constructing an appropriate Lagrangean function. The dual problem of minimizing the Lagrangean is related to finding a solution to the Existence Problem, or proving it has no feasible solution. In the following section, we provide an economic interpretation of efficient solutions generated by the method. In the section after that, we present a subgradient

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** The approach developed in this paper was suggested by Fred Shepardson.

optimization algorithm for analyzing the Existence Problem by optimizing the implied dual problem. This algorithm provides a sequence of solutions which converge to a solution to the Existence Problem if the problem has a solution. An illustrative example is given, and the paper concludes with a brief discussion of areas of future research and applications.

Statement of the Existence Problem and Lagrangean Formulation

We formulate the multi-objective linear programming model as the Existence Problem: Does there exist an $x \varepsilon R^n$ satisfying

$$Ax \le b \tag{1}$$

$$c_k x \ge t_k \quad \text{for } k = 1, 2, \ldots, K \tag{2}$$

$$x \ge 0 \tag{3}$$

In this formulation, the matrix A is $m \text{x} n$, each c_k is a $1 \text{x} n$ objective function vector, and each t_k is a target value for the kth objective function. We assume for convenience that the set

$$X = \{x \mid Ax \le b, \; x \ge 0\} \tag{4}$$

is non-empty and bounded. We let x_r for $r=1, \ldots, R$ denote the extreme points of X . For future reference, we define the $K \text{x} n$ matrix

$$C = \begin{pmatrix} c_1 \\ \bullet \\ \bullet \\ \bullet \\ c_K \end{pmatrix}$$

and the $K \text{x} 1$ vector t with coefficients t_k . We say that the vector t in the Existence Problem is attainable if there is an $x \varepsilon X$ such that $Cx \ge t$; otherwise, the vector t is unattainable.

Letting $\pi = (\pi_1, \pi_2, \ldots, \pi_K) \geq 0$ denote Lagrange multipliers associated with the K objectives, we price out the constraints (2) to form the Lagrangean

$$\begin{aligned} L(\pi) = -\pi t + \text{maximum } \pi C x \\ \text{Subject to} \quad Ax \leq b \\ x \geq 0 \end{aligned} \tag{5}$$

We let $x(\pi)$ denote an optimal solution to (5). The following definition and result, which is well known and therefore stated without proof, characterizes these solutions.

DEFINITION: The solution $x \varepsilon X$ is efficient (undominated) if there does not exist a $y \varepsilon X$ such that $Cy \geq Cx$ with strict inequality holding for at least one component.

THEOREM 1: Any solution $x(\pi)$ that is optimal in the Lagrangean (5) is efficient if $\pi_k > 0$ for $k=1, 2, \ldots, K$.

We say that the solution $x(\pi)$ spans the target vector $Cx(\pi)$; if π has all positive components, $x(\pi)$ is an efficient solution for the Existence Problem with this target vector.

The two possible outcomes for the Existence Problem (the Problem is either feasible or infeasible) can be analyzed simultaneously by optimizing the Multi-Objective Dual Problem (MODP)

$$D = \text{minimize } L(\pi)$$

$$\text{Subject to} \qquad \pi \geq 0 \tag{6}$$

THEOREM 2: If the Existence Problem has a feasible solution, $L(\pi) \geq 0$ for all $\pi \geq 0$ and $D = 0$. If the Existence Problem has no feasible solution, there exists a $\pi^* \geq 0$ such that $L(\pi^*) < 0$ implying $L(\theta\pi^*) \rightarrow -\infty$ as $\theta \rightarrow +\infty$ and therefore $D = -\infty$.

Proof:

It is easy to show that $L(\pi) \geq 0$ for all $\pi \geq 0$ when there exists an $\hat{x} \geq 0$ satisfying $A\hat{x} \leq b$ and $C\hat{x} \geq t$. For then, we have $\pi(C\hat{x} - t) \geq 0$ for all $\pi \geq 0$ which implies $L(\pi) \geq 0$ since $L(\pi) \geq \pi(C\hat{x} - t)$.

To complete the proof, we consider the phase one linear program for evaluating the existence problem

$$W = \text{minimize} \sum_{k=1}^{K} s_k$$

$$\text{Subject to} \qquad Ax \leq b \tag{7}$$

$$c_k x + s_k \geq t_k \qquad \text{for } k=1, \ldots, K$$

$$x \geq 0, \; s_k \geq 0 \qquad \text{for } k=1, \ldots, K \; .$$

The linear programming dual to this problem is

$$W = \text{maximize} \; -\sigma b + \pi t$$

$$-\sigma A + \pi C \leq 0 \tag{8}$$

$$0 \leq \pi_k \leq 1 \qquad \text{for } k=1, \ldots, K$$

$$\sigma \geq 0 \; , \; \pi \geq 0$$

Let x^* , s^* , denote an optimal solution to the primal problem (7) found by the simplex method, and let σ^* , π^* denote an optimal solution to the dual problem (8).

By linear programming duality, we have

$$W = \sigma^* b + \pi^* t \tag{9}$$

$$\sigma^*\left(b - Ax^*\right) = 0 \tag{10}$$

$$\left(-\sigma^* A + \pi^* C\right)x^* = 0 \tag{11}$$

Thus,

$$\begin{aligned} W &= -\sigma^* A x^* + \pi^* t \\ &= -\pi^* C x^* + \pi^* t \\ &= -\pi^*\left(C x^* - t\right) \end{aligned}$$

where the first equality follows from (9) and (10), and the second equality from (11). If the Existence Problem has a feasible solution, we have $W = 0$ and $L\left(\pi^*\right) = \pi^*\left(Cx^* - t\right) = 0$. This completes the first part of the proof because π^* must be optimal in (6) and $D = 0$.

If the Existence Problem does not have a solution, we have $W > 0$, or

$$\pi^*\left(Cx^* - t\right) = -W < 0 \tag{12}$$

Our next step is to show $L\left(\pi^*\right) = -W$.

To this end, consider any $\hat{x} \geq 0$ satisfying $A\hat{x} \leq b$. We have

$$\left(-\sigma^* A + \pi^* C\right)\hat{x} \leq 0$$

since $-\sigma^* A + \pi^* C \leq 0$, implying

$$\pi^* C\hat{x} \leq \sigma^* A\hat{x} \leq \sigma^* b$$

where the second inequality follows because $\sigma^* \geq 0$. Adding $-\pi^* t$ to both sides of the inequality, we obtain

$$\pi^* (C\hat{x} - t) \leq \sigma^* b - \pi^* t = -W = \pi^* \left(Cx^* - t\right) \tag{13}$$

Thus, $L\left(\pi^*\right) = \pi^* \left(Cx^* - t\right) < 0$. Moreover, (13) implies for any $\Theta > 0$ that

$$L\left(\Theta\pi^*\right) = -\Theta W = \left(\Theta\pi^*\right)\left(Cx^* - t\right) \geq \Theta\pi^* (C\hat{x} - t) .$$

This establishes the desired result in the case when the Existence Problem is infeasible. ■

It is well known and easy to show that L is a piecewise linear convex function that is finite and therefore continuous on R^K. Although L is not everywhere differentiable, a generalization of the gradient exists everywhere. A K-vector τ is called a <u>subgradient</u> of L at π if

$$L\left(\hat{\pi}\right) \geq L(\pi) + \left(\hat{\pi} - \pi\right)\tau \qquad \text{for all } \hat{\pi}$$

A necessary and sufficient condition for π to be optimal in (MODP) is that there exist a subgradient τ of L at π such that

$$\tau_k = 0 \qquad \text{if } \pi_k > 0$$

$$\tau_k \geq 0 \qquad \text{if } \pi_k = 0 \tag{14}$$

Algorithms for determining an optimal π are based in part on exploiting this condition characterizing optimality. The K-vectors $Cx(\pi) - t$ are the subgradients with which we will be working.

It is clear from the definition of the Lagrangean that $L(\lambda \pi) = \lambda L(\pi)$ for any $\pi \geq 0$ and any $\lambda \geq 0$; that is, L is homogeneous of degree one. Equivalently, for each extreme point $x_r \in X$, the set

$$\{\pi \mid \pi \geq 0 \text{ and } L(\pi) = \pi(Cx_r - t)\}$$

is a cone. The geometry is depicted in Figure 1 where $\tau_r = Cx_r - t$ for $r = 1, 2, 3, 4$.

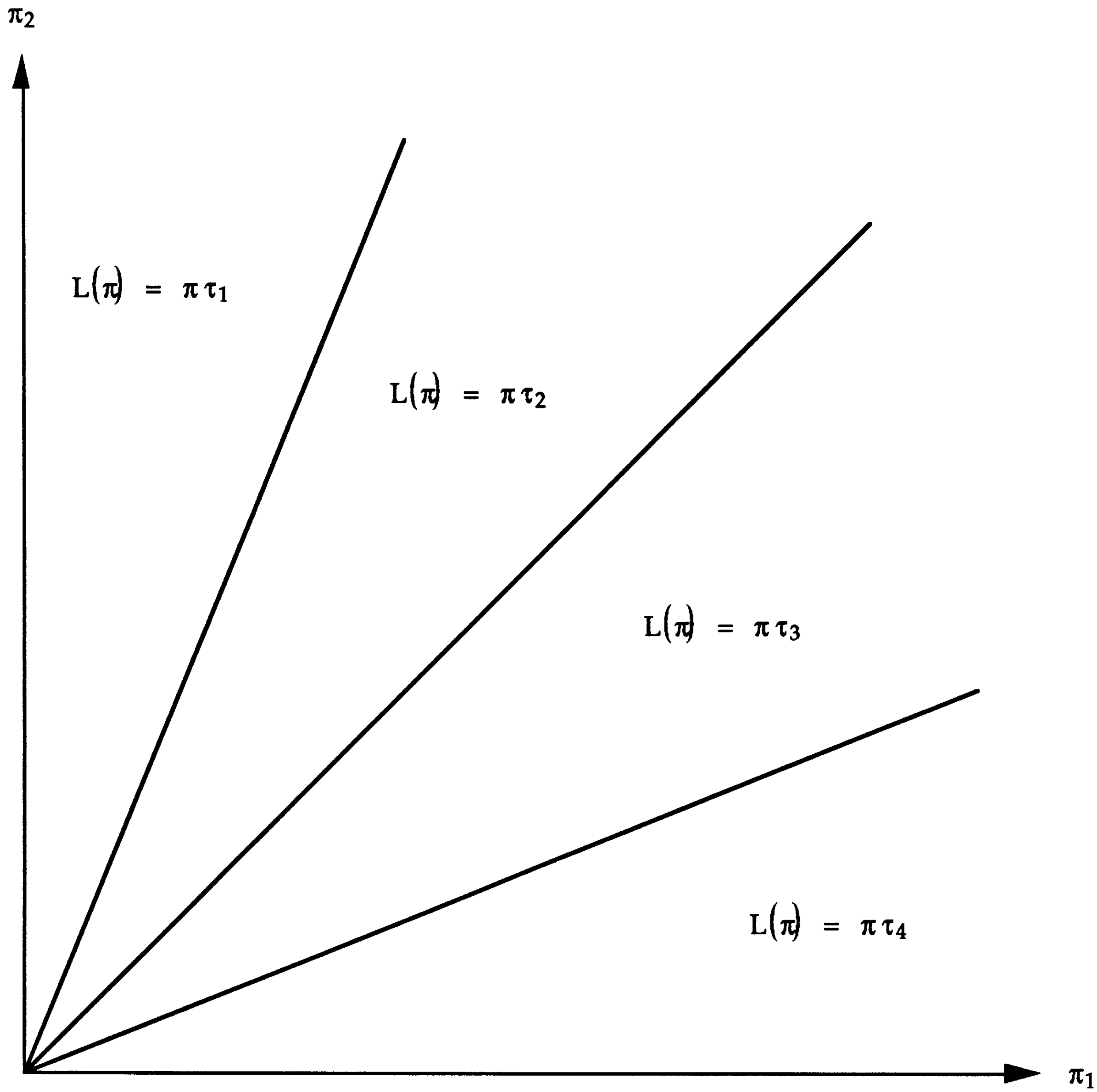
The implication of this structure to analysis of the Existence Problem is that we could restrict the vectors π in the MODP (6) to lie on the simplex

$$S = \left\{ \pi \,\middle|\, \sum_{k=1}^{K} \pi_k = 1, \pi_k \geq 0 \right\}.$$

Theorem 2 can be re-stated as

Corollary 1: Suppose the multipliers π are chosen to lie on the simplex S. If the Existence Problem has a feasible solution, $L(\pi) \geq 0$ for all $\pi \in S$ and $D = 0$. If the Existence Problem has no feasible solution, there exists a $\pi^* \in S$ such that $L(\pi^*) < 0$.

For technical reasons, we choose not to explicitly add this constraint to the MODP. When reporting results, however, we will normalize the π_k so that $\sum_{k=1}^{K} \pi_k = 1$. The normalization makes it easier for the decision-maker to compare solutions.



Conic Structure of the Lagrangean

Figure 1

Of course, the reader may have asked him(her)self why we need the Lagrangean method when we can test feasibility of the Existence Problem simply by solving the phase one linear program (7). The answer is that the decision-maker is usually unsure about the specific target values t_k that he (she) wishes to set as goals. The Lagrangean formulation and the algorithm discussed in the next section allow him (her) to interactively generate efficient solutions (assuming all the $\pi_k > 0$) spanning target values in a neighborhood of given targets t_k if these given targets are attainable. If the targets prove unattainable, undesirable, or simply uninteresting, the decision-maker can adjust them and re-direct the exploration to a different range of efficient solutions.

Economic Interpretation of Efficient Solutions

Frequently, one of the objective functions in the Existence Problem, say c_1, refers to money (e.g., maximize net revenues). In such a case, each efficient solution generated by optimizing the Lagrangean function (5) lends itself to an economic interpretation. Consider π^* with $\pi_k^* > 0$ for all k, and let x^* denote an optimal solution to (5). Furthermore, let $t^* = Cx^*$. It is easy to show that x^* is optimal in the linear program

$$\max \quad c_1 x \tag{15.1}$$

$$\text{s. t.} \quad c_k x \geq t_k^* \qquad k = 2, \ldots, K \tag{15.2}$$

$$Ax \leq b \tag{15.3}$$

$$x \geq 0 \tag{15.4}$$

THEOREM 3: The quantities

$$\hat{\pi}_k = \frac{-\pi_k^*}{\pi_1^*} \qquad k = 2, \ldots, K \tag{16}$$

are optimal dual variables for the constraints (15.2) in the linear program (15).

Proof: The dual to (15) is

$$\min \sum_{k=2}^{K} \pi_k t_k^* + \sigma b$$

$$\text{s.t.} \quad \sum_{k=2}^{K} \pi_k c_k + \sigma A \leq c_1 \tag{17}$$

$$\pi_k \leq 0, \quad \sigma \geq 0 .$$

Now the vector π^* is optimal in the Lagrangean (5) which is a linear program. Let $\sigma^* \geq 0$ denote the vector of optimal dual variables on the constraints $Ax \leq b$. By linear programming duality,

$$\sigma^* A \leq \pi^* C$$

or

$$-\sum_{k=1}^{K} \pi_k^* c_k + \sigma^* A \leq 0 \cdot$$

Dividing by $\pi_1^* > 0$, and rearranging c_1, we obtain

$$-\sum_{k=2}^{K} \frac{\pi_k^*}{\pi_1^*} c_k + \frac{1}{\pi_1^*} \sigma^* A \leq c_1 .$$

Thus, the solution $\hat{\pi}_k \leq 0$ for $k = 2, \ldots, K$, and $\frac{1}{\pi_1^*} \sigma^*$ is feasible in (17).

To prove that this dual solution is optimal in (17), we establish that complimentary slackness holds for the dual solution and the primal solution x^*. First, by optimizing (5), we have

$$\left(\pi^* C - \sigma^* A\right) x^* = 0$$

and dividing by π_1^*, we have

$$\left(c_1 - \sum_{k=2}^{K} \hat{\pi}_k c_k - \frac{1}{\pi_1^*} \sigma^* A \right) x^* = 0$$

which is the first of the two required complementary conditions.

Second, we have

$$\sum_{k=2}^{K} \left(\frac{-\pi_k^*}{\pi_1^*} \right) \left\{ c_k x_k^* - t_k^* \right\} = 0$$

since $t_k^* = c_k x_k^*$, and

$$\frac{1}{\pi_1^*}\sigma^*\left(b - Ax^*\right) = 0$$

because $\sigma^*\left(b - Ax^*\right) = 0$ from optimization in (5). ■

Thus, when the objective function c_1 refers to revenue maximization or some other money quantity, each time the Lagrangean is optimized with all $\pi_k^* > 0$, the quantities $\hat{\pi}_k$ for $k = 2, ..., K$ given by (16) have the usual linear programming shadow price interpretation. Namely, the rate of increase of maximal revenue with respect to increasing objective k at the value t_k^* spanned by the efficient solution x^* is approximated by

$$\frac{-\pi_k^*}{\pi_1^*}$$

The quantity is only approximative because the $\hat{\pi}_k$ may not be unique optimal dual variables in the linear program (15) (see Shapiro [1979; pp. 34-38] for further discussion of this point.)

Subgradient Optimization Algorithms for Selecting Lagrange Multipliers

Subgradient optimization generates a sequence of K-vectors $\{\pi_w\}$ according to the rule:

1. If $L\left(\pi_w\right) < 0$ or π_w and τ_w satisfy the optimality conditions (14), stop. In the former case, the Existence Problem is infeasible. In the latter case π_w is optimal in MODP and $L\left(\pi_w\right) = 0 = D$. Otherwise, go to Step 2.

2. For $k = 1, \ldots, K$

$$\pi_{w+1,k} = \text{maximum} \left\{ 0, \pi_{w,k} - \frac{\sigma_w L(\pi_w)}{\|\tau_w\|^2} \tau_{w,k} \right\} \tag{18}$$

where τ_w is any subgradient of L at π_w , $\varepsilon_1 < \sigma_w < 2 - \varepsilon_2$ for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, and $\| \ \|$ denotes Euclidean norm. The subgradient typically chosen in this method is $\tau_w = Cx_w - t$ where x_w is the computed optimal solution to the Lagrangean at π_w .

At each π_w , the algorithm proceeds by taking a step in the direction $-\tau_w$; if the step causes one or more components of π to go negative, rule (18) says set that component to zero. The only specialization of the standard subgradient optimization algorithm for the MODP is the assumption in the formula (18) that its minimum value $D = 0$, and therefore that the step length is determined by the value $L(\pi_w)$ which is assumed to be positive.

The following theorem characterizes convergence of the algorithm as we have stated it. The proof is a straightforward extension of a result by Polyak [1967] (see also Shapiro [1979]). We present it here for completeness.

THEOREM 4: If the Existence Problem is feasible (t is attainable), the subgradient optimization algorithm will converge to a π^* such that $L(\pi^*) = 0$. If the Existence Problem is infeasible (t is unattainable), the algorithm will converge to a π^* such that $L(\pi^*) \leq 0$.

Proof: We need only consider the situation when the algorithm generates an infinite sequence $\{\pi_w\}$ by Step 2. Let $\pi^* \geq 0$ be any dual solution such that $L(\pi^*) \leq 0$. First we show that $\|\pi_w - \pi^*\|$ is monotonically decreasing. Let ψ_{w+1} be defined by

$$\psi_{w+1,k} = \pi_{w+1,k} - \frac{\sigma_w L(\pi_w)}{||\tau_w||^2} \tau_{w,k}$$

and let $\theta_w = \frac{\sigma_w L(\pi_w)}{||\tau_w||^2}$ for future reference. We have

$$\pi^* - \pi_{w+1} = \pi^* - \psi_{w+1} + \psi_{w+1} - \pi_{w+1}$$

where $\psi_{w+1} - \pi_{w+1} \leq 0$ by construction.

Thus,

$$\pi^* - \pi_{w+1} \leq \pi^* - \psi_{w+1}$$

or

$$||\pi_{w+1} - \pi^*||^2 \leq ||\psi_{w+1} - \pi^*||^2 . \quad (19)$$

It suffices then to show that $||\psi_{w+1} - \pi^*||^2 \leq ||\pi_w - \pi^*||^2$. Consider

$$\begin{aligned} ||\psi_{w+1} - \pi^*||^2 &= ||\pi_w - \theta_w \tau_w - \pi^*||^2 \\ &= ||\pi_w - \pi^*||^2 + \theta_w^2 ||\tau_w||^2 - 2\theta_w (\pi_w - \pi^*) \tau_w \\ &\leq ||\pi_w - \pi^*||^2 + \theta_w^2 ||\tau_w||^2 - 2\theta_w [L(\pi_w) - L(\pi^*)] \\ &\leq ||\pi_w - \pi^*||^2 + \theta_w^2 ||\tau_w||^2 - 2\theta_w L(\pi_w) \end{aligned}$$

$$= \left\|\pi_w - \pi^*\right\|^2 + \frac{\left(\sigma_w^2 - 2\,\sigma_w\right)\left[L\left(\pi_w\right)\right]^2}{\left\|\tau_w\right\|^2} \tag{20}$$

where the first inequality follows because τ_w is a subgradient of the convex function L, the second inequality because $L\left(\pi^*\right) \leq 0$ and $\theta_w > 0$, and the final inequality from the definition of θ_w.

Letting

$$\Delta = \frac{\left[L\left(\pi_w\right)\right]^2}{\left\|\tau_w\right\|^2} \geq 0$$

we have

$$\left(\sigma_w^2 - 2\,\sigma_w\right)\Delta = \sigma_w\left(\sigma_w - 2\right)\Delta \leq -\in_1\in_2\Delta \tag{21}$$

Since $\sigma_w \geq \in_1$ and $\sigma_w \leq 2 - \in_2$. Combining (20) and (21), we have

$$\left\|\psi_{w+1} - \pi^*\right\|^2 \leq \left\|\pi_w - \pi^*\right\|^2 - \in_1\in_2\frac{\left[L\left(\pi_w\right)\right]^2}{\left\|\tau_w\right\|^2}$$

or from (19),

$$\left\|\pi_{w+1} - \pi^*\right\|^2 \leq \left\|\pi_w - \pi^*\right\|^2 - \in_1\in_2\frac{\left[L\left(\pi_w\right)\right]^2}{\left\|\tau_w\right\|^2} \tag{22}$$

The implication of (22) is that the sequence of non-negative numbers $\left\|\pi_w - \pi^*\right\|^2$ is monotonically decreasing, which in turn implies

$$\lim_{w \to \infty} \left\| \pi_w - \pi^* \right\|^2$$

exists. The existence of this limit implies from (22) that

$$\lim_{w \to \infty} \frac{\left[L(\pi_w)\right]^2}{\left\| \tau_w \right\|^2} = 0$$

because otherwise there would be an infinite subsequence of the $\left\| \pi_w - \pi^* \right\|^2$ decreasing at each step by at least some $\epsilon_3 > 0$ which is clearly impossible.

Since there are only a finite number of extreme points of the bounded polyhedron $\{ x \mid Ax \leq b, \ x \geq 0 \}$, the $\left\| \tau_w \right\|^2$ are uniformly bounded. Thus, we can conclude that

$$\lim_{w \to \infty} L(\pi_w) = 0 .$$

Finally, the sequence $\{\pi_w\}$ must have at least one converging subsequence because the π_w are restricted to the bounded set

$$\left\{ \pi \,\middle|\, \left\| \pi - \pi^* \right\| \leq \left\| \pi_0 - \pi^* \right\| \right\}$$

If $\{\pi_q\}$ is such a subsequence converging to π^{**}, we have since L is a continuous function that

$$\lim_{w \to \infty} L(\pi_w) = L\left(\lim_{w \to \infty} \pi_w\right) = L\left(\pi^{**}\right) = 0. \tag{23}$$

This establishes the desired result. ■

The reader may note that if the Existence Problem is infeasible and the subgradient algorithm does not converge finitely, then the proof of Theorem 4 indicates that the algorithm generates subsequences converging to π^{**} such that $L\left(\pi^{**}\right) = 0$. Thus, in this case, the algorithm fails to indicate infeasibility. Of course, it is likely that the procedure will terminate finitely by finding a π_w such that $L\left(\pi_w\right) < 0$. Alternatively, if we knew that the Existence Problem were infeasible, we could replace the term $L\left(\pi_w\right)$ in (18) by $L\left(\pi_w\right) + \delta$ for $\delta > 0$ and the algorithm would converge to π^{**} such that $L\left(\pi^{**}\right) \leq -\delta$. In effect, this is equivalent to giving the subgradient optimization algorithm a target of $-\delta < 0$ as a value for L. The danger is that if we guess wrong and the Existence Problem is feasible, then the algorithm will ultimately oscillate and fail to converge.

Railroad Car Redistribution Problem

We illustrate the Lagrangean method for multi-optimization of linear programs with a specific example drawn from the railroad industry. A railroad company wishes to minimize the cost of relocating its railroad cars for the coming week. Distribution areas 1 through 6 are forecasted to have a surplus (supply exceeds demand) of cars whereas distribution areas 7 through 14 are forecasted to have a deficit (demand exceeds supply). Unit transportation costs are shown in Table 1, surpluses for distribution areas 1 through 6 in Table 2, and deficits for distribution areas 7 through 14 in Table 3. Storage of excess cars at each of the 14 locations are limited to a maximum of 20.

	7	8	9	10	11	12	13	14
1	58	86	150	100	130	110	80	85
2	77	58	62	90	92	114	110	125
3	170	142	112	114	100	97	127	128
4	160	130	72	140	120	145	150	175
5	160	135	55	75	60	75	90	103
6	150	130	141	92	94	70	80	58

Unit Transportation costs c_{ij}

Table 1

Distribution Area	1	2	3	4	5	6
Surpluses	102	85	60	25	78	44

Surpluses S_i

Table 2

Distribution Area	7	8	9	10	11	12	13	14
Deficits	48	31	30	6	27	25	44	39

Deficits D_i

Table 3

Management is also concerned with other objectives for the week's redistribution plan. First, they would like to *minimize the flow on the link from location 2 to location 8* because work is scheduled for the roadbed. Second, they would like to *maximize the flow to locations 7* because they anticipate added demand there.

The relocation problem can be formulated as the following multi-objective linear program.

Indices:

i = 1, ..., 6

j = 7, ..., 14

Decision Variables:

x_{ij} = number of cars to be transported from distribution area i to distribution area j.

E_i = number of excess cars at distribution area i.

E_j = number of excess cars at distribution area j.

Constraints:

$$\sum_{j=7}^{14} x_{ij} + E_i = S_i \qquad \text{for } i = 1, \ldots, 7$$

$$\sum_{i=1}^{6} x_{ij} - E_j = D_j \qquad \text{for } j = 7, \ldots, 14$$

$$0 \leq E_i, E_j \leq 20 \qquad \text{for } i = 1, \ldots, 6;\ j = 7, \ldots, 14.$$

$$x_{ij} \geq 0 \qquad \text{for } i = 1, \ldots, 6;\ j = 7, \ldots, 14.$$

Objective functions:

- Minimize cost

$$Z_1 = \sum_{i=1}^{6} \sum_{j=7}^{14} c_{ij} x_{ij}$$

- Minimize flow on link (2, 8)

$$Z_2 = x_{28}$$

- Maximize flow to distribution area 7

$$Z_3 = \sum_{i=1}^{6} x_{i7}$$

We begin our analysis by optimizing the model with respect to the first objective function. The result is

Objective: Minimize Z_1 (cost)

Solution:

$Z_1^* = 19222$

$Z_2 = 51$

$Z_3 = 48$

This data is used by the decision-maker in setting reasonable targets on the three objectives:

$$\sum_{i=1}^{6} \sum_{j=7}^{14} c_{ij} x_{ij} \leq 20000$$

$$x_{28} \leq 30$$

$$\sum_{i=1}^{7} x_{i7} \geq 58$$

Taking into account that the cost and flow objectives are minimizing ones of the form $c_k x \leq t_k$, we multiply by –1 to put the Existence Problem in standard form. In addition, to enhance computational efficiency and stability, we scale the cost targets and objective function by .001 to make them commensurate with the other two. We now form the Lagrangean as detailed in the previous chapter and apply the subgradient optimization algorithm. (Actually, we applied a modified and heuristic version of the algorithm outlined above that ensures π vectors with positive components are generated at each iteration. For further details, see Ramakrishnan [1990]).

The results of nine steps are given in Table 4. Each row corresponds to a solution. The first column gives the value of $L(\pi)$ and the next three columns contain Z_1 (cost), Z_2 (flow on link (2, 8)) and Z_3 (flow to area 7) respectively. The percentage increase over minimal cost (PIC) is also provided to help the decision-maker's evaluation.

No.	$L(\pi)$	Z_1	Z_2	Z_3	π_1	π_2	π_3	PIC
1	12.976	21029	0	68	0.334	0.333	0.333	9.4
2	6.569	21029	0	68	0.516	0.113	0.371	9.4
3	4.175	19556	35	68	0.608	0.001	0.391	1.7
4	7.347	21029	0	68	0.654	0.228	0.118	9.4
5	3.711	21029	0	68	0.757	0.103	0.140	9.4
6	1.878	21029	0	68	0.809	0.040	0.151	9.4
7	1.919	19572	31	68	0.835	0.008	0.157	1.8
8	1.937	21029	0	68	0.887	0.086	0.027	9.4
9	0.979	21029	0	68	0.914	0.053	0.033	9.4
10	0.721	19572	31	68	0.928	0.036	0.036	1.8

RR Car Redistribution Problem – Second Run

Table 4

The results in Table 4 point out a deficiency of the subgradient optimization algorithm for minimizing the piecewise linear convex function L. Among the ten distinct dual vectors π that were generated during the descent, we find only three distinct efficient solutions to the Existence Problem. This is partially, but not entirely, the result of the small size of the illustrative example.

An alternative descent algorithm for MODP that would largely eliminate this deficiency is one based on a generalized version of the primal-dual simplex algorithm (see Shapiro [1979]). The generalized primal-dual is a local descent algorithm that converges finitely and monotonically to a π^* optimal in MODP. Moreover, it easily allows the constraints

$$\sum_{k=1}^{K} \pi_k = 1$$

to be added to MODP. It has, however, two disadvantages: (1) it is complicated to program; and (2) for MODP's where L has a large or dense number of piecewise linear segments, the algorithm would entail a large number of small steps. Given the intended exploratory nature of the multi-objective proceeding, it appears preferable to use the subgradient optimization algorithm and present only distinct solutions to the decision-maker.

The generalized primal-dual algorithm can be viewed as a constructive procedure for finding a subgradient satisfying the optimality conditions (14) by taking convex combinations of the subgradients derived from extreme points to X. Indeed, we may only be able to meet all our targets by taking such a combination of extreme point solutions. This suggests a heuristic for choosing an appropriate convex combination of the last two solutions in Table 4. For example, if we weight the solution on row 9 by .032 and the solution on row 10 by .968, we obtain a solution satisfying all three targets with the values

$$Z_1 = 19619$$
$$Z_2 = 30$$
$$Z_3 = 68$$

Taking the same convex combination of the multipliers associated with solutions 9 and 10, and applying the result of Theorem 3, we obtain

$\hat{\pi}_2 = \$48.30$ as the (approximate) rate of increase of minimal cost with respect to decreasing the flow on link (2, 8) at a flow level of 30, and

$\hat{\pi}_3 = \$37.50$ as the (approximate) rate of increase of minimal cost with respect to increasing the flow to distribution area 7 at a delivered flow level of 68.

Future Directions

We envision several directions of future research for the Lagrangean approach to multi-objective optimization developed in this paper. From a practical viewpoint, the approach needs testing as an effective decision support tool for analyzing multi-objective problems in an interactive mode. In this regard, the railroad car distribution example presented above is an actual application where we hope the technique will be used. The technique was successfully applied in the construction of a pilot optimization model for allocating budgets to acquire, install, and maintain new systems for the U.S. Navy submarine fleet (Manickas [1988]). For this class of problems, the multiple objectives were various measures of submarine performance with and without specific system upgrades. Unfortunately, the project did not continue beyond the pilot stage to the implementation of an interactive system for supporting decision making in this area at the Pentagon.

Interactive analysis of the Existence Problem would allow the underlying preference structure, or utility function, of the decision-maker to be assessed by asking him/her to compare the most recently generated efficient solution with each of the previously generated ones. The information about preferences gleaned from these comparisons could be represented as constraints on the decision vector x (see Zionts and Wallenius [1983] or Ramesh et al [1989]). Alternatively, we could apply the method of cojoint analysis developed by Srinivasan and Shocker [1973] to identify the decision-maker's ideal target vector t from the pairwise preferences.

Although we dismissed the generalized primal-dual algorithm as a method for computing the Lagrange multipliers, further experience might indicate that the primal-dual is more effective than subgradient optimization for some applications. Finally, research remains to be performed on the application of Lagrange multiplier methods to multiple objective problems with mixed integer and nonlinear structures. Multiple objective optimization involving measures of cost, time and customer service is highly appropriate for mixed integer programming models of production scheduling (Shapiro [1989]).

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