Networks of Non-homogeneous M/G/∞ Systems

by

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ABSTRACT

For a network of G/∞ service facilities, the transient joint distribution of the facility populations is found to have a simple Poisson product form with simple explicit formula for the means. In the network it is assumed that: a) each facility has an infinite number of servers; b) the service time distributions are general; c) external traffic is non-homogenous in time; d) arrivals have random or deterministic routes through the network possibly returning to the same facility more than once; e) arrivals use the facilities on their route sequentially or in parallel (as in the case of a circuit switched telecommunication network). The results have relevance to communication networks and manufacturing systems.

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Section 1. Introduction

Networks of M/G/∞ queues with probabilistic routing are BCMP networks [Baskett et. al. (1975)] with simple known ergodic distributions . However, the dynamics of such systems in the presence of traffic whose rates vary with time has not been well documented although the analysis is not as intractable as one might guess.

The tractability of stationary BCMP networks arises from the Poisson character of input arrival processes and the resulting Poisson character at ergodicity of the exit processes from each service facility. The key to our analysis is that the transient exit process from an infinite server facility is also simple in form even when arrival rates vary with time. To show this, one must first characterize a Poisson processes with time dependent arrival rate. This inhomogeneous Markov point processes, to be called an *M(t)-Poisson process* for brevity,will be characterized in Section 2. Using very simple arguments it will be shown that an M/G/∞ queue with an M(t)-Poisson input has two properties:

i) the output process is also M(t)-Poisson with a hazard rate that is simply stated;

ii) the population in the system is Poisson with a mean that is simply stated.

These properties serve as stepping stones for the analysis of a network of M/G/∞ queues. For networks where customers visit a specified sequence of facilities, the transient joint distribution of the number of customers at each facility is of product form. Each marginal distribution is Poisson distributed with a simple mean. For communication networks where arriving customers (calls) use a set of facilities (trunk lines) simultaneously, the transient joint distribution of the number of calls at each facility is again Poisson.

Related work has been considered by others. In Kambo and Bhalaik (1979) and Ramakrishnan (1980) the transient population is found for a single infinite server facility with exponential or deterministic service times, respectively. In Brown and Ross (1969) the transient population distribution of a non-homogeneous M/G/∞ queue with batch sizes and service times that depend on the arrival time is shown to be a product form Poisson distribution. By studying the measure on carefully constructed Borel sets, Foley (1982) applies the work of Renyi (1969) to find the transient population distribution and inter-departure time of a non-homogeneous M/G/∞ queue. Foley then extends the analysis to tandem M/G/∞ queues to derive a product form Poisson population distribution.

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In this paper new and simple methods are employed to address more general networks and to exhibit means in an intuitive form.

Section 2: The M(t)/G/∞ System

2.1 The Output Process

For what follows, some notation and nomenclature are needed. In particular the idea of a time-inhomogeneous Poisson point process must be formalized. One also wishes to speak of an input process with i.i.d. service times associated with these epochs.

Definition 1. An inhomogeneous Markov point process on the index set $(0,\infty)$ will be said to be *M(t)-Poisson* if it is governed by hazard rate $\lambda(t)$ for occurrence of an epoch at time t. The associated counting process K(t) giving the total number of epochs in the interval (0,t] will also be said to be *M(t)-Poisson*.

Definition 2. A service facility with infinitely many servers whose arrival process is M(t)-Poisson with arriving customers having i.i.d. service times will be said to be an *M(t)/G/∞ system*. Its input process will be said to be *M(t)/G*, and the service facility will be said to be *G/∞*.

As will be seen (cf Lemma 4 and Theorem 5) when the arrival process to an infinite server facility is M(t)-Poisson with i.i.d service times, the departure process from the facility is also M(t)-Poisson An important ingredient for demonstrating this is the following lemma.

Lemma 3. The superposition of independent M(t)-Poisson processes with hazard rates $\lambda_j(t)$ is a M(t)-Poisson process with hazard rate $\lambda(t) = \sum_j \lambda_j(t)$.

Proof: Let $K_j(t)$ be the number of epochs in the interval (0,t] for the j'th M(t)-Poisson process with hazard rate $\lambda_j(t)$. The process $K_j(t)$ is a markov analogue of a pure birth process on the lattice of non-negative integers. The independent sum $K(t) = \sum_j K_j(t)$ is also markov and is an M(t)-Poisson counting process. Since the hazard rate for competing independent hazard rates is the sum of these hazard rates, the lemma follows¶¶

Lemma 4. Suppose that one has an M(t)-Poisson point process of rate $\lambda(t)$. Let each epoch τ_j of the point process be given a random i.i.d. delay T_j with generalized p.d.f. $a_T(x)$ so that for the resulting point process one has epochs $\tau_j^* = \tau_j + T_j$. Then the resulting process is M(t)-Poisson with hazard rate

(2.1)
$$\theta(t) = \lambda(t) * a_T(t) = \int_0^t \lambda(t-x)\, a_T(x)\, dx$$

where * denotes the convolution operator.

Proof. Suppose that T_j is a mixture of K (deterministic) values with $\text{Prob}[T_j = x_k] = p_k$ The M(t)-process may then be viewed as the superposition of K independent M(t)-processes, the k'th having hazard rate $\lambda_k(t) = p_k\lambda(t)$ and deterministic delay x_k. Clearly, the output of the k'th M(t)-process having a deterministic fixed delay x_k is an independent M(t)-process with a hazard rate $p_k\lambda(t-x_k)$ (c.f. Ramakrishnan (1979)). From Lemma 3, the output process is M(t)-Poisson with hazard rate $\theta(t) = \sum_{k=1}^{K} p_k\lambda(t-x_k)$. Since the limit of a sequence of such M(t)-Poisson processes with rate $\lambda_i(t)$ is clearly M(t)-Poisson with rate $\lambda(t) = \lim_{i \to \infty} \lambda_i(t)$, the lemma follows.¶¶

Theorem 5 Let the system M(t)/G/∞ be such that :
        a) its input process is M(t)-Poisson with hazard rate $\lambda(t)$;
        b) its service time has general distribution with generalized pdf $a_T(x)$;
        c) the system is initially empty.
Then the output process is also M(t)-Poisson with hazard rate $\theta(t)$ given by (2.1).

Proof: An M(t)/G/∞ system is equivalent to a delay system so the result follows directly from Lemma 4. ¶¶

2.2 A Single M(t)/G/∞ Facility

It is well known that the ergodic number in system for M/G/∞ has a Poisson distribution. We show below that $N_0(t)$, the number in the system at time t of an M(t)/G/∞ facility that is initially empty, is also Poisson for all t .

Theorem 6. Let an M(t)/G/∞ facility, initially empty, have an input process with hazard rate $\lambda(t)$ and let $N_0(t)$ be the number in system at time t. Then the distribution of $N_0(t)$ is Poisson for all t with pgf

$$(2.2) \qquad\qquad E[u^{N_0(t)}] = \exp(-\zeta(t)(1-u))$$

where

$$(2.3) \qquad\qquad \zeta(t) = E[N_0(t)] = \lambda(t) * \bar{A}_T(t)$$

where $\bar{A}_T(t)$ is the survival function of the service time.

Proof. It is known that the number of Poisson arrivals in (a,b) is Poisson with mean $\int_a^b \lambda(\tau)d\tau =$

$L(a) - L(b)$ where $L(t) = \int_0^t \lambda(\tau)d\tau$. When the service time T is deterministic, $N_0(t)$ is the number of arrivals in (t-T,t) and has Poisson distribution with expectation $L(t) - L(t-T)$. When the service time has a discrete distribution, i.e., Prob $[T = x_k] = p_k$ then the input M(t)/G process of rate $\lambda(t)$ may be viewed as a superposition of *independent* M(t)/D processes with rates $\lambda(t)\ p_k$ and deterministic service times x_k. For each such M(t)/D process, the number in system is Poisson and hence the number in the system for the superposition process is also Poisson. But the parameter for a Poisson variate is its expectation. This expectation is given by the sum of the expectations of the substreams, i.e., $\sum_k [L(t) - L(t-x_k)]\ p_k$. A general service time distribution with cdf $A_T(x)$ may be viewed as the limit of a sequence of discrete distributions. In the limit, $N_0(t)$ has Poisson distribution with mean

$$(2.4) \qquad E[N_0(t)] = \int_0^\infty [L(t) - L(t-x)] \, dA_T(x).$$

The equivalence of (2.4) with (2.3) then follows from Laplace transformation since

$$L[E[N_0(t)]] = \frac{\lambda(s)}{s} - \frac{\lambda(s)}{s} \alpha_T(s) = \lambda(s) \frac{1 - \alpha_T(s)}{s} = L[\lambda(t) * \bar{A}_T(t)]$$

where $\alpha_T(s)$ is the Laplace-Stieltjes transform of $A_T(x)$ and $\lambda(s) = L[\lambda(t)]$. ¶¶

Remark: From (2.3), $E[N_0(t)]$ is also equal to $\rho(t) * a_T^*(t)$ where $a_T^*(t) = \frac{\bar{A}_T(t)}{E[T]}$ is the pdf of the forward recurrence time of the service time and $\rho(t) = \lambda(t) \, E[T]$.

Corollary 7:

For $M(t)/G/\infty$ with $N(0) = 0$, $N(t)$ increases stochastically with t when $\lambda(t)$ is non-decreasing with t.

Proof: The proof is immediate from Theorem 6, since $\zeta(t)$ is non-decreasing with t, and a Poisson variate increases stochastically with its parameter.¶¶

Note that Corollary 7 applies to $M/G/\infty$ as a special case when $\lambda(t) = \lambda$.

2.3 Initial conditions

When the system does not start empty, there will be instead a set of customers $\{C_1, C_2, C_3, \ldots C_{n0}\}$ present with residual service times $y_1, y_2, y_3, \ldots y_{n0}$ and n_0 and the y_i's will be randomly distributed. These customers may be assigned to special servers that will have no subsequent arrivals. As they complete service, their population will decrease and go to zero as t goes to infinity. If that part of the population is denoted by $N_I(t)$ and its pgf by $\pi_I(u,t)$, the pgf of the total population will be given by

$$(2.5) \qquad \pi(u,t) = \pi_I(u,t) \, \exp[-\zeta(t)(1-u)]$$

with $\zeta(t)$ given by (2.3).

The next theorem and corollary show that Poisson distributions play a broader role in the evolution of the distribution of system population over time.

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Theorem 8. For the system $M(t)/G/\infty$, let the initial population have Poisson distribution with parameter α and let all residual service times be independent with $P[Y_j > t] = \bar{A}_R(t)$. Then the distribution of the number $N(t)$ in system is Poisson for all time with parameter $\zeta(t)$ given by

$$(2.6) \qquad \zeta(t) = \alpha \bar{A}_R(t) + \lambda(t) * \bar{A}_T(t)$$

Proof: If at $t = 0$, there are m customers present, the pgf of $N_I(t)$ is $[1 - \bar{A}_R(t) + \bar{A}_R(t)u]^m$. By assumption, the probability of m present at $t = 0$ is $\dfrac{e^{-\alpha}\alpha^m}{m!}$. Hence the pgf of $N_I(t)$ is

$$\pi_I(u,t) = \sum_{m=0}^{\infty} \frac{e^{-\alpha}\alpha^m}{m!}[\ 1 - \bar{A}_R(t) + \bar{A}_R(t)u]^m = \exp[-\alpha \bar{A}_R(t)\ (1\text{-}u)]$$

and, from (2.5) the result follows. ¶¶

Corollary 9. For the system $M(t)/G/\infty$: a) let the initial population be a mixture of Poisson distributions with $P[N_I(0) = m] = \displaystyle\sum_{i=1}^{m} \dfrac{p_i\, e^{-\alpha_i}\, \alpha_i^m}{m!}$; b) let all residual service times be independent with $P[Y_j > t] = \bar{A}_R(t)$. Then the number in system at time t is also a mixture of Poisson distributions with

$$(2.7) \qquad P[N(t) = m] = \sum_{i=1}^{m} \frac{p_i\, e^{-\zeta_i(\tau)}\, \zeta_i(t)^m}{m!}$$

where

$$(2.8) \qquad \zeta_i(t) = \alpha_i \bar{A}_R(t) + \lambda(t) * \bar{A}_T(t)$$

Proof: If $N_I(0)$ is a mixture of Poisson distributions then the infinite servers can be partitioned into m infinite sets each of which is associated with one of the mixtures of $N_I(0)$. From Theorem 8, $N_I(t)$ associated with each partition will be Poisson and the theorem follows.¶¶

Remark: The class of mixtures of Poisson distributions is very broad. It contains the negative binomial distributions and is clearly closed under mixing and independent summation . For any variate N(t) in this class, one has $\sigma^2_{N(t)} \geq \mu_{N(t)}$. See for example [Keilson 1977] -

Section 3: Networks of M(t)/G/∞ systems

The simplest example of a G/∞ network is that with two G/∞ facilities in tandem. The service times of a customer at the two facilities need not be independent. The answers are still simple in structure. The D/∞ case will be considered first.

Lemma 11.

Suppose an M(t)-Poisson arrival stream goes through two D/∞ service stages in tandem having service times x_1 and x_2 . Then,the occupancy levels $N_1(t)$ and $N_2(t)$ are independent and the joint occupancy distribution has product form

$$(3.1) \qquad \pi(u_1,u_2,t) = \exp[\ -E[N_1(t)](1-u_1)]\ \exp[\ -E[N_2(t)](1-u_2)]$$

where
$$(3.7) \qquad\qquad E[N_1(t)] = L(t) - L(t-x_1)$$
and
$$(3.8) \qquad\qquad E[N_2(t)] = L(t-x_1-x_2) - L(t-x_1)$$
where $L(t) = \int_0^t \lambda(\tau)d\tau.$

Proof.

Let the service time at the first stage be x_1 and the second be x_2. Then, at time t, $N_1(t)$ is the number of arrivals to stage 1 in the interval (t - x_1,t] and $N_2(t)$ is the number of arrivals to stage 2 in the interval (t - x_2,t], i.e. the number of arrivals to stage 1 in the interval (t - x_1 - x_2, t - x_1]. Since the latter interval is disjoint from (t - x_1,t], $N_1(t)$ is independent of $N_2(t)$. But, the random number of arrivals of an M(t)-Poisson stream in an interval (a,b] has a Poisson distribution with mean L(b) -L(a) . Hence, the lemma follows.¶¶

The next theorem is a special case of the main result, Theorem 15. It is given for clarity of exposition.

Theorem 12.

Suppose an M(t)-Poisson arrival stream goes through two G/∞ service stages in tandem with service times X_1 and X_2. Then, for arbitrary joint service time distribution, equation (3.1) is still valid, i.e., the occupancy levels $N_1(t)$ and $N_2(t)$ are independent so the joint occupancy distribution has product form and each distribution is Poisson. However, now

$$(3.4) \qquad E[N_1(t)] = \lambda(t) * \bar{A}_{T1}(t)$$

and

$$(3.5) \qquad E[N_2(t)] = \lambda(t) * a_{T1}(t) * \bar{A}_{T2}(t),$$

where $a_{T1}(t)$ is the pdf of X_1 and $\bar{A}_{T1}(t)$ and $\bar{A}_{T2}(t)$ are survival functions of X_1 and X_2, respectively.

Proof: Suppose the joint distribution of the service times X_1 and X_2 is discrete with a finite number of points $\underline{x}_k = (x_{1k}, x_{2k})$ of probability p_k, $k = 1, 2, .. K$. The Poisson input stream of rate $\lambda(t)$ may be regarded as a superposition of K independent M(t)-Poisson input streams, the k'th having rate $\lambda(t)p_k$, deterministic service times x_{1k} and x_{2k}, a population $N_{1k}(t)$ and $N_{2k}(t)$ at stages 1 and 2 respectively, and a joint pgf $\pi_k(u_1, u_2, t)$. For each input streams the populations $N_{1k}(t)$ and $N_{2k}(t)$ are independent from Lemma 11. Moreover the bivariate populations $[N_{1k}(t), N_{2k}(t)]$ are independent for different k since the input processes are independent and there are infinitely many servers. Let the total joint population be $[N_1(t), N_2(t)]$ with $N_1(t) = \sum_k N_{1k}(t)$, and $N_2(t) = \sum_k N_{2k}(t)$. Let the joint total pgf be $\pi(u_1, u_2, t)$. Then, from Lemma 11,

$$\pi(u_1, u_2, t) = \prod_k \pi_k(u_1, u_2, t) = \prod_k \exp\left\{ -E[N_{1k}(t)](1-u_1) - E[N_{2k}(t)](1-u_2) \right\}$$
$$= \exp\left\{ -\sum_k E[N_{1k}(t)](1-u_1) \right\} \exp\left\{ -\sum_k E[N_{2k}(t)](1-u_2) \right\}$$
$$= \exp[-E[N_1(t)](1-u_1)] \exp[-E[N_2(t)](1-u_2)]$$

The assumption that the joint service time distribution is finite and discrete may be removed by viewing the joint distribution as the limit of a sequence of joint discrete distributions of finite support. Hence, (3.1) follows.

Equation (3.4) follows from (2.3). The hazard rate of the M(t)-Poisson process into the second stage is the hazard rate of the process out of the first state. This, from (2.1) is given by $\lambda(t)$ * $a_{T_1}(t)$. Hence, (3.5) follows from (2.3). ¶¶

Remark: Alternatively (3.5) could be derived first by applying (2.5) to the total tandem system to find that $E[N_1(t) + N_2(t)] = \lambda(t)$ * $P[X_1 + X_2 > t]$ and then use this and (3.4) to compute (3.5).

As will be seen, the theorem above and c) generalize to finite networks of G/∞ servers and any set of independent M-processes passing through the network.

Definition 13. Number the service facilities 1,2, . . ,M and define $\underline{R} = (m_1, T_1; m_2, T_2; . .; m_J, T_J)$ to be a *route* where m_i is the number of the ith service facility visited in the route and T_i is the service time required during the ith visit. Note that a route may start at any facility and that a facility can be visited more than once.

Definition 13. A sequence $\underline{R}(\omega) = (m_1(\omega), T_1(\omega); m_2(\omega), T_2(\omega); . .; m_J(\omega) T_J(\omega))$ of facilities visited and service times spent at each facility will be said to be a tour sample. The corresponding random vector $\underline{R} = (m_1, T_1; m_2, T_2; . .; m_J, T_J)$ will be said to be a random tour. The associated random vector $\underline{m} = (m_1, m_2, . ., m_J)$. will be said to be the *route* of the tour. Note that a route may start at any facility and that a facility can be visited more than once.

Theorem 14. Suppose one has a set of random tours, each with deterministic route and random service times which may be correlated. Let the k'th random tour be initiated at epochs of an independent M(t)-Poisson arrival process with rate $\lambda_k(t)$. Let $\underline{N}(t) = (N_1(t), N_2(t), N_3(t), . . N_M(t))$ be the vector of populations at the service facilities with $\underline{N}(0) = \underline{0}$. Then the multivariate population distribution of $\underline{N}(t)$ is Poisson with all facility populations independent, i.e.

(3.6) $\qquad \pi(u_1, u_2, . . , u_M, t) = \exp\left[-\sum_k E[N_k(t)] (1 - u_k)\right]$

where

(3.7) $\qquad E[N_m(t)] = \sum_n \sum_k \lambda_k(t) * a_k^{mn}(t) * \bar{A}_k^{mn}(t) \text{ for all } m.$

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Here $a_k^{mn}(t)$ is the generalized pdf for tour k of the time from initiation of the tour until facility m is visited for the nth time and $\overline{A}_k^{mn}(t)$ is the survival function of the service time at facility m when visited for the nth time.

Proof: Suppose that there is only one tour. An infinite server facility revisited in the tour can be viewed as being split into more than one infinite server facility each of which is visited only once in the tour. By construction this modified tour has no facility revisited and is equivalent to that for a multiple stage tandem queue. This can be analyzed using the method of Theorem 12 to show that the multivariate population distribution *for that route* is the product of independent Poisson distributions. When the split revisited facilities are recombined this result is maintained. For more than one tour, equation (3.6) is still true since the associated input processes are independent and there are infinitely many servers at each facility.

The proof of (3.5) can be used to show that the average number of customers from the kth route that are visiting facility m for the nth time is $\lambda_k(t) * a_k^{mn}(t) * \overline{A}_k^{mn}(t)$. Hence Equation (3.7) follows.

Theorem 15 Suppose one has a network of M/G/∞ facilities with:

(i)   an external independent M(t)-Poisson arrival process with hazard rate $\lambda_m(t)$ to facility m

(ii)   a service time with pdf $a_{Tm}(t)$ and survival function $\overline{A}_{Tm}(t)$ dependent only on facility and not on the entry point into the network

(iii)   routing probabilities p_{jm} specifying the probability that facility m will be visited after facility j and p_{jout} for the probability of leaving the network after service at facility j.

Let $\underline{N}(t) = (N_1(t), N_2(t), N_3(t), .. N_M(t))$ be the vector of populations at the service facilities with $\underline{N}(0) = \underline{0}$. Then the multivariate population distribution of $\underline{N}(t)$ is Poisson with all facility populations independent, i.e. equation (3.6) is still valid. Here,

(3.8) $E[N_m(t)] = \phi_m(t) * \overline{A}_{Tm}(t)$ for all m,

where $\phi_m(t)$ is a solution to

(3.9) $\phi_m(t) = \lambda_m(t) + \sum_j p_{jm} [\phi_j(t) * a_{Tj}(t)].$

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Proof: For each facility m construct the countably infinite set of (deterministic) routes R_{km} consisting of all possible tours that begin at facility m. The system defined by (i) - (iii) is equivalent to one where the initiation of tour R_{km} is an M(t)-Poisson with hazard rate $\lambda_m(t)p_k$ where p_k is the probability that a customer entering facility m will follow route R_{km}. In this sense the system defined by (i) - (iii) is a special case of Theorem 14 and hence (3.6) follows. Let $\phi_m(t)$ be the hazard rate of the total, (the internal and external) traffic entering facility m. From (2.1), $p_{jm}\,\phi_j(t) *$ $a_{Tj}(t)$ is the hazard rate of traffic entering facility m immediately after leaving facility j and $\lambda_m(t)$ is the hazard rate of traffic entering facility m externally. Hence, (3.9) follows and, from (2,3) equation (3.8) follows.¶¶

Until now it has been assumed that service had a sequential character, i.e., that a customer used a facility until service completion and then left the facility and entered a new facility. In a *circuit-switched* telecommunication network, all facilities (trunk lines) on the route of a call are used simultaneously. As shown in the next theorem, such a system employing sets of servers simultaneously can be accommodated within the framework of the tools developed.

Theorem 16: Suppose, for $k = 1,2,...,K$, $\lambda_k(t)$ is the hazard rate of arrivals of calls that *simultaneously* use the *set* of facilities P_k for a duration with survival function $\overline{A}_{Tk}(t)$. Then , if the system is initially empty, the joint probability generating function of the number $N_m(t)$ of customers using facility m at time t is given by

$$(3.10) \qquad \pi(\,u_1,\,u_2,\,...,\,u_M,t) = E[\,\prod_{m=1}^{M}u_m{}^{N_m(t)}\,] \;=\; \prod_{k=1}^{K}\exp[-\zeta_k(t)(1-\prod_{m\varepsilon P_k}u_m)]$$

where

$$(3.11) \qquad\qquad \zeta_k(t) = \lambda_k(t) * \overline{A}_{Tk}(t).$$

Proof: From Theorem 6, the number of calls of type k in progress at time t, $C_k(t)$ is Poisson distributed with mean $E[C_k(t)] = \zeta_k(t)$ where $\zeta_k(t)$ is defined in (3.11). Hence,

$$f(\,z_1,\,z_2,\,...,\,z_K) = E[\,\prod_{k=1}^{K}z_k{}^{C_k(t)}\,] = \prod_{k=1}^{K}\exp[-\zeta_k(t)(1-z_k)].$$

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But $C_k(t) = \sum_{m\epsilon P_k} N_m(t)$ so $\pi(u_1, u_2, ..., u_M) = E[\prod_{m=1}^{M} u_m^{N_m(t)}] = f(\prod_{m\epsilon P_1} u_m, \prod_{m\epsilon P_2} u_m, ..., \prod_{m\epsilon P_K} u_m)$

so the theorem follows. ¶¶

Remark: A steady state version of this theorem is known as well as the remarkable result that, in the presence of a finite number of servers at each facility, the multivariate population distribution is a renormalized form of the truncated distributions.

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