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A Note on Node Aggregation and Benders Decomposition*

by

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In [4], Zipkin presents procedures for node aggregation of network optimization problems, including an analysis of a priori and a posteriori error bounds. The purpose of this note is to show how node aggregation can be combined with locational decisions in a natural way using Benders decomposition method.

We will illustrate the main ideas by considering a relatively simple model; namely, the capacitated plant location problem for a single product. This problem is:

$$R = \min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{m} f_i y_i \qquad (1a)$$

$$\text{s.t.} \sum_{i=1}^{m} x_{ij} = d_j \qquad \text{for } j = 1,\ldots,n \qquad (1b)$$

$$\sum_{j=1}^{n} x_{ij} - K_i y_i \leq 0 \qquad \text{for } i = 1,\ldots,m \qquad (1c)$$

$$x_{ij} \geq 0, \; y_i \geq 0 \qquad (1d)$$

First, we need to write out the master problem and the subproblem for Benders' method (see Shapiro [2] for a review of the method). Let u_j^t, v_i^t for $t = 1,\ldots,T$, denote a collection of extreme points of the set

$$u_j - v_i \leq c_{ij} \qquad \text{for } i = 1,\ldots,m; \; j = 1,\ldots,n$$

$$v_i \geq 0.$$

These solutions are used to construct the master problem

$$V = \min \nu$$

$$\nu \geq \sum_{j=1}^{n} u_j^t d_j + \sum_{i=1}^{m} (f_i - v_i^t K_i) y_i \qquad \text{for } t = 1,\ldots,T \tag{2a}$$

$$\sum_{i=1}^{m} K_i y_i \geq \sum_{j=1}^{n} d_j \tag{2b}$$

$$y_i = 0 \text{ or } 1 \qquad \text{for } i = 1,\ldots,m$$

The constraints (2b) have been added to ensure feasibility of the location variables y_i. Given an m-vector $y = \tilde{y}$ that is optimal in (2), the method proceeds by solving the transportation subproblem

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \sum_{i=1}^{m} x_{ij} = d_j \qquad \text{for } j = 1,\ldots,n$$

$$\sum_{j=1}^{n} x_{ij} \leq K_i \tilde{y}_i \qquad \text{for } i = 1,\ldots,m \tag{TP}$$

$$x_{ij} \geq 0$$

An optimal solution $\tilde{x}_{ij}$ to (TP), along with the $\tilde{y}_i$ from the master problem, gives a feasible solution to (1). This solution is optimal if $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \tilde{x}_{ij} + \sum_{i=1}^{m} f_i \tilde{y}_i = V$.

Otherwise, the Benders cut

$$\nu \geq \sum_{j=1}^{n} \tilde{u}_j d_j + \sum_{i=1}^{m} (f_i - K_i \tilde{v}_i) y_i$$

is added to the master problem, and it is resolved, where $\tilde{u}_j$, $\tilde{v}_i$ are optimal in

$$\begin{aligned} \max \ & \sum_{j=1}^{n} u_j d_j - \sum_{i=1}^{m} v_i (K_i \tilde{y}_i) \\ \text{s.t. } & u_j - v_i \leq c_{ij} \qquad \text{for } i = 1,\ldots,m; \ j = 1,\ldots,n \\ & v_i \geq 0 \end{aligned} \qquad \text{(TD)}$$

Suppose now that we decide there are too many distinct customers j represented in the transportation subproblem (TP). Zipkin [4] suggests that it be replaced by an aggregated problem which may be a close approximation to the original. Specifically, this is accomplished by partitioning the node set

$$N = \{1,\ldots,n\}$$

into the subsets N_s, $s = 1,\ldots,S$, and aggregating the demand in each N_s to a single point. In particular, an aggregated transportation subproblem is constructed from the data

$$\begin{aligned} \bar{d}_s &= \sum_{j \in N_s} d_j \qquad && s = 1,\ldots,S \\ \bar{c}_{is} &= \min_{j \in N_s} c_{ij} \qquad && \text{for } i = 1,\ldots,m; \ s = 1,\ldots,S. \end{aligned} \qquad (3)$$

The resulting aggregated transportation subproblem is

$$\min \sum_{i=1}^{m} \sum_{s=1}^{S} \bar{c}_{is} w_{is}$$

$$\text{s.t. } \sum_{i=1}^{m} w_{is} = \bar{d}_s \qquad \text{for } s = 1,\ldots,S$$

$$\sum_{s=1}^{S} w_{is} \leq K_i \tilde{y}_i \qquad \text{for } i = 1,\ldots,m \qquad (TP_A)$$

$$w_{is} \geq 0$$

and its dual is

$$\max \sum_{s=1}^{S} p_s \bar{d}_s - \sum_{i=1}^{m} q_i (K_i \tilde{y}_i)$$

$$\text{s.t. } p_s - q_i \leq \bar{c}_{is} \qquad (TD_A)$$

$$q_i \geq 0$$

Let $\tilde{w}_{is}$ denote an optimal solution to (TP_A) and let $\tilde{p}_s$, $\tilde{q}_i$ denote an optimal solution to (TD_A). Solutions for the original (unaggregated) subproblem (TP) and its dual (TD) are computed from these solutions by the formulas

$$\tilde{x}_{ij} = \tilde{w}_{is} \frac{d_j}{\bar{d}_s} \qquad \text{for all } i, \text{ for all } j \in N_s \text{ and for } s = 1,\ldots,S$$

$$\tilde{u}_j = \tilde{p}_s \qquad \text{for all } j \in N_s \text{ and for } s = 1,\ldots,S \qquad (4)$$

$$\tilde{v}_i = \tilde{q}_i \qquad \text{for } i = 1,\ldots,m$$

Lemma 1: The primal solution $\tilde{x}_{ij}$, $\tilde{y}_i$, where the $\tilde{x}_{ij}$ are computed by (4), is feasible in the capacitated plant location problem (1). The dual solution $\tilde{u}_j$, $\tilde{v}_i$ computed by (4) is feasible in the dual (TD) to the (unaggregated) subproblem (TP).

Proof: If $\tilde{y}_i = 0$, then $\tilde{w}_{is} = 0$ for $s = 1,\ldots,S$ in (TP_A) implying $\tilde{x}_{ij} = 0$ for all j. If $\tilde{y}_i = 1$, then

$$\sum_{j=1}^{n} \tilde{x}_{ij} = \sum_{s=1}^{S} \sum_{j \varepsilon N_s} \tilde{w}_{is} \frac{d_j}{\bar{d}_s} = \sum_{s=1}^{S} \tilde{w}_{is} \sum_{j \varepsilon N_s} \frac{d_j}{\bar{d}_s} = \sum_{s=1}^{S} \tilde{w}_{is}$$

where the rightmost equality follows because $\bar{d}_s = \sum_{j \varepsilon N_s} d_j$. From ($TP_A$), we have $\sum_{s=1}^{S} \tilde{w}_{is} \leq K_i$ for these i. Feasibility of $\tilde{x}_{ij}$, $\tilde{y}_i$ is established by observing

$$\sum_{i=1}^{m} \tilde{x}_{ij} = \sum_{i=1}^{m} \tilde{w}_{is} \frac{d_j}{\bar{d}_s} = d_j$$

where the rightmost inequality follows because $\sum_{i=1}^{m} \tilde{w}_{is} = \bar{d}_s$. The feasibility of $\tilde{u}_j$, $\tilde{v}_i$ in (TD) follows from our choice of $\bar{c}_{is}$ in (3); namely;

$$\tilde{u}_j - \tilde{v}_i = \tilde{p}_s - \tilde{q}_i \leq \bar{c}_{is} \leq c_{ij} \qquad \text{for all } j \varepsilon N_s. ||$$

The implication of lemma 1 is that Benders method applied to the capacitated plant location problem (1) can be effectively integrated with node aggregation of the embedded network. Each solution of the master problem (2) produces a trial solution to the locational variables $\tilde{y}$ and a lower bound V on the optimal objective function cost of (1). We proceed by replacing the transportation subproblem (TP) by an aggregated transportation subproblem (TP_A) according to the formulas given in (3). The optimal solution to (TP_A) is then disaggregated according to (4) to provide a feasible solution $\tilde{x}_{ij}$, $\tilde{y}_i$ to (1), and a dual feasible solution $\tilde{u}_j$, $\tilde{v}_i$ to (TD). The dual solution is used in the usual way to write the Benders cut

$$\nu \geq \sum_{j=1}^{n} \tilde{u}_j d_j + \sum_{i=1}^{m} (f_i - \tilde{v}_i K_i) y_i$$

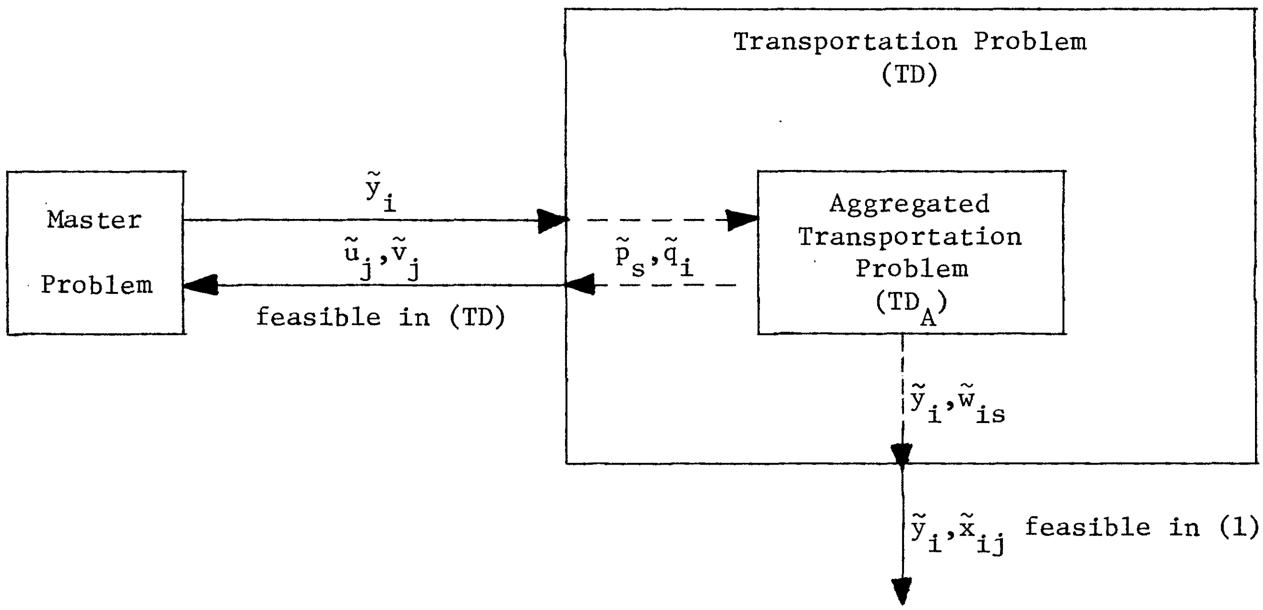
to add to the master problem. The modified method is summarized in Figure 1.

The only hitch in this modified Benders method is the possibility that the cut does not cut off the value V when $\tilde{x}_{ij}$, $\tilde{y}_i$ is not optimal in (1) because we used a dual feasible solution $\tilde{u}_j$, $\tilde{v}_i$ for (TD) that is not optimal for (TD) with $y = \tilde{y}$. The following result shows that, if the aggregated nodes are uniformly close to one another, then the best feasible solution to (1) produced by the modified Benders method will be within an a priori objective function error of being optimal.

Theorem 1: Suppose at each iteration of the modified Benders method applied to (1) that we choose a node aggregation N_s for $s = 1,\ldots,S$, which satisfies for $\varepsilon > 0$

$$c_{ij} - \varepsilon \leq \bar{c}_{is} \qquad \text{for all } j \in N_s \text{ and for all } s$$

If the modified Benders method terminates because a non-binding Benders cut is generated for the master problem, then the best known feasible solution for (1) has an objective function value within $\varepsilon \sum_{j=1}^{n} d_j$ of the optimal value.



Modified Benders Method

Figure 1

Proof: Termination of the modified Benders method occurs when

$$\sum_{j=1}^{n} \tilde{u}_j d_j + \sum_{i=1}^{m} (f_i - K_i\tilde{v}_i)\tilde{y}_i \leq V \leq R \leq \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\tilde{x}_{ij} + \sum_{T=1}^{m} f_i\tilde{y}_i$$

where $\tilde{x}_{ij}$, $\tilde{y}_i$ is the last calculated feasible solution for (1), $\tilde{u}_j$, $\tilde{v}_i$ is the last calculated dual feasible for (TD), V is the last calculated master problem objective function value, and R is the minimal objective function value for (1). We will establish that the solution $\tilde{x}_{ij}$, $\tilde{y}_i$ satisfies the conclusion of the theorem. A better solution to (1) may have been previously calculated.

To this end, consider

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\tilde{x}_{ij} + \sum_{i=1}^{m} \tilde{f}_i\tilde{y}_i - R$$

$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\tilde{x}_{ij} - \sum_{j=1}^{n} \tilde{u}_j d_j + \sum_{i=1}^{m} (K_i\tilde{v}_i)\tilde{y}_i .$$

Substituting

$$d_j = \sum_{i=1}^{m} \tilde{x}_{ij}$$

$$\sum_{s=1}^{S} \tilde{w}_{is} = K_i\tilde{y}_i \qquad \text{if } \tilde{v}_i = \tilde{q}_i > 0$$

and

$$\sum_{s=1}^{S} \tilde{w}_{is} = \sum_{j=1}^{n} \tilde{x}_{ij},$$

we obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\tilde{x}_{ij} + \sum_{i=1}^{m} f_i\tilde{y}_i - R$$
$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij} - \tilde{u}_j + \tilde{v}_i)\tilde{x}_{ij} \tag{5}$$

Now $\tilde{x}_{ij} > 0$ implies $\tilde{w}_{is} > 0$ for s such that $j \varepsilon N_s$ which in turn implies $\tilde{u}_j - \tilde{v}_i = \bar{c}_{is} \geq c_{ij} - \varepsilon$. This permits us to conclude that the right hand side in (5) is no greater than $\sum_{i=1}^{m}\sum_{j=1}^{n} \varepsilon\, \tilde{x}_{ij} = \varepsilon \sum_{j=1}^{n} d_j$ which is what we wanted to show.||

Theorem 1 provides the desired characterization of the modified Benders method when and if it terminates. However, the approximation inherent in the calculation of the new Benders cut at each iteration destroys the usual convergence argument based on the calculation of a dual extreme point solution. The simplest solution to this theoretical difficulty would be to occasionally solve an unaggregated subproblem.

More generally, we can envision a wide variety of aggregation and approximation schemes for richer mixed integer programming problems containing embedded networks and locational decision variables. For example, the analysis performed on (1) could be readily extended to multi-commodity capacitated plant location models. Related research on approximation and parametric methods for these and other models has already appeared in the literature (Bitran et al [1], Van Roy [3]). A final point is that aggregation and approximation schemes for large scale mathematical programming logistics planning models would be particularly attractive for mini computers where computation is slow and less accurate than computation on manframes.

References:

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