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DUALITY BASED CHARACTERIZATIONS OF EFFICIENT FACETS

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Abstract Most practical applications of multicriteria decision making can be formulated in terms of efficient points determined by preference cones with polyhedral closure. Using linear approximations and duality from mathematical programming, we characterize a family of supporting hyperplanes that define the efficient facets of a set of alternatives with respect to such preference cones. We show that a subset of these hyperplanes generate maximal efficient facets. These characterizations permit us to devise a new algorithm for generating all maximal efficient facets of multicriteria optimization problems with polyhedral structure.

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1. INTRODUCTION

We consider a cone dominance problem: given a "preference" cone $\mathbb{P}$ and a set $X \subseteq R^n$ of available, or feasible, alternatives, the problem is to identify the nondominated elements of X. The nonzero elements of $\mathbb{P}$ are assumed to model the dominance structure of the problem so that $y \in X$ dominates $x \in X$ if $y = x + P$ for some nonzero $p \in \mathbb{P}$. Consequently, $x \in X$ is nondominated if, and only if,

$$(\{x\} + \mathbb{P}) \cap X = \{x\} . \tag{1.1}$$

We will also refer to nondominated points as efficient points (in X with respect to $\mathbb{P}$) and we will let $EF(X,\mathbb{P})$ denote the set of such efficient points.

This cone dominance problem draws its roots from two separate, but related, origins. The first of these is multi-attribute decision making in which the elements of the set X are endowed with various attributes, each to be maximized or minimized. For example, if each component x_i of x is an attribute, then $\mathbb{P} = R^n_+$, the positive orthant, encodes the dominance structure in which "more is better." Among the approaches for analysing multi-attribute problems are (i) multi-attribute utility theory (see Keeney and Raiffa [18]) which attempts to order the alternatives by developing a utility function defined on the attributes, and (ii) vector optimization methods which attempt to view each attribute as a separate criterion function defined on the set of alternatives and to determine the set of pareto-optimal, or efficient, alternatives. From a computational point of view both approaches are limited by problem size. Nevertheless, both approaches have been applied successfully in practice, particularly in highly aggregated settings involving a relatively small number of alternatives. For example, several actual and potential applications of multi-attribute decision making have been reported recently in areas such as water resource planning [7], facility location [4], scheduling of nursing resources [8], Employee motivation [3], evaluation of urban policy [2], investment decision making [11], resources allocation [17], energy planning [30], macroeconomic policy [27], forest management [25], location of public facilities [23], activity planning [20], and corporate financial management [19].

The second origin of the cone dominance problem is mathematical programming. A well-known device in mathematical programming reduces any optimization problem to maximizing a single variable—replace maximize $\{f(z) : z \in Z \subseteq R^{\ell}\}$ by maximize $\{y : y \leq f(z) \text{ and } z \in Z\}$. The problem thus can be viewed as finding an efficient point $\bar{x} = (\bar{y},\bar{z})$ from the feasible region with respect to the cone $\mathbb{P} = \{p \in R^{\ell+1} : p_1 > 0\} \cup \{0\}$.

In this paper, we study a class of cone dominance problems from the mathematical programming point of view. We use linear approximations and duality constructs typical of nonlinear programming to study the extremal structure of the set of efficient points and present a new algorithm for determining the efficient facets of problems with polyhedral structure. This method of analysis is certainly not new. Nevertheless, we hope that the characterizations that we provide lead to a better understanding of the structure of multi-attribute decision making problems and that the algorithm might be useful in those aggregate decision making situations where a multi-criterion approach seems to be most useful.

We limit our discussion in this paper to cones $\mathbb{P}$ that are "nearly" polyhedral and whose closure only adds "lines" to the cone as formulated formally below in our first assumption. These restrictions permit us to use linear programming duality results to obtain our characterizations. In a forthcoming paper, we will show how the results generalize, but at the expense of using more sophisticated duality correspondences.

For any two sets A and B in R^n, we let $A\setminus B$ denote set theoretic difference, i.e., $A\setminus B = \{x \in A : x \notin B\}$. We also let $-A$ denote the set $\{x : -x \in A\}$. Recall that the maximal subspace L contained in a convex cone C satisfies $L = C \cap (-C)$.

ASSUMPTION 1: *$\mathbb{P}$ is a nonempty and nontrivial convex cone, its closure (denoted* $\mathrm{cl}\,\mathbb{P}$*) is polyhedral and* $\mathrm{cl}\,\mathbb{P}\setminus\mathbb{P} = L\setminus\{0\}$ *where* L *is the maximal subspace contained in* $\mathrm{cl}\,\mathbb{P}$.

Most practical applications of multiple criteria optimization can be formulated as a cone dominance problem with respect to a preference cone $\mathbb{P}$ satisfying assumption 1. For example, consider the vector optimization problem

$$\text{(VOP)} \qquad \text{"max"}\ \{f(z) = (f_1(z), f_2(z), \ldots, f_k(z)) : z \in Z \subseteq R^{n-k}\}\ .$$

Using the transformation introduced earlier for nonlinear programs, we define $X = \{x = (y,z) \in R^n : z \in Z \text{ and } y \leq f(z)\}$. Then a point $\bar{x} \in X$ is efficient in X with respect to the cone

$$\mathbb{P}_k \equiv \{0\} \cup \{(\lambda_1, \ldots, \lambda_k, \gamma_{k+1}, \ldots, \gamma_n) : \lambda_i \geq 0 \quad i = 1,2,\ldots,k \text{ and } (\lambda_1, \lambda_2, \ldots, \lambda_k) \neq 0\}$$

if, and only if, $\bar{x} = (\bar{y},\bar{z})$, $\bar{y} = f(\bar{z})$, and $\bar{z}$ is efficient in (VOP); that is, there is no $z \in Z$ satisfying $f(z) \geqq f(\bar{z})$ with at least one strict inequality.

As we will see later, when X is polyhedral, the cone dominance problem that we are considering is equivalent to the linear multiple objective program (that is, to a vector optimization problem with linear objectives and a polyhedral constraint set). This last problem has been studied extensively by Ecker, Hegner, and Kouada [9], Philip [21], Evans and Steuer [10], Gal [12], Yu and Zeleny [29], Iserman [16], and several other authors. The cone dominance problem has been analyzed in quite general contexts by Bitran and Magnanti [5], Benson [1], Borwein [6], Hartley [15], and Yu [28].

The plan of this paper is as follows. In section two, we present a characterization of efficient facets of X. We then use this characterization in section three to derive an algorithm to determine all efficient points and efficient facets of X.

We conclude this introduction by summarizing notation to be used later and by recalling one basic result concerning cones. Given a cone $\mathbb{P}$ we let $\mathbb{P}^+ = \{p^+ \in R^n : p^+p \geqq 0 \text{ for all } p \in \mathbb{P}\}$ denote its positive polar and let $\mathbb{P}_s^+ = \{p_s^+ : R^n : p_s^+p > 0$ for all nonzero $p \in \mathbb{P}\}$ denote its cone of strict supports. When $\mathbb{P}_s^+ \neq \phi$ we say that $\mathbb{P}$ is *strictly supported*. It is worth observing that assumption 1 implies that $\mathbb{P} \cap L = \{0\}$ and, therefore, by proposition A.1 of Bitran and Magnanti [5] that $\mathbb{P}$ is strictly supported.

The following adaptation of a well-known result in Rockafellar [22], Stoer and Witzgall [26], and Yu [28] is useful in this study.

PROPOSITION 1.1: *If the closure of* $\mathbb{P}$ *is polyhedral, then so is* $\mathbb{P}^+$.

2. CHARACTERIZATION OF EFFICIENT FACETS

We begin this section by deriving, at each efficient point, a family of supporting half spaces to the feasible set X. When the feasible set is polyhedral, we show that these supports characterize efficient facets. We also obtain a characterization of the maximal efficient facets incident to an efficient point. (An efficient facet is *maximal* if it is not contained in another efficient facet.) For situations involving polyhedral feasible sets, Ecker, Hegner, and Kouada [9] and Yu and Zeleny [29] have obtained results similar to those presented here, but using different arguments.

Consider the family of optimization problems, one defined at each point $x^o \in X$:

$$Q'(x^o) : \max \{p_s^+ x : x \in X \cap (x^o + \mathbb{P})\}$$

where p_s^+ is a fixed, but generic, element of the cone of strict supports $\mathbb{P}_s^+$ of $\mathbb{P}$. Note that x^o is efficient according to definition (1.1) if, and only if, it solves $Q'(x^o)$.

We will show that the analysis of this family of problems and their linear approximations leads to the characterizations that we seek. Actually, it is more convenient to consider a slightly modified version of this family of problems which we define next.

Let H denote the q by n matrix whose rows are a system of generators of $\mathbb{P}^+$; that is, $\mathbb{P}^+ = \{\lambda H : \lambda \in R_+^q\}$ which implies by Farkas' lemma that $\text{cl}\,\mathbb{P} = \{x \in R^n : Hx \geqq 0\}$. Although the condition $H(x - x^o) \geqq 0$ leads to the conclusion that $x \in x^o + \text{cl}\,\mathbb{P}$, we shall show that assumption 1 permits us to replace $\mathbb{P}$ in $Q'(x^o)$ by its closure and still derive useful results. That is, instead of analyzing the family of problems $Q'(x^o)$ directly we consider the related family:

$$Q(x^o) : \max \{p_s^+ x : H(x - x^o) \geqq 0, \ x \in X\} .$$

ASSUMPTION 2: $X = \{x \in R^n : g(x) \geqq 0, \ x \geqq 0\}$ *where* $g(\cdot) : R^n \to R^m$, $g_i(\cdot)$ *is differentiable and concave for* $i = 1,2,\ldots,m$.

We have introduced the differentiability assumption for the purpose of simplifying our presentation. Similar results can be obtained using subdifferential properties of the functions g_i for $i = 1,2,\ldots,m$ by replacing every appearance of the gradient of any g_i by a subgradient.

DEFINITION 2.1: $x \in EF(X,\mathbb{P})$ is said to be *strictly efficient* if it maximizes a linear functional $p_s^+ x$ over X for some $p_s^+ \in \mathbb{P}_s^+$.

DEFINITION 2.2: $Q(x^o)$ is said to be *regular* if x^o solves the linear approximation.

$$\begin{aligned} LQ(x^o):\quad & \max\ p_s^+(x - x^o) + p_s^+x^o \\ \text{subject to:}\quad & H(x - x^o) \geqq 0 \\ & \nabla g_i(x^o)(x - x^o) \geqq 0 \quad i \in I \\ & x_j \geqq 0 \quad j \in J \end{aligned}$$

where $I = \{i \in \{1,2,...,m\} : g_i(x^o) = 0\}$ and $J = \{j \in \{1,2,...,n\} : x_j^o = 0\}$. $LQ(x^o)$ is a linear programming problem and its dual is:

$$\begin{aligned} DLQ(x^o):\quad & \min\ p_s^+x^o \\ \text{subject to:}\quad & -\pi H - \mu\nabla g(x^o) - \tau I_n = p_s^+ \qquad (2.1) \\ & \mu \geqq 0,\ \tau \geqq 0,\ \pi \geqq 0 \\ & \mu_i = 0,\ i \notin I,\ \tau_j = 0,\ j \notin J\ . \end{aligned}$$

PROPOSITION 2.3: *Let* $Q(x^o)$ *be regular. Then* x^o *is strictly efficient.*

PROOF: The regularity of $Q(x^o)$ implies that there is a $(\bar\pi,\bar\mu,\bar\tau)$ solving $DLQ(x^o)$. Multiplying (2.1) by $(x - x^o)$ for a generic $x \in X$, we obtain

$$(p_s^+ + \bar\pi H)(x - x^o) = -\bar\mu\nabla g(x^o)(x - x^o) - \bar\tau(x - x^o) \leqq 0\ . \qquad (2.2)$$

The inequality follows from the concavity of $g(\cdot)$ and the fact that $\bar\mu_i = 0$ for $i \notin I$, and the fact that $\bar\tau_j = 0$ for $j \notin J$. Therefore, from (2.2), $(p_s^+ + \bar\pi H)x \leqq (p_s^+ + \pi H)x^o$ for all $x \in X$. Since $(p_s^+ + \bar\pi H)p > 0$ for all $p \in \mathbb{P}$, the proof is completed. □

Strictly efficient points play an important role in cone dominance problems. For example, Bitran and Magnanti (theorem 3.1 in [5]) proved that only mild conditions need be imposed upon the cone dominance problem to insure that any efficient point x can be written as

$$x = x^* - \bar p$$

where x^* is in the closure of the set of strictly efficient points and $\bar p \in \mathrm{cl}\,\mathbb{P}\backslash\mathbb{P}$ (the cone $\mathbb{P}$ is assumed to be strictly supported and convex). Bitran and Magnanti [5] have also shown that for strictly supported closed convex cones, a point $x^o \in X$ is strictly efficient if, and only if, x^o is efficient in some conical support $L(x^o)$ to X at x^o, i.e., $L(x^o) - \{x^o\}$ is a closed cone and $X \subseteq L(x^o)$. The concept of proper efficiency introduced by Geoffrion [14] and its consequences are also intimately related to strict efficiency.

PROPOSITION 2.4: *Let* $Q(x^o)$ *be regular. Denote by* $(\bar\pi,\bar\mu,\bar\tau)$ *an optimal solution to* $DLQ(x^o)$. *Then*

$$\{x \in R^n : \phi_{x^o}(x) \equiv \bar\mu\nabla g(x^o)x^o + (p_s^+ + \bar\pi H)x = 0\}$$

is a supporting hyperplane to X *at* x^o *and* $\phi_{x^o}(x) \leqq 0$ *for all* $x \in X$.

PROOF: The concavity of $g(\cdot)$ implies that, for any $x \in X$,

$$\bar\mu\nabla g(x^o)(x - x^o) \geqq 0,\ \text{i.e.,}\ \bar\mu\nabla g(x^o)x \geqq \bar\mu\nabla g(x^o)x^o\ . \qquad (2.3)$$

From (2.1) and the complementary slackness conditions of linear programming, $\phi_{x^o}(x^o) = 0$. Also from (2.1), for any $x \in X$,

$$-\bar{\mu}\nabla g(x^o)x - \bar{\tau}x = (p_s^+ + \bar{\pi}H)x .$$

By (2.3) and the fact that $\bar{\tau}x \geqq 0$,

$$-\bar{\mu}\nabla g(x^o)x^o \geq (p_s^+ + \bar{\pi}H)x .$$

Consequently, $\phi_{x^o}(x) \leqq 0$ for all $x \in X$. □

COROLLARY 2.5: *Let* $Q(x^o)$ *be regular. Then* $\{x \in R^n : \phi_{x^o}(x) = 0\}$ *separates* X *and* $\{x^o\} + \mathbb{P}$ *at* x^o.

PROOF: If $x^o + p$ is an arbitrary element of $\{x^o\} + \mathbb{P}$, then, by substitution, $\phi_{x^o}(x^o + p) = \phi_{x^o}(x^o) + (p_s^+ + \bar{\pi}H)p$. The previous proposition shows that $\phi_{x^o}(x^o) = 0$ and since $(p_s^+ + \bar{\pi}H)p \geq 0$, $\phi_{x^o}(x^o + p) \geq 0$. □

Proposition 2.4 and corollary 2.5 show that any solution to $DLQ(x^o)$ generates a hyperplane $\{x \in R^n : \phi_{x^o}(x) \geqq 0\}$ that separates X and $\{x^o\} + \mathbb{P}$ at x^o. To be precise, we should index these hyperplanes with the corresponding solution (π,μ,τ) of $DLQ(x^o)$. However, to simplify notation, we will not adopt such a representation. In the remainder of this section, we show that the family of hyperplanes, $\{x \in R^n : \phi_{x^o}(x) = 0\}$, or equivalently, all alternative optimal solutions to $DLQ(x^o)$, characterize the efficient facets incident to x^o when X is polyhedral. Moreover, proposition 2.14 characterizes the maximal efficient facets.

PROPOSITION 2.6: $x^o \in EF(X,\mathbb{P})$ *if, and only if, it solves* $Q(x^o)$.

PROOF: If x^o solves $Q(x^o)$, then it is clearly efficient. Assume $x^o \in EF(X,\mathbb{P})$ and that there is an $\bar{x} \in X$, $\bar{x} \neq x^o$, satisfying $H(\bar{x} - x^o) \geq 0$ and $p_s^+\bar{x} > p_s^+x^o$. The condition $H(\bar{x} - x^o) \geq 0$ implies that $(\bar{x} - x^o) \in \text{cl}\,\mathbb{P}$. However, if $(\bar{x} - x^o) \in \text{cl}\,\mathbb{P}\backslash\mathbb{P} = L - \{0\}$, we would have $p_s^+(\bar{x} - x^o) = 0$. Hence $(\bar{x} - x^o) \in \mathbb{P}$; but this conclusion contradicts the assumption $x^o \in EF(X,\mathbb{P})$. Therefore, x^o solves $Q(x^o)$. □

At first sight proposition 2.6 might seem to apply in all circumstances. However, the facts that the cone $\mathbb{P}$ is not closed and that $H(x - x^o) \geq 0$ implies $x \in \{x^o\} + \text{cl}\,\mathbb{P}$ instead of $x \in \{x^o\} + \mathbb{P}$ add some meaningful complications. To illustrate this point, we give an example that shows that the conclusion of proposition 2.6 need not be valid if $\mathbb{P}$ is not closed and assumption 1 is violated.

EXAMPLE 1:

Let $\mathbb{P} = \{P = (p_1,p_2) \in R^2 : p_1 \geqq 0,\ p_2 > 0\} \cup \{0\}$ and let $X = \{x = (x_1,x_2) \in R^2 : x_2 = 0,\ 0 \leqq x_1 \leqq 1\}$. Then, $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, i.e., $\mathbb{P}^+ = \{p^+ = (p_1^+,p_2^+) \in R^2 : (p_1,p_2) \geq 0\}$. Also, $x^o = (0,0) \in EF(X,\mathbb{P})$, but $x^1 = (1,0) \in X$ is such that $H(x^1 - x^o) \geqq 0$ with one strict inequality, and if we let $p_s^+ = (1,1)$, we have $p_s^+x^1 > p_s^+x^o$. Figure 2.1 illustrates the example.

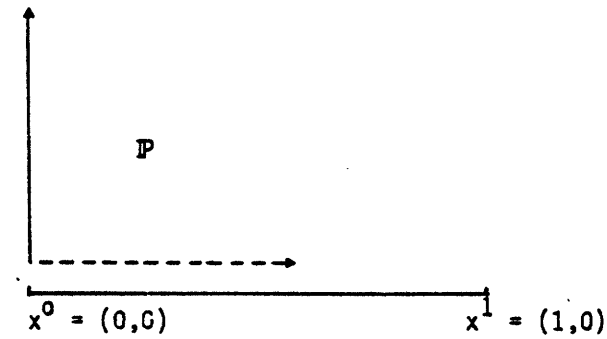

Figure 2.1 Proposition 2.6 Can Fail Without Assumption 1

For the remainder of this paper, we make the following assumption:

ASSUMPTION 3: $g(x) = b - Ax$ *for some given column vector b and matrix* A.

Note that assumption 3 implies that X is polyhedral and by virtue of proposition 2.6 that the cone dominance problem is equivalent to the linear multiple criteria problem whenever $\mathbb{P}$ satisfies assumption 1. Moreover, solving $Q(x^o)$ is equivalent to solving $LQ(x^o)$. Similarly, solving $DQ(x^o)$ defined by:

$$\begin{aligned} DQ(x^o): \quad & \min \quad -\pi H x^o + \mu b \\ \text{subject to:} \quad & -\pi H + \mu A - \tau = p_s^+ \\ & \pi, \mu, \tau \geqq 0 \end{aligned}$$

is equivalent to solving $DLQ(x^o)$.

A direct consequence of proposition 2.6 and assumption 3 is:

COROLLARY 2.7: $x^o \in EF(X,\mathbb{P})$ *if, and only if, it is strictly efficient.*

Several authors, including Bitran and Magnanti [5], Gal [13], Evans and Steuer [10], Ecker, Hegner, and Kouada [9], Philip [21], and Yu and Zeleny [29] have obtained the same result. In fact, the result is true when $\mathbb{P}$ is *any* closed convex and strictly supported cone (see [5]).

The reader should note that throughout our discussion we have not required that x^o be an extreme point of X. It is well-known (see Yu and Zeleny [29]) that, if x^o is efficient and is contained in the relative interior of a facet, then the entire facet is efficient.

PROPOSITION 2.8: *Let* x^o *be efficient and let* $(\bar{\pi},\bar{\mu},\bar{\tau})$ *solve* $DQ(x^o)$. *If* $x^1 \in X$ *and* $\phi_{x^o}(x^1) = -\bar{\mu}Ax^o + (p_s^+ + \bar{\pi}H)x^1 = 0$, *then* $x^1 \in EF(X,\mathbb{P})$. *That is, the support* $\{x \in R^n : \phi_{x^o}(x) = 0\}$ *intersects* X *only at efficient points.*

PROOF: The result follows from the fact that $(p_s^+ + \bar{\pi}H) \in \mathbb{P}_s^+$ and $\phi_{x^o}(x) \leqq 0$ for all $x \in X$. Since $\phi_{x^o}(x^1) = 0$ we have that x^1 maximizes $\bar{p}_s^+ x$ over X for some $\bar{p}_s^+ \in \mathbb{P}_s^+$. □

It is worth noting that if x^o is in the relative interior of a maximal efficient facet then any extreme point $(\bar{\pi},\bar{\mu},\bar{\tau})$ that solves $DQ(x^o)$ generates a hyperplane $\{x \in R^n : \phi_{x^o}(x) = 0\}$ that supports the maximal efficient facet.

COROLLARY 2.9: *If* $(\bar{\pi},\bar{\mu},\bar{\tau})$ *solves* $DQ(x^o)$ *and* $\phi_{x^o}(x^1) = 0$, *then* $(\bar{\pi},\bar{\mu},\bar{\tau})$ *also solves* $DQ(x^1)$.

Proposition 2.8 and Corollary 2.9 imply that by considering only efficient extreme points of X and all solutions of the corresponding dual problem, we are able to generate all efficient facets. The reason is that, if $x \in EF(X,\mathbb{P})$ is in the relative interior of a facet, then the entire facet is efficient. Also, by corollary 2.9, for all extreme points x^o in the facet, the optimal solutions to $DQ(\bar{x})$ are optimal in $DQ(x^o)$. Therefore, if one considers all triples $(\bar{\pi},\bar{\mu},\bar{\tau})$ optimal in $DQ(x^o)$, and each $\phi_{x^o}(x)$ is maximized over X, the corresponding efficient facet incident to x^o is obtained. This observation and the comment prior to corollary 2.9 establish the following result.

COROLLARY 2.10: *Let x^o be an extreme point of X and let $\{(\pi^e,\mu^e,\tau^e)\}_{e\in E}$ be the set of extreme points optimal in $DQ(x^o)$. Then the set of hyperplanes $\{x \in R^n : (p_s^+ + \pi^e H)x^o - \mu^e b = 0\}$ for $e \in E$ supports all efficient facets of X that contain x^o. Moreover, every maximal efficient facet of X can be obtained in this way for some extreme point x^o of X.*

This corollary and the results to be given below will be used in the next section as the basis for an algorithm to determine $EF(X,\mathbb{P})$ and the maximal efficient facets of X.

From the definition of $\phi_{x^o}(x)$ and the dual problem $DQ(x^o)$, we have the following result:

PROPOSITION 2.11: *The gradient of the supporting hyperplane $\{x \in R^n : \phi_{x^o}(x) = 0\}$ is a linear combination of the gradients of the active constraints at x^o.*

PROPOSITION 2.12: *Let $x^o \in EF(X,\mathbb{P})$ and assume that $(\bar{\pi},\bar{\mu},\bar{\tau})$ generates $\phi_{x^o}(x)$. Then a necessary and sufficient condition for $x^1 \in X$ to satisfy $\phi_{x^o}(x^1) = 0$ is that $\bar{\mu}(Ax^1 - b) = 0$ and $\bar{\tau}x^1 = 0$.*

PROOF: Multiplying (2.1) by x^1, we obtain

$$(p_s^+ + \bar{\pi}H)x^1 - \bar{\mu}Ax^1 + \bar{\tau}x^1 = 0 \tag{2.4}$$

Sufficiency: $\bar{\mu}(Ax^1 - b) = 0$, $\bar{\tau}x^1 = 0$, and (2.4) imply that $\phi_{x^o}(x^1) = (p_s^+ + \bar{\pi}H)x^1 - \bar{\mu}b = 0$ (recall that $\bar{\mu}Ax^o = \bar{\mu}b$).

Necessity: Assume $\phi_{x^o}(x^1) = (p_s^+ + \bar{\pi}H)x^1 - \bar{\mu}b = 0$. (2.5)

Subtracting (2.5) from (2.4) gives,

$$\bar{\mu}(b - Ax^1) + \bar{\tau}x^1 = 0 .$$

However, since $x^1 \in X$, it follows that $\bar{\mu}(b - Ax^1) \geqq 0$ and $\bar{\tau}x^1 \geqq 0$. Consequently, $\bar{\mu}(b - Ax^1) = 0$ and $\bar{\tau}x^1 = 0$.

This last proposition is essentially a statement of the linear programming complementarity conditions applied to the problem

$$\max \quad \{\phi_{x^o}(x) : x \in X\} .$$ □

In our subsequent discussion, we make use of the fact that x^o is an extreme point of X if, and only if, there is an $x^o \in R^m$ such that (x^o,s^o) is an extreme point of $\{(x,s) \in R^{n+m} : Ax + Is = b,\ x \geqq 0,\ s \geqq 0\}$. Every extreme point in the second representation is a basic solution to the equations $Ax + Is = b$ corresponding to a basis matrix from [A,I]. Recall that any edge in X corresponds to a basis matrix from [A,I] together with one additional column of the matrix, or, in terms of variables, together with one nonbasic variable from (x,s).

Proposition 2.12 is useful algorithmically. Let x^o be an extreme point of X and let x^1 be an adjacent extreme point to x^o lying on the hyperplane $\{x \in R^n : \phi_{x^o}(x) = 0\}$. The proposition provides conditions that guarantee that movement from x^o to x^1 on the edge $[x^o,x^1]$ does not leave the hyperplane.

COROLLARY 2.13: *Let* $x^o \in EF(X,\mathbb{P})$ *be an extreme point of* X *and let* B *be a corresponding basis from* [A,I]. *Also, let* $(\bar{\pi},\bar{\mu},\bar{\tau})$ *be an optimal solution to* $DQ(x^o)$. *Then every efficient edge incident to* x^o *and that is contained on the hyperplane* $\{x \in R^n : \phi_{x^o}(x) = (p_s^+ + \bar{\pi}H)x - \bar{\mu}b = 0\}$, *corresponds to a nonbasic variable from* (x,s) *whose corresponding dual variable* $\bar{\tau}_i$ *or* $\bar{\mu}_i$ *is zero.*

To conclude this section we show that the results obtained so far can be used to characterize the maximal efficient facets of X.

PROPOSITION 2.14: *Let* (π^1,μ^1,τ^1) *and* (π^2,μ^2,τ^2) *be two alternative optimal solutions to* $DQ(x^o)$. *Let* $F_1 = \{x \in X : (p_s^+ + \pi^1H)x - \mu^1b = 0\}$ and $F_2 = \{x \in X : (p_s^+ + \pi^2H)x - \mu^2b = 0\}$ *(i.e.,* F_1 *and* F_2 *are the faces of* X *generated by the two optimal solutions to* $DQ(x^o)$*). Let* Q_j, j = 1,2 *denote the set of indices of the components of* (μ^j,τ^j) *that are strictly positive. Then, if* $Q_1 \subseteq Q_2$, *it follows that* $F_2 \subseteq F_1$ *and, therefore, that* $\dim F_1 \geqq \dim F_2$.

PROOF: Suppose that $\bar{x} \in X$ and that $(p_s^+ + \pi^2H)\bar{x} - \mu^2b = 0$. Multiplying (2.1) with $(\pi,\mu,\tau) = (\pi^2,\mu^2,\tau^2)$ by $\bar{x}$, we have

$$-\pi^2H\bar{x} + \mu^2A\bar{x} - \tau^2\bar{x} = p_s^+\bar{x}$$

so that $\mu^2(b - A\bar{x}) + \tau^2\bar{x} = 0$ and, therefore, $\mu^2(b - A\bar{x}) = 0$ and $\tau^2\bar{x} = 0$. Also, since $Q_1 \subseteq Q_2$, we have $\mu^1(b - A\bar{x}) = 0$ and $\tau^1\bar{x} = 0$. Multiplying (2.1) with $(\pi,\mu,\tau) = (\pi^1,\mu^1,\tau^1)$ by $\bar{x}$ and considering the results above, it follows that

$$(p_s^+ + \pi^1H)\bar{x} - \mu^1b = 0 .$$

Hence $F_2 \subseteq F_1$ and $\dim F_1 \geqq \dim F_2$. □

Note that proposition 2.14 does *not* state that the number of positive components of (μ,τ) is the same for every maximal efficient facet. In fact, as shown by the following example, the number of positive components can vary from one efficient facet to another. The example also shows that all maximal efficient facets do not have the same dimension.

EXAMPLE 2:

Consider the system

$$x_1 + x_2 + x_3 \leqq 3 \qquad (2.6)$$
$$6x_1 - 3x_2 + x_3 \leqq 4 \qquad (2.7)$$
$$-3x_1 + 6x_2 + x_3 \leqq 4 \qquad (2.8)$$
$$x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0$$

with the preference cone $\mathbb{P} \equiv \{p \in R^3 : p \geqq 0\}$. That is, H = I, the identity matrix.

Let X denote the set of feasible solutions to this system.

The facet $F_1 \equiv \{x \in X : x_1 + x_2 + x_3 = 3\}$ is efficient since perturbing any point on this facet by a nonzero element of $\mathbb{P}$ must violate (2.6). No point $\bar{x}$ in the relative interior of the facet $F_2 = \{x \in X : 6x_1 - 3x_2 + x_3 = 4\}$ is efficient since $\bar{x} + (0,\varepsilon,0) \in X$ for $\varepsilon > 0$ sufficiently small and $(0,\varepsilon,0) \in \mathbb{P}$. Similarly, no point in the relative interior of $F_3 = \{x \in X : -3x_1 + 6x_2 + x_3 = 4\}$ is efficient. Note, though, that every point $\bar{x}$ on the facet $F_4 = F_2 \cap F_3$ is efficient, since adding (2.7) and (2.8) gives

$$3x_1 + 3x_2 + 2x_3 \leqq 8 \ . \qquad (2.9)$$

If $\bar{x} \in F_4$ then it satisfies (2.9) as an equality. Consequently, if $\bar{x} + p \in X$ and $p \in \mathbb{P}$, then, from (2.9), $p_1 + p_2 + p_3 \leqq 0$ implying that $p = 0$.

These observations show that F_1 and F_4 are maximal efficient facets, but with different dimensions. One hyperplane that supports F_4 is given by setting $\tau = 0$, $\mu = (0,\frac{5}{9},\frac{4}{9})$ and $\pi = (1,0,0)$, which corresponds to an extreme point of the dual problem $DQ(x^o)$ for any $x^o \in F_4$. A hyperplane supporting F_1 is obtained by setting $\tau = 0$, $\mu = (1,0,0)$ and $\pi = (0,0,0)$. Note that the number of positive components of (μ,τ) differs for these two hyperplanes. □

Another fact worth noting is that none of our previous results guarantee that *every* extreme point solution to the dual problem $DQ(x^o)$ corresponds to a maximal efficient facet. The following example illustrates this point and further illustrates proposition 2.14.

EXAMPLE 3:

Let $H = I$, the identity matrix in R^2, and let X be defined by the system

$$x_1 + x_2 \leqq 2 \qquad (2.10)$$

$$4x_1 + x_2 \leqq 5 \qquad (2.11)$$

$$x_1 \geqq 0, \ x_2 \geqq 0 \ .$$

The extreme point $x^o = (1,1)$ is defined by the intersection of the first two constraints. By setting $p_s^+ = (1,\frac{1}{2})$ and by solving $DQ(x^o)$, we find among the extreme point solutions:

(i) $\mu = (0,\frac{1}{2})$, $\tau = 0$, and $\pi = (1,0)$

(ii) $\mu = (1,0)$ $\tau = 0$, and $\pi = (0,\frac{1}{2})$, and

(iii) $\mu = (\frac{1}{3},\frac{1}{6})$ $\tau = 0$, and $\pi = (0,0)$.

The facets defined by these three solutions are, respectively, $F_1 = \{x \in X : 4x_1 + x_2 = 5\}$, $F_2 = \{x \in X : x_1 + x_2 = 2\}$, and $F_3 = F_1 \cap F_2$. Note that although (iii) is an extreme point solution of $DQ(x^o)$, it defines F_3 which is not a maximal efficient facet. This conclusion is a direct consequence of proposition 2.14 since the set of indices corresponding to positive components of (μ,τ) in (i) [or also (ii)] is contained in the set of indices corresponding to positive components of (μ,τ) in (iii).

3. AN ALGORITHM TO DETERMINE ALL EFFICIENT EXTREME POINTS AND EFFICIENT FACETS

The algorithm presented in this section is based on the results obtained above. It allows us to obtain adjacent efficient extreme points as alternative optimal solutions of linear problems. Other algorithms have been developed (Evans and Steuer [10], Philip [21], Ecker, Hegner, and Kouada [9], Yu and Zeleny [29], and Gal [12]) and consist essentially of checking nonbasic columns by solving subproblems and carrying the q rows of H as objective functions.

The algorithm presented below is based upon

i) Proposition 2.8 which guarantees that the support of the hyperplanes $\{x \in R^n : \phi_{x^o}(x) = 0\}$ are contained in $EF(X,\mathbb{P})$,

ii) the connectedness of $EF(X,\mathbb{P})$ (see [5] and [29]), and

iii) proposition 2.14 which reduces considerably the number of supporting hyperplanes to be generated.

We assume that $EF(X,\mathbb{P})$ is nonempty and that we have at the outset some efficient extreme point x^o. Several authors have considered the existence and determination of an efficient solution for linear multiple objective problems and for more general settings as well (Yu [28], Bitran and Magnanti [5], Hartley [15], Evans and Steuer [10], Philip [21], Soland [24], and others). An effective way to generate an initial efficient solution, in the linear case, is to select a feasible point $x^* \in X$ (for example, by phase I of the simplex method) and solve

$$Q(x^*): \quad \max \{p_s^+ x : H(x - x^*) \geqq 0,\ x \in X\} .$$

It is not difficult to show that $EF(X,\mathbb{P})$ is nonempty if, and only if, $Q(x^*)$ has an optimal solution. Moreover, any feasible solution to $Q(x^*)$ is efficient in X with respect to $\mathbb{P}$.

With a choice of an efficient extreme point x^o in hand, the algorithm can be described as follows:

Given x^o in $EF(X,\mathbb{P})$ generate a supporting hyperplane, $\{x \in R^n : \phi_{x^o}(x) = 0\}$, to X at this point. Since the support of the hyperplane on X is contained in $EF(X,\mathbb{P})$, move on this support from x^o to an adjacent extreme point. Due to the connectedness of the set of efficient points and the fact that the set of supporting hyperplanes at x^o, obtained when solving $Q(x^o)$ (or $DQ(x^o)$), characterize the efficie t facets incident to it, the algorithm will generate all the efficient extreme points and facets. Formally:

Algorithm

Initialization: Let L_1 be a list of efficient extreme points encountered; initially only x^o is in L_1. Let L_2 be a list of dual extreme points that correspond to maximal efficient facets; initially L_2 is empty. Let L_3 be a list of extreme points x^i such that $DQ(x^i)$ does not generate any optimal dual extreme points not already in L_2; initially L_3 is empty. Set k = 0.

Step 1: Solve the linear problem $DQ(x^k)$:

$$\begin{aligned} \min \quad & -\pi H x^k + \mu b \\ \text{subject to:} \quad & -\pi H + \mu A - \tau = p_s^+ \\ & (\pi,\mu,\tau) \geq 0 . \end{aligned}$$

Obtain all alternative solutions representing the maximal facets according to proposition 2.14, (i.e., those not contained in another efficient facet). Insert any of these dual extreme points in L_2 if it is not already included, and, for each of these points, $(\bar{\pi},\bar{\mu},\bar{\tau})$, solve the linear program:

Step 2: $H(\bar{\pi},\bar{\mu},\bar{\tau})$: $\max \quad \phi_{x^o}(x) = (p_s^+ + \bar{\pi}H)x + \bar{\mu}b$

subject to: $x \in X$.

By proposition 2.4, $\phi_{x^o}(x) \leq 0$ for all $x \in X$ and $\phi_{x^o}(x^o) = 0$. Hence, the optimal value of the problem is zero. By proposition 2.8, all alternative optimal solutions to $\max_{x \in X} \phi_{x^o}(x)$ are efficient. Insert any alternative optimal extreme point of $H(\bar{\pi},\bar{\mu},\bar{\tau})$ in L_1 if it is not already included.

Step 3: Add x^k to L_3. If $L_3 = L_1$, then terminate the algorithm. L_3 is the set of efficient extreme points and L_2 is the set of dual extreme points that define the maximal efficient facets. Otherwise, $L_3 \subsetneq L_1$; choose a point $x^k \in L_1 \setminus L_3$. Return to step 1.

The generation of all alternate optimal extreme point solutions to either $DQ(x^k)$ or $H(\bar{\pi},\bar{\mu},\bar{\tau})$ requires a careful enumeration and bookkeeping scheme based upon the characterization of alternate optima to linear programs.

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