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FUTURE DIRECTIONS OF LARGE SCALE MIXED INTEGER PROGRAMMING

by

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There are a growing number of large scale MIP models being built and optimized by practitioners. These models are usually investment planning problems, for example, investments in new facilities (e.g., Kazmi and Shapiro (1977)) or in new technologies (e.g., Nordhaus and van der Heyden (1977)), involving fixed costs, returns to scale and logical constraints on investment alternatives which require MIP modeling techniques. They can range in size up to 2000 or more constraints and several hundred integer, usually zero-one, variables.

The proliferation of MIP models is due in part to the existence of commercial computer codes, such as MPSX/370 on the IBM 370 computer, which make it possible for the practitioner to try to solve his/her model once it has been formulated without a separate project to develop an MIP computer code. The optimization of MIP models remains difficult and unpredictable in spite of these codes, and research into MIP algorithms continues. The main thrust of MIP research is not in algorithms,however, but rather in the development of new and interesting ways to generate and use MIP models. In spite of algorithmic inefficiencies, the greatest part of the time and money of a study requiring the construction of a large scale MIP model is spent on data collection and creating the model from the data, rather than optimizing it once it has been created.

The following is a list of new research areas in MIP which could have important payoffs in the next three to five years. I confess to some wishful thinking about scientifically interesting areas where breakthroughs may be difficult to achieve and leave it to the reader to decide for him/herself where scientific reality ends and science fiction begins. The topics we will discuss briefly are:

(1) MIP duality theory
(2) Benders' method for MIP
(3) Heuristics
(4) Multi-objective MIP
(5) MIP right hand side parametric analysis and Lagrangean techniques
(6) MIP model aggregation
(7) MIP investment planning models and econometrics
(8) Mathematical programming modeling languages

(1) MIP Duality Theory

MIP duality theory is concerned mainly with algorithmic methods based on Lagrangean techniques and group theory (see Shapiro (1971), Fisher and Shapiro (1974), Bell and Shapiro (1977); computational experience is reported in Fisher Northup and Shapiro (1975), D'Aversa (1978)). The theory was developed and implemented originally for pure IP problems, and we are currently extending the theory and implementation to large scale MIP problems (Northup and Shapiro (1978)).

Space does not permit a detailed development of MIP duality theory. The MIP dual problems are used to provide objective function lower bounds for use in MIP branch and bound methods, and they will sometimes provide optimal MIP solutions as well. The lower bounds are stronger than conventional LP lower bounds. The MIP dual problems are nondifferentiable optimization problems and they can be solved by a variety of methods including subgradient optimization, generalized linear programming, etc. If a given MIP dual problem fails to yield an optimal solution to the given MIP problem, then in most cases it can be strengthened to provide a new and stronger MIP dual problem.

(2) Benders' Method For MIP

Benders' method decomposes an MIP problem into a pure IP master problem and an LP subproblem. For large scale investment planning problems, it often is the case that the master problem involves only investment decision variables, such as the locations and sizes of plants, while the subproblem involves only operating variables such as commodity flows from manufacturing plants to markets, or energy delivered from electric power plants. Benders' method has the desirable feature that it readily produces feasible MIP solutions and lower bounds on the minimal MIP cost. This is in contrast to the branch and bound methods used in commercial codes which can take a long time to produce the first feasible MIP solution.

The computational efficiency of Benders' method is relatively unknown although a few successful applications can be cited. Geoffrion and Graves (1974) used it on a model to study the location of warehouses in a multi-commodity distribution system. For their model, the LP subproblem separated into a distribution problem for each commodity. Noonan and Giglio (1977) used Benders' method to optimize a model for the long range planning of electric power generation; Bloom (1978) has extended the model and the decomposition for solving it to study the reliability of power generating systems. For these problems, the

LP subproblem separates into an operating subproblem for each period of the planning horizon. Magnanti and Wong (1978) have used it successfully in the study of network design problems.

Some definitive experimentation with Benders' method is desirable, perhaps identifying the class of problems for which it is appropriate and effective means for integrating it with branch and bound. Another area of useful research is to extend Benders' method to permit MIP sensitivity analyses; for example, to characterize the set of demands in the operating subproblems of an electric power generation problem for which an optimal investment plan remains optimal.

(3) Heuristics

This is an area of considerable current research interest. The approach is to develop heuristics with known properties to generate feasible solutions to MIP problems with special structure. Cornuejols, Fisher and Nemhauser (1977) discovered a "greedy" heuristic for a class of uncapacitated plant location problems. They use Lagrangean techniques to direct the heuristic and to calculate an upper bound on the objective function error due to non-optimality. More work along these lines would clearly be very desirable, but there is a question about the effectiveness of heuristics for complex MIP models. Perhaps MIP model users should be encouraged to select as much as possible those models with simple structure for which effective heuristics can be derived.

(4) Multiobjective MIP

Multiple objectives are often appropriate for large scale investment planning. For example, electric power generation expansion plans should be studied using a variety of curves for construction costs or the relative costs of oil, gas and coal. Moreover, when we consider the amount of work that goes into collecting the data and generating a large scale MIP model, it makes sense to analyze the model in such a way that many "promising" solutions are generated, rather than a single "optimal" solution. Multi-objective optimization methods are a systematic means for describing the characteristics of promising solutions.

Relatively little theoretical or applied work on multi-objective MIP has been done in spite of its promise as a useful decision making tool (Shapiro (1976), Bitran (1977). The methodological approach is relatively straightforward and we give a few indications here.

Suppose we have K objective functions $c^k x + f^k y$ for an MIP problem with constraints

$$Qx + Py \leq r \tag{1a}$$

$$Ax \quad = b \tag{1b}$$

$$x_j = 0 \text{ or } 1 \qquad y \geq 0 \ . \tag{1c}$$

A feasible solution $\bar{x}$, $\bar{y}$ is called <u>efficient</u> (Pareto optimal) if there does not exist a feasible $\tilde{x}$, $\tilde{y}$ such that $c^k\tilde{x} + f^k\tilde{y} \leq c^k\bar{x} + c^k\bar{y}$ for all k, with strict inequality for some k. Efficient solutions can be generated as follows:

let

$$S = \{\lambda \varepsilon R^K \mid \sum_{k=1}^{K} \lambda_k = 1, \ \lambda_k \geq 0\},$$

and define the MIP problem

$$v(\lambda) = \min \sum_{k=1}^{K} \lambda_k (c^k x + f^k y) \tag{2}$$

$$\text{s.t. (1a), (1b), (1c) .}$$

The function $v(\lambda)$ is concave and if all λ_k are positive, then any optimal solution to (2) is efficient. There are, however, efficient solutions which cannot be generated this way; see Shapiro (1976).

The idea of multi-objective MIP would be to estimate $v(\lambda)$ for all $\lambda \varepsilon S$, for all subproblems generated during branch and bound. Specifically, we would use an upper bound function $\bar{v}(\lambda) \geq v(\lambda)$ generated from previously discovered feasible solutions to the MIP subproblem, and a lower bound function $\underline{v}(\lambda) \leq v(\lambda)$ generated from an MIP dual problem. The given subproblem would be analyzed parametrically until $\max \{|\bar{v}(\lambda) - v(\lambda)|, |\underline{v}(\lambda) - v(\lambda)| \;\big|\; \lambda \varepsilon S\}$ is sufficiently small.

(5) MIP Parametric Right Hand Side Analysis and Lagrangean Techniques

A number of large scale MIP investment models are demand driven; that is, the presence of constraints requiring exogenous demand to be satisfied forces the expense of investment to be incurred. Since demand is not known with certainty, it would be useful to perform parametric analysis with respect to it. Systematic procedures for such parametrics are not available for MIP as they are for LP, but Lagrangean techniques can be used to implicitly drive the right hand side parametric analysis.

To be precise, consider the following capacitated plant location problem

$$v(d) = \min \sum_i \sum_j (c_{ij} + v_i)x_{ij} + \sum_i f_i y_i \tag{3a}$$

$$\text{s.t.} \quad \sum_i x_{ij} = d_j \qquad j = 1,\ldots,n \tag{3b}$$

$$\sum_j x_{ij} - K_i y_i \leq 0 \qquad i = 1,\ldots,m, \tag{3c}$$

$$x_{ij} \geq 0, \qquad y_i = 0 \text{ or } 1 . \tag{3d}$$

We dualize on the demand constraints and form the Lagrangean

$$L(u) = \sum_j u_j d_j + \min \sum_i \sum_j (c_{ij} + v_i - u_j)x_{ij} + \sum_i f_i y_i$$

$$\text{s.t. (3c) and (3d) .}$$

The quantity L(u) is easier to compute because

$$\{\min_j (c_{ij} + v_i - u_j)\} K_i + f_i \geq 0 \Rightarrow y_i = 0 .$$

The solution $\bar{x}_{ij}$, $\bar{y}_i$ satisfying (3c) and (3d) and the dual variables $\bar{u}_j$ are said to satisfy the <u>global optimality conditions</u> for (3) if

(i) $L(\bar{u}) = \sum_j \bar{u}_j d_j + \sum_i \sum_j (c_{ij} + v_i - \bar{u}_j) \bar{x}_{ij} + \sum_i f_i \bar{y}_i$

(ii) $\sum_i \bar{x}_{ij} = d_j$, $j = 1,\ldots,n$.

The idea behind the parametric right hand side analysis is to change d if global optimality condition (ii) is not satisfied. Specifically, let $\bar{d}_j = \sum_i \bar{x}_{ij}$, $j = 1,\ldots,n$; then $\bar{x}_{ij}$, $\bar{y}_i$ is optimal in (3) with $d_j = \bar{d}_j$ and moreover $v(\bar{d}) = L(\bar{u})$. We have called this approach to MIP <u>inverse optimization</u> and we are studying its application to problems such as (3) (Bitran, Elliott and Shapiro, (1978)). Typical questions being studied are:

- What are the points $d \geq 0$ such that $v(d) = L(u)$ for some u?
- How can we strengthen the dual analysis for those points $d \geq 0$ such that $v(d) > L(u)$
- How should the dual variables u be selected?

(6) Aggregation and MIP Models

Investment planning models can easily become unmanageably large because of the multiplicative nature of the factors to be considered such as the number of time periods, types of plants, levels in the manufacturing or operating system, markets, etc. Some aggregation is usually required, and research has begun on systematic theories for doing it (Zipkin (1977), Geoffrion (1976)). The relationship of aggregation to decomposition theory needs elucidation. In addition, there may be some important relationships to be discovered and exploited between theories for aggregation of econometric forecasting models (Ijiri (1971)) and for aggregation of mathematical programming models.

(7) MIP Investment Planning Models and Econometrics

Thus far, we have discussed MIP investment planning models with exogenous (fixed) demand for output from the system under study. Many public and private investment studies involve endogenous (variable) demand (e.g., see Erlenkotter (1977)). Endogenous demand is described by an econometric model F which predicts demand d as a function of price; that is, $d = F(p)$.

A typical equilibrium problem is

$$\begin{aligned} &\max \{f(d) - cx\} \\ \text{s.t.} \quad & Ax - d \geq 0 \\ & Qx \leq r \\ & x \geq 0 \ , \ d \geq 0 \end{aligned} \tag{4}$$

where $f(d)$ is a consumer surplus function measuring the benefit of satisfied demand to the consumer. The Kuhn-Tucker optimality conditions for this problem are called economic equilibrium conditions by economists. The function f is related to the econometric model F by $f(d) = \int_0^d F^{-1}(\xi)\, d\xi$ where we have assume for convenience that F is invertable and interchangeable. In other words, $\nabla f(d) = F^{-1}(d)$. An iterative scheme for solving the equilibrium problem is shown in figure 1. The vector Π are the shadow prices on the demand row in (5) for d fixed. Equilibrium has been reached if it equals the commodity price vector p such that $F(p) = d$; we compute p by inverting the function F at d, possibly by numerical means. Kennedy (1974) gives a model similar to (4) for studying the world oil market. Florian and Nguyen (1976) present a more distantly related model for traffic equilibrium.

There are a number of MIP extensions of the equilibrium model just described. For example, suppose there are new technologies with associated fixed costs

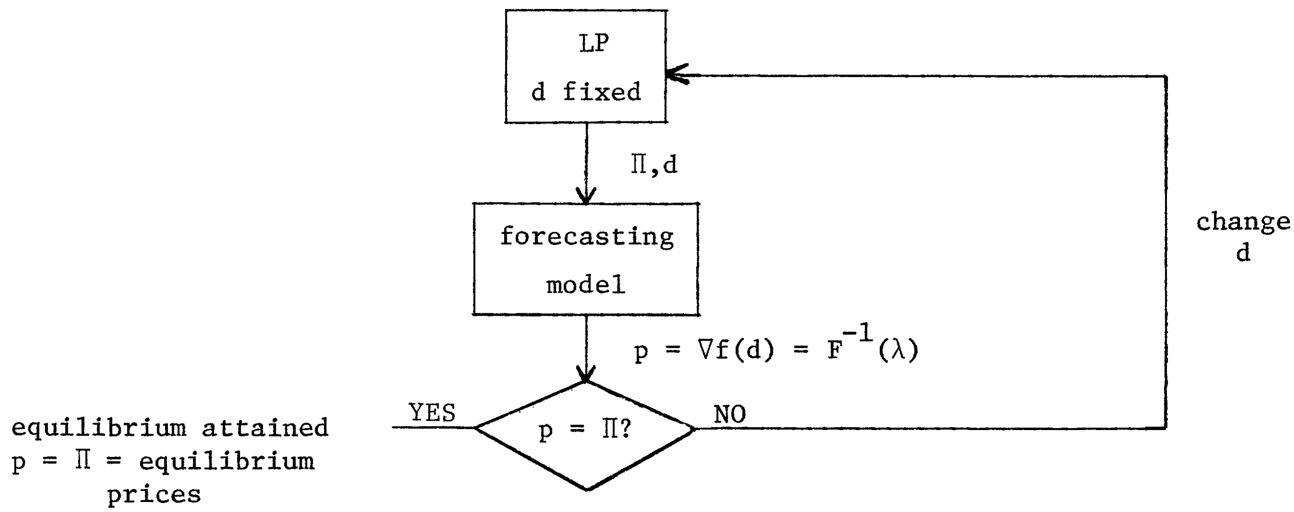

Figure 1

available to be used in meeting demand. An MIP model describing this situation is

$$\begin{aligned} &\max \{f(d) - cx - fy - dz\} \\ \text{s.t.} \quad & A^0 z + Ax - d \geq 0 \\ & Q^0 z + Qx \leq r \\ & z_j - K_j y_j \leq 0 \\ & z \geq 0,\ x \geq 0,\ d \geq 0, \quad y_j = 0 \text{ or } 1 \end{aligned} \tag{5}$$

A Benders' type decomposition of such a model is depicted in figure 2. The investment subproblem is an IP that is used to select the new technologies on the basis of post equilibrium price information. This type of model would be appropriate for studying new energy technologies or new configurations of road network systems each inducing its own traffic equilibrium. It has been used by Armstrong and Willis (1977) to study water resource investment planning and allocation decisions.

(8) Mathematical Programming Modeling Language

The size and form of a large scale MIP problem is critically important in determining the difficulty of optimizing it. We have seen that decomposition and

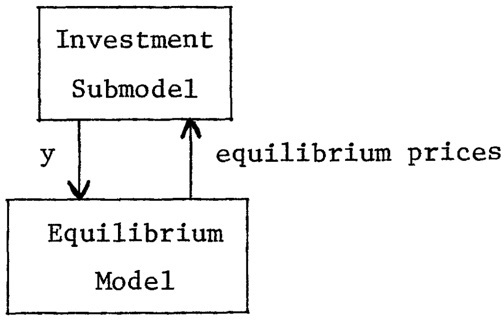

Figure 2

aggregation methods can be very effective in modularizing and transforming MIP problems to useful forms. Mathematical programming languages currently under development should greatly facilitate these model manipulations and analyses (Kuh and Zisman (1978)). In a similar vein, IBM has developed an Extended Control Language (ECL) for MPSX/370 which permits recursive use of the LP and MIP algorithms (Slate and Spielberg (1978)). Computer language developments are perhaps the most important direction of future research in large scale mathematical programming.

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