OPERATIONS RESEARCH CENTER

working paper

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

SOLUTION OF THE RETIRING SEARCH PROBLEM

by

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OR 070-77 December 1977

ABSTRACT

The well-known variational principle of the optimal distribution of search effort is applied to the case where the sought-for target has been located momentarily but the search cannot begin until a time T_o later, during which interval the target may or may not move. The optimal search path, called the retiring search curve, is a spiral starting at the point of original location, with the relationship between r and θ or L, the length of path, given by a differential equation derived from the variational principle. This equation is solved for a range of values of the parameters of greatest practical interest. The results are tabulated and also displayed in graphical form for use in practice. General conclusions emerge, of considerable practical value in searches of this kind.

Introduction.

It is fairly evident that much of the mathematical elaboration of search theory has contributed little to solving the practical problem of conducting an actual search, whether for a submarine, a ship in trouble or a lost child. Indeed, the rough and ready formulas[1] developed during World War II are still the basis for most search plans. Among these formulas, the most important was the logarithmic rule[2] for the distribution of search effort among various regions having different probability of presence of the target.

One search problem, that of search for a possibly moving target that is known to be at a given point at some given time prior to the commencement of search, was not solved quantitatively. The appropriate tactic, called retiring search ,was discussed qualitatively, but a solution utilizing the rule of optimal distribution of search effort was not obtained. The present paper is intended to satisfy that lack. It is not expected that the details of the retiring search spirals, developed here, can be followed exactly in practice, but it is hoped that the quantitative conclusions presented here can serve to discourage the expenditure of unprofitable search effort and can serve to emphasize some of the basic characteristics, such as the rapid decline in chance of success with increase of the lost time interval T_o between the initial location of the target and the start of the search, and the quickly diminishing returns from search carried on longer than time T_o (i.e., beyond time $2T_o$ after the initial location of the target).

The problem is one often encountered in anti-submarine warfare but also in many other circumstances, such as small-boat rescue or the search for a lost child, for example. The target (U-boat, small-craft, child) is known to be at a given point (called the origin) at a given time but the search vessel (ship, aircraft, helicopter, etc.) cannot arrive at the origin until a time T_o (called the lost time) later. The target may have remained at the origin, or it may have moved in any direction at any velocity up to an estimated maximum speed u. The initial location could come from the last signal from the target itself or from the observation of a passer-by who could not stop. Therefore the area within which the target may be found (the area of presence) is, in the simplest case, bounded by a circle with radius which expands at rate u. (This assumes an unlimited field of action; for cases with limits on the area of presence or with preconceptions as to preferred directions, see the comments at the end of this paper). By the time the search can start this area of presence has radius uT_o and area $A(0) = \pi(uT_o)^2$; at time t after the search has started the area is $A(t) = \pi u^2(T_o + t)^2$.

Incidentally, the initial location of the target need not be known precisely; the analysis will not be appreciably altered as long as the error of location of the origin is less than about half the radius uT_o of the area of presence at the time the search begins. Because of this lack of knowledge and because of the uncertainty as to the motion of the target during the lost time, the probability density $p(r)$ of presence of the target is a maximum at the origin and decreases to zero for $r > u(T_o + t)$. A reasonable assumption is the following:

$$p(r) = \begin{cases} \frac{2}{A(t)}\left[1 - \frac{\pi r^2}{A(t)}\right] & (\pi r^2 < A(t)) \\ 0 & (\pi r^2 > A(t)) \end{cases} \tag{1}$$

The search vessel has speed v and its detection equipment has effective <u>search width</u> W, so it can search an effective area $dE = vW\,dt$ in time dt, where $E(t) = Wvt$ is the <u>search effort</u> spent in time t. If this search is to be spread over an area dq in time dt, the <u>density of search</u> coverage is then $\phi = dE/dq = vW(dt/dq)$. At any time t after the start of the search this density of search is to be related to the probability of presence of the target in accord with the principle of optimal distribution of search[2].

The search procedure would be to start from the origin, since the probability of presence is greatest there, and to spiral out, with the search density being $\phi(r)$ when the search vessel has reached a distance r from the origin. The search track need not be a simple spiral, as long as the search density. in the annulus between r and $r + dr$ is $\phi(r)$. However a spiral path is probably simplest (see comments at the end of this paper) with a spacing S between turns of the spiral such that $\phi(r) = W/S$.

For convenience we list the definitions of the various quantities to be dealt with, before proceding further:

T_o ≡ Lost Time = Time between initial location of target at the origin and the commencement of search.

t = Time after start of search; $t + T_o$ = time after initial location.

u = Estimated maximum speed of target.

v = Speed of search vessel.

W = Effective search width of searcher.

$L \equiv vt$ = Length of search path since start of search.

$E(t) \equiv Wvt$ = Search effort used up in time t.

$z \equiv (t/T_o) = (L/vT_o)$ = Time spent in search in units of lost time.

$A(t) \equiv \pi(uT_o)^2(1+z)^2$ = Area of presence of target at time t after start of search.

$k \equiv \pi(u/v)(uT_o/W) = A(0)/E(T_o)$ = Ratio between area of presence at start of search to search effort that could have been made during lost time T_o.

$r(t)$ = Radial distance of search vessel from origin at time t after start of search, required by principle of optimum distribution of search effort.

$q(t) \equiv \pi r^2 = A(0)x$ = area already searched over by time t.

$x \equiv q(t)/A(0) = (r/uT_o)^2$ = Ratio of searched-over area to area of presence of target at start of search.

$\phi(t) \equiv dE/dq = (1/k)(dz/dx)$ = density of search coverage at time $t = T_o z$ and radius $r = uT_o\sqrt{x}$.

S = Spacing between successive tracks of search vessel.

$p(r) \equiv \frac{2}{A(t)}\left[1 - \frac{q(t)}{A(t)}\right]$ = probability density of presence of target at time $t = T_o z$ and radius $r(t) = uT_o\sqrt{x}$.

$t_m \equiv T_o z_m = E_m/Wv$ = Total time allotted to search effort. Optimal allocation principle requires that $\phi(t_m) = 0$.

$\wp$ = Probability of discovery of target by end of search (by t_m).

r_p See Eq.(11).

The Equation of Motion.

The conditional probability of finding the target in area element dq, if the target is present, is related to the density of search ϕ at dq. If the search path is piecewise random[3] this probability is equal to $1 - e^{-\phi}$. Very careful spacing of sequential search tracks can result in a somewhat larger value than this but, in view of the difficulty of spacing, it is safer to use this simpler formula.

If this formula is used, the principle of optimal distribution of search effort requires that the search start at the region of highest probability of presence, the origin, and proceed monotonically to lower probabilities, with the density of search ϕ everywhere related to the probability of presence p(r) according to the formula

$$\phi = \begin{cases} \ln(p/G) & (p > G) \\ 0 & (p < G) \end{cases} \qquad (2)$$

The value of the constant G is adjusted so that the search ends when the predetermined search effort E_m has been expended.

However, from the definitions listed above,

$$\phi = \frac{dE}{dq} = \frac{WvT_0}{A(0)}\frac{dz}{dx} = \ln(\frac{p}{G}) \qquad (p > G) \qquad \text{or}$$

$$k\phi = \frac{dz}{dx} = \begin{cases} k\ln\left[F(x,z)/\lambda\right] & (F > \lambda) \\ 0 & (F < \lambda) \end{cases} \qquad (3)$$

where $\lambda = \frac{1}{2}\pi(uT_0)^2G$ and $F(x,z) = (1+z)^{-2}\left[1 - x(1+z)^{-2}\right]$

This differential equation can be solved numerically[4]. For each step in the integration dz/dx must be adjusted, by successive trials, so that the integrated value of $z_n = z_{n-1} + (dz/dx)_{av}dx$ yields the value of $(dz/dx)_n$ required by Eq.(3). Integration is continued until $dz/dx \equiv k\phi = 0$, when the search stops.[5]

The integration results, for each trial value of λ, in a numerical relationship between z, which is proportional to the search effort already expended (or to the time already spent searching) and x, which is proportional to the area already searched over (or to the square of the distance from the origin). In other words the solution gives us the distance r the search vessel should be from the origin at time t after the start. If the search path is a spiral, with spacing between successive turns equal to S, then the first part of Eq.(3) shows that

$$\phi \equiv \frac{W}{S} = \frac{1}{k}\frac{dz}{dx} = \frac{WvT_o}{\pi(uT_o)^2}\frac{d(L/vT_o)}{d(r/uT_o)^2} = \frac{W\,dL}{2\pi r\,dr} \quad \text{or} \quad \frac{dr}{dL} = \frac{S}{2\pi r} \qquad (4)$$

indicating that the rate of increase of r with length of search path is such as to lead to a spacing, between successive turns of the spiral, of S, in conformity with the definition of ϕ.

Equation (3) has been integrated from $x = 0$ to x_m, where $(dz/dx) = 0$. Successive trial integrations were made to find the values of $\lambda = \frac{1}{2}\pi(uT_o)^2 G$ for which

$$z_m/k \equiv E(t_m)/A(0) \equiv vWt_m/\pi(uT_o)^2 \qquad (5)$$

$$= 0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.6, 0.8, 1.0, 1.2, 1.6, 2.0, 2.4, 3.0$$

and

$$k \equiv A(0)/E(T_o) = 0,\ 0.5,\ 1.0,\ 1.5 \qquad (6)$$

This inverts the solution, to provide x_m and λ as functions of z_m/k, the parameter measuring the maximum expendible search effort.

The special case of $k = 0$, where the search speed v is much greater than the maximum target speed u and/or the search width W is much greater than the radius uT_o of the circle of presence at the start of the search, can be worked out by setting $k = 0$ in Eq.(3) and solving the resulting equation for dy/dx $(y = z/k)$,

$$\frac{dy}{dx} = \ln\left\{\frac{1}{\lambda(1+ky)^2}\left[1 - \frac{x}{(1+ky)^2}\right]\right\} \longrightarrow \ln(1-x) - \ln\lambda$$

$$y \longrightarrow -(1-x)\ln(1-x) - (1+\ln\lambda)x \qquad (7)$$

Search ceases when $dy/dx \equiv \phi = 0$, so $x_m = 1 - \lambda$ for this case, and

$$z_m/k = -\lambda\ln\lambda - (1-\lambda)(1+\ln\lambda) = -1 + \lambda - \ln\lambda \qquad (8)$$

This last equation can be solved to find λ as function of z_m/k for the $k = 0$ case.

Values of x_m and $\sqrt{x_m}$ ($= r_m/uT_o$) are tabulated in Table I for the specified values of z_m/k and k. Values of λ and $\ln(1/\lambda)$ ($= \phi$ for $r = 0$) are given in Table II. The large values of $\phi(0)$ are not surprising; when total search effort is large the principle of optimal distribution of search often specifies search coverage ϕ greater than unity for the most promising regions.

Good approximations to the solution of Eq.(3), giving exact values of (z_m/k) for $x = 0$ and $x = x_m$ and discrepancies of 4 percent or less for intermediate values, are

$$\frac{z}{k} \simeq \left(x - \frac{x_m}{1-\lambda}\right)\ln\left[1 - \frac{x}{x_m}(1-\lambda)\right] - x(1 + \ln\lambda) - \frac{ax^2}{6}\left(3 - 2\frac{x}{x_m}\right)$$

$$\frac{1}{k}\frac{dz}{dx} = \phi(x) \simeq \ln\left[1 - \frac{x}{x_m}(1-\lambda)\right] - \ln\lambda - ax\left(1 - \frac{x}{x_m}\right) \qquad (9)$$

$$a = -\frac{6}{x_m^2}\left[x_m + (z_m/k) + x_m\frac{\ln\lambda}{1-\lambda}\right]$$

Note that for $k = 0$, when $x_m = 1 - \lambda$ and $a = 0$, the formulas are exact solutions. Values of a are given in Table III.

The other quantity of interest is the probability $\mathcal{P}$ of discovering the target during the search. From the earlier discussion we see that $\mathcal{P}$ is the integral of the probability of presence $p\,dq$ of the target in the area element dq, times the conditional probability $(1 - e^{-\phi})$ of finding the target if it is in dq. Using the relationships already developed we have

$$\mathcal{P} = \int(1 - e^{-\phi})p\,dq = \int_0^{x_m}(1 - e^{-\phi})(Ge^{\phi})A(0)dx$$

$$= \pi(uT_o)^2G\int_0^{x_m}(e^{\phi} - 1)dx = 2\lambda\int_0^{x_m}\left[\exp\left(\frac{1}{k}\frac{dz}{dx}\right) - 1\right]dx$$

$$= 2\int_0^{x_m}\left[F(x) - \lambda\right]dx \qquad (10)$$

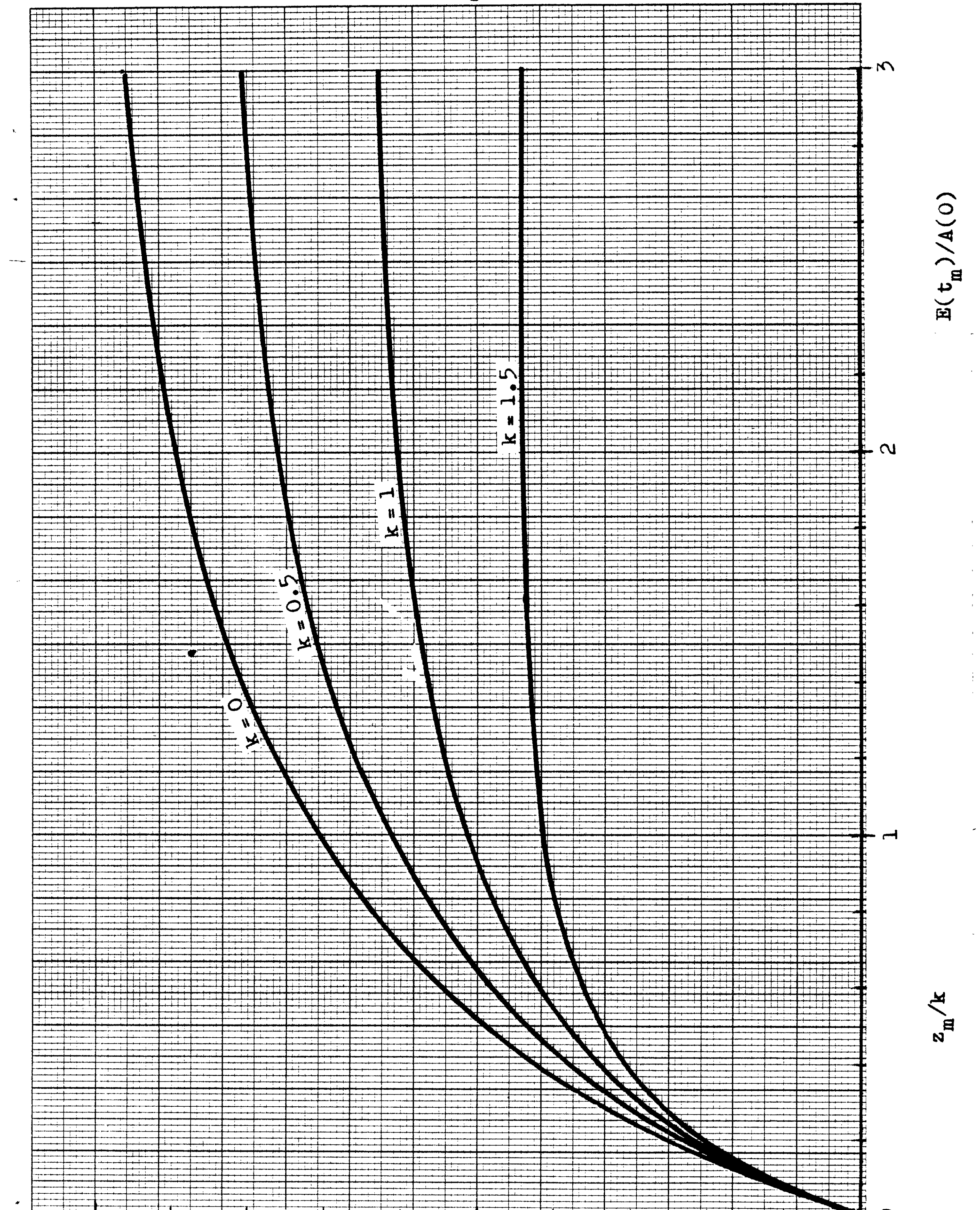
where F(x) is F(x,z) with z(x) being a solution of Eq.(3).

Values of $\mathcal{P}$ are given in Table III.

Curves of $\mathcal{P}$ as function of $(z_m/k) = E(t_m)/A(0)$ (total area covered by search divided by area of presence at start of search), for different values of search slowness parameter k) are shown in Fig. 1. We see that as k increases (as the lost time T_o increases and/or as the search speed v decreases) the chance of ever catching the target decreases. The asymptotic value of $\mathcal{P}$, for unlimited search effort, is roughly equal to $2/(2+k^2)$, at least for the range of k tabulated. For slow search vessels and/or for long delay in starting, the search, spiralling out from the origin, has less and less chance of covering the expanding area of presence of the target.

We also see that the law of diminishing returns sets in quickly after $(z_m/k) = (1/k)$. In other words, spending more time in search than an amount roughly equal to the lost time T_o yields very little additional chance of detection. This can be seen in terms of a cruder model, where we assume the target is equally likely to be anywhere within the area of presence $A(t) = \pi u^2(T_o + t)^2$. The searched area is $E(t) = vWt$ and the fraction of the area of presence covered by the search at any time is $\Phi = \frac{E(t)}{A(t)} = \frac{vW}{\pi u^2} \frac{t}{(T_o+t)^2}$. This has its maximum value Φ_m at $t = T_o$; beyond that, the search effort cannot keep up with the expansion of the area of presence. This maximum value $\Phi_m = (vW/4\pi u^2 T_o^2)$, which is equal to $(1/4k)$ in the notation of this paper. The larger k is, the smaller is Φ_m. Use of the more sophisticated model discussed in this paper predicts a somewhat larger chance of detection when the search starts at

Figure 1.



the center and spirals out according to the solution of Eq.(3). However the general conclusions remain; delay in starting the search diminishes seriously the chance of success and extending the search time longer than T_o does not add much to that chance.

Discussion of Results.

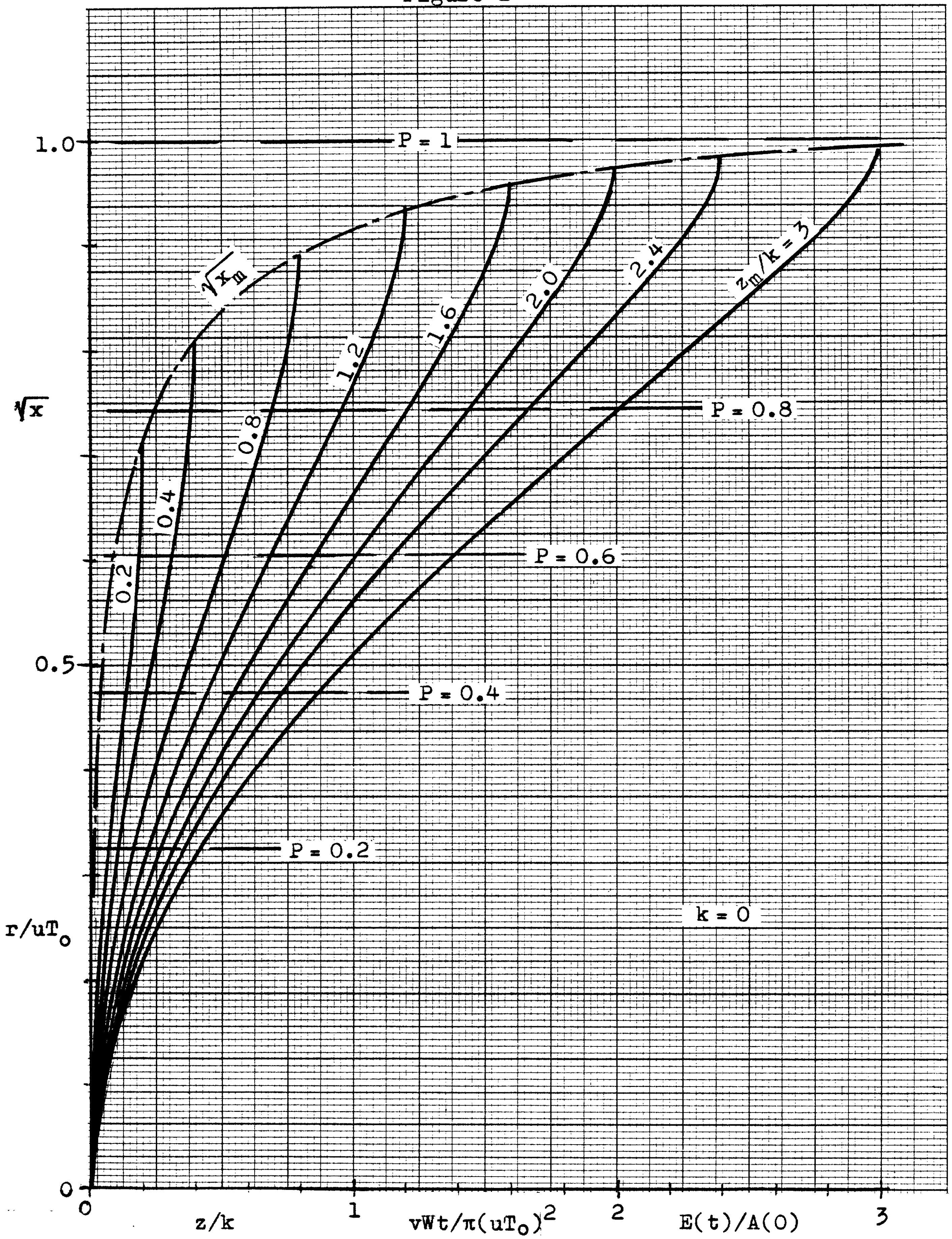
The solutions of Eq.(3) can most usefully be displayed by plotting $\sqrt{x} = r/uT_o$ as function of $z/k = E(t)/A(0)$, indicating the distance from the origin the search vessel should be after it has been searching for a time t. These plots are shown in Figs. 2 to 5, each for a different value of k. The curves are fairly similar; r increases rapidly at first because the small area near the origin can be covered fairly quickly. Then, if more search effort has been planned (if z_m/k is larger than 0.4) the increase in r slows down for a while; more effort is needed to cover the next annulus. Near the end of the run r increases more rapidly again in an effort to cover, at least partially, the expanding area of presence.[5]

The dashed lines in each Figure show the radius r_p, within which the probability of presence is P, according to the equation

$$P = 2w^2 - w^4 \quad ; \quad w = \left[r_p/u(T_o + t)\right] \qquad (11)$$

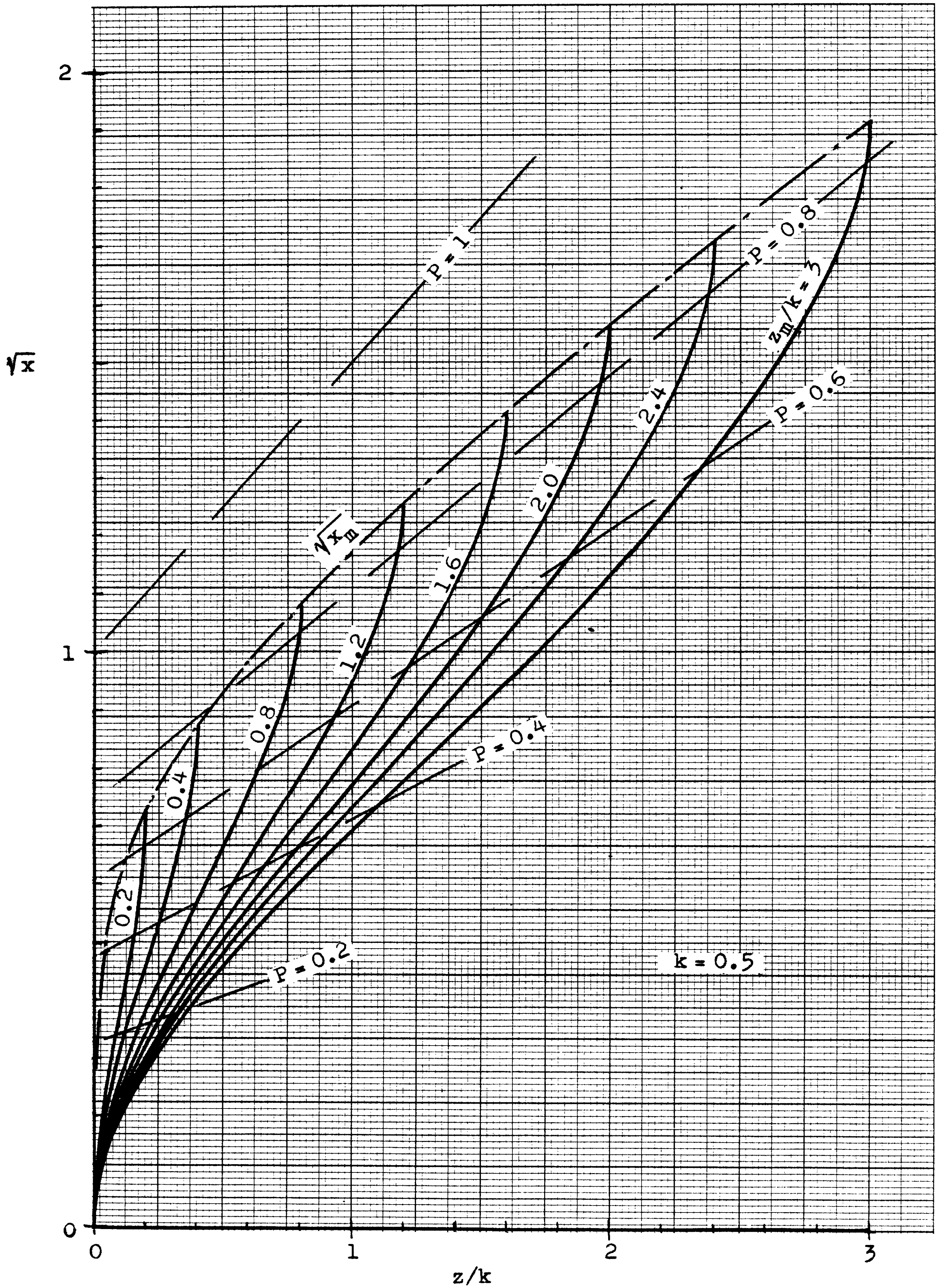
These lines are horizontal for $k = 0$ because here the search rate vW is so much greater than the initial rate of increase $2\pi u^2 T_o$ of the area of presence that the search is completed before the area has a chance to expand. For the other three values of k the area of presence expands appreciably during the search, so r_p expands linearly with z/k, the faster the larger k is.

Figure 2



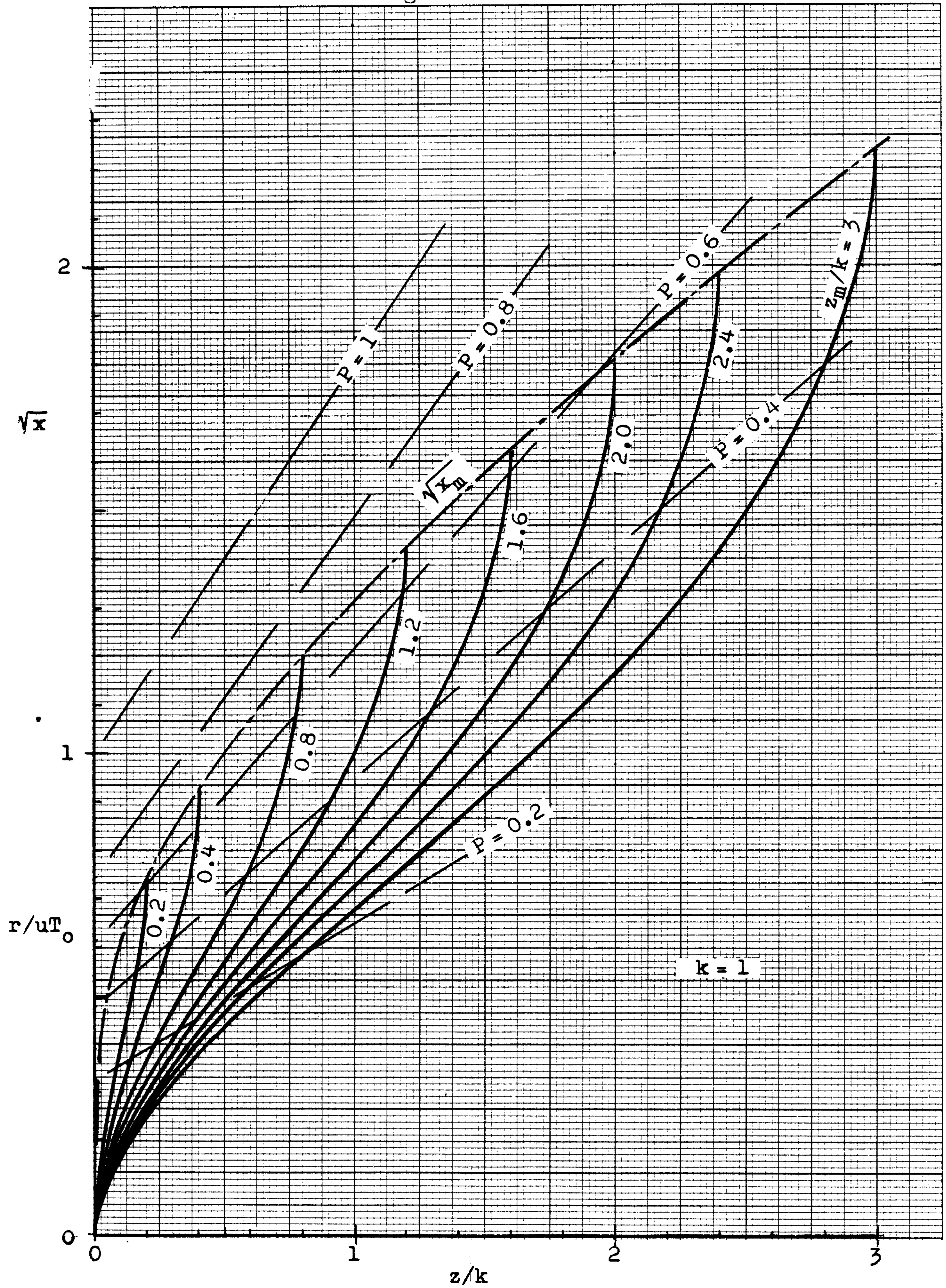
1.0
0.5
0
$\sqrt{x}$
r/uT_o
P = 1
P = 0.8
P = 0.6
P = 0.4
P = 0.2
$\sqrt{x_m}$
0.2
0.4
0.8
1.2
1.6
2.0
2.4
$z_m/k = 3$
k = 0
0
1
2
3
z/k
$vWt/\pi(uT_o)^2$
E(t)/A(0)

Figure 3



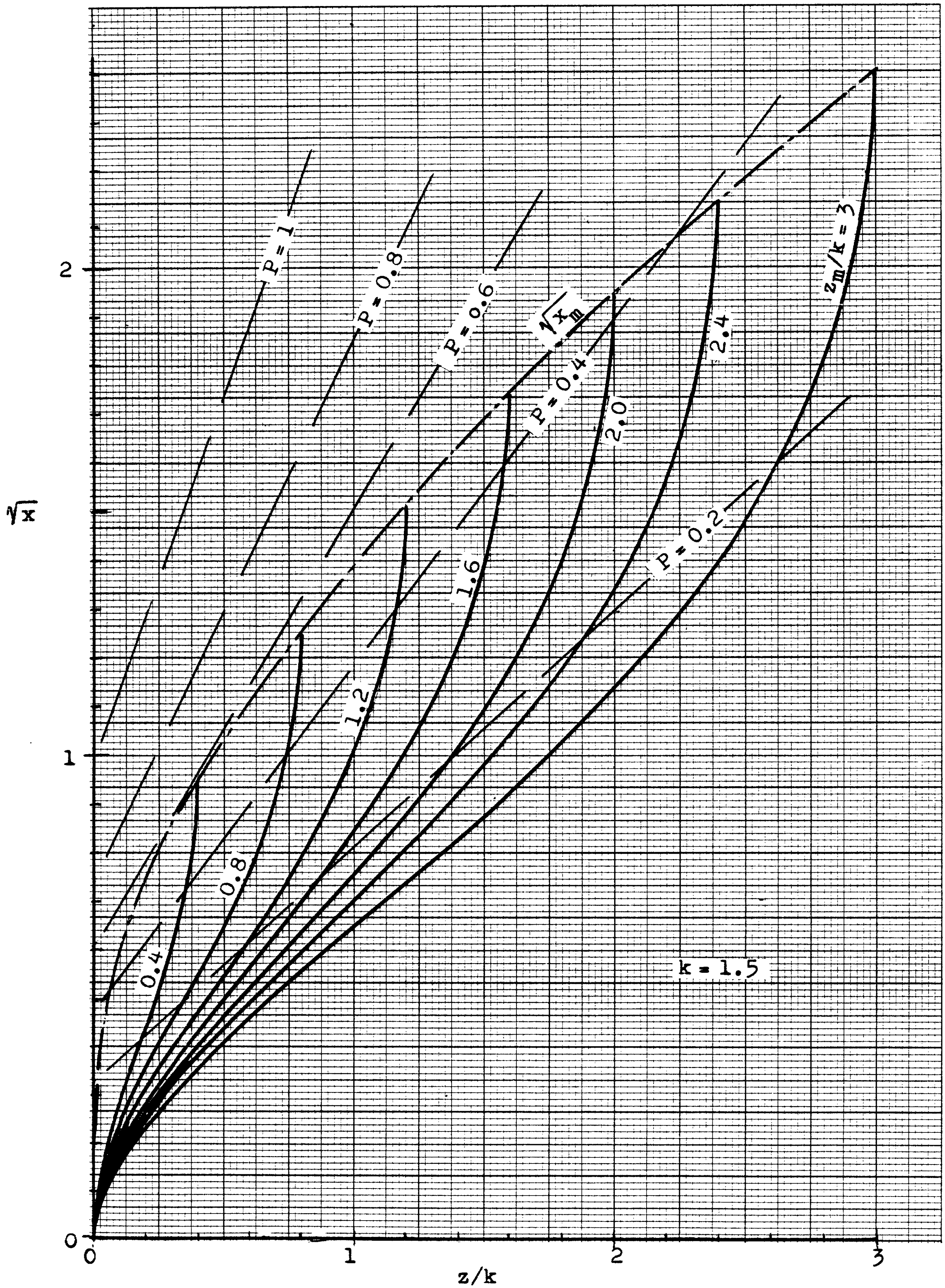
2
1
0
$\sqrt{x}$
0
1
2
3
z/k
P = 1
P = 0.8
P = 0.6
P = 0.4
P = 0.2
$z_m/k = 3$
2.4
2.0
1.6
1.2
0.8
0.4
0.2
$\sqrt{x_m}$
k = 0.5

Figure 4.



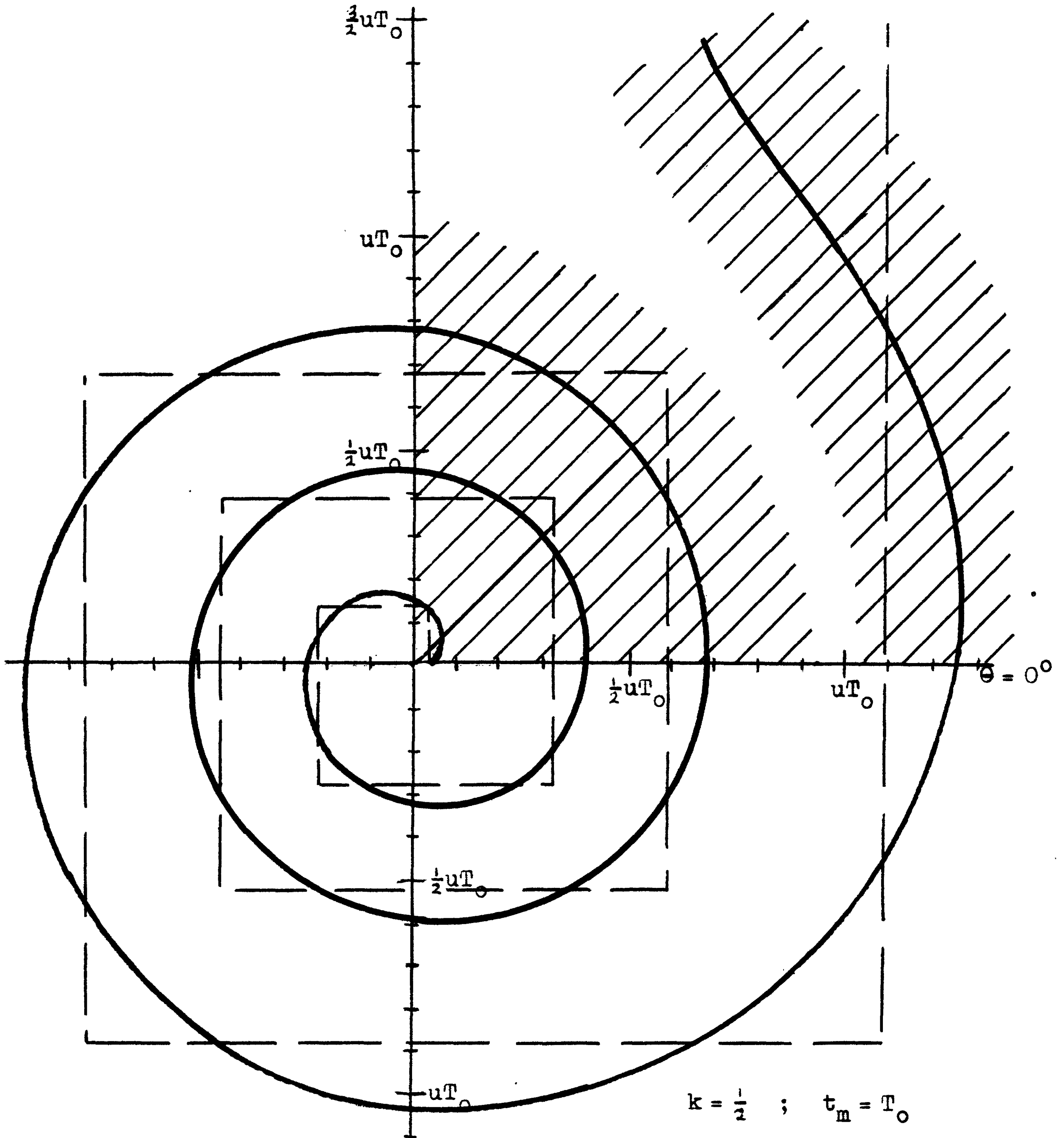
2
1
0
0
1
2
3
z/k
$\sqrt{x}$
r/uT_o
P = 1
P = 0.8
P = 0.6
P = 0.4
P = 0.2
$\sqrt{x_m}$
$z_m/k = 3$
2.4
2.0
1.6
1.2
0.8
0.4
0.2
k = 1

Figure 5



P = 1
P = 0.8
P = 0.6
P = 0.4
P = 0.2
$\sqrt{x_m}$
$z_m/k = 3$
2.4
2.0
1.6
1.2
0.8
0.4
k = 1.5
$\sqrt{x}$
z/k
0
1
2
3

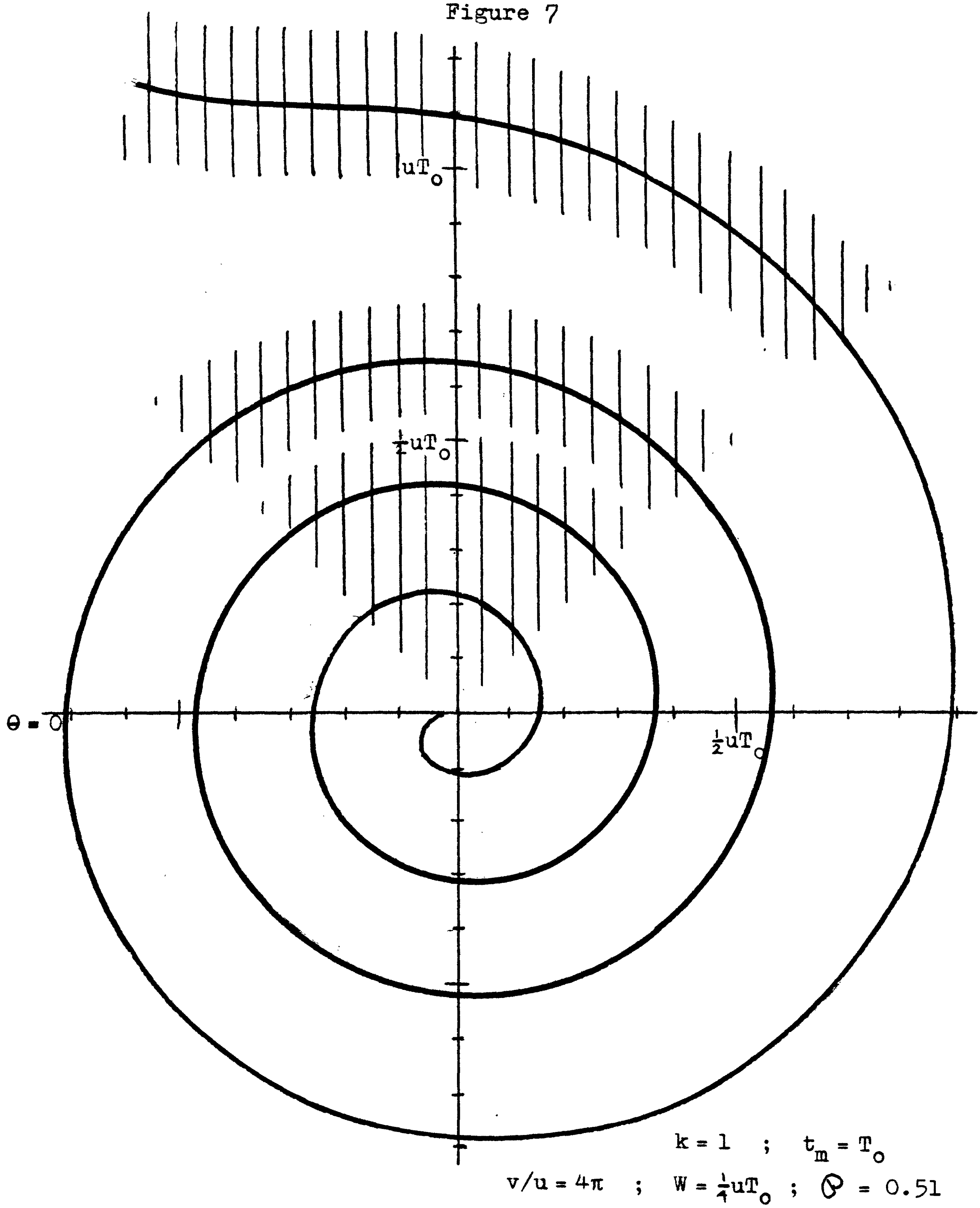
Figure 6



$k = \frac{1}{2}$; $t_m = T_o$

$v/u = 4\pi$; $W = \frac{1}{2}uT_o$; $\wp = 0.76$

Figure 7



For $k = 0$ (Fig.2) the search, if it continues long enough, can cover nearly all of the area of presence before this area expands too far. Therefore the curve for the probability of success in Fig.1 for $k = 0$ approaches unity asymptotically. On the other hand the dashed lines for $k > 0$ (Figs.3, 4, 5) are slanted upward, the larger the value of k the steeper. In these cases the search cannot reach to the edge of the area of presence if the search is to be as thorough as needed near the origin, where the chance of detection is the greatest. Those r against t curves for $k = 0.5$ stop just beyond the $P = 0.8$ line, whereas those for $k = 1.5$ fall short of the $P = 0.6$ line. By searching thoroughly near the origin (the most rewarding region, especially at first) the searcher has lost the chance of ever catching up with the expanding area of presence. Additional search time just falls farther behind.

Figures 6 and 7 are typical examples of retiring search spirals, one for $k = \frac{1}{2}$ and $z_m/k = 2$ ($v/u = 4\pi$ and $W = uT_o/2$, for example) and one for $k = 1$ and $z_m/k = 1$ ($v/u = 4\pi$ and $W = uT_o/4$, for example, though it could be for $v/u = 2\pi$ and $W = uT_o/2$). The differential equation relating r and θ for the spiral[5] comes from the usual one relating the differential length of search path $dL = v\,dt = vT_o\,dz$ to $r\,d\theta$ and dr, where $r = uT_o\sqrt{x}$,

$$(dL)^2 = (r\,d\theta)^2 + (dr)^2 \quad \text{or} \quad \left(\frac{d\theta}{dx}\right)^2 = \frac{1}{x}\left[\left(\frac{v}{u}\right)^2\left(\frac{dz}{dx}\right)^2 - \frac{1}{4x}\right] \qquad (12)$$

Approximate values of (dz/dx) can be obtained from Eqs.(9) and the Tables. The search coverage is shown as a shaded band of width $W/2$ on each side of the search path, for one quadrant. If the bands overlap the shading is continuous.

In Fig. 6 ($k = \frac{1}{2}$) the search speed is great enough and/or the sweep width is large enough so that the spacing S between spiral turns is less than W (and thus the coverage ϕ is greater than unity) for about $2\frac{1}{2}$ circuits, out to where P is about 0.6. Beyond this there are gaps in the coverage as the search path attempts to cover at least part of the rapidly expanding circle of presence. Thus the probability of success is about 3/4, as seen in Table III. In Fig. 7 the sweep width is half as large so that, although the time spent in search is the same as for Fig. 6, the total search effort $E(t_m)$ is half as large. Thus more time has to be spent in the inner regions, in order to cover it as completely as required by the optimal principle. The tighter spiral winds about 300° more around the origin than does that of Fig. 6, though the total length of path is the same. In Fig. 7, however, gaps in the swept path (as shown by the unshaded strips) begin to occur by the time P has reached 0.4 (see also Fig. 4) and the non-searched gap grows rapidly during the last circuit. Consequently the total chance of success for this case is only about 1/2, as indicated in Table III.

As mentioned in the first section, Fig. 1 shows that by the time the search has lasted a time equal to the lost time T_o, at least 80 percent of the ultimate chance of success has been achieved. Also, no matter how much additional search effort is expended, the largest probability of success one can hope to reach is roughly $1/(1 + k^2)$ (at least for $k \leq 1.5$).

The retiring search path can of course be a rectangular spiral if this is easier to execute. The successive straight elements should equal in length the curved path per quadrant, which means that the perpendicular distance from each segment to the origin should be about 0.8 times the intercept of the curved path with this perpendicular. The dashed line in Fig. 6 shows a possible exemplar.

Finally we should note that the basic equations (1) and (3) have primarily to do with the area $A(t)$ of presence of the target and thus can be modified without much change if there is some directional asymmetry in the presumed motion of the target or if there are limits to the expansion of the area in some directions. For example, if there is a greater likelihood that the target moves in one direction than in others, the area of presence might have an expanding elliptical boundary and the variable x in Eq.(3) would then be related to the dimensions of this elliptical area. Modification of the definition of x could also be made if the presence of a "shore line" prevents the expansion of the area of presence in a range of directions.

References and Notes.

1 Preliminary Report on the Submarine Search Problem by the ASW Operations Research Group, USN, (ORG Memo M1) dated 1 May, 1941. Copies may be obtained from the Center for Naval Analyses.

2 Koopman, B.O."Theory of Search, III" Opns.Res.5, 613-626 (1957). This and the succeeding reference were excerpts from OEG Report 56, "Search and Screening" dated 1946.

3 Koopman, B.O."Theory of Search, II", Opns.Res.4, 519-520 (1956).

4 A preliminary discussion of this equation was contributed, by the present author, in 1973 to the Handbook of Operations Research, van Nostrand Reinhold, It is to be part of Section II, Chapter 6. To date (Jan.1978) the Handbook has not yet been published.

5 The situation is complicated near the end of the search, for the following reason. Quantity dz/dx is inversely proportional to the radial component dr/dt of the velocity of the search vessel, which has speed v. As long as dz/dx is larger than $(u/v)(1/2\sqrt{x})$ (see Eq.12) this radial component is smaller than v, allowing the tangential component $r(d\theta/dr)$ to make up the rest. When dz/dx is less than $(u/v)(1/2\sqrt{x})$ however, Eq.(3) requires the search vessel to proceed radially at a speed greater than v, which is impossible. If v is greater than u, however, this comes close to the end of the search, where little is being added to the chance of success. Therefore if the rest of the search is continued at speed v, in any outward direction, the result will be essentially the same. Similar considerations indicate that the search should not start exactly at the origin $(x = 0)$. Again, when $v \gg u$, this nicety can be ignored in practice.

TABLE I

	$x_m = (r_m/uT_o)^2$				$\sqrt{x_m} = (r_m/uT_o)$				
z_m/k	k = 0	0.5	1.0	1.5	k = 0	0.5	1.0	1.5	z_m/k
.05	0.284	0,280	0.277	0.273	0.533	0.530	0.526	0.522	.05
0.1	.383	.386	.386	.386	.619	.621	.621	.621	0.1
0.2	.507	.535	.555	.569	.712	.732	.745	.754	0.2
0.3	.589	.657	.704	.737	.767	.810	.839	.858	0.3
0.4	.650	.767	.847	.902	.806	.876	.921	.950	0.4
0.6	.738	.973	1.131	1.233	.859	.986	1.063	1.111	0.6
0.8	.798	1.174	1.418	1.571	.893	1.083	1.191	1.254	0.8
1.0	.841	1.376	1.712	1.918	.917	1.173	1.308	1.385	1.0
1.2	.874	1.581	2.013	2.273	.935	1.258	1.419	1.508	1.2
1.6	.920	2.005	2.634	2.999	.959	1.416	1.623	1.732	1.6
2.0	.948	2.449	3.276	3.820	.973	1.565	1.810	1.954	2.0
2.4	.965	2.913	3.939	4.540	.983	1.707	1.985	2.131	2.4
3.0	.981	3.639	5.002	5.744	.991	1.908	2.237	2.397	3.0

TABLE II

	$\lambda = \pi(uT_o)^2(G/2)$				$\ln(\frac{1}{\lambda}) = \frac{1}{k}(\frac{dz}{dx})_{x=0}$				
z_m/k	k = 0	0.5	1.0	1.5	k = 0	0.5	1.0	1.5	z_m/k
0.05	0.716	0.698	0.679	0.661	0.334	0.360	0.386	0.414	.05
0.1	.617	.600	.568	.536	.483	.511	.575	.624	0.1
0.2	.493	.461	.427	.393	.707	.775	.852	.935	0.2
0.3	.411	.381	.345	.309	.889	.966	1.064	1.175	0.3
0.4	.350	.325	.290	.253	1.050	1.125	1.239	1.374	0.4
0.6	.262	.251	.218	.182	1.338	1.382	1.523	1.702	0.6
0.8	.202	.205	.174	.140	1.598	1.587	1.751	1.969	0.8
1.0	.158	.173	.143	.111	1.841	1.757	1.945	2.199	1.0
1.2	.1256	.1493	.1207	.0906	2.074	1.902	2.114	2.402	1.2
1.6	.0805	.1176	.0903	.0641	2.520	2.140	2.405	2.748	1.6
2.0	.0525	.0969	.0707	.0476	2.948	2.334	2.650	3.045	2.0
2.4	.0346	.0823	.0570	.0371	3.365	2.498	2.864	3.294	2.4
3.0	.0187	.0668	.0430	.0268	3.981	2.705	3.147	3.620	3.0

TABLE III.

	ρ (see Eq. 10)				a (see Eq. 9)				
z_m/k	k = 0	0.5	1.0	1.5	k = 0	0.5	1.0	1.5	z_m/k
0.05	0.081	0.079	0.076	0.074	0	0.267	0.540	0.834	.05
0.1	.147	.142	.136	.130	0	.278	.869	1.322	0.1
0.2	.257	.241	.225	.209	0	.710	1.356	1.980	0.2
0.3	.347	.318	.292	.264	0	.938	1.690	2.382	0.3
0.4	.423	.380	.343	.304	0	1.126	1.928	2.637	0.4
0.6	.544	.478	.418	.359	0	1.409	2.213	2.894	0.6
0.8	.636	.554	.472	.392	0	1.601	2.347	2.977	0.8
1.0	.708	.612	.512	.411	0	1.729	2.402	2.978	1.0
1.2	.765	.659	.542	.424	0	1.808	2.410	2.938	1.2
1.6	.845	.723	.581	.435	0	1.879	2.360	2.806	1.6
2.0	.898	.762	.604	.439	0	1.881	2.272	2.629	2.0
2.4	.932	.787	.618	.440	0	1.843	2.175	2.500	2.4
3.0	.963	.807	.627	.441	0	1.772	2.026	2.295	3.0