Doubling or Splitting: Strategies for Modeling and Analyzing Survivable Network Design Problems

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Abstract

Survivability is becoming an increasingly important criterion in network design. This paper studies formulations, heuristic worst-case performance, and linear programming relaxations for two classes of survivable network design problems: the low connectivity Steiner (LCS) problem for graphs containing nodes with connectivity requirement of 0, 1, or 2, and a more general multi-connected network with branches (MNB) that requires connectivities of two or more for certain (critical) nodes and single connectivity for other secondary nodes. We consider both unitary and nonunitary MNB problems that respectively require a connected design or permit multiple components. Using a doubling argument, we first show how to generalize heuristic bounds of the Steiner tree and traveling salesman problems to LCS problems. We then develop a disaggregate formulation for the MNB problem that uses fractional edge selection variables to split the total connectivity requirement across each critical cutset into two separate requirements. This model, which is tighter than the usual cutset formulation, has three special cases: a "secondary-peeling" version that peels off the lowest connectivity level, a "connectivity-dividing" version that divides the connectivity requirements for all the critical cutsets, and a "secondary-completion" version that attempts to separate the design decisions for the multi-connected network from those for the branches. We examine the tightness of the linear programming relaxations for these extended formulations, and then use them to analyze heuristics for the LCS and MNB problems. Our analysis strengthens some previously known heuristic-to-IP worst-case performance ratios for LCS and MNB problems by showing that the same bounds apply to the heuristic-to-LP ratios using our stronger formulations.

1. Introduction

Cost and survivability are essential criteria for designing communication networks. Network planners need to configure backbone, interoffice, and local access networks that are cost effective but also meet differential service requirements for different customer segments. For instance, typically business and government customers have more stringent connectivity requirements than individual households. Service interruptions, which include lost revenues of the customers, can easily run into millions of dollars (Cosares et al. [1994]). Since failures of individual links or nodes cannot be completely eliminated (e.g., cables might be cut inadvertently during road construction work), telecommunication companies require network designs that have alternate routing capabilities between "critical" customer sites, providing these customers with guaranteed minimal interruptions and quick circuit restoration. Alternate routing is possible when the network contains redundant paths and "intelligent" switching elements (such as SONET-based digital cross-connect switches and add/drop multiplexers) to reroute traffic in case a particular link fails. Providing the same high level of protection to all customers makes the networks prohibitively expensive; instead, network planners must judiciously select a topology that contains redundancy only when necessary to meet a particular segment's connectivity requirement.

Motivated by this need to simultaneously consider cost and survivability issues, operations researchers have attempted to understand and solve a core optimization model—the survivable network design (SND) problem—that incorporates both cost and survivability criteria. Given a graph $G = (N,E)$ with a nonnegative fixed cost c_{ij} for each edge $\{i,j\} \in E$ and nonnegative (symmetric) integer connectivity parameters r_{ij} for every pair of nodes i and $j \in N$, the SND problem seeks, for all node pairs i and $j \in N$, the minimum cost network containing at least r_{ij} edge-disjoint paths between these nodes. The connectivity requirement r_{ij} reflects the relative importance of traffic between nodes i and j. For instance, if $r_{ij} = 4$, nodes i and j can communicate even if three links of the network fail simultaneously; since this contingency is relatively rare, less critical traffic might not require as stringent protection. For each node i, we define the node connectivity parameter $\rho_i = \max_{j \neq i} \{r_{ij}\}$. We refer to nodes with connectivity parameters ρ_i equal to 0, 1, and greater than 1 as *Steiner*, *secondary*, and *critical* nodes, respectively, and refer to the secondary and critical nodes together as *terminal* nodes. Steiner nodes are optional intermediate points that the design might use to connect the terminal nodes. The connectivity requirements r_{ij} induce a requirement graph with edges $\{\{i,j\}: r_{ij}>0\}$. If this graph has a single connected component, we say that the SND problem is *unitary* (since the optimal design would be a single component); otherwise, we say that the SND problem is *nonunitary*. (In the latter case, we assume that each component contains at least one secondary node).

The SND model has two variants: with or without edge duplication. Edge duplication refers to the option of choosing multiple (parallel) copies of any edge $\{i,j\} \in E$, assuming that the parallel copies are edge-disjoint for purposes of alternate routing. If the graph represents a physical rather than logical configuration, a model with edge duplication might not be appropriate since circuit protection requires physically diverse paths.

For SND problems containing only Steiner and critical nodes, the optimal network configuration is a multi-connected subgraph(s) in which each node belongs to at least one cycle. Secondary nodes, if available, might serve as intermediate nodes in the multi-connected subgraph(s); or, the solution must span them via branches emanating from the subgraph(s). We, therefore, refer to SND problems with secondary nodes as the *multi-connected network with branches (MNB)* problem. This paper focuses on the (nonunitary) MNB problem, and its two special cases—the unitary MNB problem, and the *low connectivity Steiner (LCS)* problem in which all nodes have connectivity parameters of 0, 1, or 2.

Survivable network design problems are very difficult to solve optimally. Even the unitary MNB and LCS special cases are NP-hard since they generalize the classical Steiner tree problem and the traveling salesman problem. Part of the enormous literature on Steiner network problems (see Winters [1988] for a survey of this literature) is devoted to analyzing the worst-case performance of heuristics, and developing good problem formulations that can improve the performance of relaxation-based solution methods such as branch-and-bound. Following this trend, several recent papers have studied the polyhedral structure of the SND problem (e.g., Monma, Munson and Pulleyblank [1990], Gröetchel, Monma, and Stoer, [1992]) have analyzed the worst-case performance of heuristics (e.g., Goemans and Bertsimas [1993], Goemans and Williamson [1992]), and have developed optimization algorithms (e.g., Gröetchel, Monma, and Stoer [1992]). Most of this work studies unitary LCS problems. Recently, Williamson et al. (1993) and Goemans et al. (1994) have developed heuristic bounds for nonunitary problems without edge duplication. Both papers use a primal-dual heuristic that builds the network in phases. Williamson et al. first satisfy the lowest connectivity requirement and then successively satisfy the next higher connectivity requirements by adding selected edges. The worst-case bound of $2\rho_K$ for this procedure depends on the highest connectivity requirement ρ_K. Using a converse primal-dual heuristic, and by improving the lower bounds on the optimal value of the linear programming relaxation, Goemans et al. decrease this bound to $\approx 2 \ln(\rho_K)$.

This paper examines modeling and heuristic performance bounds for MNB and LCS problems with or without edge duplication. We first explore whether certain worst-case results developed for the Steiner tree and traveling salesman problems extend to multi-connected design problems. For instance, a well-known Steiner tree result states that the minimum cost tree spanning just the terminal nodes is no more than twice as expensive as the optimal Steiner tree (Takahashi and Matsuyama [1980]). Does a similar result apply to the LCS problem, i.e., can we show that the solution to the "terminal" version of the LCS problem, ignoring the Steiner nodes, costs at most twice the optimal value? We confirm this conjecture using both a familiar "solution doubling" argument (in Section 2) and linear programming-based arguments (in Section 4).

Section 3 examines ways to improve the basic "cutset" formulation that several authors have used in their analyses of SND heuristics. We first propose a family of stronger (linear programming) formulations for nonunitary MNB problems, obtained by introducing additional edge variables and splitting the connectivity requirement across each critical cutset into two sub-requirements. Depending on how we split the connectivity requirements, we obtain different versions of the extended problem formulation. We consider three classes of formulations: a "secondary-peeling" formulation that decomposes the original problem into a single connectivity subproblem and a reduced connectivity residual problem, a "connectivity-dividing" formulation that allocates an equal proportion of the connectivity requirement for every critical cutset to two subproblems, and a "secondary-completion" formulation that distinguishes between edges in the multi-connected graph and those belonging to the branches.

Section 4 analyzes the worst-case performance of two classes of heuristics. For SND problems with edge duplication, Goemans and Bertsimas [1983] analyzed the worst-case performance (relative to the optimal integer value) of a heuristic strategy that adds a sequence of minimum cost trees and matchings to meet the connectivity requirements of critical nodes. We show that their worst-case bound applies to the linear programming solution value as well as the optimal integer value. Using the secondary-completion formulation, we also express the worst-case performance of a generic Forest Completion heuristic (for problems with or without edge duplication) in terms of the performance of an embedded multi-connected heuristic.

Our discussion highlights some differences in the analysis of models that permit or prohibit edge duplication. The results in this paper can serve as building blocks for more general models. Balakrishnan, Magnanti, and Mirchandani [1992, 1994a] have used these "single-level" results to analyze multi-level, multi-connected models that incorporate multiple service and facility types (for example, fiber or copper cables).

Notation:

Let $Q = \{\rho_0=0, \rho_1, ..., \rho_K\}$ denote the set of distinct node connectivity parameters indexed in increasing order, and let $Q^{k+} = \{\rho_q \in Q: \rho_q \geq k\}$. When we consider special connectivity sets Q^*, we will denote the corresponding subclass of SND problems as Q^*-connecitivity problems. Using this convention, the LCS problem is a $\{0,1,2\}$-connectivity problem, and the MNB model is the $\{0,1,Q^{2+}\}$-connectivity problem.

For any set N' of nodes of the graph G, the induced graph G(N') is a graph with the node set N' and containing edges of G with both endpoints in N'. By "triangularizing" a graph G, we mean constructing a complete graph G' with edge costs a'_{ij} equal to the length of the shortest path from node i to node j in G. Let MST(N') and TSP(N') denote a minimum cost tree and traveling salesman tour in the triangularized graph G' spanning just the nodes in N'. We use MST(N') and TSP(N') to also refer to the MST and TSP problems defined on the graph G(N').

If P is any optimization problem, we let Z_P denote its optimal objective value. If S is any feasible solution to P, we let Z(S) denote the objective value of this solution. If M is any solution method (typically a heuristic procedure or a linear programming relaxation of an integer programming model) for solving P, we let Z^M denote the objective value of the solution that the method produces.

2. Bounding by Doubling

In this section, we show how to use a graphical procedure—a doubling argument—to establish bounds on the optimal value of some unitary MNB problems. In particular, using this argument we show that a Tree Completion heuristic has a worst-case bound of 2 for

LCS problems as well as a "Ring on Steiner tree" problem and certain other MNB special cases. We begin with a general bounding result.

Proposition 1:

Given a graph G with nonnegative edge costs and a subset T of terminal nodes, for any connected subgraph SG containing the nodes of T and any Eulerian subgraph EG of SG,

$$Z_{MST(T)} + Z(EG) \leq Z_{TSP(T)} + Z(EG) \leq 2\, Z(SG).$$

Proof:

Let DSG be the Eulerian graph (with multiple edges) formed by doubling the edges of SG, and let RG be the residual graph formed by removing one copy of EG from DSG. RG is a connected Eulerian graph. Let i_1-i_2- ...-i_K-i_1 be the node sequence of an Eulerian walk W containing all the edges of RG. For convenience assume i_1 is a terminal node. Let $Z'(W) \leq Z(W)$ denote the cost of this walk in the triangularized graph. Form a traveling salesman tour TOUR of the terminal nodes T by deleting from this node sequence every Steiner node and every second and later occurrence of every terminal node (except for the final occurrence of i_1). The deletion of any such node j short-circuits the cycle, that is, replaces two edges (i,j) and (j,k) by the edge (i,k). If Z'(TOUR) denotes the cost of this tour in the triangularized graph, $Z'(TOUR) \leq Z'(W)$. By removing the largest cost edge from this tour, we obtain a tree TREE spanning the terminal nodes T whose cost $Z(TREE) \leq (1 - 1/|T|)\, Z'(TOUR)$. These arguments, and the fact that $Z_{MST(T)}$ ($Z_{TSP(T)}$) denotes the cost of the minimum spanning tree (traveling salesman tour) over the terminal nodes T in the triangularized graph G'(T), imply $Z_{MST(T)} + Z(EG) \leq Z_{TSP(T)} + Z(EG) \leq Z'(TOUR) + Z(EG) \leq Z(W) + Z(EG) \leq Z(DSG) = 2\, Z(SG)$. ♦

Note that this proposition remains valid both with and without duplicated edges.

Observations:

(i) If EG is the null graph, then the proof of Proposition 1 shows that $Z_{MST(T)} \leq 2(1 - 1/|T|)\, Z(SG)$. In particular, if SG is an optimal Steiner tree spanning the terminal nodes, then this proposition becomes the familiar Steiner tree result

$$Z_{MST(T)} \leq 2\left(1 - \frac{1}{|T|}\right) Z_{ST}.$$

(ii) If EG is the null graph, G has triangular edge costs and no Steiner nodes, and SG is a minimum spanning tree on G, then this proposition becomes the familiar TSP result

$$Z_{TSP} \leq 2\, Z_{MST}.$$

Given any LCS problem instance, we call the version of the problem restricted to just the terminal nodes (i.e., defined over the subgraph of G induced by the terminal nodes) as the *terminal low connectivity (TLC)* problem. How much do we sacrifice in solution quality by ignoring the Steiner nodes?

Corollary 2:

For any graph with triangular costs,

$$Z_{LCS} \leq Z_{TLC} \leq Z_{TSP(T)} \leq 2\, Z_{LCS}.$$

Proof:

Select EG in Proposition 1 as the null graph and note that since TSP(T) is a feasible solution to the terminal low connectivity problem $Z_{TLC} \leq Z_{TSP(T)}$. ♦

Corollary 2 has several implications. First, it implies that any polynomial-time heuristic with a worst-case bound of α for the TLC problem is a polynomial-time heuristic with a worst-case bound of at most 2α for the LCS problem defined on a graph with triangular edge costs.

In some situations, it is possible to solve the TLC problem optimally. Consider the *dual path tree (DPT)* problem: given an undirected graph with triangular edge costs and two critical nodes 1 and 2, find the minimum cost connected subgraph that spans all the nodes and contains two edge-disjoint paths between nodes 1 and 2. The *dual path Steiner tree (DPST)* problem contains additional Steiner nodes that the solution can optionally use to reduce total design cost. Balakrishnan, Magnanti and Mirchandani [1994b] describe a polynomial, matroid intersection-based algorithm for solving the DPT problem. Corollary 2 implies that the optimal DPT solution costs no more than twice the optimal DPST solution.

Finally, consider the nonunitary version of the LCS problem. For this problem, the optimal solution might have more than one connected component, each containing at least two terminal nodes. In this case, when we double and then short circuit an optimal solution to the problem, the resulting TOUR need not be connected, but might be a set of subtours.

In this case, the Corollary 2 inequalities $Z_{LCS} \leq Z_{TLC} \leq Z_{TSP(T)} \leq 2\, Z_{LCS}$ are still valid though we must now interpret TSP(T) as a multi-tour version of the TSP that satisfies the property that the terminal nodes i and j lie on the same subtour whenever $r_{ij} > 0$.

Suppose that the requirement graph for the nonunitary LCS problem has M components. As an approximate solution procedure for the problem, we could solve M separate TSP problems, one defined over the terminal nodes within each component; we could then use the union U of these TSPs as a heuristic solution to the problem. If the costs satisfy the triangle inequality, the arguments in the proof of Proposition 1 show that the solution U costs no more than 2M times the cost of an optimal LCS tree solution. Examples (which we will not provide here) show that this bound is the best possible if the LCS problem does not have any critical nodes.

In the next subsection we use Proposition 1 to analyze other heuristics.

2.1 EG-Tree problems

Suppose we wish to find a minimum cost connected graph that contains a subgraph EG from a given class $\mathbf{C}_{EG}$ of Eulerian subgraphs and spans all the terminal nodes T. The subgraphs in $\mathbf{C}_{EG}$ need not span all the terminal nodes, and might contain Steiner nodes. We also permit Steiner nodes to be part of the additional branches used to span the terminal nodes. We refer to this class of problems as **EG-tree problems**. Here are two examples:

(i) The **k-path Steiner tree** problem is the same as the dual-path Steiner tree problem except that we now require k edge disjoint paths connecting the designated terminal nodes 1 and 2. We assume that k is even; otherwise, we can add a zero cost edge from node 1 to node 2 (with an intermediate dummy node, if necessary) and increase the connectivity parameter for nodes 1 and 2 to k+1. For this problem, the class of Eulerian graphs is the set of all k edge-disjoint paths in G connecting nodes 1 and 2.

(ii) The **Ring-on-Steiner tree** problem is a constrained LCS problem in which all the critical nodes must lie on a common simple circuit. The class of Eulerian graphs is, therefore, the set of all simple circuits containing the critical nodes and possibly one or more Steiner or secondary nodes.

These EG-tree problems are special cases of the following problem. Given a graph G with terminal nodes $T \subseteq N$ and a collection $\mathbf{C}$ of subgraphs of G, find the minimum cost subgraph of G that spans all the nodes in T and contains a subgraph belonging to **C**. For EG-tree problems, the collection **C** contains only Eulerian subgraphs. Consider the following heuristic procedure for solving this problem.

2.1.1 Tree Completion heuristic:

Step 1: Find an approximate or optimal graph OG from the given collection **C** of subgraphs of G.

Step 2: Contract OG into a single node, 0; if this contraction creates multiple edges between a pair of nodes, replace the multiple edges by the lowest-cost edge between that pair of nodes. Eliminate all self-loop edges. Next, create a graph G^* by triangularizing the edge costs. Find the minimum spanning tree TREE spanning the nodes $\{0\} \cup T$ in the triangularized graph G^*.

Step 3: As a heuristic solution, choose the edges in OC plus the edges in the shortest path in G connecting the nodes i and j for every edge $\{i,j\}$ in TREE.

The complexity of Step 1 depends upon the nature of the class of subgraphs **C**. For example, for the Ring on Steiner tree problem, finding the optimal graph OG requires solving the "Steiner" traveling salesman problem over the terminal nodes with optional intermediate Steiner nodes. In contrast, for the k-path Steiner tree problem without edge duplication, the minimum cost k edge-disjoint 1-to-2 paths in Step 1 is easy to solve as a minimum cost network flow problem for routing k units of flow from node 1 to node 2 with unit edge capacities (with edge duplication, the optimal k-path solution is k replications of the shortest 1-to-2 path).

The next proposition uses Proposition 1 to bound the worst-case performance of the Tree Completion heuristic for EG-tree problems as a function of the worst-case performance of the method used to solve Step 1. In Section 4.4, we generalize the Tree Completion heuristic to solve nonunitary MNB problems in which **C** is a collection of multi-connected subgraphs of G spanning the critical nodes.

Proposition 3:

For EG-tree problems with nonnegative costs, if we find an α-approximate solution to the problem in Step 1, then the cost of the solution produced by the Tree Completion heuristic is no more than 2α times the optimal EG-tree cost.

Proof:

If the costs are nonnegative, the cost of the edges that the Tree Completion heuristic adds to OG in Step 2 is no more than $Z_{MST(T)}$. Let SG* be the optimal solution to the EG-tree problem. By definition, this solution contains a subgraph C from the given collection $\mathbf{C}_{EG}$. Since the costs are nonnegative, $Z(C) \le Z(SG^*)$, and since our embedded heuristic in Step 1 selects a subgraph OG that costs no more than α times the minimum cost subgraph in $\mathbf{C}_{EG}$, $Z(OG) \le \alpha Z(C)$. The cost $Z^{TC\text{-heur}}$ of the heuristic solution is the cost of OG plus the cost of the added edges, and so Proposition 1 with EG = C and SG = SC* implies that $Z^{TC\text{-heur}} \le Z_{MST(T)} + Z(OG) \le 2\,\alpha\, Z_{EG\text{-tree}}$. ♦

Proposition 3 implies that for the k-path Steiner tree problem, the Tree Completion heuristic is a polynomial-time algorithm with a worst-case performance of at most 2. Note that this analysis does not require triangular costs, and if the given class of Eulerian graphs does not duplicate any edges, then neither does the heuristic solution. If we permit edge duplication, then other heuristics are possible. For instance, to heuristically solve the dual path Steiner tree problem with edge duplication: (i) select the edges of the shortest path from node 1 to 2, and (ii) add the edges of the minimum spanning tree MST(T). This heuristic also has a worst-case bound of 2.

Worst-case example:

The DPST example in Figure 1 shows that the worst-case bound of 2 in Proposition 3 is best possible. In Figure 1(a), nodes 1 and 2 are the critical nodes; all the other nodes are secondary nodes. Four paths connect nodes 1 and 2, each of length q. Two of these paths have one intermediate node, while the other two paths have q–1 intermediate nodes each. Figure 1(b) shows the Tree Completion heuristic solution. The first step chooses the two paths having a single intermediate node. Step 2 greedily connects the remaining secondary nodes to this dual path. This solution costs $2q + 2(q-1) = 4q - 2$. The optimal solution (Figure 1(c)) costs 2q; as q becomes large, the ratio of these costs approaches 2.

2.2 A structural property for LCS problems

The doubling argument also permits us to establish a structural property for the broader class of LCS problems without requiring Eulerian subgraphs. Consider the {0,1,2}-connectivity problem and its counterpart without secondary nodes, i.e., the {0,2}-connectivity problem. If we can establish a bound on the optimal value of one of these problems, can we establish a bound for the other problem? For a set of nodes N', let CLASS(N') denote any class of subgraphs that are defined on the given graph G and that span (at least) the nodes N'. For LCS problems, CLASS(C) and CLASS(T) might represent classes of subgraphs used to construct heuristic {0,2}-connectivity and {0,1,2}-connectivity solutions.

Proposition 4:

Let $\beta(\cdot)$ be a nondecreasing, nonnegative real-valued function of the positive integers. For any graph G with nonnegative costs, if we permit edge duplication, the following two results are equivalent.

(a) $Z_{CLASS(C)} \leq \beta(|C|)\, Z_{02}$ for all {0,2}-connectivity problems.

(b) $Z_{CLASS(T)} + Z_{CLASS(C)} \leq 2\, \beta(|T|)\, Z_{012}$ for all {0,1,2}-connectivity problems.

Proof:

If (b) is true, then since the {0,2}-connectivity problem is a special case of the {0,1,2}-connectivity problem with T = C, substituting C for T in (b) gives (a), i.e., (b) implies (a).

To establish the converse, choose any optimal solution OS to the {0,1,2}-connectivity problem. Between every pair of critical nodes, the solution contains 2 edge-disjoint paths. Let Γ be the union of the edges in these paths. Consider a "doubled" solution containing two copies of each edge in OS. From this solution, extract one copy of Γ. The residual graph Δ contains one copy of Γ. Γ contains two paths joining the critical nodes, and Δ contains two paths joining every pair of terminal nodes. Therefore, for some constant value $\beta(|T|)$ of the function $\beta(\cdot)$, property (a) with C = T implies that $Z_{CLASS(T)} \leq \beta(|T|)\, Z_{02} \leq \beta(|T|)\, Z(\Delta)$, and the monotonicity of $\beta(\cdot)$ implies that $Z_{CLASS(C)} \leq \beta(|C|)\, \bar{Z}_{02} \leq \beta(|T|)\, Z(\Gamma)$. Our use of the bar on the second Z_{02} emphasizes the fact that Z_{02} and $\bar{Z}_{02}$ are defined on different graphs and so typically have different values. But by construction, $2\, Z_{012} = Z(\Delta) + Z(\Gamma)$ which, together with the previous inequalities, implies (b). ♦

Since the graph defined by the edges Γ need not be Eulerian, this result uses the doubling argument in a slightly different way than we have previously. The following three results are special cases of Proposition 4.

Proposition 5:

(a) $Z_{MST(C)} \leq \left(1 - \frac{1}{|C|}\right) Z_{02}$ for all {0,2}-connectivity problems with edge duplication.

(b) $Z_{MST(T)} + Z_{MST(C)} \leq 2\,(1 - \frac{1}{|T|})\, Z_{012}$ for all {0,1,2}-connectivity problems with edge duplication.

Proposition 6:

(a) $Z_{TSP(C)} \leq \frac{4}{3} Z_{02}$ for all {0,2}-connectivity problems with edge duplication.

(b) $Z_{TSP(T)} + Z_{TSP(C)} \leq \frac{8}{3} Z_{012}$ for all {0,1,2}-connectivity problems with edge duplication.

Proposition 7: If C and T have even cardinalities,

(a) $Z_{Match(C)} \leq \frac{1}{2} Z_{02}$ for all {0,2}-connectivity problems with edge duplication.

(b) $Z_{Match(T)} + Z_{Match(C)} \leq Z_{012}$ for all {0,1,2}-connectivity problems with edge duplication.

In Proposition 5, CLASS(N') is the set of all spanning trees on the induced graph G(N') (since we permit edge duplication, without loss of generality we assume that G is triangular). The function $\beta(|T|) = (1-1/|T|)$ is nonincreasing in $|T|$ so that Proposition 4 applies. Both statements in Proposition 5 are special cases of more general results established by Goemans and Bertsimas [1993] and of those we establish later in this paper. In Proposition 6, CLASS(N') is the set of all hamiltonian circuits through the nodes N', and $\beta(\cdot)$ is a constant. Monma, Munson, and Pulleyblank [1990] established part (a) of Proposition 6. Part (b) of this proposition is a consequence of Proposition 4. Notice that whenever $Z_{TSP(C)} \geq \frac{1}{3} Z_{TSP(T)}$, part (b) of Proposition 6 provides a sharper lower bound on Z_{012} than does the bound $Z_{TSP(T)} \leq 2\, Z_{012}$ from Proposition 1. Finally, in Proposition 7, CLASS(N') is the set of all matchings over N'. In this case, $\beta(\cdot)$ equals 1/2 and, as noted by Goemans and Bertsimas [1993], part (a) of this proposition follows from Edmonds' [1965] perfect matching polytope result and a parsimonious property that they establish.

In Section 3, we provide a strengthening of each of the statements in Proposition 5 (and their generalizations to $\{0,\rho\}$-connectivity and $\{0,1,\rho\}$-connectivity nonunitary MNB problems for $\rho \geq 2$). We also show that we can tighten the bounds by replacing the optimal IP values Z_{02} and Z_{012} in the right-hand sides of these expressions by the optimal values of linear programming relaxations of certain problem formulations.

3. Modeling MNB problems

This section presents improvements to a standard cutset formulation of MNB problems. After reviewing the standard formulation, we describe a stronger "connectivity-splitting" formulation that (i) replaces the single connectivity constraint for each critical cutset with two constraints, and (ii) uses one integer and two continuous variables for each edge to replace the single integer edge selection variable. After proving the validity of this reformulation, we strengthen it and examine three special cases, obtained by considering particular schemes for splitting the connectivities. One formulation *peels* the lowest connectivity level, another *divides* the connectivity requirement in the same proportion for all critical cutsets, and the third *completes* a multi-connected solution to a problem with secondary nodes treated as Steiner points. In Section 4, we use these formulations to analyze the worst-case performance of tree-based heuristics for MNB problems with and without edge duplication.

Developing strong formulations with improved linear programming relaxations has proven useful both to develop better heuristic bounds and to improve solution performance for several classical discrete optimization problems (e.g., Nemhauser and Wolsey [1988]). Typically, these *extended* formulations introduce additional variables from a different space (e.g., flow or node variables for Steiner network models; see, for example, Beasley [1984], Wong [1984], and Goemans and Myung [1991]), and add appropriate linking constraints. Our reformulation strategy for the MNB problem also introduces additional variables, but they all belong to the original space of edge variables. We present formulations for MNB problems without edge duplication: if we eliminate upper bounds on the edge section variables, these models are valid for problems with edge duplication.

Let us first introduce some notation and conventions. For any subset of nodes $S \subset N$ and $T = N \backslash S$, let $\{S,T\}$ denote the edge-cutset defined by S and T, i.e., $\{S,T\} = \{\{i,j\} \in E$ with $i \in S$ and $j \in T\}$. If x_{ij} is any quantity (decision variable, given data) imposed upon the edges E of the graph, we let $X_{ST} = \sum_{\{i,j\} \in \{S,T\}} x_{ij}$ denote the sum of that quantity over all the edges in the cutset $\{S,T\}$. Given (symmetric) connectivity requirements r_{ij} for all $i, j \in N$ and a cutset $\{S,T\}$, we refer to the maximum value of r_{ij} over all node pairs $\{i,j\} \in \{S, T\}$ as the *crossing requirement* of cutset $\{S,T\}$. For each $q \in Q^{1+}$, let σ_q denote the collection of all cutsets of the graph G with crossing requirement equal to q.

3.1 Cutset formulation for SND problems

The standard cutset formulation, Problem [CUT], for the SND formulation without edge duplication uses binary edge selection variables u_{ij} for all edges $\{i,j\} \in E$. The variable u_{ij} is 1 if the network design includes edge $\{i,j\}$, and is 0 otherwise.

Problem [CUT]:

$$Z_{CUT} = \text{minimize} \sum_{\{i,j\}\in E} c_{ij}\, u_{ij} \tag{3.1}$$

subject to

$$U_{ST} \geq q \qquad \text{for all cutsets } \{S,T\}\in \sigma_q,\ q \in Q^{1+}, \tag{3.2}$$

$$u_{ij} = \text{integer} \qquad \text{for all } \{i,j\}\in E, \text{ and} \tag{3.3}$$

$$u_{ij} \leq 1 \qquad \text{for all } \{i,j\}\in E. \tag{3.4}$$

If $Q = \{0,1\}$, [CUT] is a standard cutset formulation [ST] for the Steiner tree problem or [SF] for the Steiner forest problem depending on whether the connectivity requirements pattern in unitary or nonunitary.

3.2 Critical connectivity-splitting formulation for MNB problems

We consider a general reformulation scheme for nonunitary MNB problems that models the crossing requirement $q \geq 2$ for every critical cutset $\{S,T\} \in \sigma_q$ as two complementary sets of requirements $q\phi_q$ and $q(1-\phi_q)$ across the cutset. For all $q \in Q^{2+}$, $\phi_q \in [0,\frac{1}{2}]$ is a pre-specified "connectivity fraction" for critical cutsets with connectivity requirement q. To perform this decomposition, we introduce two additional *continuous* edge variables x_{ij} and y_{ij}, and a separate set of 0-1 edge selection variables b_{ij} that represents edges belonging to the branches (connecting secondary nodes to the multi-connected graph) in the solution. Let $\phi = \min_{q\in Q^{2+}} \phi_q$ and let $\mu \leq \frac{1}{2\phi}$ be a nonnegative parameter. We define $\alpha_1 = \frac{1-2\mu\phi}{2(1-\phi)}$, and $\alpha_q = \frac{\phi_q-\phi}{(1-\phi)}$ for all $q \in Q^{2+}$. Note that $0 \leq \alpha_q \leq 1$ for all $q \in Q^{1+}$. We choose the connectivity fractions ϕ_q so that the parameters satisfy the condition $q\alpha_q \leq 1$. Note that choosing either $\phi_q = 1/q$, or $\phi_q = 1/\delta$ for all q and any $\delta \geq 1$, satisfies these assumptions.

For a given value of μ and the vector $\Phi = (\phi_q)$ of connectivity fractions, consider the following "critical connectivity-splitting" formulation [CCS(μ,Φ)] for the nonunitary MNB problem:

Problem [CCS(μ,Φ)]:

$$Z_{CCS} = \min \sum_{\{i,j\}\in E} \{c_{ij}\, b_{ij} + c_{ij}\, z_{ij}\} \tag{3.5}$$

subject to

$$
\begin{array}{rcllr}
B_{ST} + \mu X_{ST} + \alpha_1 Y_{ST} & \geq & 1 & \text{for all } \{S,T\} \in \sigma_1, & (3.6)\\
X_{ST} + \alpha_q Y_{ST} & \geq & q\phi_q & \text{for all } \{S,T\} \in \sigma_q,\ q \in Q^{2+}, & (3.7)\\
Y_{ST} & \geq & q(1-\phi_q) & \text{for all } \{S,T\} \in \sigma_q,\ q \in Q^{2+}, & (3.8)\\
z_{ij} & \geq & x_{ij} + y_{ij} & \text{for all } \{i,j\} \in E, & (3.9)\\
x_{ij}, y_{ij} & \geq & 0 & \text{for all } \{i,j\} \in E, & (3.10)\\
b_{ij} & = & 0 \text{ or } 1 & \text{for all } \{i,j\}\in E, & (3.11)\\
z_{ij} & = & \text{integer} & \text{for all } \{i,j\}\in E, \text{ and} & (3.12)\\
z_{ij} + b_{ij} & \leq & 1 & \text{for all } \{i,j\}\in E. & (3.13)
\end{array}
$$

Eliminating the upper bound (3.13) gives a model for MNB problems with edge duplication.

Theorem 8:

Suppose $\mu \in [0,1]$ and $\Phi = \{\phi_q\}$ with $\phi_q \in [0,1/2]$ satisfies the conditions $q\,\alpha_q \leq 1$. Then [CCS(μ,Φ)] is a valid formulation for the MNB problem without edge duplication.

Proof:

Given any feasible solution to formulation [CCS(μ,Φ)], as the following argument shows, we obtain an equal-cost feasible solution to the cutset formulation [CUT] of the MNB problem by setting $u_{ij} = b_{ij} + \lceil x_{ij}+y_{ij} \rceil$. Since the given CCS solution satisfies constraints (3.6) and both $\mu \leq 1$ and $\alpha_1 \leq 1$, the derived solution satisfies constraint (3.2) for every cutset $\{S,T\} \in \sigma_1$. Consider a cutset $\{S,T\} \in \sigma_q$ for some $q \in Q^{2+}$. If the given solution satisfies $X_{ST} > q\phi_q - 1$ for this cutset, then constraints (3.8) imply that $U_{ST} \geq \sum_{\{S,T\}} \lceil x_{ij}+y_{ij} \rceil \geq \lceil X_{ST} + Y_{ST} \rceil \geq q$ and so the derived u-solution satisfies the crossing requirement for this cutset in formulation [CUT]. On the other hand, suppose $X_{ST} \leq q\phi_q-$

1. Since the given solution satisfies (3.7), $\alpha_q Y_{ST} \geq 1$, i.e., $Y_{ST} \geq \frac{1}{\alpha_q} \geq q$. Therefore, the derived u-solution is feasible in formulation [CUT] and has the same cost as the original CCS solution.

Conversely, given any feasible (integer) solution u to formulation [CUT], we use the following "allocation" procedure to obtain feasible values of the b, x, y, and z variables in formulation [CCS(μ,Φ)]. The given u solution contains at least q edge-disjoint paths connecting every pair of q-critical nodes; let E_q denote the union of the edges contained in those paths and let $E_{2+} = \cup_{q \geq 2} E_q$. For each edge $\{i,j\} \in E_{2+}$, we set $x_{ij} = \phi u_{ij}$, $y_{ij} = (1-\phi)u_{ij}$, and $z_{ij} = u_{ij}$. For edges in $E \backslash E_{2+}$, we set $b_{ij} = u_{ij}$ and $x_{ij} = y_{ij} = z_{ij} = 0$.

This solution [b,x,y,z] satisfies constraints (3.9) to (3.13). Consider any cutset $\{S,T\} \in \sigma_q$ with $q \in Q^{2+}$. The u-solution selects at least q edges in this cutset. By construction, on each such edge $\{i,j\}$, $x_{ij} = \phi\, u_{ij}$ and $y_{ij} = (1-\phi)\, u_{ij}$. Therefore, the left-hand side of (3.7) is at least $q\,\phi + q\alpha_q\,(1-\phi) = \; q\,\phi + q\,\frac{\phi_q-\phi}{(1-\phi)}\,(1-\phi) = q\,\phi_q$, and the left-hand side of (3.8) is at least $q(1-\phi) \geq q(1-\phi_q)$.

Finally, for a cutset $\{S,T\} \in \sigma_1$, if the given u-solution contains at least one edge from $E \backslash E_{2+}$, then $U_{ST} = B_{ST} \geq 1$. Otherwise, the u-solution contains at least two edges from E_{2+}. But this observation and the definition of α_1 implies that the left-hand side of (3.6) in the derived solution is at least 1. ♦

Notice that, unlike the original [CUT] formulation, the connectivity requirements in constraints (3.7) and (3.8) of [CCS(μ,Φ)] might be fractional. The following {1,2}-connectivity example shows that the CCS model can provide a strictly tighter optimal LP bound on the IP value than the cutset formulation. Consider a triangle with 3 critical nodes. Duplicate each edge of the graph and on each edge (including the duplicate edges), introduce a secondary node. This construction creates a graph with 3 critical nodes, 6 secondary nodes, and 12 edges, each with a cost of 1/6. The optimal solution to the LP relaxation of [CUT] sets $u_{ij} = 1/2$ on all edges; this solution costs 1. An optimal solution to the LP relaxation of [CCS(μ,Φ)], with $\mu = 1$ and $\phi_1 = \phi_2 = 1/2$, sets $x_{ij} = 1/2$ and $b_{ij} = 0$ for all edges, and $y_{ij} = 1/2$ for all edges on the outer ring; this solution costs 3/2. So for this example, $Z^{LP}_{CUT} < Z^{LP}_{CCS}$.

We next describe a way to strengthen the CCS formulation.

3.2.1 Stronger connectivity-splitting formulation:

The x and y values that we derive using the allocation scheme to transform a given u-solution in the proof of Theorem 8 satisfy the following constraints:

$$Y_{ST} \geq q(1-\phi) \quad \text{for all } \{S,T\} \in \sigma_q,\ q \in Q^{2+}, \text{ and} \tag{3.8a}$$

$$(1-\phi)\, x_{ij} \geq \phi\, y_{ij} \quad \text{for all } \{i,j\} \in E. \tag{3.14}$$

Constraints (3.8a) are stronger than (3.8). Replacing constraints (3.8) with (3.8a) and adding constraints (3.14) retains the validity of the reformulation. We will denote this stronger connectivity-splitting formulation as **[SCS(μ,Φ)]**.

3.3 Special cases of connectivity-splitting

We obtain three intuitive special cases of formulation [SCS(μ,Φ)] by selecting certain special values for μ and Φ. First, consider the formulation [SCS(μ,Φ)] with $\mu = 1$. Note that, in this case, both the integer formulation and its LP relaxation must have optimal solutions with $b_{ij} = 0$ for all edges $\{i,j\} \in E$. For, given an optimal solution with some $b_{ij} > 0$, we can obtain an equal or lower cost feasible solution by setting $x_{ij} \leftarrow x_{ij} + b_{ij}$. Therefore, we can drop the b variables from formulation [SCS(1,Φ)]. Within this class of formulations, we consider two special connectivity-splitting vectors Φ, namely, $\phi_q = 1/q$ for all $q \in Q^{2+}$, and $\phi_q = 1/\delta$ for some $\delta \geq 2$, for all $q \in Q^{2+}$.

When $\boldsymbol{\phi_q = 1/q}$ **for all** $\boldsymbol{q} \in Q^{2+}$, the right-hand side of constraints (3.7) is 1 for all critical cutsets, and the right-hand side of constraints (3.8) (in the original CCS model) is $(q-1)$. Intuitively, this disaggregation strategy attempts to separate or peel a single connectivity subproblem (constraints (3.6) and (3.7)) over all the terminal nodes from a "reduced connectivity" subproblem (constraints (3.8)) with the connectivity of each critical node reduced by 1. We, therefore, refer to this special case of [CCS(μ,Φ)] as the **secondary-peeling** formulation **[PEEL]**. This formulation is potentially useful for analyzing a heuristic that first finds a Steiner forest spanning all the terminal nodes and then adds edges belonging to a reduced connectivity solution.

Given a parameter $\delta \geq 2$, if $\boldsymbol{\phi_q = \frac{1}{\delta}}$ **for all** $\boldsymbol{q} \in Q^{2+}$, formulation [SCS(1,$\Phi$)] becomes a **connectivity-dividing** formulation **[DIV(δ)]**. In this formulation, the right-hand side values of constraints (3.7) and (3.8a) are q/δ and $q(1-1/\delta)$, i.e., the disaggregation strategy now "divides" the connectivity requirement in the same proportion for every critical cutset.

Note that since $\phi_q = \phi$, $\alpha_q = 0$ for all $q \in Q^{2+}$. Therefore, constraints (3.7) contain only the x variables. The connectivity-dividing formulation might be useful for analyzing a "divide and conquer" heuristic strategy that solves two problems with, say, half the original connectivities and treats the union of these two solutions as the heuristic solution to the original problem (this strategy assumes edge duplication). Note that for $\{0,1,\rho\}$-connectivity problems, the secondary-peeling formulation is the same as the connectivity-dividing formulation with $\delta = 1/q$.

We obtain a particular connectivity-dividing formulation [DIV(δ)] by setting $\delta = 2$ (i.e., $\phi = 1/2$). In this case, $\alpha_1 = 0$ and constraints (3.7) and (3.8) both have equal (possibly fractional) connectivity requirements of q/2. In essence, this "connectivity-halving" formulation contains two fractional (i.e., the variables can be fractional) connectivity subproblems, one each corresponding to the x and y variables: each provides half the required connectivity for every critical node. However, only one of these subproblems includes the unit requirement of secondary nodes. In particular, for the $\{0,1,2\}$-connectivity or LCS problem, the X subproblem corresponds to a "fractional Steiner forest" over all the secondary and critical nodes, and the Y subproblem is a fractional Steiner forest over the critical nodes. The forcing constraints (3.14) require the y solution to be "overlayed" on the x solution, i.e., $x_{ij} \geq y_{ij}$ for all edges $\{i,j\}$. We can show that for LCS problems with edge duplication, if we use a heuristic with worst-case performance ratio of θ (relative to the optimal LP value) to solve the Steiner subproblems, the union of these trees is a feasible LCS solution and has a worst-case performance ratio of θ.

We obtain a third special case of the SCS model by setting $\mu = \frac{1}{2}$ and $\phi_q = 0$ for all $q \in Q^{2+}$. In this case, we cannot drop the b variables, but since $\alpha_q = 0$ for all $q > 1$, constraints (3.7) and (3.14) are redundant. Furthermore, since $\alpha_1 = 1/2$, both the integer version and linear programming relaxation of formulation [SCS($\frac{1}{2}$,**0**)] have optimal solutions with $x_{ij} = 0$ for all edges $\{i,j\}$ (otherwise, we obtain an equal or lower cost feasible solution by setting $y_{ij} \leftarrow y_{ij} + x_{ij}$). Therefore, we can drop the x variables and replace z_{ij} with y_{ij} in the objective function, the integrality constraints (3.12), and the upper bounds (3.13). Constraints (3.8) of this reformulation require that the y variables define a multi-connected graph containing the required number of edge-disjoint paths connecting all the critical nodes. Constraints (3.6) ensure that every secondary node either belongs to the multi-connected component or is spanned by the branches emanating from this component, i.e., each cutset $\{S,T\} \in \sigma_1$ contains either at least two edges of a multi-connected component or one edge belonging to a branch. Notice that, unlike the secondary-peeling and connectivity-dividing formulations, this third reformulation does not reduce the connectivities of the critical nodes but instead ensures that the solution completes the multi-connected graph to span all the remaining secondary nodes. Another difference is that all the variables in this formulation must be integer valued. We refer to this special version of the SCS model as the **secondary-completion** model **[COMPL]**. We subsequently use this formulation to analyze the worst-case performance of the Forest Completion heuristic.

In passing, we note that it is possible to derive this formulation by strengthening the original cutset formulation via a coefficient reduction procedure. To do so, we write any feasible solution to the problem (3.1)-(3.4) as $u = g+h$ with (i) $h_{ij} = u_{ij}$ for all edges $(i,j) \in E_{2+}$ that belong to some subset σ_q, $q \geq 2$ and $h_{ij} = 0$ otherwise, and (ii) $g_{ij} = u_{ij}$ on the remaining edges $(i,j) \in E \backslash E_{2+}$ and $g_{ij} = 0$ otherwise. With this notation, the constraints (3.2) for the cutsets with a crossing requirement of one are $g_{ST} + h_{ST} \geq 1$ for all $\{S,T\} \in \sigma_1$. (Note that the variables g do not appear in the crossing constraints (3.2) for cutsets $\{S,T\} \in \sigma_q$ for any $q \geq 2$.) Since either $h_{ST} = 0$ or $h_{ST} \geq 2$ in any feasible solution, we can tighten the constraints to $g_{ST} + \frac{1}{2} h_{ST} \geq 1$ for all $\{S,T\} \in \sigma_1$ and $h_{ST} \geq q$ for all $\{S,T\} \in \sigma_q$. This model, with $b = g$ and $y = h$, is equivalent to [COMPL] .

3.4 Tightness of extended formulations

In formulation [SCS(μ,Φ)], since $Y_{ST} \geq q(1-\phi)$ for all $\{S,T\} \in \sigma_q$, $q \in Q^{2+}$, the constraint $(1-\phi)x \geq \phi y$ implies that $X_{ST} \geq q\phi$. Moreover, since $\alpha_q(1-\phi) = \phi_q - \phi$, the inequalities $Y_{ST} \geq q(1-\phi)$ and $X_{ST} \geq q\phi$ imply $X_{ST} + \alpha_q Y_{ST} \geq q\phi_q$. These observations imply that constraints (3.7) are redundant in the SCS model; therefore, for any vector Φ, we obtain an equivalent [DIV(δ)] model by choosing $\delta = 1/\min\{\phi_q: q \in Q^{2+}\}$. We next show that the value of δ does not influence the optimal LP value of the formulation [DIV(δ)].

Proposition 9:

For any value of $\delta > 2$, the formulation [DIV(δ)] is LP-equivalent to the formulation [DIV(2)].

Proof:

Let x and y denote a generic solution to formulation [DIV(2)] and x' and y' denote a generic solution for any fixed, but arbitrary value of $\delta > 2$. To show the equivalence between the formulations, we use the transformation

$$y = (1-\alpha_1)\, y' \quad \text{or} \quad y' = \frac{y}{1-\alpha_1}$$

$$\text{and} \quad x = x' + \alpha_1 y' \quad \text{or} \quad x' = x - \frac{\alpha_1 y}{1-\alpha_1}.$$

Note that $x + y = x' + y'$ so we choose $z \geq x + y = x' + y'$ as having the same value in both problems.

Let x' and y' be a feasible solution to [DIV(δ)] and let x and y be defined by the transformation. Then clearly, $x \geq 0$ and $y \geq 0$ and by definition of x,

$$X_{ST} \quad \geq \quad 1 \text{ for all } \{S,T\} \in \sigma_1.$$

Since $(\delta-1)\, x' \geq y'$, and $1-2\alpha_1 = \frac{1}{(\delta-1)}$

$$x = x' + \alpha_1 y' \quad \geq \quad \frac{1}{\delta-1} y' + \alpha_1 y' = (1-\alpha_1)\, y' = y.$$

Moreover, for all $\{S,T\} \in \sigma_q$, $q \in Q^{2+}$,

$$X_{ST} \quad = X'_{VW} + \alpha_1 Y'_{VW} \geq \frac{q}{\delta} + q\alpha_1 (1 - \frac{1}{\delta}) = \frac{q}{2}, \text{ and}$$

$$Y_{ST} \quad = (1-\alpha_1)\, Y'_{VW} \geq q(1-\alpha_1)(1-\frac{1}{\delta}) = \frac{q}{2}.$$

Therefore, x and y are feasible for the linear programming relaxation of [DIV(2)].

Now suppose x and y are feasible for [DIV(2)]. Then since $\alpha_1 < 1$, $y' = \frac{y}{1-\alpha_1} \geq 0$, and since $x \geq y$,

$$x' = x - \frac{\alpha_1}{1-\alpha_1} y = x - \frac{\delta-2}{\delta} y = (x-y) + \frac{2}{\delta} y \geq \frac{2}{\delta} y \geq 0.$$

The last expression implies that for all $\{S,T\} \in \sigma_q$, $q \in Q^{2+}$,

$$X'_{VW} \geq \frac{2}{\delta} Y_{ST} \geq \frac{q}{\delta}.$$

Moreover,

$$Y'_{VW} = \frac{1}{1-\alpha_1} Y_{ST} \geq \frac{q}{2(1-\alpha_1)} = q\,(1 - \frac{1}{\delta}).$$

For any $\{S,T\} \in \sigma_1$,

$$X'_{VW} + \alpha_1 Y'_{VW} = X_{ST} - \frac{\alpha_1}{1-\alpha_1} Y_{ST} + \frac{\alpha_1}{1-\alpha_1} Y_{ST} = X_{ST} \geq 1.$$

Finally, $x \geq y$, and the fact that $\delta \geq 2$ implies that

$$(\delta-1)\, x' = (\delta-1)\,(x - \frac{\alpha_1}{1-\alpha_1} y) \geq \frac{2(\delta-1)}{\delta} y \geq \frac{1}{1-\alpha_1} y = y'.$$

Therefore, x' and y' are feasible in [DIV(δ)]. ♦

In this section, we have considered several valid mixed integer programming models for the MNB problem. These extended formulations have tighter linear programming relaxations than the original [CUT] formulation. Our observations prior to Proposition 9 show that for any vector Φ of connectivity fractions, the linear programming relaxations of the [SCS(1,ϕ)] model with the constraints (3.6), (3.8a), (3.14) and constraints (3.9) to (3.13) is LP-equivalent to the formulation [SCS(1,1/Δ)] with $\Delta = 1/\min\{\phi_q : q \in Q^{2+}\}$. This model is the same as [DIV(Δ)]. Therefore, letting Z_M^{LP} denote the linear programming relaxation of model M, we have established the following result.

Proposition 10:

Let $\Phi = \{\phi_q\}$ with $\phi_q \in [0,1/2]$ be any vector of connectivity fractions satisfying the conditions $q\,\alpha_q \leq 1$ and let $\Delta = 1/\min\{\phi_q : q \in Q^{2+}\}$. Then $Z_{CUT}^{LP} \leq Z_{CCS(1,\Phi)}^{LP} \leq Z_{SCS(1,\Phi)}^{LP} = Z_{DIV(\Delta)}^{LP} = Z_{DIV(\delta)}^{LP}$ for all $\delta \geq 2$.

The results shows that, in general, the connectivity-dividing formulation provides a tighter linear programming relaxation than the secondary-peeling formulation, but that if we use the strengthened connectivity-splitting model, the linear programming relaxations for these models are equivalent; moreover, Proposition 9 shows that every such model (independent of Φ and δ) is LP equivalent. As we might expect, and as we show in the next section, the worst-case ratio of heuristic to optimal LP values of the splitting formulations are generally smaller than those for the aggregate [CUT] formulation.

4. Worst-case Analysis of Heuristics for MNB problems

This section analyzes the worst-case performance, relative to the optimal LP value of our extended formulation, of an "overlay" heuristic strategy for MNB problems with edge duplication, and the Tree Completion heuristic for unitary MNB problems without edge duplication. Unless otherwise specified, all of our results apply to nonunitary MNB problems. Section 4.1 introduces some additional notation. Section 4.2 describes an overlay heuristic for MNB problems with edge duplication and analyzes its worst-case performance relative to the optimal LP value of the connectivity-dividing formulation. This analysis shows that the heuristic-to-IP worst-case bound that Goemans and Bertsimas [1993] derived for a particular version of the overlay method also applies to the heuristic-to-LP ratio. Section 4.3 analyzes the performance of the Tree Completion heuristic for unitary MNB problems without edge duplication, generalizing some of the results we obtained in Section 2 via doubling arguments.

4.1 Preliminaries

Let $T \subseteq N$ denote the set of terminal nodes of a Steiner forest problem, i.e., T is the set of all critical and secondary nodes (which might belong to different components for the nonunitary MNB problem) of the problem. Let $Z_{01}(T)$ and $Z_{01}^{LP}(T)$ denote, respectively, the cost of the optimal Steiner forest with T as terminal nodes and the optimal LP value of the cut formulation [SF] of this Steiner forest problem. Let C_q denote the set of all nodes with connectivity requirement equal to q. C_{q+} is the set of all nodes with requirement q or higher. Note that C_{1+} is the set of all terminals nodes T, and C_{2+} is the set of all critical nodes C. For any integer $q \geq 1$, consider any $\{0,q\}$-connectivity (nonunitary) problem with edge duplication. The nodes in this problem have a connectivity requirement of 0 (Steiner nodes) or q (the set C_q). Let $Z_{0q}(C_q)$ and $Z_{0q}^{LP}(C_q)$ be the optimal integer value and optimal LP value of formulation [CUT] for this problem. For problems with nonnegative edge costs, $Z_{0q}^{LP}(C_{q+}) = q\, Z_{01}^{LP}(C_{q+})$.

4.2 MNB with edge duplication: Analysis of Overlay heuristic

For situations with edge duplication, we consider the following general subgraph overlay heuristic which successively meets the connectivity requirements of critical nodes in order of increasing criticality.

Overlay Heuristic for MNB problems with edge duplication:

Step 1: Find a heuristic or optimal solution S_1 to the Steiner forest problem with all the secondary and critical nodes as terminals.

Step 2: For k = 2 to K, find a heuristic or optimal solution S_k to the $\{0, \rho_k - \rho_{k-1}\}$-connectivity (nonunitary) problem with edge duplication over the terminal nodes C_{k+}.

Step 3: The union of solutions S_k for k = 1, 2, ..., K is the overlay heuristic solution to the MNB problem.

We next analyze the worst-case performance of this heuristic relative to the optimal LP value of the connectivity-splitting formulation for MNB problems with edge duplication.

4.2.1 Worst-case analysis of Overlay Heuristic

For k = 1, ..., K, let $\eta_k = \rho_k - \rho_{k-1}$. For MNB problems, $\rho_0 = 0$ and $\rho_1 = 1$, and so $\eta_1 = 1$. Let θ_k be an upper bound on the worst-case ratio of the cost $Z(S_k)$ of the heuristic solution at stage k of the overlay procedure relative to the optimal LP value $Z^{LP}_{0\eta_k}(C_{k+})$ of the cutset formulation for the $\{0,\eta_k\}$-connectivity problem with C_{k+} as critical nodes. Then, the cost Z^{Ovl} of the Overlay heuristic solution satisfies the inequality

$$\begin{aligned} Z^{Ovl} &= Z(S_1) + Z(S_2) + \ldots + Z(S_K) \\ &\leq \theta_1 Z^{LP}_{01}(T) + \theta_2 Z^{LP}_{0\eta_2}(C) + \sum_{3\leq k\leq K} \theta_k Z^{LP}_{0\eta_k}(C_{k+}). \end{aligned} \tag{4.1}$$

Since $Z^{LP}_{0\eta_k}(C_{k+}) = \frac{\eta_k}{\rho_k} Z^{LP}_{0\rho_k}(C_{k+})$ and since $Z^{LP}_{0\rho_k}(C_{k+})$ is a lower bound on the optimal LP value Z^{LP}_{CUT} of the CUT formulation of the MNB problem, inequality (4.1) leads to the following upper bound on the ratio ω of Z^{Ovl} to Z^{LP}_{CUT}:

$$\omega \leq \theta_1 + \theta_2 \frac{\eta_2}{\rho_2} + \sum_{3\leq k\leq K} \theta_k \frac{\eta_k}{\rho_k}. \tag{4.2}$$

Goemans and Bertsimas [1993] previously obtained this bound for certain values of the θ's (see the next section). We next show how to reduce this bound (in particular, reduce the first two terms on the right-hand side of (4.2)) by examining the MNB problem's LP relaxation.

Consider the **connectivity-halving** formulation [DIV(2)] of the MNB problem with edge duplication. This formulation consists of the objective function (3.5) and constraints (3.6) to (3.10) and (3.12) but with the following modifications: (i) $\mu = 1$ and so we ignore the b variables in (3.5) and (3.6), and (ii) $\delta = 2$ or $\phi_q = 1/2$ for all $q \in Q^{2+}$. Consequently, $\alpha_q = 0$ for all $q \in Q^{2+}$, i.e., we drop the y variables in constraints (3.6) and (3.7). Since the costs are nonnegative, the LP relaxation obtained by dropping the integrality restrictions (3.12) on the z variables has an optimal solution satisfying constraints (3.9) as equalities for all edges $\{i,j\}$. Substituting $z_{ij} = x_{ij} + y_{ij}$ in the objective function (3.5) and relaxing constraints (3.9) decomposes the LP relaxation into two subLPs: LP1 containing only the x variables and constraints (3.6) and (3.7) and nonnegativity, and LP2 containing only the y variables with constraints (3.8) and the nonnegativity requirements. Both x_{ij} and y_{ij} have the original cost c_{ij} as their objective function coefficients.

Note that if we "downgrade" all the cutset requirements from q/2 to 1 in constraints (3.7), the first subLP reduces to the Steiner Forest problem over all the terminal nodes $T = C_{1+}$. Therefore, $Z^{LP1} \geq Z^{LP}_{01}(T)$. Similarly, if we downgrade the cutset requirements from $q/2$ to $\rho_2/2$ in the second subLP, we obtain a relaxation whose optimal value $Z^{LP}_{0\rho_2/2}(C_{2+}) = \frac{1}{2} Z^{LP}_{0\rho_2}(C_{2+})$ underestimates Z^{LP2}. Therefore,

$$Z^{LP}_{DIV} \geq Z^{LP1} + Z^{LP2}$$

$$\geq \quad Z^{LP}_{01}(T) + \frac{1}{2} Z^{LP}_{0\rho_2}(C). \tag{4.3}$$

(Since the formulation [DIV(δ)] is LP-equivalent for all values of δ, we drop the argument δ in the notation for its optimal LP value.) The bounds (4.1) and (4.3), and our previous observation

$$Z^{LP}_{0\eta_k}(C_{k+}) = \frac{\eta_k}{\rho_k} Z^{LP}_{0\rho_k}(C_{k+}) \leq \frac{\eta_k}{\rho_k} Z^{LP}_{CUT} \leq \frac{\eta_k}{\rho_k} Z^{LP}_{DIV},$$

implies that

$$\begin{aligned} Z^{Ovl} &\leq \theta_1 Z^{LP}_{01}(T) + \frac{1}{2}\theta_1 Z^{LP}_{0\rho_2}(C) - \frac{1}{2}\theta_1 Z^{LP}_{0\rho_2}(C) + \theta_2 \frac{\eta_2}{\rho_2} Z^{LP}_{0\rho_2}(C) + \sum_{3\leq k\leq K} \theta_k \frac{\eta_k}{\rho_k} Z^{LP}_{0\rho_k}(C_{k+}) \\ &\leq \theta_1 Z^{LP}_{DIV} + \{\theta_2 \frac{\eta_2}{\rho_2} - \frac{1}{2}\theta_1\} Z^{LP}_{DIV} + \sum_{3\leq k\leq K} \theta_k \frac{\eta_k}{\rho_k} Z^{LP}_{DIV} \qquad \text{if } \theta_2 \frac{\eta_2}{\rho_2} \geq \frac{1}{2}\theta_1. \end{aligned}$$

Theorem 11:

For MNB problems with edge duplication, the overlay heuristic procedure produces a solution with the following worst-case bound relative to the optimal LP value Z^{LP}_{DIV} of the connectivity-dividing formulation:

$$\begin{aligned} \frac{Z^{Ovl}}{Z^{LP}_{DIV}} &\leq \frac{1}{2}\theta_1 + \sum_{2\leq k\leq K} \theta_k \frac{\eta_k}{\rho_k} && \text{if } \theta_2 \frac{\eta_2}{\rho_2} \geq \frac{1}{2}\theta_1, \text{ and} \\ &\leq \theta_1 + \sum_{3\leq k\leq K} \theta_k \frac{\eta_k}{\rho_k} && \text{otherwise.} \end{aligned}$$

4.2.2 Bounds for tree and forest-based overlay heuristics

We now specialize the bound in Theorem 11 to tree-based and forest-based versions of the overlay heuristic for unitary and nonunitary MNB problems with edge duplication. For unitary MNB problems, Goemans and Bertsimas [1993] analyzed two specialized versions of the overlay heuristic—a *tree+tree* heuristic and a *tree+matching* (our terminology) heuristic—that use particular heuristic solutions as S_1 and S_k. If T denotes the set of terminal nodes, both methods select the minimum spanning tree MST(T) as the heuristic solution S_1. In Step 2, the tree+tree heuristic selects $(\rho_k - \rho_{k-1})$ copies of a minimum spanning tree $MST(C_{k+})$ as S_k. Whenever the tree+tree heuristic selects two identical trees, the tree+matching heuristic improves the solution by replacing the second tree with a minimum cost matching over the odd-degree nodes of the tree. Thus, in the tree+matching heuristic, S_k consists of $\lceil (\rho_k - \rho_{k-1})/2 \rceil$ copies of a minimum spanning tree $MST(C_{k+})$ and $\lfloor (\rho_k - \rho_{k-1})/2 \rfloor$ copies of an optimal matching $MATCH(C_{k+})$ on the nodes C_{k+}.

The natural extension of tree+tree heuristic to nonunitary MNB problems would overlay heuristic Steiner forest solutions instead of spanning trees. Step 1 finds an approximate Steiner forest problem with all secondary and critical nodes as terminal nodes. In Step 2, for each k = 2, ..., K, the algorithm sets S_k equal to $(\rho_k - \rho_{k-1})$ copies of the Steiner forest over the terminal nodes C_{k+}. Since this method solves a Steiner forest problem in each step, we refer to this heuristic as the *forest+forest* heuristic.

Goemans and Williamson [1992] developed a dual-based heuristic FOREST(N') for the Steiner forest problem defined on the set N' of terminal nodes. They showed that

$$Z^{\mathrm{FOREST(N')}} < 2\, Z_{01}^{LP}(N'). \tag{4.4}$$

This result generalizes the following Steiner tree bound (Goemans and Bertsimas [1993]) relating the minimum spanning tree heuristic and the optimal LP value $Z_{ST}^{LP}(N')$ of the cutset formulation [ST] of the Steiner tree problem over the terminal nodes N', i.e.,

$$Z_{MST(N')} < 2\, Z_{ST}^{LP}(N'). \tag{4.5}$$

If we include a multiplicative factor of $(1 - \frac{1}{|N'|})$ in the right hand sides of either (4.4) or (4.5), then the strict inequalities become inequalities.

Suppose we use the forest+forest heuristic for general MNB problems. This heuristic selects all the edges belonging to FOREST(T) at step 1, and $\rho_k - \rho_{k-1}$ copies of FOREST(C_{k+}) at step k, k = 2, 3, ..., K. In this case, (4.4) and (4.5) imply that $\theta_k = 2$ for all k. Observe that the first inequality of Theorem 11 holds for these values of θ_k. If Z^{F+F} denotes the cost of the forest+forest heuristic solution, then Theorem 11 implies the following result.

Corollary 12:

For nonunitary MNB problems with edge duplication, the forest+forest heuristic has the following worst-case bound relative to the linear programming relaxation of [DIV(2)]:

$$\frac{Z^{F+F}}{Z_{DIV}^{LP}} < 1 + 2 \sum_{2 \le k \le K} \frac{\eta_k}{\rho_k}.$$

For unitary MNB problems, we use the tree+matching heuristic. In this case, $\theta_1 = 2$. In Step k, the tree+matching heuristic has the following worst-case ratios (Goemans and Bertsimas [1993]): if $\rho_k - \rho_{k-1} \ge 2$ is even, then $\theta_k = 3/2$, and if $\rho_k - \rho_{k-1} \ge 3$ is odd, then θ_2

$= \frac{3}{2} + \frac{1}{2(\rho_k - \rho_{k-1})}$. Let Z^{T+M} denote the cost of the tree plus matching heuristic solution. Theorem 11 implies the following result:

Corollary 13:

For unitary MNB problems with edge duplication, the tree+matching heuristic has the following worst-case bound relative to the linear programming relaxation of [DIV(2)]:

$$\frac{Z^{T+M}}{Z^{LP}_{DIV}} < 1 + \sum_{2\leq k\leq K} \frac{\eta_k}{2\rho_k} \{3 + \frac{|\sin(\frac{\eta_k \pi}{2})|}{\eta_k}\}.$$

As we noted previously, Goemans and Bertsimas developed the same bound, but relative to the optimal IP value of the MNB problem.

For specific connectivity values, we can compute the right hand sides of Corollaries 12 and 13 and demonstrate that they are asymptotically tight. Let us first consider $\{0,1,\rho\}$-connectivity problems first. Using Corollary 12, we obtain

$$\frac{Z^{F+F}}{Z^{LP}_{01\rho}} < 3 - \frac{2}{\rho}. \tag{4.6}$$

In particular, for unitary LCS problems, the heuristic method selects all the edges belonging to MST(T) and MST(C). Proposition 5 showed, using a doubling argument, that the overlay heuristic has a worst-case performance ratio of $2(1-1/|T|)$ relative to the optimal integer value of the LCS problem. The inequality (4.6) strengthens this result by showing that the same bound applies asymptotically to the ratio of the heuristic cost to LP value of the connectivity-dividing formulation.

The bound of 2 for unitary LCS problems is tight. Consider a ring with $|C|$ equally spaced critical nodes on its circumference and a secondary node in the center. Suppose the total ring cost is 1, and the secondary node is connected to one of the critical nodes with a zero cost edge. The linear programming solution (which is also the optimal solution) chooses all the ring edges and the spoke edge; the cost of this solution is 1. In contrast, the tree+tree heuristic chooses the spoke edge and two copies of all but one ring edge, incurring a total cost of $2(1-1/|C|)$ which approaches 2 as $|C|$ increases.

Next, consider the performance of the tree+matching heuristic for unitary $\{0,1,\rho\}$-connectivity problems with edge duplication. In this case, Corollary 13 implies the following worst-case bound relative to the optimal LP value $Z_{01\rho}^{LP}$ of formulation [DIV(δ)]:

$$\frac{Z^{T+M}}{Z_{01\rho}^{LP}} < \frac{5}{2} - \frac{3}{2\rho} \qquad \text{if } \rho > 2 \text{ is odd, and} \qquad (4.7a)$$

$$\frac{Z^{T+M}}{Z_{01\rho}^{LP}} < \frac{5}{2} - \frac{1}{\rho} \qquad \text{if } \rho > 2 \text{ is even.} \qquad (4.7b)$$

In particular, when $\rho = 3$, this corollary implies a bound of 2 for large values of $|T|$. Appendix A shows that this bound of 2 is tight. Goemans and Bertsimas showed that the tree+matching heuristic has a worst-case ratio of 3 relative to the LP relaxation of formulation [CUT], but a bound of 2 relative to the optimal integer value. The example in Appendix A also shows that the bound of 3 relative to the LP value of formulation [CUT] is tight.

4.3 Analysis of Tree Completion heuristic for unitary MNB problems without edge duplication

This section analyzes the worst-case performance of the Tree Completion heuristic we described in Section 2 for unitary MNB problems without edge duplication, assuming that edge costs satisfy the triangle inequality. We develop an upper bound on the cost of the Tree Completion heuristic relative to the optimal LP and IP values of the secondary-completion formulation [COMPL]. This bound depends upon the worst-case performance ratio used in Step 1 of the Tree Completion heuristic to configure a multi-connected network providing the requisite number of edge-disjoint paths between all the critical nodes.

Consider formulation [COMPL], the special version of [CCS(μ,Φ)] with $\mu = 1/2$ and $\phi_q = 0$ for all q. Recall that in this formulation, we omit the x variables and the redundant constraints (3.7). Suppose we replace y_{ij} with y_{ij}^2 in constraints (3.6) and y_{ij}^1 in constraints (3.8) and add the constraint

$$y_{ij}^1 = y_{ij}^2 \qquad \text{for all edges } \{i,j\} \in E. \qquad (4.8)$$

If we dualize these linking constraints (4.8) using a multiplier of 1/2 for all edges $\{i,j\}$, the problem decomposes into two <u>integer programs</u>: a Q^{2+}-connectivity problem (without edge duplication) over all the critical nodes with half the original edge costs, and a "Steiner-like" problem defined over the set of secondary nodes plus a dummy node representing the critical nodes as the terminals. Let [ST*] denote this problem. The Q^{2+}-connectivity problem is the same as the original MNB problem but with the connectivity requirement for all secondary nodes reduced to 0. If Z_{Q2+} denotes the optimal value of this problem (without edge duplication), Z_{ST*} denotes the optimal value of the second subproblem, and Z_{COMPL} denotes the optimal value of the problem (that is, the secondary-completion model [COMPL]), then

$$Z_{COMPL} \geq \frac{1}{2} Z_{Q2+} + Z_{ST*}. \tag{4.9}$$

Consider the second subproblem containing constraints (3.6), but with variable y_{ij}^2 instead of y_{ij}, and the integrality (binary) constraints. Unlike the standard Steiner tree problem, this problem requires not only connecting the secondary nodes to each other but also requires a path to at least one critical node. Since we have allocated half the edge cost to y_{ij}^2, this problem's LP relaxation has an optimal solution with $y_{ij}^2 = 0$ for all edges $\{i,j\}$. Otherwise, given an optimal LP solution with $y_{ij}^2 > 0$, we can obtain an equal or lower cost feasible solution by setting $b_{ij} = y_{ij}^2/2$. Thus, the LP relaxation of the second subproblem is the LP relaxation of the following extended Steiner problem:

Problem [ES]:

$$Z_{ES} = \min \sum_{\{i,j\} \in E} c_{ij}\, b_{ij} \tag{4.10}$$

subject to

$$B_{ST} \geq 1 \qquad \text{for all } \{S,T\} \in \sigma_1, \tag{4.11}$$

$$b_{ij} = 0 \text{ or } 1 \qquad \text{for all } \{i,j\} \in E. \tag{4.12}$$

We refer to this model as the "extended" Steiner problem because σ_1 contains a special class of cutsets that separate all the secondary nodes from the critical nodes. If Z_{ES}^{LP} denotes the optimal LP value of this problem, then

$$Z_{ST*} \geq Z_{ES}^{LP}. \tag{4.13}$$

4.3.1 Extended Spanning Tree (EST) heuristic:

The following adaptation of the usual minimum spanning tree (MST) heuristic for Steiner trees applies to the extended problem :

- For each critical node i, find the minimum spanning tree T(i) of the subgraph induced by the secondary nodes and node i.

- Choose the tree T(i) with the lowest cost as the heuristic solution to the Extended Steiner tree problem.

If n_s is the number of secondary nodes and the original edge costs are triangular, this method produces a heuristic solution that costs no more than $2(1-1/(n_s+1))\ Z_{ES}^{LP}$. This result is valid because any optimal solution to the extended problem must span some critical node i. Therefore, the MST heuristic applied to a regular Steiner tree problem in which we treat all the secondary nodes plus node i as the terminals produces a solution that costs no more than $2(1-1/(n_s+1))$ times the optimal value (Takahashi and Matsuyama [1980]). Since in this case the MST heuristic selects tree T(i), our heuristic solution also costs no more than $2(1-1/(n_s+1))$ times the optimal value of the ES problem.

The following alternative "Extended Spanning Tree" (EST) heuristic does not necessarily produce a feasible ES solution, but is useful for analyzing the Tree Completion heuristic.

- Consider the subgraph of G induced by all the secondary and critical nodes. Merge (contract) all the critical nodes into a single dummy node. In the contracted graph, the length of the edge connecting a secondary node j to the dummy node is the length of the smallest edge connecting a critical node to node j.
- Find the minimum spanning tree of the contracted graph, and replace the edges incident to the dummy node in this solution with the original edges.

Note that: (i) when we replace the edges of the contracted graph with the original edges in the second step, the resulting solution need not be connected, i.e., this solution is not necessarily feasible for problem [ES]; (ii) this solution costs no more than the cost of our previous heuristic solution (i.e., the cost of the least cost tree T(i)); and (iii) the EST procedure essentially solves a "reduced cost" version of problem [ES] in which we use zero cost edges to connect all pairs of critical nodes. Since we have decreased the cost of certain edges to 0, the optimal value of this version is less than or equal to Z_{ES}. Therefore, we have the following result.

Proposition 14:

If the edge costs c_{ij} satisfy the triangle inequality, the EST heuristic produces a solution that costs no more than $2\ (1-\frac{1}{(n_s+1)})\ Z_{ES}^{LP}$.

Recall that Step 2 of the Tree Completion heuristic contracts all the nodes spanned by the Q^{2+}-connected solution into a dummy node, and finds the minimum cost tree spanning this dummy node and the remaining secondary nodes (not spanned by the Q^{2+} solution). It is easy to show that the cost incurred by the tree completion step is no more than the cost of the EST heuristic solution.

Proposition 15:

Suppose the heuristic method we use to solve the triangular cost Q^{2+}-connectivity problem (without edge duplication) has IP or LP performance guarantees of θ or θ_{LP} (the LP guarantee is relative to the problem's cutset formulation). Then the Tree Completion heuristic produces a MNB solution with the following upper bounds on its IP and LP worst-case ratios ω and ω_{LP} (relative to the secondary-completion formulation):

$$\omega < \theta + 1, \text{ and}$$
$$\omega_{LP} < \theta_{LP} + 1.$$

Proof:

Let $Z^{heurQ^{2+}}$, Z^{branch}, and Z^{TC} denote, respectively, the costs of the heuristic solution to the Q^{2+}-connectivity problem, the total cost of the branches added in Step 2, and the total cost of the Tree Completion heuristic solution. Proposition 14, the inequality (4.9), and the inequalities $Z_{Q^{2+}} \leq Z_{COMPL}$ and $Z^{LP}_{ES} \leq Z_{ES} = Z_{ST^*}$ imply that

$$\begin{aligned} Z^{TC} &= Z^{heurQ^{2+}} + Z^{branch} < \theta\, Z_{Q^{2+}} + 2\, Z^{LP}_{ES} \\ &= (\theta-1)\, Z_{Q^{2+}} + 2\, (\tfrac{1}{2} Z_{Q^{2+}} + Z^{LP}_{ES}) \;\leq\; (\theta + 1)\, Z_{COMPL}. \end{aligned}$$

Let $Z^{LP}_{Q^{2+}}$ and Z^{LP}_{COMPL} denote the optimal LP values of the cutset formulation of the Q^{2+}-connectivity subproblem and the secondary-completion formulation of the original MNB problem. Since the inequality (4.9) also applies to the linear programming relaxations of problems [COMPL], the Q^{2+}-connectivity problem, and [ST*] and since $Z^{LP}_{Q^{2+}} \leq Z^{LP}_{COMPL}$,

$$\begin{aligned} Z^{TC} &= Z^{heurQ^{2+}} + Z^{branch} < \theta_{LP}\, Z^{LP}_{Q^{2+}} + 2\, Z^{LP}_{ES} \\ &= (\theta_{LP}-1)\, Z^{LP}_{Q^{2+}} + 2\, (\tfrac{1}{2} Z^{LP}_{Q^{2+}} + Z^{LP}_{ES}) \\ &\leq (\theta_{LP} + 1)\, Z^{LP}_{COMPL}. \end{aligned}$$

♦

4.3.2 Some Implications

(i) **LCS problems without edge duplication:**

To solve unitary LCS problems without edge duplication, consider the following tree+matching heuristic for the unduplicated {0,2}-connectivity problem: find the minimum cost tree spanning the critical nodes and construct an Eulerian graph by adding the edges of the minimum cost matching over the odd degree nodes in this tree. Consider an Eulerian tour in this tree+matching solution. By short-circuiting edges, we can transform this solution into an equal or lower cost hamiltonian tour over the critical nodes. As Goemans and Bertsimas have shown, $\theta_{LP} = 3/2$ for this solution. Therefore, the Tree Completion method with this embedded solution procedure in Step 1 produces a solution with an LP worst-case ratio of at most 5/2.

Alternatively, if we use the optimal TSP tour over the critical nodes as the heuristic {0,2}-connectivity solution in Step 1, then since the costs are triangular, from Proposition 6, $\theta = 4/3$ (Monma, Munson and Pulleyblank [1990]). So, the Tree Completion method with the embedded TSP solution procedure has an overall LP worst-case ratio of 7/3.

Worst-case examples:

Consider the example in Figure 3(a). This figure has two concentric rings each consisting of q critical nodes. The critical nodes on the rings are aligned. Every critical node on each ring is connected to its two neighbors on that same ring with edges of cost 1. Every critical node on a ring is also connected to 3 other nodes on the other ring: the node directly aligned with it via a (spoke) edge of cost 1, and the two nodes to the immediate right and left; each of these (spoke) edges has a cost of 2. An alternate path (of total length 1) consisting of p–1 secondary nodes also connects every pair of adjacent critical nodes on the same ring.

Figure 3(b) shows the heuristic solution if we use the Tree+Matching heuristic to connect the critical nodes. We first choose the inner ring (except one edge), and all unit-cost spoke edges in the MST. The matching step then selects q–2 of the remaining spoke edges and an edge on the outer ring. Short circuiting provides us with a two-connected solution that costs 3q–2. To obtain a heuristic solution to the {0,1,2} problem, we then connect all the secondary nodes to this subgraph and incur an additional cost of 2q(1–1/p). Thus, the total heuristic cost is 5q–2q/p–2. The optimal solution in Figure 3(c) costs 2q+2, and thus we obtain an asymptotic heuristic-to-optimal cost bound of 5/2 for large values of p and q. But then the asymptotic ratio of the heuristic-to-LP cost is at least 5/2, which is the worst-case ratio we established earlier.

The example of Figure 4 proves that the bound of 7/3 is tight if we use the TSP as a heuristic solution for the {0,2}-connectivity subproblem. This example is an extension of a problem instance proposed by Monma, Munson, and Pulleyblank [1990]. The network, shown in Figure 4(a), contains three paths, each with q–1 critical nodes, connecting two special critical nodes 1 and 2. The cost of each edge on these three paths is 1. Every pair of adjacent critical nodes is also connected by a path containing p–1 secondary nodes; the total cost of this alternate path is 1. The cost between any other pair of nodes is the shortest path cost between these nodes.

Figure 4(b) shows the heuristic solution whose the cost is $4q-1 + 3q(1-1/p)$. Figure 4(c) shows the optimal solution with a cost of 3q. This example achieves the bound of 7/3 for large values of p and q. Note that since this example does not contain a Steiner node, the worst-case bound also applies to {1,2}-connected problems.

(ii) **K-path Steiner tree problem:**

Recall that the K-path Steiner tree problem contains two critical nodes that must be connected by K edge-disjoint paths that possibly pass through secondary nodes or Steiner points; the optimal design must also contain other secondary nodes on Steiner branches connected to these K paths. In this case, the {0,K}-connectivity subproblem without edge duplication is solvable as a minimum cost flow problem. Therefore, $\theta_{LP} = 1$ and so the Tree Completion method has a worst-case LP ratio of at most 2.

(iii) **MNB problems with side constraints:**

The model extends to more general classes of MNB problems with additional configuration constraints imposed on the multi-connected network. Consider, for instance, the Ring on Steiner tree problem, which requires the two-connected subgraph of the LCS solution to be a hamiltonian tour that visits all the critical nodes (and optionally visits secondary or Steiner nodes). In this case, we have additional configuration constraints in formulation [COMPL] specify that every critical node must have degree 2. The formulation, and therefore our analysis, remains valid even with these additional constraints as long as we use an appropriate heuristic method in Step 1 of the completion procedure. So, if we find the optimal TSP tour over the critical nodes, then $\theta = 1$ and so the Tree Completion method has a worst-case ratio of $\omega = 2$. (Note that, with triangular costs, the optimal TSP tour that visits just the critical nodes is optimal, i.e., we can ignore the secondary and Steiner nodes while solving the constrained {0,2}-connectivity subproblem in Step 1.)

5. Conclusions

Since even the simplest cases of survivable network design problems are NP-hard, researchers have focused on modeling enhancements to improve the effectiveness of linear programming-based solution methods, and on analyzing tree, tour, and matching-based heuristics. In this paper, we have studied modeling and heuristic methods for unitary and nonunitary MNB problems (containing one or more single-connectivity secondary nodes) both with and without edge duplication. We also addressed an important special case, namely, LCS problems with path connectivity requirements of 0, 1, or 2. Our analysis uses two complementary approaches: a solution *doubling* argument to establish heuristic bounds relative to the optimal IP value, and a connectivity *splitting* (halving) formulation to establish bounds relatives to the optimal LP value.

We first developed a result for LCS problems that is analogous to a well-known Steiner tree result: if we solve an LCS problem without the Steiner nodes, the resulting solution costs at most twice the optimal value of the original LCS problem. The solution doubling argument used to prove this result applies to other related problems as well. For example, it permits us to use any heuristic with a worst-case bound of α for Eulerian graph optimization problems to develop a Tree Completion heuristic with a worst-case bound of 2α for an MNB version of these problems. A similar doubling argument establishes relationships among the optimal objective value of certain LCS problems and MST solution values, TSP solution values, and costs of optimal matchings over secondary and critical nodes.

Our discussion of MNB modeling issues builds upon a traditional cut formulation for modeling survivability problems. Because it is more tractable, most researchers have used the cut formulation to develop lower bounds in order to analyze the worst-case performance of SND heuristics. However, since the cut formulation has a weak linear programming relaxation, developing and guaranteeing strong worst-case bounds is difficult even though the heuristics might inherently be good. Goemans et al. (1994) have recently developed an elegant heuristic to improve earlier performance bounds (for example, Williamson et al.

(1993)) for survivability problems. Goemans et al. suggest that improving the performance guarantee any further might require a completely different solution approach.

Our modeling and analysis approach in this paper differs from previous approaches in two respects: (i) we strengthen the problem's linear programming formulation without sacrificing its tractability for heuristic analysis, and (ii) we analyze relatively simple heuristics that use MST, matching, and forest heuristics as building blocks. Even with these simple heuristics, we are able to achieve or improve upon some of the existing bounds in the literature.

Consider, for instance, the {0, 1, 2}-connectivity problem without edge duplication. Williamson et al.'s (1993) heuristic was the first polynomial time heuristic with a constant performance bound for this problem. For MNB problems without edge duplication, Section 4.3 develops a heuristic-to-IP bound of $\theta + 1$ that depends on the worst-case ratio θ of the heuristic to IP solution value for the Q_{2+}-connectivity subproblem (obtained by treating the secondary nodes in the original problem as Steiner nodes.) If the original problem is a {0,1,2}-connectivity problem, then the Q_{2+} subproblem is a {0,2}-connected problem. Consider the following Tree Completion heuristic, containing the embedded Christofides heuristic to approximately solve the {0,2}-connected subproblem.

<u>Step 1</u>:	Find an MST spanning all the critical nodes.
<u>Step 2</u>:	Find a matching of the odd-degree nodes in this tree.
<u>Step 3</u>:	Short circuit edges to obtain a tour through the critical nodes. (Steps 1 – 3 generate Christofides' heuristic solution for the critical nodes.)
<u>Step 4</u>:	Aggregate this tour into a single node, replacing any parallel edges by the least cost edge.
<u>Step 5</u>:	Find an MST spanning this aggregate node and the secondary nodes.

Monma, Munson and Pulleyblank have shown that the optimal TSP objective value is at most 16/9 of the optimal {0,2}-connected solution value (assuming triangular costs). Since the Christofides heuristic provides an approximate solution to the {0,2} connected problem whose cost is within 3/2 of the optimal TSP solution value, this heuristic solution costs at most $\frac{3}{2} * \frac{16}{9} = \frac{8}{3}$ times the optimal {0,2}-connected solution. Consequently, $\theta = \frac{8}{3}$ and using Proposition 15 we obtain a $\frac{8}{3} + 1 = \frac{11}{3}$ approximate solution for the {0,1,2} problem using the simple MST-matching heuristic. Using the more sophisticated primal-dual heuristic, Goemans et al. and Williamson et al. established a worst-case bound of 3 for this problem.

As another application of our results, consider MNB problems with edge duplication and complete connectivity (i.e., the set Q contains all integer connectivity values from 0 through K). Our LP bound of $2\,(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{K}) - 1$ for this problem is 1 less than Goemans et al.'s bound (which applies to the more complex unduplicated case as well); it is also lower than Williamson et al.'s bound of $2K - 1$. If some intermediate connectivity levels are missing, then Theorem 11 provides a bound of $2\,(1 + 2 \sum_{2 \le k \le K} \frac{\rho_{k-1} - \rho_k}{\rho_k}) - 1$ which further improves upon Goemans et al.'s bound of $2\,(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{K})$.

In developing these bounds, we have used a new connectivity-splitting mixed-integer programming formulation (3.5 – 3.13) for survivable network design problems. This formulation generalizes in two ways. First, if we change the right hand sides of (3.11) and (3.13), the formulation models a "capacitated" version of the problem. For example, by allowing b_{ij} to be any positive integer, and by changing the right hand side of (3.13) to β_{ij} allows us to choose up to β_{ij} copies of edge $\{i,j\}$.

Second, the formulation (3.5) – (3.13) applies with minor modifications even when the right hand side of (3.2) is a proper function (see Goemans et al., [1994]). (An integer valued function $f(\cdot)$ defined on the subsets of a set N is proper if $f(N) = 0$, $f(S) = f(N \backslash S)$ for $S \subseteq N$, and $f(A \cup B) \le \max\{f(A), f(B)\}$ whenever A and B are disjoint.) To incorporate this change, we alter the right hand sides of (3.7) and (3.8) to $f(S)\,\phi_{f(S)}$ and $f(S)\,(1-\phi_{f(S)})$ for some pre-specified connectivity fractions $\phi_{f(S)} \in [0, \frac{1}{2}]$ for all $f(S) \ge 2$. The parameter ϕ is defined as the minimum of all values $\phi_{f(S)}$, $f(S) \ge 2$. The proof of the validity of this formulation is similar to the proof of Theorem 8. This observation suggests the possiblity of extending this paper's approach to survivable network design problems with proper connectivity functions and without edge duplication.

APPENDIX A

The example in Figure 2(a) shows that the Tree+Matching heuristic achieves an asymptotic bound of 2 relative to the linear programming relaxation of formulation [DIV(3)]. This "honeycomb" example has m hexagons packed in a plane, with m a sufficiently large integer. All the hexagon vertices represent critical nodes of connectivity requirement 3. Each pair of adjacent critical nodes is connected by two alternate *indirect paths* containing p–1 secondary nodes. The direct edge cost, as well as the total cost of the indirect paths connecting adjacent critical nodes, is 1.

Since (i) the honeycomb has m hexagons, (ii) each hexagon has 6 edges, and (iii) each edge belongs to 2 hexagons, the honeycomb example has a total of 3m <u>direct</u> edges. (We ignore the boundary effects since the number of boundary edges grows sublinearly with m.) Similarly, the honeycomb has 2m critical nodes. For each direct edge in the network, the optimal solution chooses all edges of one of the corresponding indirect paths and all but one edge of the other indirect path. The optimal solution cost is $3m + 3m(1-1/p)$.

In Step 1, the heuristic finds an MST spanning all terminal nodes: the cost of edges chosen in this step is $3m + 3m(1-1/p)$. In Step 2, the heuristic first finds an MST spanning the critical nodes and chooses $2m-1$ direct edges. Figure 1(b) shows an MST spanning the critical nodes as bold edges. The heuristic then finds a minimum matching over the odd degree nodes of the tree. The minimum matching over the odd degree nodes of the tree duplicates the pendant edges; since each hexagon has, on average, one pendant edge, the cost of the minimum matching is m. Thus, the total heuristic cost is

$$(3m + 3m(1-1/p)) + (2m-1) + m \;=\; 9m - 3m/p - 1.$$

Next consider the linear programming solution to formulation [DIV(3)]. Setting $y_{ij} = 1/3$ and $x_{ij} = 5/12$ for each edge on all the indirect paths, we obtain a feasible solution to the linear programming relaxation of [DIV(3)]. This solution costs 4.5m, and thus asymptotically, for large values of p and m, this example achieves the desired bound of 2. We note that this example also achieves the Goemans and Bertsimas heuristic to linear programming bound of 3 for formulation [CUT]. Setting $u_{ij} = 1/2$ for each edge on all the indirect paths, we obtain an optimal solution, of cost 3m, to the linear programming relaxation of [CUT]. Since the tree+matching heuristic solution costs 9m, this example asymptotically achieves the bound of 3.

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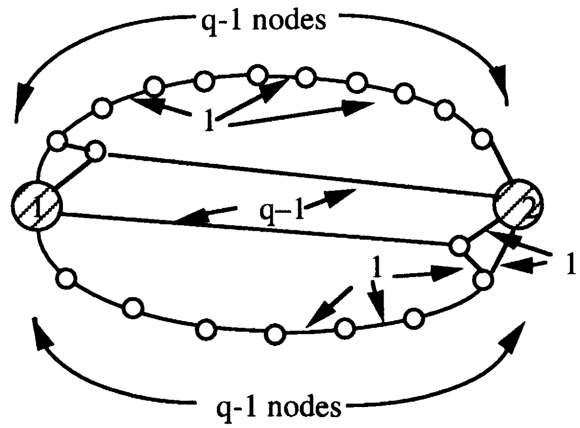

Figure 1(a): DPST example
costs shown on edges

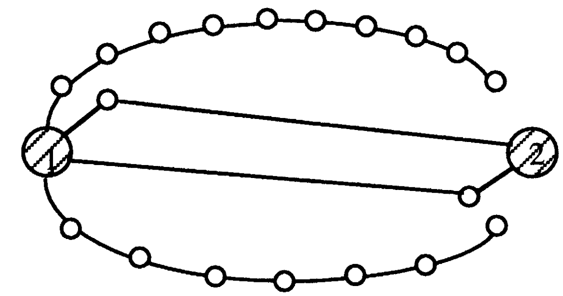

Figure 1(b): Tree Completion heuristic solution

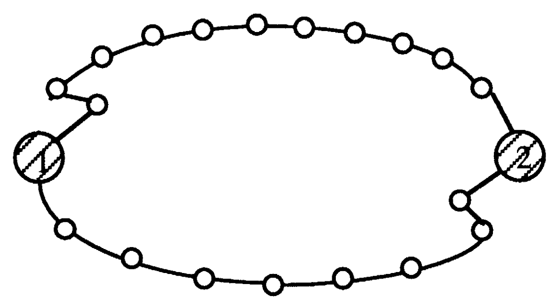

Figure 1(c): Optimal solution

Figure 1: Worst-case example for DPST problem

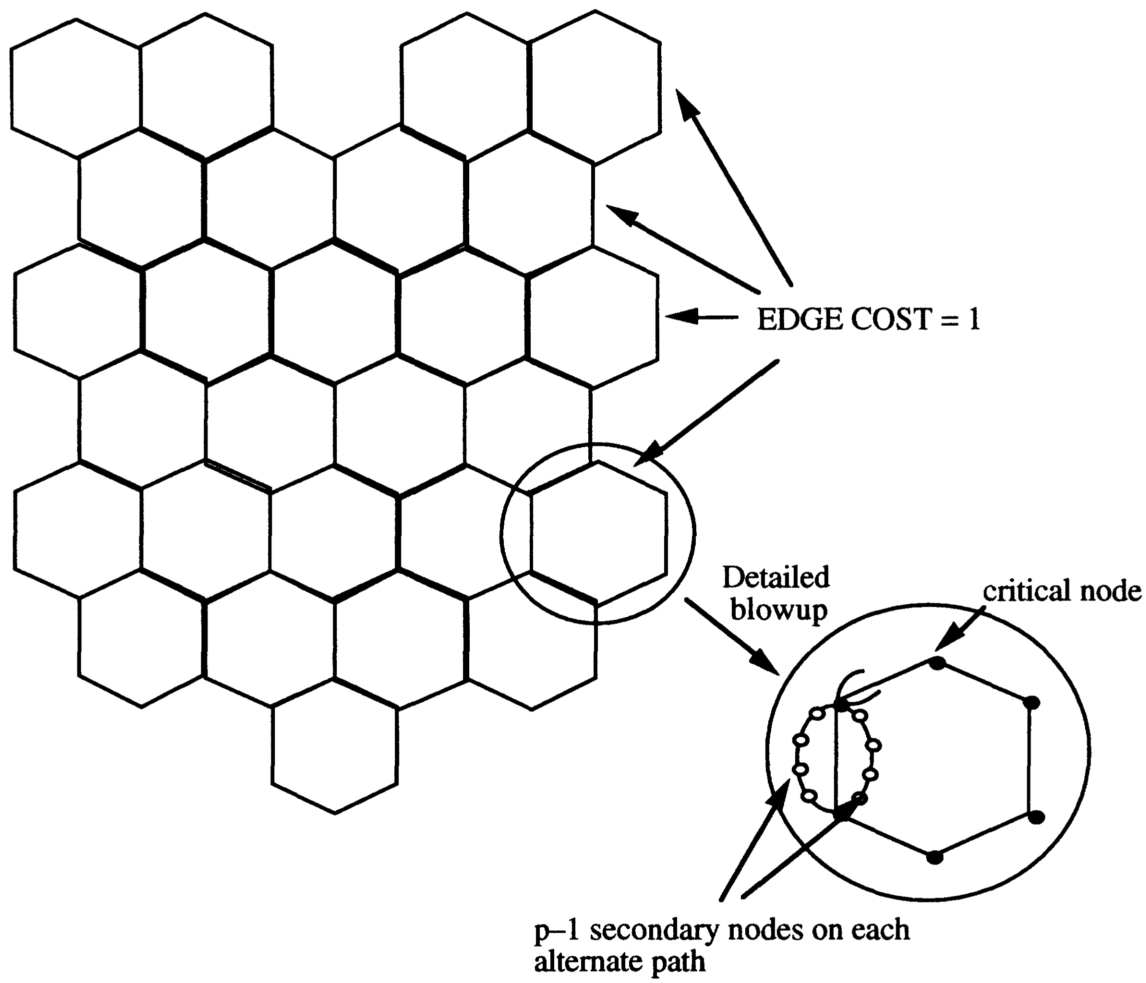

Figure 2(a): Example to show that the {0,1,3} heuristic to LP bound of 2 is tight

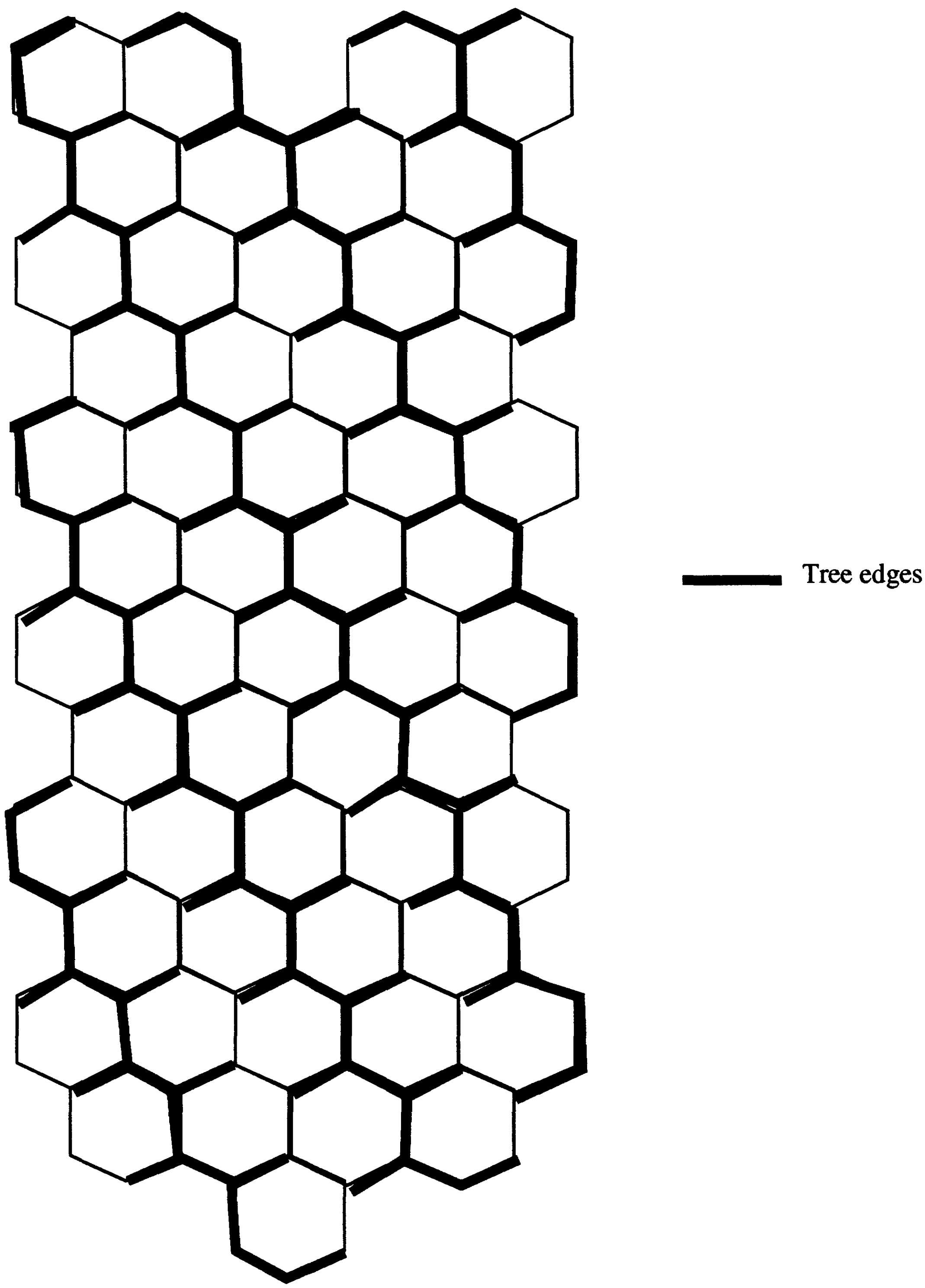

Figure 2(b) Tree edges chosen by first part of Step 2 of the Tree+Matching heuristic

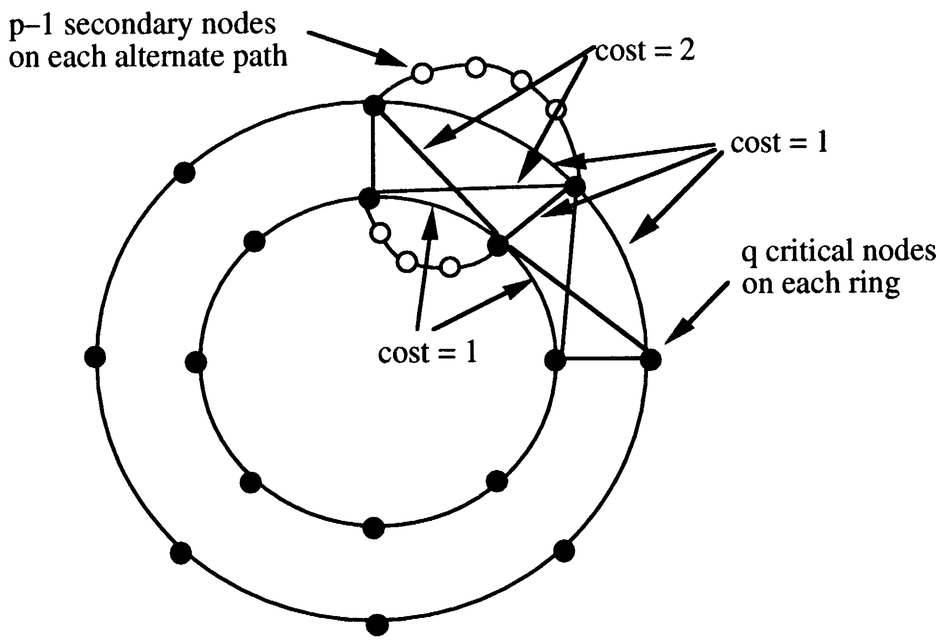

Figure 3(a) Network structure and costs

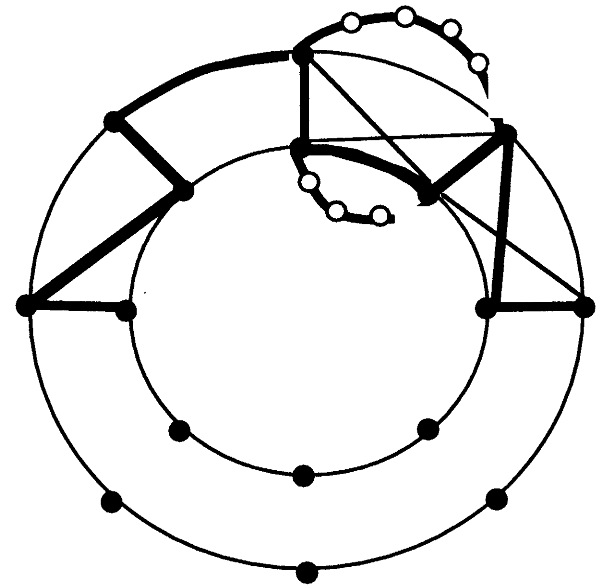

Figure 3(b) Heuristic solution shown in bold

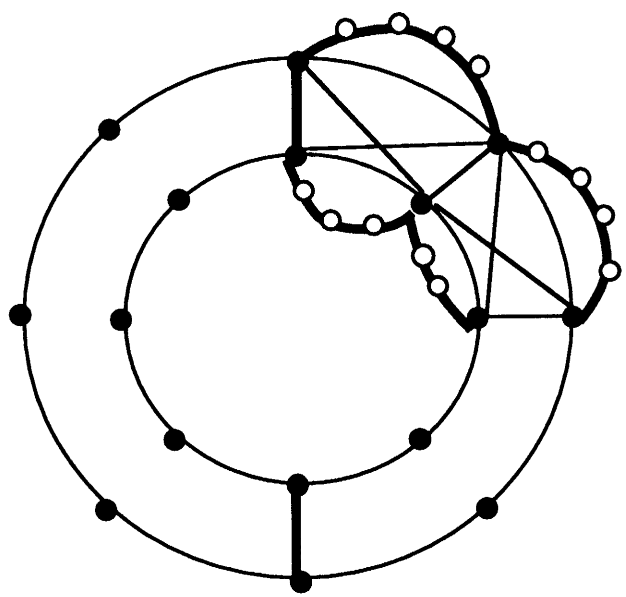

Figure 3(c) Optimal solution shown in bold

Figure 3: Example to show that the bound of 5/2 is tight for the {0,1,2} connectivity problem

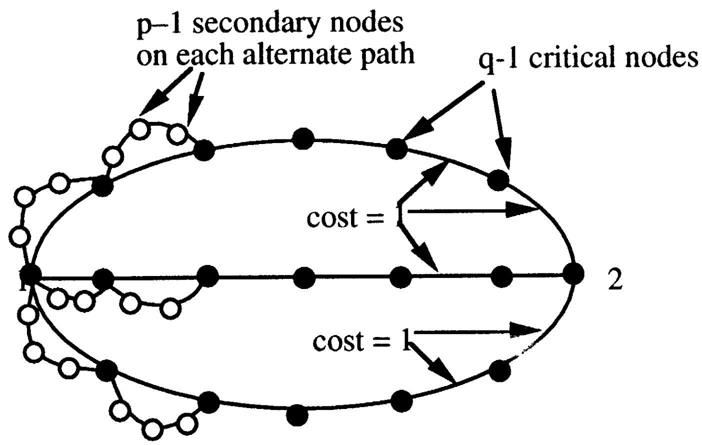

Figure 4(a) Partial network structure and costs
(Network is complete; edge costs are shortest path distances)

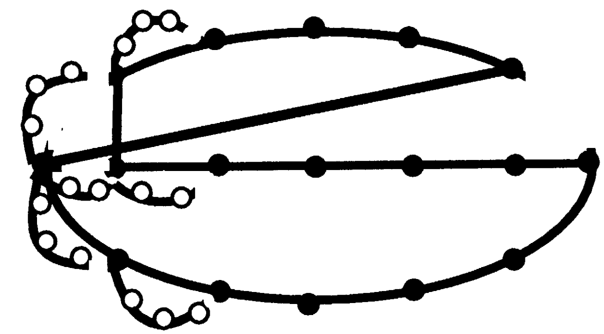

Figure 4(b) Heuristic solution shown in bold

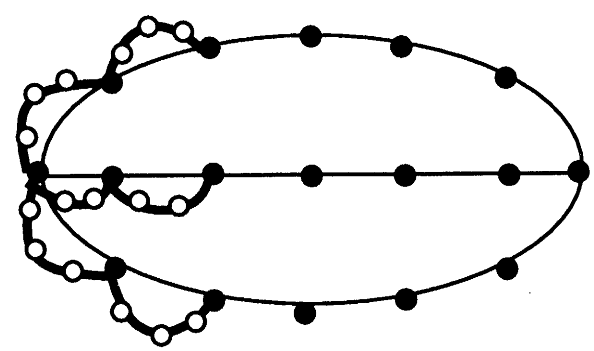

Figure 4(c) Optimal solution shown in bold

Figure 4: Example to show that the bound of 7/3 is tight for the {0,1,2} connectivity problem