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M/G/∞ with Batch Arrivals

by

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Abstract

Let $\mathbf{p}_\infty(\mathbf{n})$ be the distribution of the number $\mathbf{N}(\infty)$ in the system at ergodicity for systems with an infinite number of servers, batch arrivals with general batch size distribution and general holding times. This distribution is of impotance to a variety of studies in congestion theory, inventory theory and storage systems. To obtain this distribution, a more general problem is addressed . In this problem, each epoch of a Poisson process gives rise to an independent stochastic function on the lattice of integers, which may be viewed as a stochastic impulse response. A continuum analogue to the lattice process is also provided.

O. Introduction.

The distribution of the number in the system at ergodicity for systems with an infinite number of servers, batch arrivals with general batch size distribution and general holding times is of importance to a variety of studies in congestion theory, inventory theory and storage systems. To obtain this distribution, a more general problem is addressed. In this problem, each epoch of a Poisson process gives rise to an independent stochastic response function on the lattice of integers.

The simplicity and importance of the results suggest that they may be known, but the authors have not been able to find them in the literature. The method employed is probabilistic and succinct and may be of independent interest.

1.1 A more general problem

The work described has been motivated by M/G/∞ batch arrival system needs. The results however relate simply to a more general system context and may have value elsewhere.

For the M/G/∞ batch system, customers arrive at Poisson epochs of rate λ with random batch size and random i.i.d service times to a system having an infinite number of servers . The p.g.f. at ergodicity of **N(t),** the number in the system, is wanted. To find this distribution , we consider the more general problem where, for each arrival epoch τ_k of of the Poisson stream , there is a stochastic response function $\mathbf{N_k^*(t)}$ having integer values with $N_k^*(t) \to 0$ as $t \to \infty$. The system occupancy level N(t) is then given by

$$\mathbf{N(t)} = \sum_{k=-\infty}^{k=+\infty} N_k^*(t - \tau_k)$$

To obtain the p.g.f. of N(∞) for the general problem, one notes that because of the Poisson character of the arrival process, one may regard the input stream as a superposition of **M** independent sub-streams. Each sub-stream has

Poisson arrivals of rate λ/M and has the same stochastic response function $N^*(t)$ and each sub-stream may be thought of as associated with its own subsystem. The response function $N^*(t)$ will be assumed initially to terminate at the level 0 after a service interval of finite duration D. Because each sub-stream is thin when M is large, the probability that two or more batches are present simultaneously in a subsystem is $o(\lambda/M)$ as $M \to \infty$. During the service interval of length D, let the p.g.f. of the number in the subsystem present at time y after the service inception be

$$g(u,y) = \Sigma_n \; r_n(y) \; u^n \tag{1}$$

where $r_n(y) = P[N^*(y) = n]$. Because the arrival process for each subsystem is Poisson, the length of the intervals between the termination of service on the previous batch and the the start of service on the next batch is exponentially distributed with mean M/λ. If $I_n(y)$ is the indicator function for being at state n at time y, the fraction of time spent in level n at ergodicity is given for n >1 by

$$\frac{E[\int_0^D I_n(y)\,dy\,]}{D + \frac{M}{\lambda}} = \frac{\int_0^D r_n(y)\,dy}{D + \frac{M}{\lambda}}$$

It follows that the p.g.f of the number in the system at ergodicity is for fixed D,

$$\pi\,(u\,,\infty) = \left[\frac{\frac{M}{\lambda} + \int_0^D g(u,y)\,dy}{D + \frac{M}{\lambda}}\right]^M + o(1)\,,\; M \to \infty\,.$$

We next permit M to become infinite and then D to become infinite in turn. The latter limit is appropriate since the duration of the holding times may be infinite. From the familiar $\lim_{M\to\infty}(1+\frac{x}{M})^M = e^x$, one has

$$\pi(u,\infty) = \lim_{D \to \infty} \exp\left[-\lambda\ D \left\{ 1 - \frac{1}{D} \int_0^D g(u,y)\, dy \right\} \right]$$

$$= \lim_{D \to \infty} \exp\left[-\lambda\ D \left\{ \frac{1}{D} \int_0^D (1 - g(u,y))\, dy \right\} \right] .$$

i.e.

(2) $$\pi(u,\infty) = \exp\left[-\lambda \int_0^\infty \{1 - g(u,y)\} dy \right].$$

A more formal demonstration of (2) can be obtained from a continuous infinite product representation of $.\pi(u,\infty)$.

Note that for the ordinary M/G/∞ system with batch size K =1 and service time c.d.f A(x) , one has

(3) $$g(u, y) = A(y) + u\ \bar{A}(y)$$

Hence

$$1 - g(u, y) = \bar{A}(y)\ (1-u)$$

and one obtains from (2) the classical result $\pi(u,\infty) = \exp[-\lambda E[T]\ (1-u)]$. (See for example Tijms(1986).) Equation (2) implies at once that:

Theorem. For a process **N(t)** with batch poisson arrivals of rate λ, independent identical random impulse response **g(u,t)** for each batch and infinite number of servers, the ergodic distribution of the number of items in the system is a compound Poisson distribution with p.g.f. given by (2) .

The basic result (2) can be extended by an identical argument to the continuum case for which

(4)

$$X(t) = \sum_{k=-\infty}^{k=+\infty} X^*_k(t - \tau_k) .$$

Here the summands are i.i.d. stochastic functions which go to zero as t goes to infinity. If $\phi(s,\infty) = E[\exp\{-sX(\infty)\}]$ one has

$$\phi(s,\infty) = \exp\left[-\lambda \int_0^\infty \{1 - \psi(s,y)\}\, dy\right]$$

where $\psi(s,y) = E[\exp\{-s X_k^*(y)\}]$. In the special case where $X_k^*(y)$ is deterministic and exponential this result coincides with J. Keilson & D. Mirman, (1959). An extensive discussion of the general deterministic case may be found in S.O.Rice(1945). The result extends to multivariate processes of the form (4).

1.2 The ergodic Mean and Variance

From **(2)** one has at once

$$E[N(\infty)] = \lambda \int_0^\infty g_u(1,y)\, dy = \lambda \int_0^\infty \sum_0^\infty n\, r_n(y)\, dy .$$

If **N*(t) is** the decreasing number in the system associated with each Poisson epoch, then

$$(5) \qquad E[N(\infty)] = \lambda \int_0^\infty E[N^*(y)]\, dy .$$

In the same way, one finds that

$$\mathrm{Var}[N(\infty)] = \lambda \int_0^\infty [\, g_{uu}(1,y) + g_u(1,y)\,]\, dy$$

so that

$$(6) \qquad \mathrm{Var}[N(\infty)] = \lambda \int_0^\infty E[N^{*2}(y)]\, dy$$

1.3 $M^{batch}/M/\infty$

Until now no assumptions have been made about $N^*(t)$ other than that batches are served independently. For $M^{batch}/M/\infty$, each customer is served independently and lifetimes are exponentially distributed. Let us also suppose that batch size is exactly **K**. Then, as for (3)

$$g(u,y) = \left[(1-e^{-\theta y}) + ue^{-\theta y}\right]^{K}$$

so that for (2) one must evaluate

$$f_K(u) = \int_0^\infty \left[(1-e^{-\theta y}) + ue^{-\theta y}\right]^{K} dy$$

From the Binomial expansion, one must then evaluate

$$h(K,r) = \int_0^\infty (1-e^{-\theta y})^{K-r} e^{-r\theta y}\ dy = [\ \theta r\ {}^{K}C_r\ ]^{-1}$$

from the integral representation of the Beta function.. It follows that

(7)

$$\pi(u,\infty) = \exp\left[-\frac{\lambda}{\theta}\sum_1^K \frac{1}{r} + \frac{\lambda}{\theta}\sum_1^K \frac{u^r}{r}\right]$$

This may be seen to coincide with the p.g.f. obtained for $M^B/M/\infty$ by analysis via a birth-death process for general batch size distribution. . Let $a_n = P[K=n]$ and $p_n(t) = P[N(t) = n]$. One then has from the forward Kolmogorov equations

$$\frac{d}{dt} p_n(t) = -(\lambda+n\mu)p_n(t) + \lambda\ (p_n(t))^*(a_n) + (n+1)\mu p_{n+1}(t)$$

The use of generating functions then gives

$$\frac{\partial}{\partial t}\pi(u,t) = -\lambda[1-\alpha(u)]\ \pi(u,t) + \mu(1-u)\ \frac{\partial}{\partial u}\pi(u,t)\ ,$$

i.e.

$$(1-u)\ \frac{\partial}{\partial u}\log \pi(u) = \frac{\lambda}{\mu}\ [1-\alpha(u)]\ .$$

One has finally

(8) $$\pi(u) = \exp\left[\ \frac{\lambda}{\mu} \int_1^u \frac{[1-\alpha(w)]}{1-w} dw\ \right]\ .$$

This may be seen to coincide with **(7)** when $\alpha(u) = u^K$.

1.4 $M^K/G/\infty$

For constant batch size and general holding times, one has

$$g(u,y) = \left[A(y) + u \bar{A}(y) \right]^K$$

so that

$$\pi(u,\infty) = \exp\left[\lambda \int_0^\infty [g(u,y) - g(1,y)]\, dy \right.$$

$$= \exp\left[\lambda \int_0^\infty \{ [A(y) + u \bar{A}(y)]^K - [A(y) + \bar{A}(y)]^K \} dy \right].$$

Hence

(9) $$\pi(u,\infty) = \exp\{ -\theta[1 - \beta(u)] \}$$

where

(10) $$\theta = \lambda E[\max(T_1, T_2, \ldots, T_k)] = \lambda \int_0^\infty [1 - A^k(y)]\, dy$$

and

(11)

$$\beta(u) = \frac{\sum_1^K u^k C_k^K \int_0^\infty \bar{A}_T^k(y)\, A_T^{K-k}(y)\, dy}{\int_0^\infty [1 - A_T^K(y)]\, dy}$$

Note that only when K = 1 is the distribution of $N(\infty)$ independent of the lifetime distribution. Note also that for constant batch size K one has from differentiation

(12) $$E[N(\infty)] = \lambda K\, E[T]$$

(13) $$\mathrm{Var}[N(\infty)] = \lambda K\, E[T] + \lambda K(K-1) \int_0^\infty [1 - A(y)]^2\, dy$$

in agreement with Tijms [1986] . The numerical evaluation of the distribution of $N(\infty)$ from (9) can be obtained algorithmically from

$$\exp \{ -\theta[1-\beta(u)] \} = e^{-\theta} \sum \frac{\theta^n}{n!} \beta^n(u)$$

and n-fold convolution of the lattice distribution for $\beta(u)$ with itself. It is clear from the compound Poisson character of (9) that for any holding time distribution with finite mean, when K is fixed and λ becomes large, the distribution of $N(\infty)$ becomes normal . Normality does not set in with increasing batch size K alone since $\pi(u,\infty)$ is not the K'th power of a p.g..f. . The normality will be present when $\theta = \lambda E[\max(T_1, T_2, .., T_k)]$ is large.

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