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Connectivity-Splitting Models for Survivable Network Design.

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Abstract

The survivable network design (SND) problem seeks a minimum cost set of edges that meet prescribed node connectivity requirements. We present a new family of strong mixed integer programming formulations for this problem, examine the tightness of the associated linear programming relaxations, and then use the relaxations to analyze heuristics for several variants of the SND problem and its special cases. The new models are tighter than the usual cutset formulation when the network contains both regular nodes that must be connected to other nodes in the network and critical nodes with higher connectivity requirements. Our analysis provides stronger worst-case bounds or shows that some previously known worst-case performance ratios of heuristic to optimal (mixed integer programming) costs also hold relative to the optimal ratio of the heuristic to optimal linear programming values of these stronger formulations. The new formulations use fractional edge selection variables to split the connectivity requirements of the critical nodes into two separate requirements. We consider three versions of the model. A connectivity-peeling version peels off the lowest connectivity level, a connectivity-dividing version divides the connectivity requirements for all cutsets separating critical nodes, and a branch-addition version attempts to separate the design decisions for a multi-connected component of the network from those for the branches.

1. Introduction

Cost and survivability are primary criteria for designing telecommunication and other infrastructure networks. In configuring these networks, planners must select a configuration that is not only cost effective but also meets the service requirements of different customer segments. For telecommunication networks, business and government customers might have more stringent connectivity requirements than individual households since interruptions of service to institutions can be life-threatening or result in lost revenues and costs running into millions of dollars (Cosares et al. [1995]). To guarantee minimal interruptions for these high priority customers, the network must contain alternate paths and restoration facilities. Since providing the same high level of protection against failures to all customers is prohibitively expensive, network planners must judiciously select a topology that contains redundancy only when needed to provide adequate service for the critical customers.

Motivated by this need to simultaneously consider cost and survivability, researchers have attempted to understand and solve a core optimization model—the **survivable network design (SND) problem**. Given an undirected network $G:(N,E)$ with nonnegative *costs* c_{ij} for each edge $(i,j)\in E$ and nonnegative, symmetric (without loss of generality) integer *connectivity requirements* r_{ij} specifying the minimum number of edge-disjoint paths needed between nodes $i,j\in N, i\neq j$, the SND problem seeks the minimum cost network that meets all the connectivity requirements. We define the *node connectivity* of each node i as $\rho_i=\max\{r_{ij}: j\in N\}$. We refer to nodes with ρ_i equal to zero, one, and greater than one as *Steiner*, *regular*, and *critical* nodes, respectively. Steiner nodes are optional intermediate points that the design might use to connect the regular and critical nodes.

This paper presents a new family of extended cutset formulations for the SND problem, called *connectivity-splitting* models, and uses this family of formulations to analyze the worst-case performance ratios of several heuristics. Although our model applies to general SND problems without any restrictions on node connectivity values, it is stronger than the traditional cutset formulation only when the network contains one or more regular nodes. Our worst-case analysis also exploits the presence of regular nodes. In this case, the optimal SND solution consists of a multi-connected network that spans all the critical nodes and optionally includes some regular and Steiner nodes, plus a set of *branches* emanating from this network to span any remaining regular nodes. We, therefore, refer to SND problems with at least one regular node as the *multi-connected network with branches (MNB)* problem.

One characteristic—whether or not *edge duplication* is permitted—plays an important role in modeling, solving, and analyzing SND problems. Edge duplication refers to the option of installing multiple (parallel) copies of any edge $(i, j) \in E$ to create alternate paths. Some application contexts permit edge duplication while others do not (e.g., if customers require physically diverse paths). In this paper, we consider both problem variants—with or without edge duplication.

The literature has addressed several special cases of the SND problem, obtained by limiting the set of permissible node connectivity values, and assuming special cost structures. In some instances, (for example, if the network contains a node *i* with connectivity requirement $r_{ij} \geq 1$ to all other regular and critical nodes *j*), every feasible design must necessarily be connected. We refer to this special case as the *unitary* SND problem. In other, general, situations, the optimal solution might contain multiple components. Special cost structures, such as Euclidean or triangular edge costs (i.e., edge costs that satisfy the triangle inequality) can also lead to simplifications. SND problems with nodes connectivity values limited to two or three special values have received particular attention in the literature. Let $Q = \{q : q = \rho_i \text{ for some } i \in N\}$ be the set of node connectivity values in the network. One well-studied special case is the *Low Connectivity Steiner (LCS)* problem with $Q = \{0,1,2\}$, i.e., all critical nodes have connectivity requirements of two. Likewise, the *Equal Connectivity Steiner (ECS)* problem has $Q = \{0, \rho\}$ for some integer $\rho \geq 2$. In Section 3, we address heuristic worst-case performance for some of these special cases.

Survivable network design problems are difficult to solve optimally. Even the unitary LCS special case is NP-hard since it generalizes the classical Steiner tree problem and the traveling salesman problem. Part of the enormous literature on Steiner network problems focuses on analyzing the worst-case performance of heuristics, and developing good problem formulations that can improve the performance of relaxation-based solution methods such as branch-and-bound. Following this trend, several papers have studied the polyhedral structure of the SND problem (e.g., Monma, Munson and Pulleyblank [1990], Gröetchel, Monma, and Stoer, [1992]), analyzed the worst-case performance of heuristics (e.g., Goemans and Williamson [1992], Goemans and Bertsimas [1993]), and developed optimization algorithms (e.g., Gröetchel, et al. [1992]). Most of this work studies unitary LCS problems. Williamson et al. [1995] and Goemans et al. [1994] have developed heuristic bounds for general MNB problems without edge duplication.

This paper presents a family of models, and develops heuristic-to-LP worst-case performance ratios for MNB problems with or without edge duplication. Section 2 improves upon the basic "cutset" formulation that several authors have used in their analyses of SND heuristics. We first propose a family of stronger formulations for MNB problems, obtained by introducing additional edge variables and splitting the connectivity requirement across each critical cutset into two sub-requirements. Depending on how we split the connectivity requirements, we obtain different versions of the extended problem formulation. We consider three classes of formulations: a connectivity-peeling formulation, a connectivity-dividing formulation, and a branch-addition formulation.

Section 3 analyzes the worst-case performance of two classes of heuristics—an Overlay heuristic for MNB problems with edge duplication, and a Tree Completion heuristic for problems without edge duplication but triangular costs. For the overlay heuristic, we either provide tighter bounds or show that previous bounds with respect to the optimal integer programming value also apply relative to the optimal linear programming value of our connectivity-splitting model. Using the branch-addition formulation, we analyze the worst-case performance of the Tree Completion heuristic in terms of the performance of an embedded multi-connected heuristic, generating new bounds. The results in this paper can serve as building blocks for more general models, for instance, to analyze multi-level, multi-connected models incorporating multiple service and facility types (Balakrishnan, Magnanti, and Mirchandani [1994a, 1994b]).

2. Modeling MNB problems

To date, researchers have largely used a standard cutset formulation of MNB problems (and its special cases) to analyze heuristic worst-case performance for these problems. In this section, we describe a stronger *connectivity-splitting* formulation that splits the connectivity requirement across *critical* cutsets into two sets of constraints, and examine three special cases obtained by considering particular schemes for splitting the connectivities. One formulation *peels* the lowest connectivity level, another *divides* the connectivity requirement in the same proportion for all critical cutsets, and the third *adds branches* to a multi-connected solution. In Section 3, we use these formulations to analyze the worst-case performance of overlay and tree completion heuristics for the MNB problem.

Developing strong formulations with improved linear programming relaxations has proven useful both to develop better heuristic bounds and to improve solution performance for several classical discrete optimization problems (see Nemhauser and Wolsey [1988]). Typically, these *extended* formulations introduce additional variables from a different space (e.g., flow or node variables for Steiner network models; see, for example, Beasley [1984], Wong [1984], Goemans and Myung [1993], and Magnanti and Raghavan [1999]), and add appropriate linking constraints. Our reformulation strategy for the MNB problem also introduces additional variables, but they all belong to the original space of edge variables.

For notation, we will let Z_M denote the optimal value of any (mixed) integer programming model M and let L_M denote the optimal value of its linear programming relaxation.

2.1 Cutset formulation for SND problems

Given a graph G, for any subsets $S \subset N$ and $T = N \setminus S$ of nodes, let $\{S,T\} = \{(i,j) \in E : i \in S, j \in T\}$ denote the undirected edge cutset defined by S and T. If a_{ij} is any quantity (decision variable, given data) associated with edge (i,j) of the graph, we let $A_{ST} = \sum_{(i,j)\in\{S,T\}} a_{ij}$. For any cutset $\{S,T\}$, we refer to the maximum value of r_{ij} over all node pairs i,j with $i \in S, j \in T$ as the *crossing requirement* of cutset $\{S,T\}$. We refer to cutsets with crossing requirement of one as *regular cutsets* and those with crossing requirement of two or more as *critical cutsets*. For each $q \in Q$, let σ_q denote the collection of all cutsets of the underlying graph G with crossing requirement equal to q. For any $k \in Q$, let $Q^{k+} = \{q \in Q : q \geq k\}$.

The following standard cutset formulation [CUT] for the SND problem without edge duplication uses binary edge selection variables u_{ij} for all edges $(i,j) \in E$. The variable u_{ij} is 1 if the network design includes edge (i,j) and is 0 otherwise.

Problem [*CUT*]:

$$Z_{CUT} = \min \sum_{(i,j)\in E} c_{ij} u_{ij} \tag{2.1}$$

subject to:

$$U_{ST} \geq q \quad \text{for all cutsets } \{S,T\} \in \sigma_q, q \in Q^{1+}, \tag{2.2}$$

$$u_{ij} = \text{integer} \quad \text{for all } (i,j) \in E \text{, and} \tag{2.3}$$

$$u_{ij} \leq 1 \quad \text{for all } (i,j) \in E. \tag{2.4}$$

Omitting the upper bounds (2.4) models situations with edge duplication. Jain [1998] has examined an integer rounding heuristic that iteratively solves the [CUT] formulation with certain variables fixed at value one (those with value $\geq \frac{1}{2}$ in the previous iteration). He shows that the ratio of objective value of this heuristic to the objective value of the linear programming relaxation of [CUT] does not exceed 2. In contrast, the heuristics we consider in this paper are combinatorial (and simple to implement), but do not lead to as sharp performance guarantees.

2.2 Critical connectivity-splitting formulation for MNB problems

To strengthen the formulation [*CUT*], we consider a general reformulation scheme for MNB problems that splits the connectivity requirement q for each *critical* cutset $\{S,T\} \in \sigma_q, q \in Q^{2+}$ into two complementary requirements $\phi_q q$ and $(1-\phi_q)q$, for some given nonnegative fractions ϕ_q. To meet these requirements, we introduce two new *continuous* edge variables, x_{ij} and y_{ij}, and replace the previous edge selection variables u_{ij} with two sets of binary variables z_{ij} and b_{ij}. The z_{ij} variables select edges belonging to a multi-connected network connecting the critical nodes, while the b_{ij} variables choose branches emanating from the multi-connected network to connect the remaining regular nodes (those not spanned by the multi-connected network).

We consider a family of such critical connectivity-splitting formulations [*CCS*(μ,Φ)], parameterized by a constant μ, with $0 \le \mu \le 1$, and a vector $\Phi = (\phi_q)$ of *connectivity fractions* ϕ_q, with $0 \le \phi_q \le \frac{1}{2}$ for all $q \in Q^{2+}$. Let $\phi = \min\{\phi_q : q \in Q^{2+}\}$, and define:

$$\alpha_1 = \frac{1-2\mu\phi}{2(1-\phi)}, \text{ and } \alpha_q = \frac{\phi_q - \phi}{(1-\phi)} \text{ for all } q \in Q^{2+}.$$

These definitions imply that $0 \le \alpha_q \le 1$ for all $q \in Q$.

For a given constant μ and a vector Φ of connectivity fractions, consider the following *C*ritical *C*onnectivity-*S*plitting model [*CCS*(μ,Φ)] for the MNB problem without edge duplication.

Problem [*CCS*(μ,Φ)]:

$$Z_{CCS} = \min \sum_{(i,j)\in E} c_{ij}(b_{ij} + z_{ij}) \tag{2.5}$$

subject to:

$$B_{ST} + \mu X_{ST} + \alpha_1 Y_{ST} \ge 1 \quad \text{for all } \{S,T\} \in \sigma_1, \tag{2.6}$$

$$X_{ST} + \alpha_q Y_{ST} \ge q\phi_q \quad \text{for all } \{S,T\} \in \sigma_q, q \in Q^{2+}, \tag{2.7}$$

$$(1-\alpha_q)Y_{ST} \ge q(1-\phi_q) \quad \text{for all } \{S,T\} \in \sigma_q, q \in Q^{2+}, \tag{2.8}$$

$$z_{ij} \ge x_{ij} + y_{ij} \quad \text{for all } (i,j) \in E, \tag{2.9}$$

$$x_{ij}, y_{ij} \ge 0 \quad \text{for all } (i,j) \in E, \tag{2.10}$$

$$b_{ij} = 0 \text{ or } 1 \quad \text{for all } (i,j) \in E, \tag{2.11}$$

$$z_{ij} = \text{integer} \quad \text{for all } (i,j) \in E, \tag{2.12}$$

$$z_{ij} + b_{ij} \le 1 \quad \text{for all } (i,j) \in E. \tag{2.13}$$

To relate this model to the original cutset formulation, first note that the formulation [*CCS*(μ,Φ)] replaces the original variables u_{ij} in the objective function of formulation [CUT] with the sum

$z_{ij} + b_{ij}$. It also splits the connectivity constraints (2.2) for the critical cutsets (i.e., for $\{S,T\} \in \sigma_q, q \in Q^{2+}$) into the two constraints (2.7) and (2.8). Observe that adding constraints (2.7) and (2.8) gives $X_{ST} + Y_{ST} \geq q$, which together with constraints (2.9) and (2.12) ensure that the z_{ij} values satisfy the connectivity requirement of q across all critical cutsets $\{S,T\} \in \sigma_q, q \in Q^{2+}$. That is, the network design defined by the z solution provides the required number of edge-disjoint paths among all pairs of critical nodes. As we shall see later (in the proof of Proposition 1), constraints (2.6) ensure that any regular nodes not spanned by the z solution are connected via edges defined by the b variables. Finally, note that the formulation permits edge duplication if we eliminate constraints (2.13).

Before formally examining the validity of formulation [$CCS(\mu,\Phi)$], let us justify it intuitively by considering a specific set of parameters. Suppose $\phi_q = \frac{1}{4}$ for all $q \in Q^{2+}$ and $\mu = \frac{1}{2}$. In this case, $\alpha_1 = \frac{1}{2}$, and $\alpha_q = 0$ for all $q \in Q^{2+}$. Constraints (2.7) and (2.8) now specify that the total x-value across any critical cutset $\{S,T\} \in \sigma_q, q \in Q^{2+}$ must equal or exceed $q/4$ while the total y-value across this cutset must equal or exceed $3q/4$. Therefore, $u_{ij} = z_{ij} \geq x_{ij} + y_{ij}$ satisfies the original constraints (2.2) for this cutset. Now consider constraints (2.6). If an edge (i, j) of the regular cutset $\{S,T\} \in \sigma_1$ belongs to the multi-connected subnetwork connecting the critical nodes, then this subnetwork spans nodes i and j, and must therefore include at least one other edge of $\{S,T\}$. Therefore, the sum of the x-values and y-values over all the edges of this cutset must be at least 2. Since constraint (2.6) specifies that $B_{ST} + \frac{1}{2}X_{ST} + \frac{1}{2}Y_{ST} \geq 1$, the solution $u_{ij} = z_{ij}$ satisfies constraint (2.2) for cutset $\{S,T\}$. On the other hand, if $X_{ST} + Y_{ST} = 0$, then $\{S,T\}$ must contain at least one branch, i.e., $b_{ij} = 1$ for some $(i, j) \in \{S,T\}$.

In analyzing the worst-case performance of a tree heuristic for the SND problem, Goemans and Bertsimas [1993] have previously divided the optimal integer solution to formulation [CUT] into two fractional values (half each) corresponding to the edges in the maximal two-connected component of the optimal design. This scheme for splitting variables corresponds to the connectivity-halving special case of [$CCS(\mu,\Phi)$] that we discuss later. Goemans and Bertsimas did not explicitly develop the stronger critical connectivity-splitting (or halving) formulation, nor did they consider the implications of this variable splitting approach for the analysis of heuristics relative to the optimal linear programming values as we do in this paper.

We next show that formulation [CCS(μ,Φ)] is a valid model for the MNB problem.

Theorem 1.

Suppose $\Phi = \{\phi_q\}$, with $0 \le \phi_q \le \frac{1}{2}$ for all $q \in Q^{2+}$, $\phi = \min\{\phi_q : q \in Q^{2+}\}$, and $\alpha_q = (\phi_q - \phi)/(1-\phi)$. Then, for any $0 \le \mu \le 1$, the formulation [CCS(μ,Φ)]—with constraints (2.13) if edge duplication is not permitted, and without constraints (2.13) otherwise—is a valid formulation for the MNB problem.

Proof:

Given any feasible solution to [*CCS*(μ,Φ)], consider the derived solution to formulation [*CUT*] obtained by setting $u_{ij} = b_{ij} + z_{ij} \ge b_{ij} + \lceil x_{ij} + y_{ij} \rceil$ for all edges $(i, j) \in E$. Since the given CCS solution satisfies constraints (2.6), either $\mu x_{ij} + \alpha_1 y_{ij} > 0$ or $b_{ij} = 1$ for at least one edge (i, j) in every cutset $\{S,T\} \in \sigma_1$. In the first case, $x_{ij} + y_{ij} > 0$, and so $\lceil x_{ij} + y_{ij} \rceil \ge 1$. Therefore, the derived solution satisfies constraints (2.2). Consider a cutset $\{S,T\} \in \sigma_q, q \in Q^{2+}$. Adding constraints (2.7) and (2.8) gives $X_{ST} + Y_{ST} \ge q$ for all $\{S,T\} \in \sigma_q, q \in Q^{2+}$. Moreover, since $z_{ij} \ge x_{ij} + y_{ij}$ is integral, $U_{ST} \ge \sum_{(i,j)\in\{S,T\}} \lceil x_{ij} + y_{ij} \rceil \ge \lceil X_{ST} + Y_{ST} \rceil \ge q$, and so the derived u solution satisfies the connectivity requirement for all critical cutsets $\{S,T\} \in \sigma_q, q \in Q^{2+}$. Therefore, the derived u-solution is feasible in [*CUT*]; moreover, it has the same cost as the original CCS solution.

Conversely, given any feasible (integer) solution u to formulation [*CUT*], we use the following "allocation" procedure to obtain feasible values of the *b*, *x*, *y*, and *z* variables in formulation [*CCS*(μ,Φ)]. The given u solution contains at least q paths connecting every pair of nodes i and j with $r_{ij} = q$. Let E_{2+} denote the union of edges contained in all the paths connecting the critical nodes. For each edge $(i, j) \in E_{2+}$, we set $x_{ij} = \phi u_{ij}$, $y_{ij} = (1-\phi)u_{ij}$, and $z_{ij} = u_{ij}$. For edges in $E \setminus E_{2+}$, we set $b_{ij} = u_{ij}$ and $x_{ij} = y_{ij} = z_{ij} = 0$. As the following arguments show, this solution satisfies constraints (2.6) to (2.13). Consider any cutset $\{S,T\} \in \sigma_q$ for $q \in Q^{2+}$. The u-solution selects at least q edges in this cutset. By construction, for each of these edges (i, j), $x_{ij} = \phi u_{ij}$ and $y_{ij} = (1-\phi)u_{ij}$. Therefore, the lefthand side of the inequality (2.7) is at least $q\phi + q\alpha_q(1-\phi) = q\phi_q$, and the lefthand side of (2.8) is at least $q(1-\phi) \ge q(1-\phi_q)$. Finally, for a cutset $\{S,T\} \in \sigma_1$, if the given u-solution contains at least one edge from E / E_{2+} belonging to this cutset, then $U_{ST} = B_{ST} \ge 1$. Otherwise, the u-solution contains at least two edges from E_{2+}. On these edges, $x_{ij} = \phi u_{ij} = \phi$ and $y_{ij} = (1-\phi)u_{ij} = (1-\phi)$. Therefore, $X_{ST} \ge 2\phi$ and $Y_{ST} \ge 2(1-\phi)$, and so the lefthand side of (2.6) in the derived solution is at least

$$2\mu\phi + 2\alpha_1(1-\phi) = 2\mu\phi + 2\left(\frac{1-2\mu\phi}{2(1-\phi)}\right)(1-\phi) = 1.$$

Therefore, the derived solution is feasible in [*CCS*(μ,Φ)]. ♦

Recall that L_{CUT} and L_{CCS} represent the optimal values of the linear programming relaxations of the formulations [CUT] and [$CCS(\mu,\Phi)$].

Theorem 2.

$L_{CUT} \le L_{CCS}$, i.e., the CCS formulation is at least as strong as the cutset formulation.

Proof:

Since the edge costs are nonnegative, the linear programming relaxation of the CCS formulation has an optimal solution with $z_{ij} = x_{ij} + y_{ij}$ for all $(i, j) \in E$. Let (x, y, b) be an optimal solution to this linear programming relaxation. Consider the u solution to the linear programming relaxation of [CUT] obtained by setting $u_{ij} = x_{ij} + y_{ij} + b_{ij}$ for all edges $(i, j) \in E$. This solution is nonnegative and has the same cost as the CCS solution. Adding constraints (2.7) and (2.8) of [$CCS(\mu,\Phi)$] shows that $X_{ST} + Y_{ST} \ge q$ for all $\{S,T\} \in \sigma_q, q \in Q^{2+}$. Therefore, $U_{ST} \ge q$ for all $\{S,T\} \in \sigma_q, q \in Q^{2+}$. Since $\mu \le 1, \alpha_1 \le 1$, and all the variables are nonnegative, the constraints (2.6) imply that $B_{ST} + X_{ST} + Y_{ST} \ge 1$, and therefore $U_{ST} \ge 1$, for all $\{S,T\} \in \sigma_1$. Consequently, the derived u solution is feasible for the linear programming relaxation of [CUT]. ♦

The following example shows that the CCS model can provide a strictly tighter optimal linear programming value than the cutset formulation.

Example 1. Consider a triangle with three critical nodes. Suppose that $Q = \{1, 2\}$. Every pair of nodes is connected by two parallel, edge-disjoint paths, each with an intermediate regular node. Thus, the network contains 3 critical nodes, 6 regular nodes, and 12 edges. Suppose each edge has unit cost. The optimal solution to the linear programming relaxation of [CUT] sets $u_{ij} = \frac{1}{2}$ on all edges, with optimal value of 6. An optimal solution to the linear programming relaxation of [$CCS(\mu,\Phi)$], with $\mu = 1$ and $\phi_1 = \phi_2 = \frac{1}{2}$, sets $x_{ij} = \frac{1}{2}$ and $b_{ij} = 0$ for all edges, and $y_{ij} = \frac{1}{2}$ for 6 edges, two each on one of the parallel paths connecting every pair of critical nodes. The cost of this solution is 9, strictly exceeding the optimal value of the linear programming relaxation of [CUT]. Moreover, for this example, the optimal value of the linear programming relaxation of the CCS formulation equals the optimal value of the integer solution. ♦

2.3 Types of connectivity splitting

We obtain three intuitive versions of the critical connectivity-splitting formulation by selecting certain special values for the parameters μ and Φ.

First, suppose $\mu = 1$. In this case, given any optimal solution with $b_{ij} > 0$ for some edge (i, j), we can obtain an equal or lower cost feasible solution by setting $x_{ij} \leftarrow x_{ij} + b_{ij}$. Therefore, both the integer formulation and its linear programming relaxation must have optimal solutions with $b_{ij} = 0$ for all edges $(i, j) \in E$, and we can drop the b variables from the model. Within this class of formulations (with $\mu = 1$), we consider two special connectivity-splitting vectors Φ, namely, $\phi_q = 1/q$ for all $q \in Q^{2+}$, and $\phi_q = 1/\delta$ for all $q \in Q^{2+}$ for a given constant $\delta \geq 2$.

Connectivity-peeling formulation [*PEEL*]. When $\phi_q = 1/q$ for all $q \in Q^{2+}$, and $\mu = 1$, the righthand side of constraint (2.7) is 1, and the righthand side of constraint (2.8) is $q(1-1/q) = q-1$, for all critical cutsets. Intuitively, this disaggregation strategy attempts to separate or peel a single connectivity subproblem (constraints (2.6) and (2.7)) over all the regular and critical nodes from a "reduced connectivity" subproblem (constraints (2.8)) in which the connectivity of each critical cutset is reduced by one (or less). We, therefore, refer to this special case of [$CCS(\mu,\Phi)$] as the connectivity-peeling formulation [*PEEL*]. This formulation is potentially useful for analyzing a heuristic that first finds a Steiner forest spanning all the regular and critical nodes, and then adds edges belonging to a reduced connectivity solution.

Connectivity-dividing formulation [*DIV*(δ)]. For any $\delta \geq 2$, if $\phi_q = 1/\delta$ for all $q \in Q^{2+}$ and μ = 1, formulation [$CCS(1,\Phi)$] becomes a connectivity-dividing formulation [*DIV*(δ)]. The righthand side values of constraints (2.7) and (2.8) are now q/δ and $q(1-1/\delta)$, i.e., this disaggregation strategy "divides" the connectivity requirement in the same proportion for every critical cutset. Since $\phi_q = \phi$ and $\alpha_q = 0$ for all $q \in Q^{2+}$, constraints (2.7) contain only the x variables. For the special connectivity-dividing formulation [***DIV***(2)] obtained by setting $\delta = 2$, constraints (2.7) and (2.8) both have equal (possibly fractional) connectivity requirements of $q/2$. In essence, this **connectivity-halving** formulation contains two connectivity subproblems, one each corresponding to the x and y variables: each provides half the required connectivity for every critical node. However, only one of these subproblems includes the unit connectivity requirement of the regular nodes. Thus, for the LCS special case, the X subproblem corresponds to a "fractional Steiner forest" over all the regular and critical nodes, and the Y subproblem is a fractional Steiner forest with only the critical nodes as terminals. We later use this connectivity-halving model to analyze the worst-case performance of an overlay solution strategy that obtains heuristic solutions to MNB problems with edge duplication by combining integer solutions to the two subproblems.

Branch-addition formulation. We obtain a third version of the connectivity-splitting model, called the branch-addition formulation [*BRANCH*], by setting $\phi_q = 0$ for all $q \in Q^{2+}$ and $\mu = \frac{1}{2}$. In this case, we cannot drop the b variables, but constraints (2.7) are redundant. Furthermore, since $\alpha_1 = \frac{1}{2}$, both the integer program [$CCS(\frac{1}{2},\mathbf{0})$] and its linear programming relaxation have optimal solutions with $x_{ij} = 0$ for all edges (i, j) (otherwise, we obtain an equal or lower cost feasible solution by setting $y_{ij} \leftarrow y_{ij} + x_{ij}$). Therefore, we can drop the x variables, replace z_{ij} with y_{ij} in the formulation, and impose integrality on the y variables. Constraints (2.8) require that the y variables define a multi-connected network that meets the connectivity requirements of all the critical nodes. Constraints (2.6) ensure that every regular node either belongs to the multi-connected network or is spanned by the branches emanating from this component, i.e., each cutset $\{S,T\} \in \sigma_1$ contains either two or more edges of the multi-connected network or one edge belonging to a branch. Thus, instead of reducing the connectivities of critical nodes, this version of the model ensures that the solution completes the multi-connected network by adding branches to span all the remaining regular nodes. In Section 3, we use the branch-addition formulation to analyze the worst-case performance of a Tree Completion heuristic.

2.4 Tightness of the extended formulations

First, we show that the connectivity-dividing formulation is the strongest in the family of [$CCS(1,\Phi)$] formulations. This result, combined with our earlier observations, helps to rank the various formulations in terms of the tightness of their linear programming relaxations.

Proposition 3.

Given any vector $\Phi = \{\phi_q\}$, with $0 \le \phi_q \le \frac{1}{2}$, the formulation [$DIV(\delta)$] with $\delta = 1/\phi$ has an optimal linear programming value that equals or exceeds the linear programming value of formulation [$CCS(1,\Phi)$].

Proof:

Let (x, y) be an optimal solution to the linear programming relaxation of [$DIV(1/\phi)$]. Since

$$X_{ST} \ge q\phi \text{ and } Y_{ST} \ge q(1-\phi),\ X_{ST} + \alpha_q Y_{ST} \ge q\phi + \frac{\phi_q - \phi}{(1-\phi)} q(1-\phi) = q\phi_q \text{ for all } \{S,T\} \in \sigma_q, q \in Q^{2+},$$

i.e., the solution satisfies constraints (2.7) in [$CCS(1,\Phi)$]. Moreover, $(1-\alpha_q)Y_{ST} \ge \frac{1-\phi_q}{1-\phi} q(1-\phi) = q(1-\phi_q)$. All of the other constraints in [$CCS(1,\Phi)$] are the same as those in [$DIV(1/\phi)$]. Therefore, (x, y) is a feasible solution to the linear programming relaxation of [$CCS(1,\Phi)$], implying that the optimal linear programming value of formulation

$[CCS(1,\Phi)]$ can be no greater than the cost of (x, y) which is also the optimal linear programming value of $[DIV(1/\phi)]$. ♦

Proposition 4.

For any value of $\delta > 2$, the formulation $[DIV(\delta)]$ is LP-equivalent to formulation $[DIV(2)]$.

Proof:

Let (x', y', z') be any feasible solution to the linear programming relaxation of $[DIV(\delta)]$. Consider the solution (x, y, z) with $x_{ij} = x'_{ij} + \alpha_1 y'_{ij}$, $y_{ij} = (1-\alpha_1)y'_{ij}$, and $z_{ij} = z'_{ij}$ for all $(i, j) \in E$. Clearly, $x \geq 0$ and $y \geq 0$, and $X_{ST} \geq 1$ for all $\{S,T\} \in \sigma_1$. Moreover,

$$X_{ST} = X'_{ST} + \alpha_1 Y'_{ST} \geq (q/\delta) + q\alpha_1(1-1/\delta) = q/2, \text{ and}$$
$$Y_{ST} = (1-\alpha_1)Y'_{ST} \geq q(1-\alpha_1)(1-1/\delta) = q/2, \text{ for all } \{S,T\} \in \sigma_q, q \in Q^{2+}.$$

Therefore, (x, y, z) is feasible for the linear programming relaxation of $[DIV(2)]$.

Now suppose (x, y, z) is feasible for the linear programming relaxation of $[DIV(2)]$. Without loss of generality, assume $x \geq y$. Otherwise, if (x'', y'', z'') is a feasible solution with $x'' \not\geq y''$, the solution $x_{ij} = \max\{x''_{ij}, y''_{ij}\}$, $y_{ij} = \min\{x''_{ij}, y''_{ij}\}$, and $z_{ij} = z''_{ij}$ is feasible. Consider the solution (x', y', z') with $y'_{ij} = y_{ij}/(1-\alpha_1)$, $x'_{ij} = x_{ij} - \alpha_1 y'_{ij}$, and $z'_{ij} = z_{ij}$ for all $(i, j) \in E$. Since $\alpha_1 < 1$, $y' = y/(1-\alpha_1) \geq 0$. Therefore,

$$x' = x - \{\alpha_1/(1-\alpha_1)\}y = x - \{(\delta-2)/\delta\}y = (x-y) + (2/\delta)y \geq (2/\delta)y \geq 0,$$
i.e., $X'_{ST} \geq (2/\delta)Y_{ST} \geq q/\delta$, for all $\{S,T\} \in \sigma_q, q \in Q^{2+}$.

Similarly, $Y'_{ST} = \{1/(1-\alpha_1)\}Y_{ST} \geq q/\{2(1-\alpha_1)\} = q(1-1/\delta)$, for all $\{S,T\} \in \sigma_q, q \in Q^{2+}$.

For any $\{S,T\} \in \sigma_1$, $X'_{ST} + \alpha_1 Y'_{ST} = X_{ST} - \{\alpha_1/(1-\alpha_1)\}Y_{ST} + \{\alpha_1/(1-\alpha_1)\}Y_{ST} = X_{ST} \geq 1$.

Therefore, (x', y', z') is a feasible solution to the linear programming relaxation of $[DIV(\delta)]$. ♦

In this section, we have considered several valid mixed-integer programming models for the MNB problem that have tighter linear programming relaxations than the traditional $[CUT]$ formulation. Propositions 3 and 4 and our prior observations have established the following result.

Theorem 5.

Let $\Phi = \{\phi_q\}$, with $0 \leq \phi_q \leq \frac{1}{2}$, be any vector of connectivity fractions, and let $\phi = \min\{\phi_q : q \in Q^{2+}\}$. Then,

$$L_{CUT} \leq L_{CCS(1,\Phi)} \leq L_{DIV(1/\phi)} = L_{DIV(\delta)} \text{ for all } \delta \geq 2.$$

In the next section, we exploit the tighter linear programming relaxations of the connectivity-splitting models to develop better (smaller than previous) bounds on the worst-case performance ratio of two broad families of MNB heuristics.

3. Worst-case Analysis of Heuristics for MNB problems

This section analyzes the worst-case performance, relative to the optimal linear programming value of the connectivity-splitting formulation, of two heuristic solution methods for the MNB problem: an *Overlay heuristic* for MNB problems with edge duplication, and a *Tree Completion heuristic* for unitary MNB problems without edge duplication.

3.1 Preliminaries

For any integer $\rho \geq 1$ and a specified subset of (terminal) nodes $T \subseteq N$, let $Z_{0\rho}(T)$ and $L_{o\rho}(T)$ denote the optimal values of formulation $[CUT]$ and its linear programming relaxation for an equal connectivity (ECS) version of the survivable network design problem. In this $\{0,\rho\}$-connectivity problem, all nodes of *T* have the same connectivity requirement ρ, and all other nodes are Steiner nodes. In particular, $Z_{01}(T)$ and $L_{01}(T)$ denote the cost of the optimal Steiner forest with *T* as terminal nodes, and the optimal linear programming value of this problem's cut formulation.

We will use the $\{0,\rho\}$-connectivity problem in our analysis in the following way. Suppose that in a given MNB problem, all the positive connectivity requirements *q* equal or exceed ρ. Let Z_{DIV} and L_{DIV} denote the optimal mixed integer programming and linear programming values for the connectivity-halving formulation [*DIV*(2)] of this problem. Then the $\{0,\rho\}$-connectivity problem is a relaxation of the MNB problem, and so $Z_{0\rho}(T) \leq Z_{CUT} = Z_{DIV}$ and $L_{0\rho}(T) \leq L_{CUT} \leq L_{DIV}$. Note that if edge duplication is permitted, $L_{0\rho}(T) = \rho L_{01}(T)$.

For a given MNB problem, suppose the set *Q* contains *K* distinct connectivity values, $q_0 = 0, q_1 = 1, \ldots, q_K$, indexed in increasing order. For $k = 1, \ldots, K$, let N_k and N_{k+} denote the set of all nodes with connectivity requirement equal to q_k and greater than or equal to q_k, and define $\eta_k = q_k - q_{k-1}$.

3.2 MNB problems with edge duplication: Overlay heuristic

The *Overlay heuristic* generates a feasible solution for MNB problems with edge duplication by successively satisfying the connectivity requirements of critical nodes in order of increasing criticality. For $k = 1, 2, \ldots, K$, the method selects a heuristic or optimal solution S_k to the $\{0, \eta_k\}$-connectivity problem, with edge duplication, assuming that all the nodes in N_{k+} require connectivity of η_k. Note that, for MNB problems, since $q_1 = \eta_1 = 1$, S_1 is a Steiner forest (or Steiner tree, for the unitary special case) over all the regular and critical nodes. The union of the K solutions S_k, for $k = 1,\ldots,K$, is the overlay heuristic solution to the MNB problem.

3.2.1 Worst-case analysis of Overlay heuristic

Let $Z(S_k)$ be the cost of the solution S_k generated at each stage k of the overlay procedure. Suppose θ_k is a known upper bound on the worst-case ratio of $Z(S_k)$ to $L_{0\eta_k}(N_{k+})$. Then, the cost Z^{Ovl} of the Overlay heuristic solution satisfies the inequality:

$$\begin{aligned} Z^{Ovl} &= Z(S_1) + Z(S_2) + \ldots + Z(S_k) \\ &\le \theta_1 L_{01}(N_{1+}) + \theta_2 L_{0\eta_2}(N_{2+}) + \sum_{3 \le k \le K} \theta_k L_{0\eta_k}(N_{k+}) \end{aligned} \qquad (3.1)$$

Since $L_{0\eta_k}(N_{k+}) = (\eta_k / q_k) L_{0q_k}(N_{k+})$ and $L_{0\eta_k}(N_{k+}) \le L_{CUT}$, inequality (3.1) leads to the following upper bound (obtained previously by Goemans and Bertsimas [1993] for certain specific values of the θ parameters) on the ratio of Z^{Ovl} to L_{CUT}:

$$\frac{Z^{Ovl}}{L_{CUT}} \le \theta_1 + \theta_2 \frac{\eta_2}{q_2} + \sum_{3 \le k \le K} \theta_k \frac{\eta_k}{q_k}. \qquad (3.2)$$

We next show how to reduce this bound (in particular, the first two terms in the righthand side of (3.2)) by considering the tighter connectivity-halving formulation.

Recall that the connectivity-halving formulation [*DIV*(2)] of the MNB problem selects: (i) $\mu = 1$, and so we drop the b variables, and (ii) $\delta = 2$ or $\phi_q = \frac{1}{2}$ for all $q \in Q^{2+}$, and so $\alpha_q = 0$ for all $q \in Q$, i.e., we drop the y variables in constraints (2.6) and (2.7). Since the costs are nonnegative, the linear programming relaxation of [*DIV*(2)], with edge duplication, has an optimal solution satisfying constraints (2.9) as equalities for all edges (i, j). Substituting $z_{ij} = x_{ij} + y_{ij}$ in the objective function (2.5) decomposes this linear programming relaxation into two subLPs: *LP1* containing only the x variables with constraints (2.6), (2.7), and nonnegativity, and *LP2* containing only the y variables with constraints (2.8) and the nonnegativity requirements. Both x_{ij} and y_{ij} have the original cost c_{ij} as their objective function coefficients. Let L_{LP1} and L_{LP2} denote the optimal values of *LP1* and *LP2*.

Note that if we reduce all the cutset requirements from $q/2$ to 1 in constraints (2.7), the first subLP reduces to the linear programming relaxation of the Steiner Forest problem with terminal nodes N_{1+}. Therefore, $L_{LP1} \ge L_{01}(N_{1+})$. Similarly, if we downgrade the cutset requirements from $q/2$ to $q_2/2$ in the second subLP, we obtain a relaxation whose optimal value $L_{0q_2/2}(N_{2+}) = \frac{1}{2} L_{0q_2}(N_{2+})$ underestimates L_{LP2}. Therefore, the optimal linear programming value L_{DIV} of the connectivity-halving formulation satisfies the inequalities:

$$L_{DIV} \geq L_{LP1} + L_{LP2} \geq L_{01}(N_{1+}) + \tfrac{1}{2} L_{0q_2}(N_{2+}) \,. \tag{3.3}$$

The bounds (3.1) and (3.3), and our previous observation that

$L_{0\eta_k}(N_{k+}) = \frac{\eta_k}{q_k} L_{0q_k}(N_{k+}) \leq \frac{\eta_k}{q_k} L_{CUT} \leq \frac{\eta_k}{q_k} L_{DIV}$ imply that

$$Z^{Ovl} \leq \theta_1 L_{01}(N_{1+}) + \frac{1}{2}\theta_1 L_{0q_2}(N_{2+}) - \frac{1}{2}\theta_1 L_{0q_2}(N_{2+}) + \theta_2 \frac{\eta_2}{q_2} L_{0q_2}(N_{2+}) + \sum_{3 \leq k \leq K} \theta_k \frac{\eta_k}{q_k} L_{0q_k}(N_{k+})$$

$$\leq \theta_1 L_{DIV} + \{\theta_2 \frac{\eta_2}{q_2} - \frac{1}{2}\theta_1\} L_{0q_2} + \sum_{3 \leq k \leq K} \theta_k \frac{\eta_k}{q_k} L_{DIV} \,.$$

Theorem 6.

For MNB problems with edge duplication, the Overlay heuristic produces a solution with the following worst-case bound relative to the optimal value L_{DIV} of the connectivity-halving model's linear programming relaxation:

$$\frac{Z^{Ovl}}{L_{Div}} \leq \frac{1}{2}\theta_1 + \sum_{2 \leq k \leq K} \theta_k \frac{\eta_k}{q_k} \quad \text{if } \theta_2 \frac{\eta_2}{q_2} \geq \frac{1}{2}\theta_1, \text{ and}$$

$$\leq \theta_1 + \sum_{2 \leq k \leq K} \theta_k \frac{\eta_k}{q_k} \quad \text{otherwise.}$$

3.2.2 Forest Overlay method

For the general (nonunitary) MNB problem, suppose we use a Steiner forest heuristic to *approximately* solve the $\{0,\eta_k\}$-connectivity problem at each stage k of the overlay procedure. In step k, the method sets S_k equal to η_k copies of the heuristic Steiner forest solution over the terminal nodes N_{k+}. We refer to this implementation of the Overlay heuristic as the *Forest Overlay* method, and denote the cost of the heuristic solution it generates as Z^{FOvl}.

Goemans and Williamson [1992] proposed a dual-based heuristic for the Steiner forest problem. If T is the set of terminal nodes, this heuristic produces a Steiner forest solution *FOREST*(T) whose cost, $Z^{FOREST(T)}$, satisfies the upper bound:

$$Z^{FOREST(T)} < 2L_{01}(T) \,, \tag{3.4}$$

i.e., $\theta_1 < 2$. Note that if we select η_k copies of *FOREST*(η_k) as the approximate solution to the $\{0,\eta_k\}$-connectivity problem in the Forest Overlay procedure, then the inequality (3.4) and the fact that $L_{0\eta_k}(N_{k+}) = \eta_k L_{01}(N_{k+})$ imply that $\theta_k < 2$ for all $k = 2,\ldots,K$. Therefore, from Theorem 6, we obtain the following result if we use the dual-based Steiner forest heuristic in each step of the Forest Overlay procedure.

Corollary 7.

For MNB problems with edge duplication, the Forest Overlay heuristic has the following worst-case bound relative to optimal value L_{DIV} of the connectivity-halving formulation's linear programming relaxation:

$$\frac{Z^{FOvl}}{L_{DIV}} < 1 + 2 \sum_{2 \le k \le K} \frac{\eta_k}{q_k}.$$

To show that this bound is asymptotically tight, suppose all critical nodes have connectivity requirement ρ (i.e., with $Q = \{0,1,\rho\}$) for a situation permitting edge duplication. As applied to this $\{0,1,\rho\}$-connectivity problem, Corollary 7 implies that

$$\frac{Z^{FOvl}}{L_{DIV}} < 3 - \frac{2}{\rho}. \quad (3.5)$$

For unitary LCS problems (i.e., with $\rho = 2$) with edge duplication, the heuristic method selects all the edges belonging to $MST(N_{1+})$ and $MST(N_{2+})$. The overlay heuristic has a worst-case performance ratio of $2(1-1/|N_2|)$ relative to the optimal integer value of the LCS problem (Balakrishnan, Magnanti, and Mirchandani [1994c]). Inequality (3.5) strengthens this result by showing that the same bound applies asymptotically to the ratio of the heuristic cost to linear programming value of the connectivity-halving formulation.

Example 2. The bound of 2 for unitary LCS problems is tight. Consider a ring with m equally spaced critical (connectivity 2) nodes on its circumference and a regular node in the center. Suppose each edge on the ring costs $1/m$, and the regular node is connected to one of the critical nodes with a zero cost edge. The linear programming solution (which is also the optimal integer solution) chooses all the ring edges and the spoke edge; the cost of this solution is 1. In contrast, the tree + tree heuristic chooses the spoke edge and two copies of all but one ring edge, incurring a total cost of $2(1-1/m)$ which approaches 2 as m increases. ♦

3.2.3 Tree + Matching Overlay method

For the unitary MNB special case (with edge duplication), using tree-based heuristics at each step of the Overlay procedure provides better bounds. For any given subset of nodes *T*, let MST(*T*) denote the minimum cost tree spanning just *T*. Goemans and Bertsimas [1993] proposed the following method, which we call the *tree + matching* overlay heuristic, for the unitary MNB problem with edge duplication. The method first selects the minimum spanning tree $MST(N_{1+})$ spanning the terminal nodes N_{1+} as S_1, and for $k = 2,\ldots,K$, obtains S_k by adding $\lceil \eta_k/2 \rceil$ copies of a minimum spanning tree $MST(N_{k+})$ to $\lfloor \eta_k/2 \rfloor$ copies of an optimal matching $MATCH(N_{k+})$ on the nodes of N_{k+} that have odd degree in $MST(N_{k+})$. Using this method, the worst-case ratios are: (i) $\theta_1 = 2$, and (ii) for $k = 2,\ldots,K$, $\theta_k = 3/2$ if $\eta_k \geq 2$ is even, and $\theta_k = 3/2 + 1/2\eta_k$ if $\eta_k \geq 3$ is odd. Let Z^{T+M} denote the cost of the tree plus matching heuristic solution. Substituting these bounds in Theorem 6 gives the following result.

Corollary 8.

For unitary MNB problems with edge duplication, the tree + matching heuristic has the following worst-case bound relative to the optimal value L_{DIV} of the linear programming relaxation of the connectivity-halving formulation:

$$\frac{Z^{T+M}}{L_{DIV}} < 1 + \sum_{2 \leq k \leq K} \frac{\eta_k}{2q_k} \{3 + \frac{|\sin(\eta_k \pi/2)|}{\eta_k}\}.$$

Goemans and Bertsimas previously established the same bound, but relative to the cost of the optimal *integer* solution to the MNB problem.

Consider, the performance of the tree + matching heuristic for unitary $\{0,1,\rho\}$-connectivity problems with edge duplication. In this case, Corollary 8 implies the following worst-case bound relative to L_{DIV}:

$$\frac{Z^{T+M}}{L_{DIV}} < \frac{5}{2} - \frac{3}{2\rho} \quad \text{if } \rho > 2 \text{ is odd, and} \quad (3.6a)$$

$$\frac{Z^{T+M}}{L_{DIV}} < \frac{5}{2} - \frac{1}{\rho} \quad \text{if } \rho > 2 \text{ is even.} \quad (3.6b)$$

In particular, when $\rho = 3$, this corollary implies a bound of 2. Appendix A shows that this bound of 2 is tight, and also shows that the known bound of 3 (Goemans and Bertsimas [1993]) relative to the linear programming value of formulation [*CUT*] is tight.

3.3 Unitary MNB problems without edge duplication: Tree Completion Heuristic

This section analyzes the worst-case performance of a family of *Tree Completion heuristics* for unitary MNB problems without edge duplication, assuming that edge costs satisfy the triangle inequality. Given a MNB problem with connectivity set *Q*, we define the associated Q^{2+}-connectivity problem (without edge duplication) as the SND problem over the same graph but with the connectivity requirement of all regular nodes reduced to 0. Let $Z_{Q^{2+}}$ denote its optimal value.

3.3.1 Tree Completion heuristic

Step 1: Find an approximate or optimal solution *MCN* (multi-connected network) to the associated Q^{2+}-connectivity problem. Denote its cost by $Z^{HeurQ^{2+}}$.

Step 2: Contract *MCN* into a single node 0, choosing the least cost edge whenever this contraction step creates parallel edges. Find the minimum cost tree *TREE* spanning node 0 and the remaining regular nodes (not spanned by *MCN*). Denote the cost of the tree by $Z^{Addtree}$.

Step 3: The union of the edges in *MCN* and those in *TREE* is the Tree Completion heuristic solution to the MNB problem. Let $Z^{TC} = Z^{HeurQ^{2+}} + Z^{Addtree}$ denote the cost of this solution.

We now develop an upper bound on Z^{TC} relative to L_{BRANCH} and Z_{BRANCH}, the optimal linear programming and integer programming values of the branch-addition formulation [*BRANCH*]. Recall that in this formulation (see Section 2.3), $\phi_q = 0$ for all $q \in Q^{2+}$, and $\mu = \alpha_1 = \frac{1}{2}$. So, we omit the *x* variables, and substitute y_{ij} for z_{ij} but now require the *y* variables to be binary. Suppose we replace y_{ij} with y''_{ij} in constraints (2.6) and (2.13) and y'_{ij} in constraints (2.8), and add the linking constraint:

$$y'_{ij} = y''_{ij} \qquad \text{for all edges } (i,j) \in E. \tag{3.7}$$

Both y'_{ij} and y''_{ij} are required to be binary. Dualizing constraints (3.7) using multipliers of $\frac{1}{2}$ for all edges (i,j) decomposes the problem into two *integer* programs: a Q^{2+}-connectivity problem (without edge duplication) over all the critical nodes, but with half the original edge costs, and a Steiner tree-like subproblem [*ST**] that we discuss next. If Z_{ST*} and L_{ST*} denote the optimal integer programming and linear programming values of this latter subproblem, then

$$Z_{BRANCH} \geq \frac{1}{2} Z_{Q^{2+}} + Z_{ST*}, \text{ and} \tag{3.8a}$$

$$L_{BRANCH} \geq \frac{1}{2} L_{Q^{2+}} + L_{ST*}. \tag{3.8b}$$

Problem [*ST**] has the following formulation:

$$Z_{ST*} = \min \sum_{(i,j)\in E} c_{ij}(b_{ij} + \tfrac{1}{2} y''_{ij}) \tag{3.9}$$

subject to:

$$B_{ST} + \tfrac{1}{2} Y_{ST} \geq 1 \qquad \text{for all } \{S,T\} \in \sigma_1, \tag{3.10}$$

$$b_{ij} + y''_{ij} \leq 1 \qquad \text{for all } (i,j) \in E, \text{ and} \tag{3.11}$$

$$b_{ij}, y''_{ij} = 0 \text{ or } 1 \qquad \text{for all } (i,j) \in E. \tag{3.12}$$

The unitary network assumption implies that $r_{ij} = 1$ for at least one regular node *i* and a critical node *j*. Therefore, σ_1 contains a special class of cutsets $\{S,T\}$ in which all the regular nodes belong to *S* and all the critical nodes belong to *T*. The connectivity requirements across these cutsets ensure that the regular nodes are connected to at least one critical node.

Consider the linear programming relaxation of [*ST**]. We can assume, without loss of generality, that this linear programming relaxation has an optimal solution with $y''_{ij} = 0$ for all edges (i, j). Otherwise, given an optimal linear programming solution with $y''_{ij} > 0$, we can obtain an equal or lower cost feasible solution by setting $b_{ij} \leftarrow b_{ij} + y''_{ij}/2$. Therefore, the linear programming relaxation of [*ST**] is the same as the linear programming relaxation of a *Modified Steiner Tree* (*ModST*) problem that seeks a minimum cost tree (with c_{ij} as edge costs) spanning all the regular nodes and at least one critical node. If we reduce to zero the cost of all the edges connecting pairs of critical nodes, then requiring the regular nodes to be connected to at least one critical node is equivalent to requiring connectivity to all the critical nodes. Therefore, we obtain the following *Extended Steiner Tree* (*EST*) problem as a relaxation of the ModST problem. If *G'* denotes the graph obtained by contracting all the critical nodes into a single node 0, the EST problem is the Steiner tree problem defined on *G'*, treating all the regular nodes and node 0 as terminals. The minimum spanning tree heuristic applied to EST selects the minimum tree *EMST* in *G'* spanning just the regular nodes and node 0; the cost of this solution, Z^{EMST} is no more than $2(1-1/(n_r+1))$ times the optimal linear programming value L_{EST} of the EST problem (Magnanti and Wolsey [1995]). Moreover, the cost $Z^{Addtree}$ of the edges added in Step 2 of the Tree Completion heuristic does not exceed Z^{EMST}.

If L_M denotes the optimal value of the linear programming relaxation of the cutset formulation of Problem *M*, then the preceding observations establish the following bounds:

$$Z^{Addtree} \le Z^{EMST} < 2L_{EST} < 2L_{ModST} = 2L_{ST*}. \qquad (3.13)$$

Proposition 9.

Suppose, in the Tree Completion heuristic, we solve the triangular cost Q^{2+}-connectivity problem (without edge duplication) using a heuristic method that has integer programming and linear programming performance guarantees of θ and θ_{LP} (the linear programming guarantee is relative to the problem's cutset formulation). Then, the Tree Completion heuristic satisfies the following worst-case bounds:

$$\frac{Z^{TC}}{Z_{BRANCH}} < \theta + 1, \text{ and } \frac{Z^{TC}}{L_{BRANCH}} < \theta_{LP} + 1$$

Proof:

Inequalities (3.8a) and (3.13), and the observation that $Z_{Q^{2+}} \le Z_{BRANCH}$ (since the Q^{2+}-connectivity problem is a relaxation of [*BRANCH*] with certain node connectivities reduced to zero) imply that

$$Z^{TC} = Z^{HeurQ^{2+}} + Z^{Addtree} < \theta Z_{Q^{2+}} + 2L_{ST*} = (\theta - 1)Z_{Q^{2+}} + 2(\tfrac{1}{2}Z_{Q^{2+}} + L_{ST*}) \le (\theta + 1)Z_{BRANCH}.$$

Let $L_{Q^{2+}}$ denote the optimal linear programming values of the cutset formulation of the Q^{2+}-connectivity subproblem. Using inequality (3.8b) and the inequality $L_{Q^{2+}} \le L_{BRANCH}$,

$$Z^{TC} < \theta_{LP} L_{Q^{2+}} + 2L_{ES} = (\theta_{LP} - 1)L_{Q^{2+}} + 2(\tfrac{1}{2}L_{Q^{2+}} + L_{ES}) \le (\theta_{LP} + 1)L_{BRANCH}.$$ ♦

3.3.2 Special case: LCS problems without edge duplication

To solve unitary LCS problems without edge duplication, suppose we apply the following tree + matching heuristic to approximately solve the {0,2}-connectivity problem in Step 1 of the Tree Completion heuristic: find the minimum cost tree spanning the critical nodes, and construct an Eulerian graph by adding the edges of the minimum cost matching over the odd degree nodes in this tree. Consider an Eulerian tour in this tree + matching solution. As Goemans and Bertsimas have shown, $\theta_{LP} = 3/2$ for this solution. By short-circuiting edges, we can transform this solution into an equal or lower cost hamiltonian tour over the critical nodes. This transformed (called tree + matching) solution does not duplicate edges whenever the costs are triangular and the number of critical is nodes is at least 3. Therefore, Proposition 9 implies the following corollary.

Corollary 10.

For the triangular cost LCS problem without edge duplication, the Tree Completion heuristic has an linear programming worst-case ratio of 5/2 if we use the tree + matching heuristic to solve the embedded {0,2}-connectivity problem.

Observe that our bound improves upon the best previously known worst-case bound of 3 for this version of the LCS problem (Goemans et al.).

Example 3. To show that the bound of 5/2 in Corollary 10 is tight, consider the example in Figure 1(a). This figure has two concentric rings each consisting of q critical nodes. The critical nodes on the rings are aligned. Every critical node on each ring is connected to its two neighbors on that same ring with edges of cost 1. Every critical node on a ring is also connected to 3 other nodes on the other ring: the node directly aligned with it via a (spoke) edge of cost 1, and the two nodes to the immediate right and left; each of these (spoke) edges has a cost of 2. An alternate path (of total length 1) consisting of $p-1$ regular nodes also connects every pair of adjacent critical nodes on the same ring.

Figure 1(b) shows the heuristic solution if we use the Tree+Matching heuristic to connect the critical nodes. We first choose the inner ring (except one edge), and all unit-cost spoke edges in the MST. The matching step then selects $q-2$ of the remaining spoke edges and an edge on the outer ring. Short circuiting provides us with a two-connected solution that costs $3q-2$. To obtain a heuristic solution to the LCS problem, we then connect all the regular nodes to this subgraph and incur an additional cost of $2q(1-1/p)$. Thus, the total heuristic cost is $5q-(2q/p)-2$. The optimal solution in Figure 1(c) costs $2q+2$, and thus we obtain an asymptotic heuristic-to-optimal cost bound of 5/2 for large values of p and q. ♦

Instead of the tree + matching heuristic, suppose we use the optimal TSP tour over the critical nodes as the heuristic {0,2}-connectivity solution in Step 1 of the Tree Completion method. With triangular costs, Monma et al. [1990] have shown that $Z_{TSP}/L_{02} = 4/3$. Substituting $\theta_{LP} = 4/3$ in Proposition 9 gives the following result:

Corollary 11.

For the triangular cost LCS problem without edge duplication, the Tree Completion method with the embedded TSP solution procedure has an linear programming worst-case ratio of 7/3.

Example 4. To show that this bound of 7/3 is tight, consider the example shown in Figure 2(a). This example, an extension of a problem instance in Monma et al. [1990], has three paths connecting two special critical nodes 1 and 2. Each path has $q-1$ critical nodes, and the cost of each edge on these three paths is 1. A path containing $p-1$ regular nodes also connects every pair of adjacent critical nodes; the total cost of this alternate path is 1. The cost between any other pair of nodes is the shortest path cost between these nodes.

Figure 2(b) shows the heuristic solution whose the cost is $4q-1 + 3q(1-1/p)$. Figure 2(c) shows the optimal solution with a cost of $3q$. This example achieves the bound of 7/3 for large values of p and q. Note that since this example does not contain a Steiner node, the worst-case bound also applies to {1,2}-connected problems. ♦

3.3.3 Other special cases

(i) ***K*-path Steiner tree problem**

The *K-path Steiner tree problem* is a special $\{0,1,K\}$-connectivity problem containing only two critical nodes that must be connected via K edge-disjoint paths (that possibly pass through regular nodes or Steiner points). In this case, the $\{0,K\}$-connectivity subproblem without edge duplication (in Step 1) is solvable as a minimum cost flow problem. Therefore, $\theta_{LP} = 1$ and so the Tree Completion method has an linear programming worst-case ratio of at most 2.

(ii) **MNB problems with side constraints**

The model and our analysis extends to more general classes of MNB problems with additional configuration constraints imposed on the multi-connected network. Consider, for instance, the Ring on Steiner tree problem, a constrained version of the LCS problem which requires the two-connected subgraph of the LCS solution to be a hamiltonian tour that visits all the critical nodes (and optionally visits regular or Steiner nodes). In this case, we have additional configuration constraints in the formulation [BRANCH] specifying that every critical node must have degree 2. The formulation, and therefore our analysis, remains valid even with these additional constraints as long as we use an appropriate heuristic method in Step 1 of the completion procedure. So, if we find the optimal TSP tour over the critical nodes, then $\theta = 1$ and the Tree Completion method has an integer programming worst-case ratio of 2.

4. Conclusions

Since even the simplest cases of survivable network design problems are NP-hard, researchers have focused on modeling enhancements to improve the effectiveness of linear programming-based solution methods, and on analyzing tree, tour, and matching-based heuristics. In this paper, we have studied modeling and heuristic methods for the MNB problem and its special cases, both with and without edge duplication.

Our discussion of MNB modeling issues builds upon a traditional cut formulation for modeling survivability problems. Because it is more tractable, most researchers have used the cut formulation to develop lower bounds in order to analyze the worst-case performance of SND heuristics. However, since the cut formulation has a weak linear programming relaxation, developing and guaranteeing strong worst-case bounds is difficult even though the heuristics might inherently be good. Our modeling and analysis approach in this paper differs from previous approaches in two respects: (i) we strengthen the problem's linear programming formulation without sacrificing its tractability for heuristic analysis, and (ii) we analyze relatively simple heuristics that use MST, matching, and forest heuristics as building blocks.

Even with these simple heuristics, we are able to achieve or improve upon some of the existing bounds in the literature.

Consider, for instance, MNB problems with edge duplication and contiguous connectivities, i.e., $Q = \{0,1,2,...,K\}$. Our linear programming bound of $2(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{K})-1$ for this problem is one less than Goemans et al.'s bound (which does not assume edge duplication); it is also lower than Williamson et al.'s bound of $2K-1$. If some intermediate connectivity levels are missing, then Theorem 6 provides a bound of $2(1+\sum_{2\le k\le K}(q_k - q_{k-1})/q_k)-1$ which further improves upon Goemans et al.'s bound.

In developing these bounds, we have used a new connectivity-splitting mixed-integer programming formulation (2.5)–(2.13) for survivable network design problems. This formulation generalizes in two ways. First, by allowing b_{ij} to be any positive integer in (2.11), and by changing the righthand side of (2.13) to β_{ij} allows us to choose up to β_{ij} copies of edge (i,j), and thus model a "capacitated" version of the problem. Second, the formulation (2.5)–(2.13) applies with minor modifications even when the righthand side of (2.2) is a proper function (see Goemans et al. [1994]). To incorporate this change, we alter the righthand sides of (2.7) and (2.8) to $f(S)\phi_{f(S)}$ and $f(S)(1-\phi_{f(S)})$ for some pre-specified connectivity fractions $0 \le \phi_{f(S)} \le \frac{1}{2}$ for all $f(S) \ge 2$. As earlier, the parameter α_q is defined using ϕ, the minimum of all values $\phi_{f(S)}, f(S) \ge 2$. The proof of the validity of this formulation is similar to the proof of Theorem 1. This observation suggests the possibility of extending this paper's approach to survivable network design problems with proper connectivity functions and without edge duplication.

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APPENDIX A
Worst-case Example for Tree + Matching Heuristic

Example 5. The "honeycomb" example in Figure 3(a) shows that the Tree + Matching heuristic achieves an asymptotic bound of 2 relative to the linear programming relaxation of formulation [*DIV*(3)]. This example has m hexagons packed in a plane, for a sufficiently large integer m. All the hexagon vertices represent *critical* nodes with connectivity requirement 3. Each pair of adjacent critical nodes is connected by two alternate *indirect paths* containing $(p-1)$ *regular* nodes. The direct edge cost, as well as the total cost of the indirect paths connecting adjacent critical nodes, is 1.

Since the honeycomb has m hexagons, and each edge belongs to two hexagons, the netowrk contains $2m$ critical nodes and $3m$ direct edges. (We ignore boundary effects since the number of boundary edges grows sublinearly with m.) For each direct edge in the network, the optimal solution chooses all the edges in one of the corresponding indirect paths and all but one edge of the other indirect path. The cost of this optimal solution is $3m + 3m(1-1/p)$.

In Step 1, the Tree + Matching overlay heuristic finds an MST spanning all the terminal nodes, incurring a total cost of $3m + 3m(1-1/p)$. In step 2, the heuristic finds an MST spanning the critical nodes, thus selecting $2m-1$ direct edges. Figure 3(b) shows one such tree as bold edges. The heuristic then finds a minimum matching over the odd degree nodes of the tree. This matching duplicates the pendant edges; since each hexagon has, on average, one pendant edge, the cost of the minimum matching is m. Thus, the total heuristic cost is

$$(3m + 3m(1-1/p)) + (2m-1) + m \;=\; 9m - 3m/p - 1.$$

Next, consider the linear programming relaxation of formulation [*DIV*(3)]. Setting $y_{ij} = 1/3$ and $x_{ij} = 5/12$ for each edge on all the indirect paths, we obtain a feasible solution to this linear programming relaxation costing $4.5m$. Therefore, asymptotically, for large values of p and m, this example achieves the desired bound of 2. For the cutset formulation [*CUT*], setting $u_{ij} = 1/2$ for each edge on all the indirect paths, with total cost equal to $3m$, is optimal. Therefore, this example also asymptotically achieves the heuristic-to-LP (with respect to the cutset formulation) worst-case bound of 3.

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Figure 1: Worst-case example for LCS problem without edge duplication

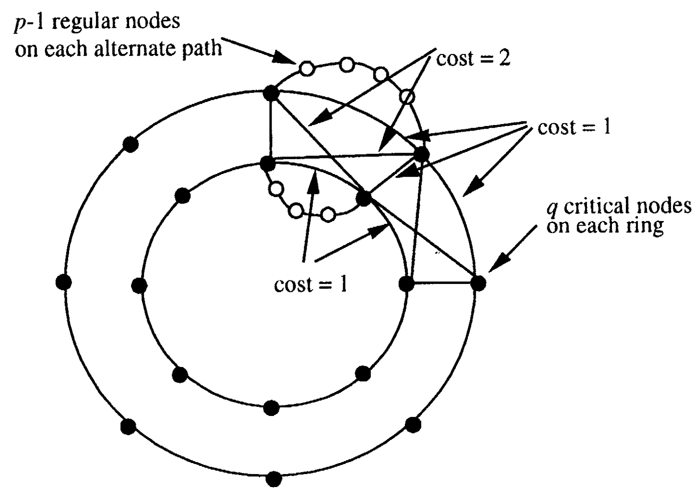

Figure 1(a) Network structure and costs

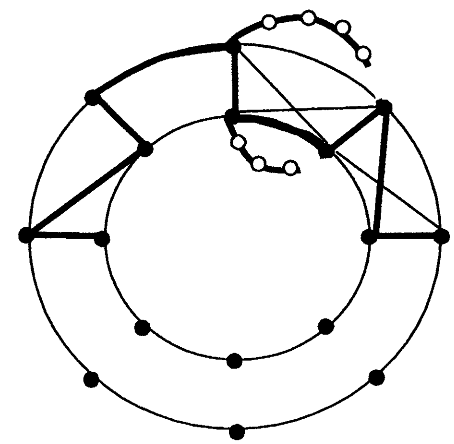

Figure 1(b) Heuristic solution shown in bold

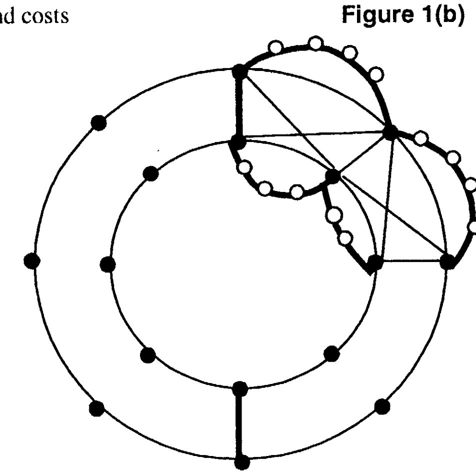

Figure1(c) Optimal solution shown in bold

Figure 2: Worst-case example using TSP-based heuristic for LCS problem without edge duplication

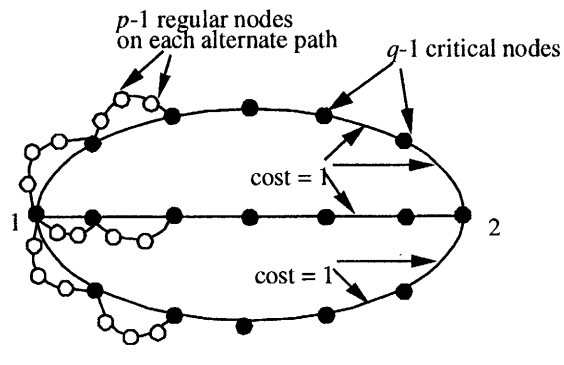

Figure 2(a) Network structure and costs (Network is complete; edge costs are shortest path distances)

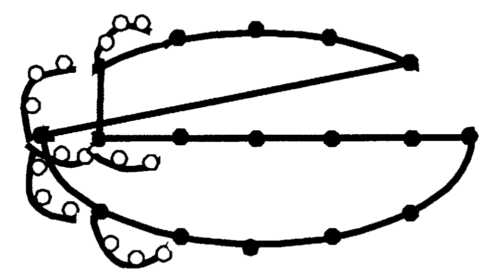

Figure 2(b) Heuristic solution shown in bold

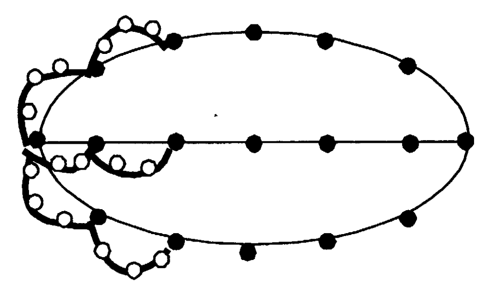

Figure 2(c) Optimal solution shown in bold

Figure 3: Worst-case example for {0,1,3}-connectivity problem

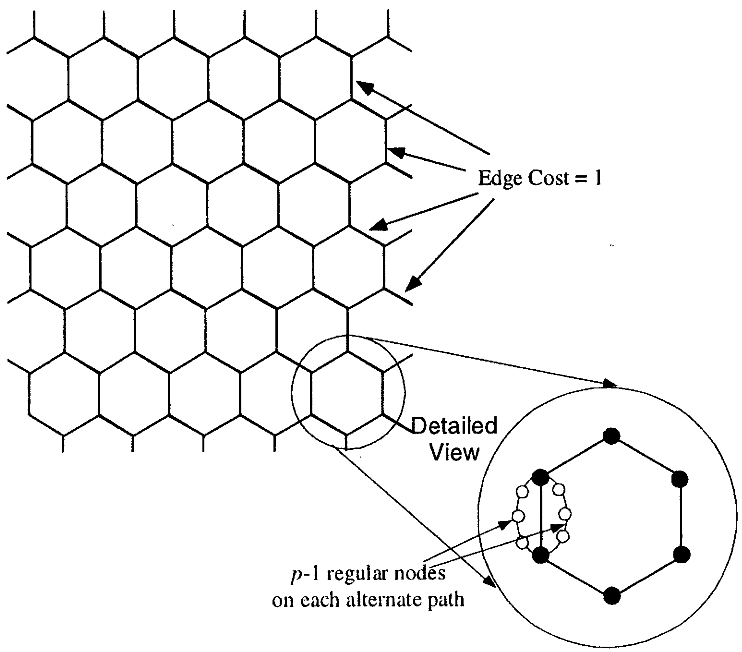

Figure 3(a) Network structure and costs

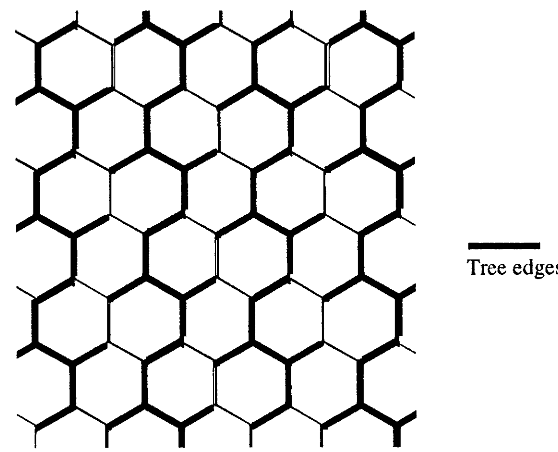

Figure 3(b) Tree edges chosen by Step 2 of Tree + Matching heuristic