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The Application of Variational Inequality Theory to the Study of Spatial Equilibrium and Disequilibrium

by

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1. Introduction

The spatial price equilibrium models of Takayama and Judge have provided the basic framework for the analysis of competitive systems over space and time. Moreover, their fundamental contributions have stimulated the development of new methodologies and uncovered vistas for applications in agriculture, energy markets, mineral economics, and finance (see, e.g., Judge and Takayama (1973), Uri (1975), Takayama and Labys (1986), Newcomb, Reynolds, and Masbruch (1988), Moore and Nagurney (1989)).

In the past decade, spatial price equilibrium problems have captured the interest of scholars from a wide spectrum of disciplines, including: operations research, mathematical programming, economics, regional science, and transportation science. The attraction has come from several factors: the richness of the problems for model development, the computational challenges posed by the large- scale nature of the problems, and the evolving connections with equilibrium problems in distinct disciplines.

Historically, spatial price equilibrium models were usually reformulated as optimization problems, provided that a certain symmetry or integrability assumption held for the underlying functions. Utilizing such an approach, Samuelson (1952) and Takayama and Judge (1964, 1971) introduced a variety of spatial price equilibrium models. Convex programming algorithms could then, at least in principle, be used for the computation of the regional commodity production, consumption, and interregional (and intertemporal) trade patterns. Analogously, Beckmann, McGuire, and Winsten (1956) reformulated traffic network equilibrium models with both fixed and elastic demands as optimization problems.

It has now been realized that equilibrium problems governed by distinct equilibrium conditions and operating under distinct behavioral assumptions - for which the integrability assumption need no longer be imposed - can be modelled and studied via the theory of variational inequalities.

The theory of variational inequalities (VI) had been introduced by Hartman and Stampacchia (1966) as a tool for the study of partial differential equations. The identification by Dafermos (1980) that the traffic network equilibrium conditions had the structure of a finite-dimensional VI problem opened new avenues for the development of more general, asymmetric multicommodity (and multimodal) models and the design of mathematically correct and convergent algorithms. Florian and Los (1982) then formulated the spatial price equilibrium conditions as a variational inequality problem (see also, e.g., Dafermos and Nagurney (1984a), Friesz, Harker, and Tobin (1984)). Later, Dafermos and Nagurney (1985) (see, also, e.g., Dafermos (1986a)) showed that the spatial price equilibrium problem could be cast into a traffic network equilibrium problem with a special, simple network structure. This special network structure in which each origin/destination pair consists of paths which are disjoint, has been exploited computationally by Dafermos and Nagurney (1989), Nagurney (1989a), Eydeland and Nagurney (1989), and Nagurney and Kim (1989) by observing that each restricted demand market (or supply market equilibrium subproblem) could be solved exactly in closed form. Interestingly, the special network structure can also be used to link spatial price equilibrium problems and constrained matrix problems. Such a connection had been postulated earlier by Stone (1951) and recently formalized by Nagurney (1989a). These constrained matrix problems include the estimation of input/output matrices, social/national accounts, origin/destination traffic flows, and demographic patterns.

Other equilibrium problems which have also been studied as VI problems include: imperfectly competitive oligopolistic market equilibrium problems, both aspatial (Gabay and Moulin (1980)), and spatial (Dafermos and Nagurney (1987), Harker (1986), Nagurney (1988)), market equilibrium problems with production (Dafermos and Nagurney (1984b)), Walrasian price and general economic equilibria (Border (1985), Dafermos (1986b), Zhao (1989)), and migration equilibria (Nagurney (1989b)). Most of these problems can be viewed as network equilibrium problems, in which, however, the nodes of the underlying abstract network representation need no longer correspond to locations in space. The principal advantage of a network formalism from a conceptual standpoint is that seemingly disparate problems can be studied in a unified fashion. On the other hand, the main advantage from a computational standpoint is that previously intractable problems can be efficiently computed.

In this paper we focus on the application of the methodology of variational inequalities, combined with network theory, to the study of spatial price equilibrium problems and - in the case of policy interventions - disequilibrium, or "constrained equilibrium" problems. We note that although our emphasis in this paper is on applications, in particular, on perfectly competitive spatial price problems, within an equilibrium/disequilibrium framework, the variational inequality problem contains, as special cases not only such problems and minimization problems, but virtually all the classical problems of mathematical programming, such as linear and nonlinear complementarity problems, fixed point problems, and minimax problems. For further discussion and a list of references, see Nagurney (1987a).

The paper is organized as follows:

In Section 2 we provide the necessary background for the theory of variational inequalities and focus on the qualitative properties of existence and uniqueness.

In Section 3 we present a synthesis of asymmetric spatial price equilibrium models in quantity variables and in price variables, and give the variational inequality formulations of the governing spatial price equilibrium conditions. We also relate the models to other models in the literature.

In Section 4 we then generalize the models described in Section 3 to handle policy interventions explicitly, again within a variational inequality framework. Policy instruments which we consider include price supports and trade restrictions. These VI formulations differ from those given in Section 3 in the defining functions and/or feasible sets.

We then show in Section 5 that both the equilibrium problems and the disequilibrium problems can be solved using a variational inequality decomposition algorithm which resolves the original variational inequality under consideration into three simpler variational inequality problems, in which the "dominant" subproblem has the structure of a network equilibrium problem. We also provide in Section 6 numerical results to illustrate the computational performance of the algorithm for both equilibrium and contrained equilibrium problems.

Finally, we conclude with a summary and discussion in Section 7.

2. Background

In this Section we briefly review the basic theory of variational inequalities. For amplified discussions, see the book by Kinderlehrer and Stampacchia (1980), the surveys of Magnanti (1984), Dafermos (1987), Nagurney (1987a), and the thesis of Zhao (1989).

The finite dimensional VI problem is to determine x such that

$$f(x) \cdot (x' - x) \geq 0, \quad \text{for all} \quad x' \in K \tag{1}$$

where K is a closed convex subset of R^n and $f(\cdot)$ is a known function from K to R^n.

In the case where the feasible set K is bounded and $f(\cdot)$ is a continuous function, there exists at least one solution x to (1). When K is not necessarily bounded, a solution to (1) exists, provided that $f(\cdot)$ is continuous and coercive, i.e.,

$$\frac{(f(\bar{x}) - f(x')) \cdot (\bar{x} - x')}{\|\bar{x} - x'\|} \to \infty, \quad \text{as} \quad \begin{matrix} \|\bar{x}\| \\ \bar{x} \in K \end{matrix} \to \infty \tag{2}$$

for some fixed $x' \in K$, where $\|\cdot\|$ denotes the Euclidean norm.

In the case where certain monotonicity conditions can be expected to hold, the theory of variational inequalities becomes particularly powerful. For example, when $f(\cdot)$ is strictly monotone, i.e.,

$$(f(\bar{x}) - f(x')) \cdot (\bar{x} - x') > 0 \tag{3}$$

for all $\bar{x}, x' \in K$, $\bar{x} \neq x'$, then VI (1) has at most one solution.

Furthermore, when $f(\cdot)$ is strongly monotone, that is:

$$(f(\bar{x}) - f(x')) \cdot (\bar{x} - x') \geq \alpha \|\bar{x} - x'\|^2, \quad \text{for every} \quad \bar{x}, x' \in K \tag{4}$$

where α is a positive constant, then there exists a unique solution x to VI (1). Necessary and sufficient conditions for (4) to hold is that the (not necessarily symmetric) Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$ is positive definite over the feasible set K. A sufficient condition for the coercivity condition (2) to hold is that the strong monotonicity condition (4) holds.

Finally, the function f is called monotone if the left-hand side of (4) is greater than or equal to zero for every $\bar{x}, x' \in K$.

The connection between variational inequality problems and minimization problems, in which monotonicity for the former plays an analogous role as convexity in the latter, is as follows:

Let $F(\cdot)$ be a continuously differentiable scalar-valued function defined on some open neighborhood of K and denote its gradient by $\nabla F(\cdot)$. If there exists an $x \in K$ such that

$$F(x) = \min_{x' \in K} F(x') \tag{5}$$

then x is a solution to the variational inequality

$$\nabla F(x) \cdot (x' - x) \geq 0, \quad \text{for all} \quad x' \in K. \tag{6}$$

On the other hand, if $f(\cdot)$, again on an open neighborhood of K, is the gradient of a convex continuously differentiable function $F(\cdot)$, then VI (1) and the minimization problem (5) are equivalent; in other words, x solves (1) when x minimizes $F(\cdot)$ over K. Note that $f(\cdot)$ is a gradient mapping if and only if its Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$ is symmetric, in which case the objective function $= \int f(y)dy$.

Moreover, if $F(\cdot)$ is convex, strictly convex, or uniformly convex, then its gradient mapping is, respectively, monotone, strictly monotone, or strongly monotone.

We note that the above "symmetry" or "integrability" condition had been utilized by Samuelson (1952) and Takayama and Judge (1971) to reformulate the equilibrium conditions of spatial price equilibrium models as the Kuhn-Tucker conditions of appropriately defined optimization problems. We note that now, in view of the above, a single inequality of the form (1) can be used to formulate the equilibrium conditions of spatial price equilibrium problems in which the symmetry condition need no longer be assumed. Hence, multicommodity spatial price equilibrium problems, either static or intertemporal, can now be modelled and studied under more realistic conditions without the restrictive symmetry assumption. However, VI (1) still contains such symmetric problems as special cases.

In the next two Sections we will present variational inequality formulations of a series of spatial price equilibrium models in quantity variables and in price variables, in the absence and then in the presence of policy interventions in the form of trade restrictions and price controls. The motivation stems from the seminal book of Takayama and Judge (1971) and the edited volume of Judge and Takayama (1973).

We now turn to a brief overview of the numerical procedures for the computation of VI (1).

VI (1) can be solved via the general iterative scheme of Dafermos (1983) which contains both projection methods (Dafermos (1980, 1982), Bertsekas and Gafni (1982)), as well as other linearization methods (Pang and Chan (1981)), and relaxation/diagonalization methods (Florian and Spiess (1982)), as special cases. The computation of the solution x to VI (1) is accomplished iteratively via the computation of the solutions to a series of simpler VI subproblems, which, typically, are mathematical programming (minimization) problems, since efficient algorithms for such problems exist. Projection methods resolve the original variational inequality problem into series of quadratic programming problems, whereas, relaxation methods resolve the original variational inequality problem into, typically, series of nonlinear programming problems. Hence. the overall efficiency of a VI algorithm depends on the efficiency of the algorithm selected for the computation of the embedded mathematical programming problems. Indeed, the desire to compute general multicommodity spatial price equilibrium problems within realistic time frames has spurred the development of special- pupose algorithms for single commodity spatial price equilibrium problems, which exploit the underlying problem structure. Such special-purpose algorithms have outperformed convex programming algorithms (see, e.g., Nagurney (1987b), Nagurney (1989a,c), Dafermos and Nagurney (1989), and Eydeland and Nagurney (1989)).

For computational comparisons of variational inequality algorithms, see Nagurney (1984, 1987b) and the references therein.

Moreover, in the case where the feasible set K (cf. (1)) can be expressed as a Cartesian product of sets, where

$$K = \prod_{a=1}^{z} K_a \tag{7}$$

where each K_a is a subset of R^{n_a}, the reformulation of VI (1) over (7) induces natural decompositions of the original variational inequality into subproblems of lower dimensions. Such decompositions are especially appealing in the case of large-scale multicommodity spatial price equilibrium problems. Recently, parallel and serial variational inequality decomposition algorithms have been applied to multicommodity spatial price equilibrium problems by Nagurney and Kim (1989) to compute solutions to problems with as many as 100 markets and 12 commodities using serial and parallel computers. For VI decomposition algorithms applied to intertemporal spatial price equilibrium problems with discounting, gains and losses, and other modelling enhancements, see, e.g., Nagurney and Aronson (1988, 1989), Nagurney (1989d). For decomposition schemes applied to spatial oligopoly models operating under the Cournot-Nash postulate of noncooperative behavior, see Nagurney (1988). For alternative parallel and serial decomposition algorithms. see Pang (1985).

In the subsequent Sections we focus on the derivation of variational inequalities over Cartesian products of sets for a variety of spatial price models and and provide a synthesis of many of the recent research results. These variational inequality formulations are not the immediately obvious ones, but are notable in that they induce efficient decomposition schemes which we then describe in Section 5.

3. Equilibrium Models

In this Section we synthesize spatial price equilibrium models within a variational inequality framework. In particular, we present both quantity and price formulations.

We consider m supply markets and n demand markets involved in the production /consumption of a commodity. We denote a typical supply market by i and a typical demand market by j. We let s_i denote the supply at supply market i and we let d_j denote the demand at demand market j. We let π_i denote the supply price associated with supply market i and ρ_j the demand price associated with demand market j. We group the supplies and supply prices into vectors $s \in R^m$ and $\pi \in R^m$, respectively. Similarly, we group the demands and demand prices into vectors $d \in R^n$ and $\rho \in R^n$, respectively.

We let Q_{ij} denote the nonnegative commodity shipment between the supply and demand market pair (i,j) and we let c_{ij} denote the nonnegative transaction cost associated with trading the commodity between (i,j). We assume that the transaction cost c_{ij} includes the transportation cost. Hence, the supply and demand markets can be spatially separated. We note that the transaction cost may also include such policy intruments as tariffs, taxes, fees, duties, or subsidies. We group the commodity shipments into a vector $Q \in R^{mn}$ and the transaction costs into a vector $c \in R^{mn}$.

The well-known market equilibrium conditions, assuming perfect competition take, following Samuelson (1952) and Takayama and Judge (1971), the following form: For all pairs of supply and demand markets (i,j); $i = 1, \ldots, m$; $j = 1, \ldots, n$:

$$\pi_i + c_{ij} \begin{cases} = \rho_j, & \text{if } Q_{ij} > 0 \\ \geq \rho_j, & \text{if } Q_{ij} = 0. \end{cases} \tag{8}$$

The conditions (8) state that a pair of markets (i,j) will trade, provided that the supply price at supply market i plus the transaction cost between the pair of markets is equal to the demand price at demand market j. Moreover, the following feasibility conditions must hold:

$$s_i \begin{cases} = \sum_j Q_{ij}, & \text{if } \pi_i > 0 \\ \geq \sum_j Q_{ij}, & \text{if } \pi_i = 0 \end{cases} \tag{9}$$

and

$$d_j \begin{cases} = \sum_i Q_{ij}, & \text{if } \rho_j > 0 \\ \leq \sum_i Q_{ij}, & \text{if } \rho_j = 0. \end{cases} \tag{10}$$

Typically, it is assumed that both supply and demand prices are positive in equilibrium. Hence, usually the equalities are assumed to hold in both (9) and (10). Here, however, we consider the above more general situation, which is in the spirit of Takayama and Judge (1971).

Introducing now the nonnegative variables u_i and w_j, where u_i denotes the possible excess supply at supply market i and w_j denotes the possible unmet demand at demand market j, we may rewrite (9) and (10), respectively, as: For every $i, i = 1, \ldots, m$:

$$u_i \begin{cases} = 0, & \text{if } \pi_i > 0 \\ \geq 0, & \text{if } \pi_i = 0 \end{cases} \tag{11}$$

and for every $j, j = 1, \ldots, n$:

$$w_j \begin{cases} = 0, & \text{if } \rho_j > 0 \\ \geq 0, & \text{if } \rho_j = 0 \end{cases} \tag{12}$$

where

$$s_i = \sum_j Q_{ij} + u_i \quad \text{and} \quad d_j = \sum_i Q_{ij} - w_j \tag{13}$$

We group the u_i's into a vector $u \in R^m$ and the w_j's into a vector $w \in R^n$.

We now discuss the supply price, demand price, and transaction cost structure.

We assume that the supply price associated with any supply market may depend upon the supply of the commodity at every supply market, that is,

$$\pi = \pi(s) \tag{14}$$

where π is a known smooth function. On the other hand, in the case where the supply function, rather than the supply price function is given, we assume that the supply can depend, in general, upon the supply price at every supply market, that is,

$$s = s(\pi). \tag{15}$$

Similarly, the demand price associated with any demand market may depend upon the demand of the commodity at every demand market, that is,

$$\rho = \rho(d) \tag{16}$$

where ρ is a known smooth function. Analogously, in the case where the demand function, rather than the demand price function is given, we assume that the demand can depend, in general, upon the demand price at every demand market, that is,

$$d = d(\rho) \tag{17}$$

The transaction cost, which includes the transportation cost between a pair of supply and demand markets may depend, in general, upon the shipments of the commodity between every pair of markets, that is,

$$c = c(Q) \tag{18}$$

where c is a known smooth function.

In the special case where the number of supply markets m is equal to the number of demand markets n, the transaction cost functions are assumed to be fixed and the supply price functions and demand price funcions are symmetric, i.e., $\frac{\partial \pi_i}{\partial s_k} = \frac{\partial \pi_k}{\partial s_i}$, for all $i = 1, \ldots, m$; $k = 1, \ldots, m$, and $\frac{\partial \rho_j}{\partial d_l} = \frac{\partial \rho_l}{\partial d_j}$, for all $j = 1, \ldots, n$; $l = 1, \ldots, n$, then the above model with supply price functions (14) and demand price functions (16) collapses to the quantity models introduced in Takayama and Judge (1971) for which an equivalent optimization formulation exists. Similarly, if the analogous symmetry assumption holds for the supply functions (15) and demand functions (17), then the above model contains as a special case the price models of Takayama and Judge (1971).

In the case where the equalities in (9) and (10) are assumed to hold the above model in quantity variables collapses to the spatial market model of Dafermos and Nagurney (1985) which has been solved as a VI problem in Nagurney (1987b). For the relationship between this model and a general spatial oligopoly model, see Dafermos and Nagurney (1987). On the other hand, the spatial model in price variables, using (15) and (17) had been introduced by Dafermos and McKelvey (1986).

Before proceeding to state the VI formulations of the spatial price equilibrium models discussed above, we first introduce some notation for simplification purposes. We define the vector $\tilde{\pi}$ in R^{mn} consisting of m vectors $\{\tilde{\pi}_i\}$ in R^n with components $\{\pi_i\}$. Similarly, we define the vector $\tilde{\rho}$ in R^{mn} consisting of m vectors $\{\tilde{\rho}_j\}$ in R^n with components $\{\rho_1, \rho_2, \ldots, \rho_n\}$. We further define the vectors $\hat{\pi} = \pi \in R^m$, and $\hat{\rho} = \rho \in R^n$. In view of the feasibility conditions (9) and (10), we can express $\hat{\pi}$ and $\hat{\rho}$ in the following manner:

$$\hat{\pi} = \hat{\pi}(Q, u) \quad \text{and} \quad \hat{\rho} = \hat{\rho}(Q, w). \tag{19}$$

We also define the vectors $\tilde{\hat{\pi}} \in R^{mn}$ consisting of m vextors $\{\tilde{\hat{\pi}}\} \in R^n$ with components $\{\hat{\pi}_i, \ldots, \hat{\pi}_i\}$ and the vector $\tilde{\hat{\rho}} \in R^{mn}$ consisting of m vectors $\{\tilde{\hat{\rho}}_j\} \in R^n$ with components $\{\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_n\}$.

We are now ready to present variational inequality formulations of a spectrum of spatial price equilibrium models in quantity variables, in price variables, and in "combined" price-quantity variables:

Assuming that we are given the supply price functions (14), the demand price functions (16), and the transaction cost functions (18). then the spatial price equilibrium conditions (8) subject to (11), (12), and (13) take on the following alternative formulations:

VI 3.1a

A pattern $(s, d, Q, u, w) \in K^{1a}$, where $K^{1a} \equiv \{(s', d', Q', u', w')$ satisfying(13) where $u' \in R^m_+$ and $w' \in R^n_+\}$ satisfies equilibrium conditions (8), (11), (12) if and only if it satisfies the VI

$$(\tilde{\pi}(s) + c(Q) - \tilde{\rho}(d)) \cdot (Q' - Q) + \pi(s) \cdot (u' - u) + \rho(d) \cdot (w' - w) \geq 0 \tag{20}$$

$$\text{for all} \quad (s', d', Q', u', w') \in K^{1a}$$

or, equivalently,

VI 3.1b

$$(\tilde{\hat{\pi}}(Q, u) + c(Q) - \tilde{\hat{\rho}}(Q, w)) \cdot (Q' - Q) + \hat{\pi}(Q, u) \cdot (u' - u) + \hat{\rho}(Q, w) \cdot (w' - w) \geq 0 \tag{21}$$

$$\text{for all} \quad (Q', u', w') \in K \equiv R^{mn}_+ \times R^m_+ \times R^n_+.$$

For detailed derivations of similar variational inequality formulations, see Nagurney and Zhao (1988), Dafermos (1982), Nagurney (1987b). We note that, in the case where the equalities in (9) and (10) are assumed to hold, then the governing VI contains only the first term in (20), where the feasible set K^{1a} is accordingly simplified.

Assuming, on the other hand, that we are given the demand functions (17), then a VI formulation akin to VI 3.1b, is given by:

VI 3.2

$$(\tilde{\hat{\pi}}(Q,u)+c(Q)-\tilde{\hat{\rho}})\cdot(Q'-Q)+\hat{\pi}(Q,u)\cdot(u'-u)-D(Q,\rho)\cdot(\rho'-\rho)\geq 0 \tag{22}$$

$$\text{for all}\quad (Q',u',\rho')\in K\equiv R^{mn}_{+}\times R^{m}_{+}\times R^{n}_{+}$$

where $D\in R^n$ consists of components $D_j\equiv\sum_i Q_{ij}-d_j(\rho)$ and $\tilde{\hat{\rho}}$ is a now a vector variable.

For details, see Nagurney and Zhao (1989a).

If now, instead, we are given the demand price functions (16) and the supply functions (15), then the VI formulation is given by:

VI 3.3

$$(\tilde{\hat{\pi}}+c(Q)-\tilde{\hat{\rho}}(Q,w))\cdot(Q'-Q)+S(\pi,Q)\cdot(\pi'-\pi)+\rho(Q,w)\cdot(w'-w)\geq 0 \tag{23}$$

$$\text{for all}\quad (Q',\pi',w')\in K\equiv R^{mn}_{+}\times R^{m}_{+}\times R^{n}_{+},$$

where $S\in R^m$ consists of components $S_i\equiv s_i(\pi)-\sum_j Q_{ij}$, and $\tilde{\hat{\pi}}$ is now a vector variable.

Finally, if we are given both the supply functions (15), and the demand functions (17), then the formulation becomes:

VI 3.4

$$(\tilde{\hat{\pi}}+c(Q)-\tilde{\hat{\rho}})\cdot(Q'-Q)+S(\pi,Q)\cdot(\pi'-\pi)-D(Q,\rho)\cdot(\rho'-\rho)\geq 0 \tag{24}$$

$$\text{for all}\quad (Q',\pi',\rho')\in K\equiv R^{mn}_{+}\times R^{m}_{+}\times R^{n}_{+}.$$

For a detailed derivation, see Dafermos and McKelvey (1986). For an excess demand. single price model, see Friesz, Harker, and Tobin (1984).

Observe that each of the above VI formulations is of the form of inequality (1), where the vector f and x are defined accordingly. Moreover, observe that each of the feasible sets K, for VI 3.1b through 3.4, is, in fact, a Cartesian product, of the form (7). In particular, each such K consists of the product of three simpler feasible sets. Hence, as intimated in Section 2, a decomposition approach is especially appealing. Indeed, as we will show in Section 5, both the above problems and the disequilibrium or "constrained equilibrium " problems which will be outlined in the subsequent Section can be solved using a variational inequality decomposition algorithm which will resolve each of the variational inequality problems over a Cartesian product of sets into three simpler variational inequality subproblems. Each of these, in turn, will have a special structure in which the "dominant" VI subproblem can be formulated and solved as a network equilibrium problem.

A qualitative analysis of the above equilibrium problems can be obtained by applying the theory described in Section 2. Illustrative and complete analyses in terms of existence and uniqueness of solutions to the above VI problems can be found in Nagurney and Zhao (1988, 1989a,b) and Dafermos and McKelvey (1986). Stability and sensitivity analysis results for spatial price equilibrium problems are described in Dafermos and Nagurney (1984). A general approach to sensitivity analysis for variational inequalities can be found in Dafermos (1988).

4. Disequilibrium or Constrained Equilibrium Models

In this Section we focus on spatial models in the case of trade restrictions and price controls. Policy interventions in the form of tariffs, subsidies, and quotas played a prominent role in applied spatial price models studied by Takayama and Judge (1971). In particular, our goal here is to demonstrate how the equilibrium models outlined in the preceding Section can be generalized within the variational inequality framework to handle policy instruments. The modifications result in changes to the governing functions and/or the feasible sets. As noted by Thore (1986), in the case of policy interventions the governing state may be one of disequilibrium. As mentioned earlier, the modelling of tariffs and subsidies can also be incorporated into the VI framework by modifying the transaction cost functions appropriately.

We denote a minimum nonnegative supply price floor for supply market i by $\underline{\pi}_i$, and the maximum supply price ceiling by $\bar{\pi}_i$. We group the supply price floors into a vector $\underline{\pi} \in R^m$ and the supply price ceilings into a vector $\bar{\pi} \in R^m$. We denote then a minimum nonnegative demand price floor for demand market j by $\underline{\rho}_j$ and the maximum demand price ceiling by $\bar{\rho}_j$. We group the demand price floors into a vector $\underline{\rho} \in R^n$ and the demand price ceilings into a vector $\bar{\rho} \in R^n$.

We also denote a nonnegative trade floor for the commodity shipment Q_{ij} by $\underline{M}_{ij}$ and the maximum trade ceiling by $\overline{M}_{ij}$. We group the trade floors into a vector $\underline{M} \in R^{mn}$ and the trade ceilings into a vector $\overline{M} \in R^{mn}$. The market condition (8), in the presence of trade restrictions, is now extended to: For all pairs of supply and demand markets $(i,j), i = 1, \ldots, m; j = 1, \ldots, n$:

$$\pi_i + c_{ij} \begin{cases} \leq \rho_j, & \text{if } Q_{ij} = \overline{M}_{ij} \\ = \rho_j, & \text{if } \underline{M}_{ij} < Q_{ij} < \overline{M}_{ij} \\ \geq \rho_j, & \text{if } Q_{ij} = \underline{M}_{ij} \end{cases} \tag{25}$$

whereas, conditions (9) and (10) now take the form:

$$s_i \begin{cases} \leq \sum_j Q_{ij}, & \text{if } \pi_i = \bar{\pi}_i \\ = \sum_j Q_{ij}, & \text{if } \underline{\pi}_i < \pi_i < \bar{\pi}_i \\ \geq \sum_j Q_{ij}, & \text{if } \pi_i = \underline{\pi}_i \end{cases} \tag{26}$$

and

$$d_j \begin{cases} \geq \sum_i Q_{ij}, & \text{if } \rho_j = \bar{\rho}_j \\ = \sum_i Q_{ij}, & \text{if } \underline{\rho}_j < \rho_j < \bar{\rho}_j \\ \leq \sum_i Q_{ij}, & \text{if } \rho_j = \underline{\rho}_j. \end{cases} \tag{27}$$

In the case where only the price floors $\underline{\pi}$ are imposed on the producers, then the analogue of condition (11) is: For every $i, i = 1, \ldots, m$:

$$u_i \begin{cases} = 0, & \text{if } \pi_i > \underline{\pi}_i \\ \geq 0, & \text{if } \pi_i = \underline{\pi}_i. \end{cases} \tag{28}$$

We now present variational inequality formulations of the constrained equilibrium counterparts of VI 3.1a through VI 3.4. The models presented below are in increasing order of generality.

We first present the VI formulations, akin to VI 3.1a and VI 3.1b, satisfying conditions (25) in the presence of supply price floors only. In particular, assuming that we are given the supply price functions (14), the demand price functions (16), and the trade cost functions (17), then the market conditions (25) take on the following formulations.

VI 4.1a

A pattern $(s, d, Q, u, w) \in \bar{K}^{1a}$, where $\bar{K}^{1a} \equiv \{(s', d', Q', u', w')| \quad \underline{M} \leq Q' \leq \overline{M}$, and satisfying (13) where $u' \in R^m_+$, and $w' \in R^n_+\}$, satisfies conditions (25), subject to (12) and (28), if and only if it satisfies the VI

$$(\tilde{\pi}(s) + c(Q) - \tilde{\rho}(d)) \cdot (Q' - Q) + (\pi(s) - \underline{\pi}) \cdot (u' - u) + \rho(d) \cdot (w' - w) \geq 0$$

$$\text{for all} \quad (s', d', Q', u', w') \in \bar{K}^{1a} \tag{29}$$

or, equivalently,

VI 4.1b

$$(\tilde{\hat{\pi}}(Q, u) + c(Q) - \tilde{\hat{\rho}}(Q, w)) \cdot (Q' - Q) + (\hat{\pi}(Q, u) - \underline{\pi}) \cdot (u' - u) + \hat{\rho}(Q, w) \cdot (w' - w) \geq 0 \tag{30}$$

$$\text{for all} \quad (Q', u', w') \in K \equiv K_1 \times R^m_+ \times R^n_+$$

where $K_1 \equiv \{Q'| \quad \underline{M} \leq Q' \leq \overline{M}\}$.

The above model generalizes the model of Greenberg and Murphy (1985). For a detailed derivation, see Nagurney and Zhao (1988).

In the case that we retain the supply price floors $\underline{\pi}$, and include now the demand price floors $\underline{\rho}$ and ceilings $\bar{\rho}$, then the VI formulation of the constrained equilibrium analogue of VI 3.2 is given by:

VI 4.2

$$(\tilde{\hat{\pi}}(Q,u) + c(Q) - \tilde{\hat{\rho}}) \cdot (Q' - Q) + (\hat{\pi}(Q,u) - \underline{\pi}) \cdot (u' - u) - D(Q,\rho) \cdot (\rho' - \rho) \geq 0 \quad (31)$$

$$\text{for all} \quad (Q', u', \rho') \in K \equiv K_1 \times R_+^m \times K_3$$

where $K_3 \equiv \{\rho' | \quad \underline{\rho} \leq \rho' \leq \bar{\rho}\}$.

For a discussion of this model, see Nagurney and Zhao (1989a).

On the other hand, if we now include the supply price ceilings, and retain only the demand price floors, we may rewrite VI 3.3 now as

VI 4.3

$$(\tilde{\hat{\pi}} + c(Q) - \tilde{\hat{\rho}}(Q,w)) \cdot (Q' - Q) + S(\pi, Q) \cdot (\pi' - \pi) + (\rho(Q,w) - \underline{\rho}) \cdot (w' - w) \geq 0 \quad (32)$$

$$\text{for all} \quad (Q', \pi', w') \in K \equiv K_1 \times K_2 \times R_+^n$$

where $K_2 \equiv \{\pi' | \quad \underline{\pi} \leq \pi' \leq \bar{\pi}\}$.

Finally, we present the most general formulation, akin to VI 3.4, in which the supply and demand functions are used, price floors and price ceilings are permitted on both the production and consumption sides and the trade restrictions remain, i.e.,:

VI 4.4

$$(\tilde{\hat{\pi}} + c(Q) - \tilde{\hat{\rho}}) \cdot (Q' - Q) + S(\pi, Q) \cdot (\pi' - \pi) - D(Q,\rho) \cdot (\rho' - \rho) \geq 0 \quad (33)$$

$$\text{for all} \quad (Q', \pi', \rho') \in K \equiv K_1 \times K_2 \times K_3.$$

See also, e.g., Nagurney and Zhao (1989b) and Dafermos and McKelvey (1986).

5. The Variational inequality Decomposition Algorithm

Recall that the variational inequality formulations of the spatial price equilibrium models, VI 3.1b - VI 3.4. and their constrained equilibrium analogues, VI 4.1b - VI 4.4, were each defined over a Cartesian product K. Each such set, in turn, consisted of three sets. Hence, we can decompose each of the variational inequalities into three simpler VI subproblems in lower dimensions. The first encountered or "dominant" VI subproblem will have a structure identical to a network equilibrium problem adjusted to the case of bounds on the transaction links to handle trade restrictions.

We state the algorithm for the computation of the disequilibrium problem VI 4.1b and then for VI 4.2. We also relate the statement of the algorithm for the computation of equilibrium problems VI 3.1b and VI 3.2. The statement for VI 4.3 and VI 4.4 and their equilibrium analogues should then be readily apparent. For proof of global convergence, see Nagurney and Zhao (1988) and Nagurney and Zhao (1989a).

Computation of VI 4.1b

The algorithm computes a sequence (Q^0, u^0, w^0), (Q^1, u^1, w^1), by solving three VI's sequentially and converges to the solution of (30).

The steps are:

Step 0: Start with any $(u^0, w^0) \in R^m_+ \times R^n_+$.

Step 1:$(t = 0, 1, 2, \ldots)$ Solve the VI

$$\left[\tilde{\hat{\pi}}(Q, u^t) + c(Q) - \tilde{\hat{\rho}}(Q, w^t)\right] \cdot (Q' - Q) \geq 0 \quad \text{for all} \quad Q' \in K_1 \tag{34}$$

The solution to (34) is Q^t.

Step 2: $(t = 0, 1, 2, \ldots,)$ Solve the VI

$$\left[\hat{\pi}(Q^t, u) - \underline{\pi}\right] \cdot (u' - u) \geq 0, \quad \text{for all} \quad u' \in R^m_+. \tag{35}$$

The solution to (35) is u^{t+1}.

Step 3: $(t = 0, 1, 2, \ldots)$ Solve the VI

$$\hat{\rho}(Q^t, w) \cdot (w' - w) \geq 0, \quad \text{for all} \quad w' \in R^n_+. \tag{36}$$

The solution to (36) is w^{t+1}.

Let $t = t + 1$, and go to Step 1.

The solution of equilibrium problem VI 3.1b can be obtained by setting $\underline{\pi}$ in (35) equal to zero and letting $K_1 = R_+^{mn}$ in (34)..

As shown in Nagurney and Zhao (1988), under the assumption that the supply price $\pi(s)$, demand price $\rho(d)$, and the transaction cost functions are strongly monotone in s, d, and Q, respectively, then each of the above subproblems (34), (35), and (36) admits a unique solution and, hence, the sequence $\{(Q^t, u^t, w^t)\}, t = 1, 2, \ldots$ is well-defined. The economic meaning of such an assumption is that the supply price at a supply market depends primarily upon the supply of the commodity at that supply market, the demand price at a demand market depends primarily upon the demand for the commodity at the demand market, and the transaction cost between a pair of supply and demand markets depends primarily upon the commodity shipment between the pair of supply and demand markets. Such a condition is not unreasonable in appropriate applications.

Moreover, the algorithm is globally convergent under conditions given in Nagurney and Zhao (1988).

The effectiveness of the decomposition algorithm is based on the fact that the first VI subproblem given in (34) is actually the one governing the well-known spatial price equilibrium problem in the case of equality constraints (see, e.g., Dafermos and Nagurney (1985), Nagurney (1987b)). Furthermore, this problem can be cast into a traffic network equilibrium problem (with bounds on the transportation links) on a network with special structure (cf. Figure 1) (see also, e.g., Dafermos and Nagurney (1985), Dafermos (1986)). This problem can be efficiently solved using a Gauss-Seidel serial linearization algorithm (or a projection method) in which each restricted demand market equilibrium subproblem can be solved exactly in closed form via the algorithm introduced in Dafermos and Nagurney (1989) (see also, e.g., Nagurney (1987a, b, 1989a), Eydeland and Nagurney (1989), and Nagurney and Kim (1989)). VI subproblems (35) and (36) are very simple and can also be computed using a serial linearization method outlined in Nagurney (1987b).

In a similar manner, we have the

Computation of VI 4.2

The algorithm computes a sequence (Q^0, u^0, ρ^0), $(Q^1, u^1, \rho^1), \ldots$, by solving three VI's sequentially and converges to the solution of (31).

The steps are:

Step 0: Start with any $(u^0, \rho^0) \in R^m_+ \times K_3$.

Step 1: $(t = 0, 1, 2, \ldots)$ Solve the VI

$$\left[\tilde{\hat{\pi}}(Q, u^t) + c(Q) - \tilde{\hat{\rho}}^t\right] \cdot (Q' - Q) \geq 0 \quad \text{for all} \quad Q' \in K_1. \tag{37}$$

The solution to (37) is Q^t.

Step 2: $(t = 0, 1, 2, \ldots)$ Solve the VI

$$\left[\hat{\pi}(Q^t, u) - \underline{\pi}\right] \cdot (u' - u) \geq 0, \quad \text{for all} \quad u' \in R^m_+. \tag{38}$$

The solution to (38) is u^{t+1}.

Step 3: $(t = 0, 1, 2, \ldots)$ Solve the VI

$$-D(Q^t, \rho) \cdot (\rho' - \rho) \geq 0, \quad \text{for all} \quad \rho' \in K_3. \tag{39}$$

The solution to (39) is ρ^{t+1}.

Let $t = t + 1$, and go to Step 1.

The solution of the equilibrium analogue, VI 3.2, follows by setting $\underline{\pi} = 0$ in (38), and letting $K_1 = R^{mn}_+$ in (37) and $K_3 = R^m_+$ in (39)

Each of the above VI subproblems (37), (38), and (39) will admit unique solutions, provided that $\pi(s), c(Q)$, and $d(\rho)$ are each strongly monotone in s, Q and ρ, respectively. Thus, the sequence $\{(Q^t, u^t, \rho^t)\}$, $t = 1, 2, \ldots$ is well-defined and can be obtained by applying any appropriate algorithm for the computation of the individual variational inequalities (37), (38), and (39). In particular, VI (38) is identical to VI (35). VI (37), on the other hand, also has a simple network structure which should be exploited and which will now be elaborated upon, cf. Figure 2.

Specifically, VI (37) is a specially-structured network problem in which there are m origins and n potential destinations, where users at each origin seek to establish their user cost- minimizing destinations, where the transportation cost associated with traveling from origin i to any destination j is given by:

$$\hat{c}_{ij}(Q) = c_{ij}(Q) - \rho_j, \quad \text{for} \quad j = 1, \ldots, n \tag{40}$$

and the attractiveness function associated with locating at origin i is defined to be $\bar{\pi}_i(Q) \equiv \hat{\pi}_i$. The paths available consist of single disjoint links (i, j). This network equilibrium problem, thus constructed, is a member of one of the classes of integrated traffic network equilibrium problems formulated by Dafermos (1976). Observe that the characteristic network representation of VI (37) is even simpler than the one encountered in the traffic network equilibrium representation of the spatial price equilibrium problem encountered in (34). In particular, VI (37) can be solved by a Gauss-Seidel serial linearization decomposition algorithm by supply markets given in Nagurney (1987b), in which the supply market equilibration algorithm introduced in Dafermos and Nagurney (1989) and further theoretically analyzed in Nagurney and Eydeland (1989) is embedded. This algorithm exploits the "disjointness" of the origin/destination paths explicitly, by solving each supply market equilibrium subproblem exactly in closed form.

A mirror image network on the demand side to the one in Figure 2 may be constructed for the dominant VI subproblem encountered in the application of the decomposition algorithm to VI 4.3 and VI 3.3. On the other hand, VI 4.4 and VI 3.4, although the most general formulations, have no apparent network structure in the principal VI subproblem which can be exploited. Nevertheless, the encountered principal VI subproblem is simple to compute using, again, a term-by-term Gauss-Seidel scheme.

In the next Section, we provide computational results for the decomposition algorithm applied to VI 4.2 and VI 3.2.

6. Numerical Experience

In this Section we consider, as an illustration, the spatial market models formulated as VI 4.2 and VI 3.2 and we provide numerical experience with the variational inequality decomposition algorithm outlined in Section 5.

Since the decomposition algorithm resolves the solution of VI 4.2 and VI 3.2 into three simpler variational inequality subproblems, the decomposition algorithm allows one the opportunity to select any appropriate algorithm for the individual VI subproblems. However, due to the special structure of the first and principal VI subproblem (37), the application of a special-purpose algorithm is appealing. Hence, as mentioned in the preceding Section, we will apply the Gauss- Seidel serial decomposition algorithm by supply markets (with the appropriate simplification since the demand prices now are fixed) in which we embed, also accordingly simplified, the supply market equilibration algorithm, proposed in Dafermos and Nagurney (1989) which solves each restricted supply market equilibrium subproblem exactly, rather than iteratively. Gauss-Seidel serial decomposition algorithms are also adapted to compute the solutions to (38) and (39). For alternative algorithms and references, see Nagurney (1987b).

In our computational test, we, hence, utilized the above described algorithms for the computation of the individual VI subproblems.

In order to illustrate how the decomposition algorithm performs computationally we considered spatial market problems with linear asymmetric functions, where the supply price functions are given by

$$\pi_i = \pi_i(s) = \sum_j r_{ij} s_j + t_i = \hat{\pi}_i(Q, u) = \sum_j r_{ij}(\sum_k Q_{jk} + u_j) + t_i, \tag{41}$$

the demand functions are given by

$$d_j = d_j(\rho) = -\sum_k p_{jk} \rho_k + l_j \tag{42}$$

and the transaction cost functions are given by

$$c_{ij} = c_{ij}(Q) = \sum_{kl} g_{ijkl} Q_{kl} + h_{ij}, \tag{43}$$

where the not necessarily symmetric Jacobians of the supply price and transaction cost functions are positive definite. whereas the Jacobian of the demand functions is negative definite.

In this Section we considered randomly generated market problems in which the supply price (41), and transaction cost functions (43) were generated uniformly in the same manner as described in Nagurney (1987b). In particular, the function term ranges were as follows: $r_{ii} \in [3, 10], t_i \in [10, 25]$ and $g_{ijij} \in [1, 15], h_{ij} \in [10, 25]$, $i = 1, \ldots, m; j = 1, \ldots, n$. The demand functions were generated so that: $-p_{jj} \in [-10, -15]$, $l_j \in [150, 650]$. The remaining $r_{ij}, -p_{jk}$, and g_{ijkl} terms were generated to ensure that the Jacobian matrices were strictly diagonal dominant and, hence, positive definite. We set the number of supply markets m equal to the number of demand markets n and varied the problem sizes from 45 supply markets and 45 demand markets (90 markets total) to 90 supply markets and 90 demand markets (180 markets total) in increments of 15 markets. These problems are larger than the equilibrium problems considered in Nagurney (1987b) and of the same size as the disequilibrium problems solved for the inverse demand models in Nagurney and Zhao (1988).

In Table 1 we fixed the number of cross-terms in the functions (41), (42), and (43) to 5, whereas in Table 2, we fixed the number of cross-terms to 10. We set $\underline{M} = 0$, and $\bar{M} = \infty$. The termination criterion utilized was $|\pi_i + c_{ij} - \rho_j| \leq \epsilon = 5$, if $Q_{ij} > 0$ and $\pi_i + c_{ij} - \rho_j \geq -\epsilon$ if $Q_{ij} = 0$ and $\pi_i \geq \underline{\pi}_i, \underline{\rho}_j \leq \rho_j \leq \bar{\rho}_j$, where $\underline{\rho}$ was set at zero and $(\pi_i - \underline{\pi}_i) \cdot u_i \leq 5$; $(\sum_i Q_{ij} - d_j(\rho)) \times \rho_j \leq 5$, if $\underline{\rho}_j < \rho_j < \bar{\rho}_j$; $(\sum_i Q_{ij} - d_j(\rho)) \leq 0$, if $\rho_j = \bar{\rho}_j$, and $(\sum_i Q_{ij} - d_j(\rho)) \geq 0$, if $\rho_j = \underline{\rho}_j$. Since verification of convergence can in itself be computationally time-consuming, especially in large-scale examples, we verified convergence for VI (37) after every other iteration.

The algorithm was coded in FORTRAN and compiled using the FORTVS compiler, optimization level 3 on the IBM 4381-14 mainframe at the Cornell National Supercomputer Facility. The CPU times reported in Tables 1 and 2 are exclusive of input and output. The initial pattern was set at $Q_{ij} = 0$ for all i and j, $u_i = \max(0, \frac{\underline{\pi}_i - t_i}{r_{ii}})$, for all i, and $\rho_j = \max(0, \frac{\bar{\rho}_j - l_j}{-p_{jj}})$ for all j.

In each of the first column examples in Tables 1 and 2, we set $\underline{\pi}_i = 0$ for all i and $\bar{\rho}_j = 1000$ for all j. (In view of the generation of functions such price floors and ceilings would generate equilibrium solutions.) Hence, the reported CPU times in these columns reflect the computational time for the decomposition algorithm to solve the market model governed by VI 3.2.

To the same problems, we then tightened the bounds on the demand side in column 2 of each Table where $\underline{\pi} = 0$, $\underline{\rho} = 0$, and $\bar{\rho} = 50$. We also report the number of supply and demand markets in which the respective prices are at one of the bounds. In column 3 of each Table we then raised the supply price floors to $\underline{\pi} = 150$ and loosened the demand price ceilings to $\bar{\rho} = 750$, but kept $\underline{\rho} = 0$, and report the number of supply and demand markets with prices at the bounds.

As can be seen from Tables 1 and 2, the decomposition algorithm was robust, converging for all the examples and requiring only seconds of CPU time on a readily available mainframe. The other models discussed in Sections 2 and 3 can now also be solved in a timely fashion using the variational inequality decomposition procedure.

The spatial price models presented and synthesized in this paper should enable the computation of a greater spectrum of problems than heretofore was possible, thus expanding the potential scope of applications for policy analyses.

7. Summary and Conclusions

In this paper we have focused on general. asymmetric, perfectly competitive spatial price equilibrium problems using as the stimulus the fundamental contributions of Takayama and Judge (1971). In particular, we have shown how variational inequality theory and networks can be utilized to formulate, study, compute, and synthesize a spectrum of spatial price problems. We first considered market models in the absence of policy instruments, and then in the presence of such interventions as price controls on the production and consumption sides and trade restrictions. The models presented were related to other models in the literature and include quantity models, price models, and combined quantity-price models.

The theory of variational inequalities, hence, can be viewed as playing the same role in the analysis of equilibrium and disequilibrium problems as mathematical programming has in optimization problems. Indeed, although we have concentrated our attention on perfectly competitive partial equilibrium models, imperfectly competitive oligopolistic market equilibrium problems operating under the Cournot-Nash behavioral postulate, Walrasian price and general economic equilibrium problems, and migration equilibrium problems, have all been formulated and studied as variational inequality problems. Moreover, since the variational inequality problem contains, as special cases: linear and nonlinear complementarity problems, fixed point problems, min/max problems, as well as, minimization problems. it provides us with a powerful unifying framework, of which we can expect to see more use in economics in the future.

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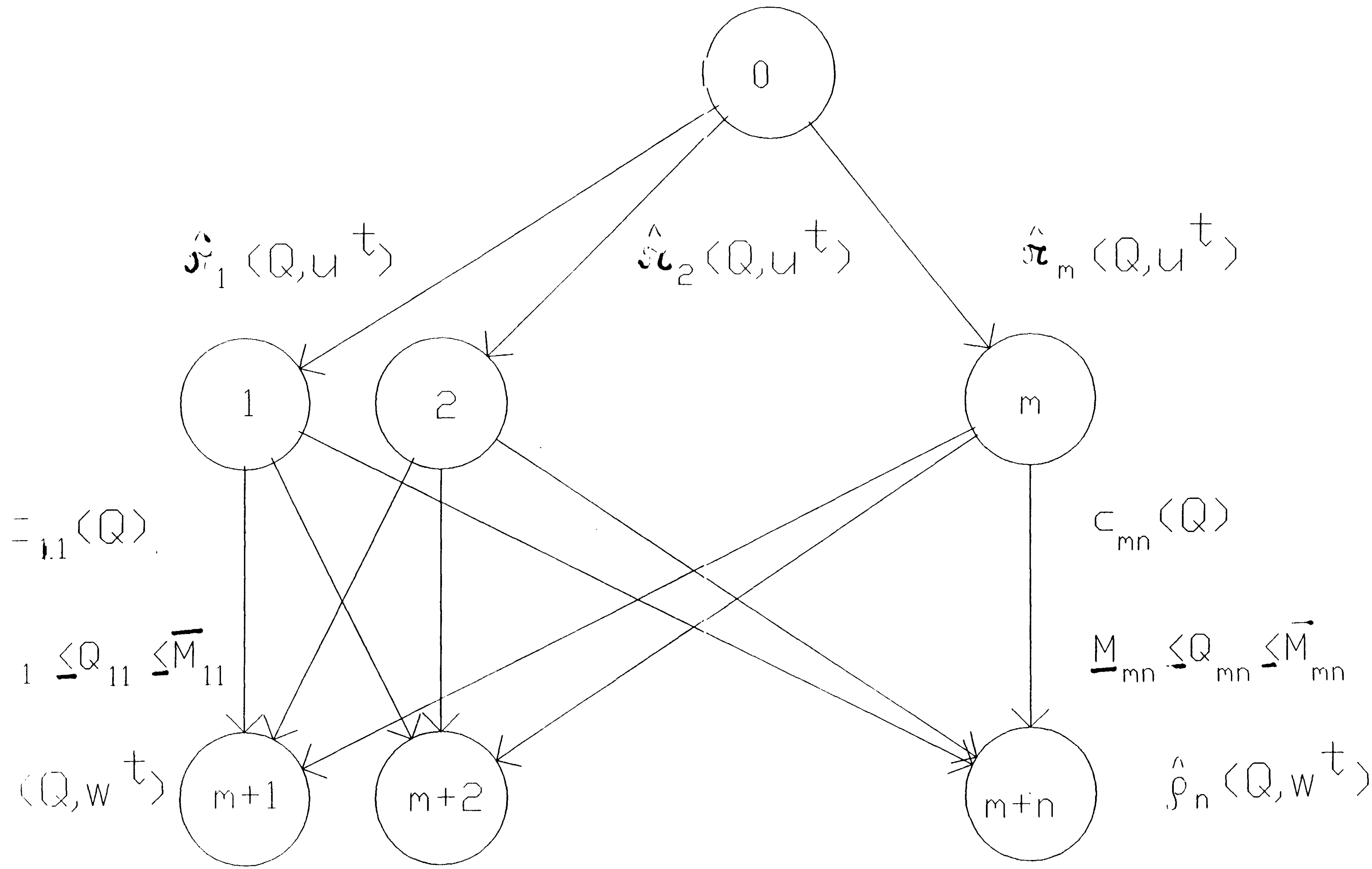

Figure 1: The Network Representation of VI(34)

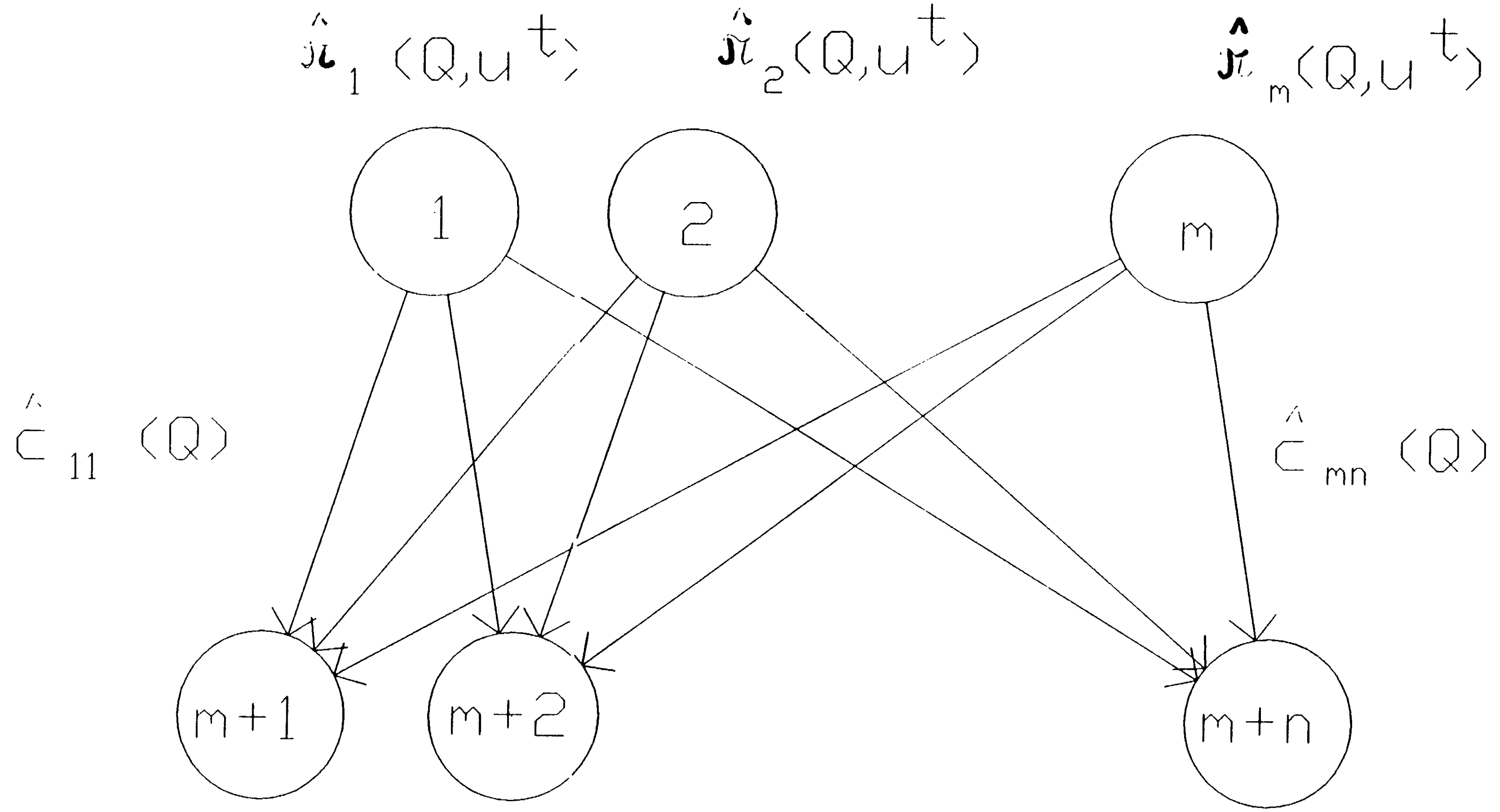

Figure 2: The Network Representation of VI(37)

Table 1 Computational Experience on

Large-Scale Spatial Market Problems

Number of Crossterms = 5

CPU time in seconds (* , ** , ***)

(m,n)	$\underline{\pi}=0, \underline{\rho}=0, \bar{\rho}=1000$	$\underline{\pi}=0, \underline{\rho}=0, \bar{\rho}=50$	$\underline{\pi}=150, \underline{\rho}=0, \bar{\rho}=750$
(45,45)	4.33(0,0,0)	1.64(0,0,33)	3.82(38,0,0)
(60,60)	7.26(0,,0,0)	3.07(0,0,50)	5.05(55,0,0)
(75,75)	12.00(0,0,0)	4.88(0,0,61)	10.64(65,0,0)
(90,90)	18.02(0,0,0)	5.74(0,0,78)	15.86((84,0,0)

Table 2 Computational Experience on

Large-Scale Spatial Market Problems

Number of Crossterms = 10

CPU time in seconds (* , ** , ***)

(m,n)	$\underline{\pi}=0, \underline{\rho}=0, \bar{\rho}=1000$	$\underline{\pi}=0, \underline{\rho}=0, \bar{\rho}=50$	$\underline{\pi}=150, \underline{\rho}=0, \bar{\rho}=750$
(45,45)	9.11(0,0,0)	2.72(0,0,34)	5.10(39,0,0)
(60,60)	14.30(0,0,0)	4.96(0,0,45)	10.41(51,0,0)
(75,75)	21.09(0,0,0)	6.29(0,0,59)	16.81(64,0,0)
(90,90)	24.66(0,0,0)	11.80(0,0,67)	5.98(79,0,0)

* Number of supply markets, i, with supply price $\pi_i = \underline{\pi}_i$.

** Number of demand markets, j, with demand price $\rho_j = \underline{\rho}_j$.

*** Number of demand markets, j, with demand price $\rho_j = \bar{\rho}_j$.