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CAVITIES AND WAVEGUIDES WITH INHOMOGENEOUS
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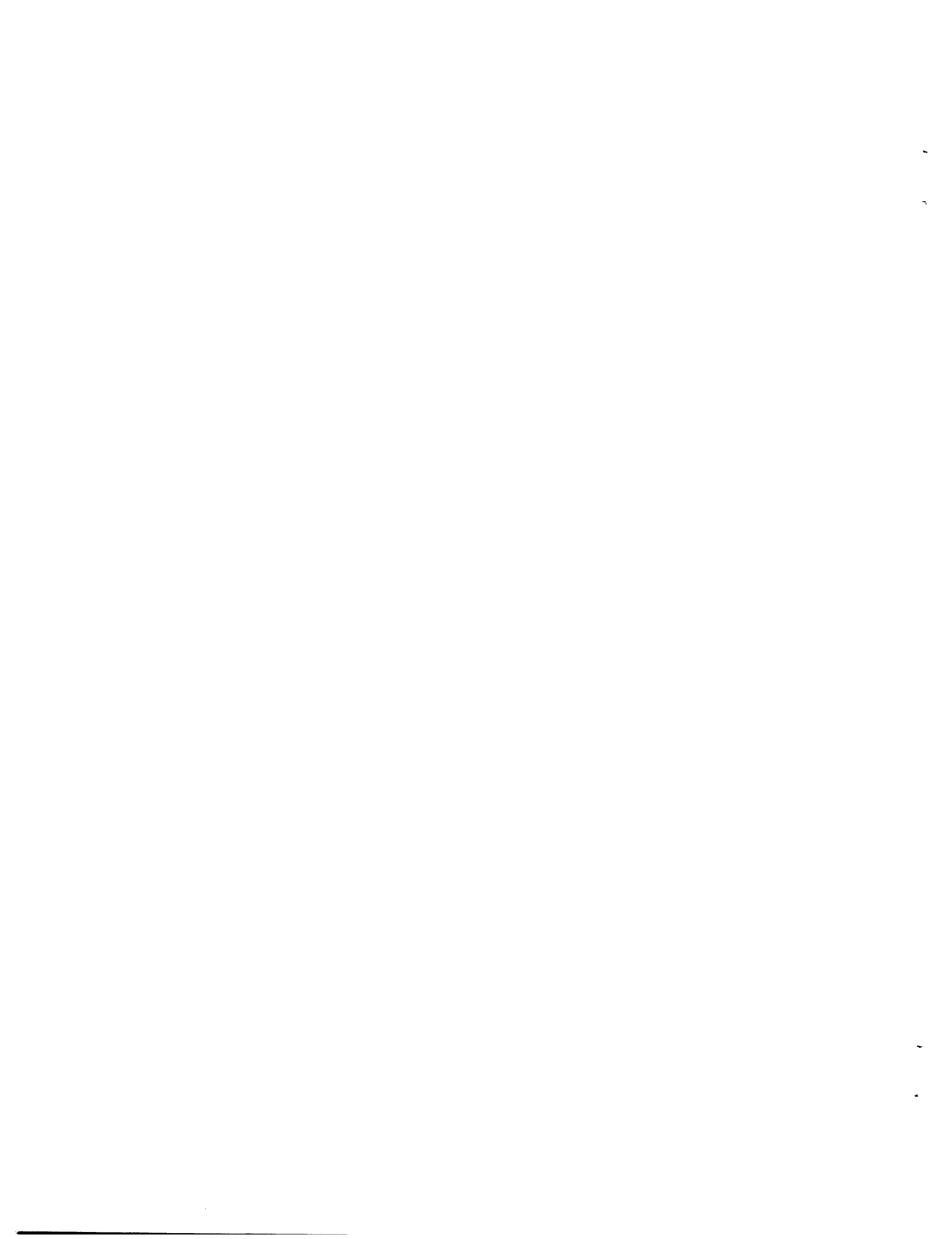
Abstract

With the advent of ferrites at microwave frequencies, the treatment of electromagnetic boundary-value problems involving anisotropic substances has become more than an academic exercise. Since exact methods of analysis often encounter formidable mathematical difficulties, it is necessary to resort to approximate calculations.

Two such methods are developed. The first one is based on a mode-expansion analysis, the second on variational calculations. The former is applied to the determination of resonant frequencies and impedance matrices of cavities and to the determination of propagation constants of waveguides. The variational method is utilized in obtaining approximate expressions for resonant frequencies of cavities and for cutoff frequencies and propagation constants of waveguides.

Several examples with emphasis on microwave components containing ferrites are worked out. The results indicate that it is often possible to obtain approximate, yet sufficiently accurate, solutions of problems of which the exact solutions are extremely difficult.

The interesting problem of the completeness of a set of cavity modes is briefly treated in Appendix I. Several points of view are reviewed and reconciled with some modification. It appears that Slater's treatment of 'empty' cavities is, for all practical purposes, complete.



Introduction

The general problem discussed in this work consists of the determination of the electromagnetic field in bounded regions. When these regions are completely bounded (cavities), special emphasis is given to the resonant frequencies; when they are only partially bounded by a cylindrical surface (waveguides), the emphasis is on the propagation constant.

The exact solution of an electromagnetic problem can be obtained, in principle, by solving Maxwell's equations, subject to the appropriate boundary conditions. Though simple in principle, this method of approach is in practice limited to special configurations where an explicit solution may be found. In all other situations, it is necessary to resort to techniques of approximation in order to avoid insuperable mathematical complications. It is these techniques of approximation with which the present work is concerned – in particular, with mode-expansion analysis and with the application of variational principles.

This investigation was motivated by a desire to treat problems associated with cavities and transmission lines containing ferrites. The basic theory developed is more general, however, and is applicable to other classes of problems, such as those involving magneto-ionic gases.



I. CAVITIES WITH INHOMOGENEOUS AND ANISOTROPIC MEDIA

We define an inhomogeneous and anisotropic medium as one whose permittivity and permeability is a tensor function of position. In this section, we deal with natural or forced oscillations in electromagnetic cavities containing inhomogeneous and anisotropic media, assuming that the reader is familiar with Slater's treatment of cavities as given in reference 1. (A brief account of this method may also be found in Appendix I.) We extend Slater's method to include the effect of magnetic 'currents' and 'charges.' This is followed by an integral-equation treatment of the same general problem with essentially the same results, and by the application of the general principles to a specific configuration.

A. AN EXTENSION OF SLATER'S METHOD

Consider a bounded region V containing distributions of electric current density J_e and magnetic current density J_m . J_e may be any form of electronic current, or it may be a polarization current accounting for the presence of a dielectric; J_m is always a magnetic polarization current density. We assume that these current densities depend linearly on the field vectors E and H in the following manner:

$$J_e = j\omega\epsilon_0\vec{\chi}_e \cdot E \quad (1)$$

$$J_m = j\omega\mu_0\vec{\chi}_m \cdot H \quad (2)$$

where ω is the angular frequency, ϵ_0 and μ_0 are the permittivity and permeability of free space and $\vec{\chi}_e$ and $\vec{\chi}_m$ are the electric and magnetic susceptibilities. These susceptibilities are assumed to be dyadics (tensors of rank two) and functions of position. The dyadic form accounts for the anisotropic nature of the medium. Incidentally, we shall often refer to inhomogeneous and anisotropic media simply as 'tensor media.'

Let the bounding surface consist of two parts, S and S' , over which arbitrary tangential components of the electric and magnetic fields, respectively, are assumed. That under such conditions there is a unique solution for the electromagnetic field is a well-known theorem; see reference 2, for example. Following Slater's method we expand Maxwell's equations

$$\text{curl } H - j\omega\epsilon_0 E = J_e \quad (3)$$

$$\text{curl } E + j\omega\mu_0 H = -J_m \quad (4)$$

in terms of sets of normal modes. For a solenoidal set of the electric type we use Slater's E_a -modes. Similarly, for a solenoidal set of the magnetic type, we utilize his H_a -modes. These are defined as

$$\nabla^2 E_a + k_a^2 E_a = 0, \text{ div } E_a = 0, \mathbf{n} \times E_a = 0 \text{ on } S, \mathbf{n} \cdot E_a = 0 \text{ on } S' \quad (5a)$$

$$\nabla^2 H_a + k_a^2 H_a = 0, \text{ div } H_a = 0, \mathbf{n} \times H_a = 0 \text{ on } S', \mathbf{n} \cdot H_a = 0 \text{ on } S \quad (5b)$$

where \mathbf{n} is the unit vector in the direction of the outward normal to the boundary and k_a is an eigenvalue.

For the irrotational part of the electric field we introduce a set F_b , which differs from Slater's. While they both satisfy the same differential equations, namely,

$$\nabla^2 F_b + k_b^2 F_b = 0, \quad \text{curl } F_b = 0 \quad (6)$$

they have different boundary conditions. Our set is subjected to

$$\mathbf{n} \times F_b = 0 \text{ on } S, \quad \mathbf{n} \cdot F_b = 0 \text{ on } S' \quad (7)$$

whereas Slater's satisfies the condition of vanishing tangential component of F_b over both S and S' . The F_b set as defined here is complete and meets the criticism regarding completeness expressed in reference 3. More will be said about the matter of completeness in Appendix I.

Finally, we introduce an irrotational set of modes of the magnetic type defined as

$$\begin{aligned} \nabla^2 G_c + k_c^2 G_c &= 0, & \text{curl } G_c &= 0 \\ \mathbf{n} \times G_c &= 0 \text{ on } S', & \mathbf{n} \cdot G_c &= 0 \text{ on } S \end{aligned} \quad (8)$$

This is omitted in reference 1. It can be easily shown that the four sets satisfy the orthogonality relations

$$\begin{aligned} \int E_m \cdot E_n dV &= \delta_{mn} & \int E_m \cdot F_n dV &= 0 & \int F_m \cdot F_n dV &= \delta_{mn} \\ \int H_m \cdot H_n dV &= \delta_{mn} & \int H_m \cdot G_n dV &= 0 & \int G_m \cdot G_n dV &= \delta_{mn} \end{aligned} \quad (9)$$

where δ_{mn} is zero for m different from n , but is unity otherwise. These relations also imply that the various modes are normalized.

We now expand each term appearing in Eq. 3 in terms of the E_a and F_b sets, and each term appearing in Eq. 4 in terms of H_a and G_c . For details, see Appendix II. The results are:

$$\begin{aligned} E &= \sum_a \left(\int E \cdot E_a dV \right) E_a + \sum_b \left(\int E \cdot F_b dV \right) F_b \\ H &= \sum_a \left(\int H \cdot H_a dV \right) H_a + \sum_c \left(\int H \cdot G_c dV \right) G_c \\ \text{curl } H &= \sum_a \left(k_a \int H \cdot H_a dV + \int_{S'} \mathbf{n} \times H \cdot E_a dS \right) E_a + \sum_b \left(\int_{S'} \mathbf{n} \times H \cdot F_b dS \right) F_b \\ \text{curl } E &= \sum_a \left(k_a \int E \cdot E_a dV + \int_S \mathbf{n} \times E \cdot H_a dS \right) H_a + \sum_c \left(\int_{S'} \mathbf{n} \times E \cdot G_c dS \right) G_c \\ J_e &= \sum_a \left(\int J_e \cdot E_a dV \right) E_a + \sum_b \left(\int J_e \cdot F_b dV \right) F_b \end{aligned}$$

$$J_m = \sum_a \left(\int J_m \cdot H_a dV \right) H_a + \sum_c \left(\int J_m \cdot G_c dV \right) G_c$$

Substituting these in Eqs. 3 and 4, we obtain the following relations among the expansion coefficients.

$$k_a \int H \cdot H_a dV - j\omega\epsilon_o \int E \cdot E_a dV + \int_{S'} n \times H \cdot E_a dS - \int J_e \cdot E_a dV = 0 \quad (10)$$

$$k_a \int E \cdot E_a dV + j\omega\mu_o \int H \cdot H_a dV + \int_S n \times E \cdot H_a dS + \int J_m \cdot H_a dV = 0 \quad (11)$$

$$j\omega\epsilon_o \int E \cdot F_b dV + \int J_e \cdot F_b dV - \int_{S'} n \times H \cdot F_b dS = 0 \quad (12)$$

$$j\omega\mu_o \int H \cdot G_c dV + \int J_m \cdot G_c dV + \int_S n \times E \cdot G_c dS = 0 \quad (13)$$

Equations 10, 11, and 12 correspond to Eqs. 2.6, 2.7 and 2.8 of reference 1. In the reference there is no analog to Eq. 13.

These four equations constitute the point of departure for any specific problem. The technique of utilizing them is exhaustively treated in reference 1 and will not be repeated here. At the end of this section, however, we briefly work out specific examples because of their current interest.

When the tangential component of the electric field only is specified over the entire boundary of the cavity, we can use a complete set of modes which is simpler than that defined by Eqs. 5(a, b) through Eq. 8; see Appendix I. These modes satisfy the same differential equations as those in Eqs. 5(a, b) through Eq. 8, but the boundary conditions are different. For both the solenoidal and irrotational electric modes we demand vanishing tangential components at the entire boundary. For the magnetic modes we impose the condition of vanishing normal component at the entire surface of the cavity. To avoid introducing a new set of symbols we still call these modes and their eigenvalues E_a , H_a , F_b , G_c , k_a , k_b , but we shall warn the reader whenever the possibility of ambiguity exists. We also use S to denote the entire bounding surface in this case.

With the latter set of modes, Eqs. 10 through 13 reduce to the system

$$k_a \int E \cdot E_a dV + j\omega\mu_o \int H \cdot H_a dV + \int J_m \cdot H_a dV + \int_S n \times E \cdot H_a dS = 0 \quad (14)$$

$$j\omega\epsilon_o \int E \cdot E_a dV - k_a \int H \cdot H_a dV + \int J_e \cdot E_a dV = 0 \quad (15)$$

$$j\omega\mu_o \int H \cdot G_c dV + \int J_m \cdot G_c dV + \int_S n \times E \cdot G_c dS = 0 \quad (16)$$

$$j\omega\epsilon_o \int E \cdot F_b dV + \int J_e \cdot F_b dV = 0 \quad (17)$$

The physical factor governing the choice between Eqs. 10 through 13 and Eqs. 14 through 17 is the following. In practice a cavity is formed by well-conducting walls except for irises, loops,

and the like, introduced for the purpose of exciting the cavity. As we shall see in Appendix I, one of the important characteristics of a cavity is the impedance or admittance it offers to a driving waveguide. If it is the impedance we wish to calculate, we assume a transverse distribution of the magnetic field at the input waveguide, solve for the electric field in the cavity, evaluate the latter at the input, and form the ratio of the tangential components of the electric and magnetic fields at the input. Now, if we used a set of electric modes with vanishing tangential components all over the boundary, the solution for the electric field would be nonuniformly convergent at the bounding surface. This would lead to all sorts of mathematical complexities. However, by introducing the normal modes with mixed boundary conditions, Slater has avoided this difficulty: At surface S' the electric normal modes have a nonzero transverse component, and the series representing the electric field solution presents no special problem.

When we wish to calculate the admittance rather than the impedance, we may use the simpler set of modes satisfying homogeneous boundary conditions; the tangential components of the magnetic modes being nonzero at the boundary, the evaluation of the magnetic field at the boundary presents no special difficulty.

B. THE INTEGRAL-EQUATION TREATMENT

The foregoing results can be obtained by introducing a tensor Green's function and formulating the problem in the form of an integral equation. This general method was used by Schwinger(4). Our treatment, although along the same general lines, differs in two respects: the mathematical formulation and the tensor Green's functions used are different; special emphasis is given here to the presence of tensor media in the region of the electromagnetic field. A brief discussion of the principal results obtained by Schwinger may be found in Appendix I.

Maxwell's equations may be combined in the usual manner to yield

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = j\omega\mu_0 \mathbf{J}_e - \frac{1}{j\omega\epsilon_0} \nabla (\nabla \cdot \mathbf{J}_e) + \text{curl } \mathbf{J}_m \quad (18)$$

and

$$\nabla^2 \mathbf{H} + k^2 \mathbf{H} = j\omega\epsilon_0 \mathbf{J}_m - \frac{1}{j\omega\mu_0} \nabla (\nabla \cdot \mathbf{J}_m) - \text{curl } \mathbf{J}_e \quad (19)$$

where $k^2 = \omega^2 \mu_0 \epsilon_0$. In these two equations, use was made of the continuity relation of electric and magnetic currents. In these inhomogeneous equations the right-hand side may be interpreted as impressed sources producing the fields \mathbf{E} and \mathbf{H} . It is logical to introduce the electric and magnetic tensor Green's functions defined as

$$\nabla^2 \mathbf{G}_E + k^2 \mathbf{G}_E = \delta \mathbf{I} \quad (20)$$

and

$$\nabla^2 \mathbf{G}_H + k^2 \mathbf{G}_H = \delta \mathbf{I} \quad (21)$$

where δ is the Dirac delta-function and I is the idem factor. At the boundary the tangential component of G_e vanishes and so does the tangential component of the curl of G_H .

The solution for the electric field may now be obtained by forming the scalar products of Eq. 18 with G_E and of Eq. 20 with E and subtracting the two. The result is

$$\begin{aligned} E(r) = & \int G_E(r,r') \cdot f(r') dV' + \int_S n \times E(r') \cdot G_E(r,r') dS' \\ & - \int_S n \cdot \left[G_E(r,r') \cdot \text{div } E(r') - E(r') \cdot \text{div } G_E(r,r') \right] dS' \end{aligned} \quad (22)$$

The two surface integrals are over the complete boundary S . Primed coordinates indicate source points; unprimed ones, field points. The symbol $f(r')$ has been used to abbreviate the right-hand side of Eq. 18. H has a similar solution which can be obtained by replacing G_E by G_H , E by H , and f by the right-hand side of Eq. 19. We can now expand the two Green's functions in terms of normal modes. Since we have assumed that the tangential component of G_E vanishes at the boundary, it is logical to expand the electric Green's function in terms of the 'short-circuit' modes E_a and F_b , previously defined. This can be done by substituting a formal expansion in Eq. 20 and evaluating the expansion coefficients from the orthonormality property of the modes. The result is

$$G_E = \sum_a \frac{E_a(r)E_a(r')}{k^2 - k_a^2} + \sum_b \frac{F_b(r)F_b(r')}{k^2 - k_b^2} \quad (23)$$

Similarly, we may expand the magnetic Green's function in terms of the magnetic modes H_a and G_c , having vanishing normal component and vanishing tangential component of their curl, as defined earlier. The resulting expression may be obtained by replacing E_a by H_a and F_b by G_c in Eq. 23.

If we substitute the two expanded Green's functions in Eq. 22 and in its companion for the magnetic field, we obtain for the electric and the magnetic fields

$$E = \sum_a \left(\int E \cdot E_a dV \right) E_a + \sum_b \left(\int E \cdot F_b dV \right) F_b$$

$$H = \sum_a \left(\int H \cdot H_a dV \right) H_a + \sum_c \left(\int H \cdot G_c dV \right) G_c$$

where we have

$$(k^2 - k_a^2) \int E \cdot E_a dV = j\omega\mu_0 \int J_e \cdot E_a dV + k_a \int_S n \times E \cdot H_a dS + k_a \int J_m \cdot H_a dV \quad (24)$$

$$j\omega\epsilon_0 \int E \cdot F_b dV = - \int J_e \cdot F_b dV \quad (25)$$

$$(k^2 - k_a^2) \int H \cdot H_a dV = j\omega\epsilon_0 \int J_m \cdot H_a dV + j\omega\epsilon_0 \int_S n \times E \cdot H_a dS - k_a \int J_e \cdot E dV \quad (26)$$

$$j\omega\mu_0 \int H \cdot G_c dV = - \int J_m \cdot G_c dV - \int_S n \times E \cdot G_c dS \quad (27)$$

These are exactly what we would get if we solved Eqs. 14 through 17 for the various expansion coefficients of the fields. The integral equation approach thus leads essentially to the same results as those in part A.

The reader should, perhaps, be reminded that the preceding treatment is useful only when $n \times E$ is specified over the entire boundary. In the more general case, when $n \times E$ is specified over part of the surface and $n \times H$ over the rest, the procedure is exactly the same except that the electric and magnetic Green's functions satisfy mixed boundary conditions. The tangential component of G_E , for example, is required to vanish over the part of the surface where $n \times E$ is given; over the rest of the surface the condition is that its normal component should vanish. The expansions of the Green's functions are now made in terms of the normal modes satisfying mixed boundary conditions and defined by Eqs. 5 through 8. Finally, the equations corresponding to Eqs. 24 through 27 and obtained by this method are the result of solving Eqs. 10 through 13 for the expansion coefficients of the fields. Equations 24 and 26 are particularly useful in computing small frequency shifts caused by perturbing substances in a cavity.

C. APPLICATION

1. Impedance Matrix of a Cavity of the Transmission Type Containing a Ferrite Sphere. To illustrate some of the general principles in the preceding sections, consider the following example. Let a circular cylindrical cavity be driven by two waveguides, as shown in Fig. 1, so that, essentially, only the two linearly polarized degenerate TE_{111} -modes are excited, each of the latter

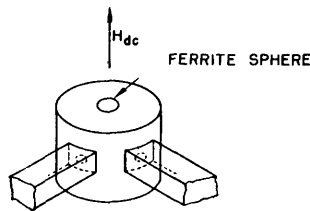


Fig. 1. Two-input cylindrical cavity with a ferrite particle.

being coupled to one and only one of the inputs. Let there be a small sphere of ferrite inside the cavity at the center of one of its bases. With the steady magnetic field as shown, the two degenerate modes of the empty cavity will be coupled and interaction will occur between the two inputs. This can best be evaluated by computing the impedance matrix of the cavity. We shall not go into the details, which can be found in reference 5 and are summarized in Appendix III, but shall briefly outline the method and describe the results. Since it is the impedance in which we are

interested, our working equations are Eqs. 10 through 13. By hypothesis, only two modes of the empty cavity are appreciably excited, the two TE_{111} -modes. Hence, we have two H_a 's, two E_a 's, and no F_b 's nor G_c 's to consider. Assuming a tangential magnetic field distribution at the two inputs and taking into account the small value of the tangential electric field at the metallic boundaries of the cavity by introducing the surface impedance Z_s , we have four unknowns and four equations. The electric field can thus be determined, and from its evaluation at the two inputs, the impedance matrix can be calculated. The result is

$$Z_{12} = v_{\alpha 1} v_{\beta 2} \frac{j\omega_o^2 \omega}{\epsilon_o \Delta} I_{\alpha\beta} \quad (28)$$

$$Z_{21} = -Z_{12}$$

$$Z_{11} = j \frac{\omega v_{\alpha 1}^2}{\epsilon_o \Delta} \left[(\omega_o^2 - \omega^2 p) p - \omega^2 I_{\alpha\beta}^2 \right] \quad (29)$$

$$\Delta = (\omega_o^2 - \omega^2 p)^2 + \omega^4 I_{\alpha\beta}^2$$

$$p = \frac{1-j}{Q_w} + 1 + I_{\alpha\alpha}$$

Z_{22} may be obtained from Z_{11} by interchanging α and β , and by replacing $v_{\alpha 1}$ with $v_{\beta 2}$. The subscripts α and β are used to distinguish between the two degenerate TE_{111} -modes; $v_{\alpha 1}$ is a coupling parameter between the α cavity mode and the first input; $v_{\beta 2}$ between the other cavity mode and the second input.* The angular frequency of excitation is denoted by ω ; the common natural angular frequency of the two modes, by ω_o . Q_w is the 'Q' of each cavity mode without the ferrite, and the remaining symbols are abbreviations for the integrals

$$I_{qr} = \int H_q \cdot \vec{\chi}_m \cdot H_r dV, \quad q, r = \alpha, \beta$$

where the integration is over the volume occupied by the ferrite particle and $\vec{\chi}_m$ is the magnetic susceptibility tensor given (see ref. 7) as

$$\vec{\chi}_m = \begin{bmatrix} \chi & -j\kappa & 0 \\ j\kappa & \chi & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (30)$$

This is, in general, complex (in order to account for losses), and we therefore have

$$\chi = \chi_1 - j\chi_2; \quad \kappa = \kappa_1 - j\kappa_2$$

* In terms of the external Q's, we have $v_{\alpha 1}^2 = \omega_o \epsilon_o / Z_{o1} Q_{\alpha 1}$; $v_{\beta 2}^2 = \omega_o \epsilon_o / Z_{o2} Q_{\beta 2}$, where $Q_{\alpha 1}$, $Q_{\beta 2}$ are external Q's and Z_{o1} , Z_{o2} are characteristic impedances of the two transmission lines.

The remarkable property about the impedance matrix of such a system is its nonreciprocal nature. Furthermore, not only are the transfer impedances unequal, but one is the negative of the other so that the system under consideration is a microwave gyrator. Note, however, that the last statement is strictly true only when our hypotheses as to the number of cavity modes and their coupling to the driving waveguides are correct. In a practical setup, these assumptions are reasonably true.

2. Input Impedance of a Cavity of the Reaction Type Containing a Ferrite Sphere. As another example, consider the calculation of the input impedance of a system that is similar to the one discussed in the preceding example in every respect except that there is only one input. Again, we refer the reader to reference 5 for details and briefly discuss only the results.

The input impedance is given by

$$\frac{Z}{Z_o} = \frac{1}{2Q_{\text{ext}}} \left[\frac{1}{j\left(\frac{\omega}{\omega_-} - \frac{\omega_-}{\omega}\right) + \frac{1}{Q_-}} + \frac{1}{j\left(\frac{\omega}{\omega_+} - \frac{\omega_+}{\omega}\right) + \frac{1}{Q_+}} \right] \quad (31)$$

where the perturbed resonant angular frequencies and Q 's are given by the expressions

$$\omega_{\mp} = \omega_o \left\{ 1 - \frac{1}{2} \left[\frac{1}{Q_w} + \text{tg}(\chi_{1\mp} \kappa_1) \right] \right\} \quad (32)$$

$$\frac{1}{Q_{\mp}} = \frac{1}{Q_w} + \text{tg}(\chi_{2\mp} \kappa_2) \quad (33)$$

t is the volume of the ferrite, g a numerical factor, and Z_o is the characteristic impedance of the input line. Equation 31 has an equivalent circuit as shown in Fig. 2 and represents two

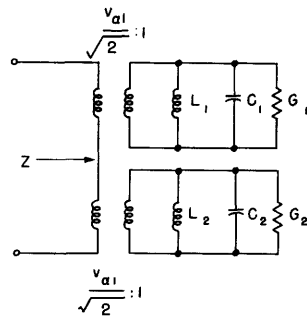


Fig. 2. Equivalent circuit of a two-input cavity with a ferrite particle.

uncoupled antiresonant circuits. Thus, the system under consideration, although it physically involves two linear cavity modes coupled by the action of the ferrite, is expressible in terms of uncoupled perturbed modes. This is analogous to writing the input impedance of two parallel

resonant circuits loosely coupled by a transformer in the form of two uncoupled but perturbed parallel circuits.

3. Perturbation of a Rotating TE₁₁₁-Mode by Means of a Small Ferrite Sphere. Let a ferrite sphere be placed on the axis of a circular cylindrical cavity where the electric field of the TE₁₁₁-mode vanishes. In the absence of the ferrite sample the field vectors of the rotating TE₁₁₁-mode are denoted by E₀ and H₀, and the resonant frequency by ω₀. We now assume that the field with the ferrite sample present can be approximated by E = e₀E₀, H = h₀H₀, where e₀ and h₀ are amplitude coefficients. At the sample the electric field vanishes, so that J_e = 0; but the magnetic field is circularly polarized: H₀ = |H₀|2^{1/2}(a_x ± ja_y), where a_x and a_y are unit vectors and the plus or minus signs correspond to the two senses of rotation. Thus

$$J_m \cdot H_0^* = (j\omega\mu_0 \vec{\chi}_m \cdot H) \cdot H_0^* = j\omega\mu_0 (\chi \pm \kappa) h_0 |H_0|^2 \quad (34)$$

Substituting in Eq. 26, we get

$$\left[(k^2 - k_0^2) + k^2 (\chi \pm \kappa) |H_0|^2 v \right] h_0 = 0 \quad (35)$$

where v is the volume of the sample. A non-trivial solution will exist if the expression within the bracket vanishes; hence for small perturbations

$$\frac{k - k_0}{k} = \frac{\omega - \omega_0}{\omega} = -\frac{1}{2} (\chi \pm \kappa) |H_0|^2 v \quad (36)$$

Writing $\omega = \omega_1 + j\omega_2$ and separating Eq. 36 into its real and imaginary parts, we find

$$\frac{\omega_1 - \omega_0}{\omega_1} = \frac{\Delta\omega}{\omega} = -\frac{1}{2} (\chi_1 \pm \kappa_1) |H_0|^2 v \quad (37)$$

$$2 \frac{\omega_2}{\omega_1} = \Delta \frac{1}{Q} = (\chi_2 \pm \kappa_2) |H_0|^2 v \quad (38)$$

The last two formulas were derived in reference 6 in a somewhat different fashion. They form a basis for the measurement of the susceptibility tensor. Note, however, that the tensor thus determined is not a quantity depending solely on the ferrite material, but an 'effective' susceptibility defined by $M = \vec{\chi}_m \cdot H_{\text{external}}$; consequently, it depends on the shape of the sample* as implied in Eq. 34.

* Added in press: For the determination of the intrinsic susceptibility see J.H. Rowen and W. von Aulock, Phys. Rev. 96, 1151-3 (1954); A.D. Berk and B.A. Lengyel, Proc. IRE 43, 1587-90 (1955).

II. WAVE PROPAGATION ALONG INHOMOGENEOUS AND ANISOTROPIC STRUCTURES WITH CYLINDRICAL SYMMETRY

We define these structures as waveguides with perfectly conducting walls which enclose substances whose permittivity and permeability are tensor functions of the cross-sectional coordinates. As examples, we cite a rectangular waveguide with a dielectric slab and a circular cylindrical waveguide with a coaxial rod of ferrite. Our general approach to the problem is similar to the one used for cavities: we expand the various quantities appearing in Maxwell's equations in terms of certain orthonormal modes which differ somewhat from the conventional TE-, TM-modes and determine the relations that must exist between the various expansion coefficients. We then work out various examples to clarify and illustrate this method of analysis which we call, for the purpose of easy reference, the mode-expansion method.* The third section is devoted to the derivation of some useful perturbation formulas. We conclude by giving a brief account of an integral-equation treatment that yields, essentially, the same results as the mode-expansion analysis.

A. THE MODE-EXPANSION METHOD

Because of the cylindrical symmetry we have assumed, we can write the following expressions for the electric and magnetic fields E and H

$$E = E(x,y)e^{-j\gamma z} \quad H = H(x,y)e^{-j\gamma z} \quad (39)$$

The time dependence is dropped and is understood to be $\exp(j\omega t)$. E and H are three-dimensional vectors, independent of the direction of propagation which is taken along the z -axis. We have similarly for the electric and magnetic current densities

$$J_e = J_e(x,y)e^{-j\gamma z} \quad J_m = J_m(x,y)e^{-j\gamma z} \quad (40)$$

respectively. Note the following difference in notation in Sections I and II: while E , H , J_e and J_m were the entire field vectors and current densities in Section I, the same symbols have the slightly different meaning expressed in Eqs. 39 and 40.

Substituting Eqs. 39 and 40 in Maxwell's equations

$$\text{curl } E + j\omega\mu_0 H = J_m \quad (41a)$$

$$\text{curl } H - j\omega\epsilon_0 E = J_e \quad (41b)$$

* The author, after completing the development of the mode-expansion method in the summer of 1953, became aware of its similarity to the approach used by Schelkunoff in reference 8. In reference 8, however, the emphasis is on formulating the problem as a set of generalized telegraphist's equations.

we obtain

$$\text{curl } \mathbf{E} - j\gamma \mathbf{a}_z \times \mathbf{E} + j\omega \mu_0 \mathbf{H} = \mathbf{J}_m \quad (42a)$$

$$\text{curl } \mathbf{H} - j\gamma \mathbf{a}_z \times \mathbf{H} - j\omega \epsilon_0 \mathbf{E} = \mathbf{J}_e \quad (42b)$$

where \mathbf{a}_z denotes the unit vector in the z-direction. The propagation constant is, for a fixed frequency, an eigenvalue.

Our next step is to expand all quantities appearing in Eqs. 42(a, b) in terms of a complete and preferably orthogonal set of modes. Here, however, we have at least a choice of two. We may choose the usual TE-, TM-set of the empty waveguide (that is, with \mathbf{J}_e and \mathbf{J}_m equal to zero), completed with a set of irrotational modes. The advantage of this choice would be that each TE- or TM-mode has a physical significance as it stands. The disadvantage is that the mathematical expressions for these modes contain more than one term and are therefore cumbersome to utilize, especially when they occur in cross products. For example, the magnetic field of the TE-modes is given by

$$\omega \mu_0 \mathbf{H}_n = (\omega^2 \mu_0 \epsilon_0 - \alpha_n^2)^{1/2} \text{grad } \psi_n + j\alpha_n^2 \mathbf{a}_z \psi_n$$

where ψ_n , α_n satisfy $\nabla^2 \psi_n + \alpha_n^2 \psi_n = 0$.

In working out actual cases (with ferrites, for example) we shall be confronted with expressions of the form $\int \mathbf{H}_n \cdot \vec{\chi}_m \cdot \mathbf{H}_m^* dS$, which become rather involved if we use such a set of modes.

We shall, therefore, choose the following set. (Its derivation is outlined in Appendix IV.) For modes of the electric type, that is, modes with vanishing tangential component at the walls of the waveguide, we have

$$\mathbf{E}_n^a = \frac{-1}{\alpha_n} \mathbf{a}_z \times \text{grad } \psi_n \quad (43a)$$

$$\mathbf{E}_n^b = \mathbf{a}_z \phi_n \quad (43b)$$

$$\mathbf{E}_n^c = \frac{1}{\beta_n} \text{grad } \phi_n \quad (43c)$$

where ψ_n and ϕ_n are scalar eigenfunctions corresponding to the eigenvalues α_n and β_n in this manner:

$$\nabla^2 \psi_n + \alpha_n^2 \psi_n = 0 ; \quad \frac{\partial \psi_n}{\partial n} = 0 \text{ on the boundary} \quad (44a)$$

$$\nabla^2 \phi_n + \beta_n^2 \phi_n = 0 ; \quad \phi_n = 0 \text{ on the boundary} \quad (44b)$$

For modes of the magnetic type we have

$$\mathbf{H}_n^a = \mathbf{a}_z \psi_n \quad (45a)$$

$$H_n^b = \frac{-1}{\beta_n} a_z \times \text{grad } \phi_n \quad (45b)$$

$$H_n^c = \frac{1}{\alpha_n} \text{grad } \psi_n \quad (45c)$$

The following observations can be made. First, we have in each case three kinds of modes, corresponding to the superscripts a, b, and c, each in a different direction; this is as it should be if we are to expand an arbitrary vector field in terms of these modes. Second, each kind can be expected to form a complete set, since the scalar functions defined in Eqs. 44(a, b) form a complete set. Third, the usual TE-, TM-modes are linear combinations of these modes. Fourth, ϕ_n and ψ_n have physical meaning in that they are equal to the axial component of the electric and magnetic fields, respectively. Last, a mode taken individually does not necessarily constitute a possible field configuration even though it has a physical meaning. H_n^a alone, for example, does not represent a physical field; yet it does constitute the z-component of the magnetic field of TE-modes. The modes defined by Eqs. 43 and 45 are orthonormal, so that

$$\int E_n^a \cdot E_m^a dS = \delta_{nm}; \quad \int E_n^a \cdot E_m^b dS = 0$$

and so on. (The integration is over the cross-sectional area of the waveguide.) They also satisfy the following relations as we can easily verify.

$$\begin{aligned} \text{curl } E_n^a &= \alpha_n H_n^a & \text{curl } E_n^b &= \beta_n H_n^b \\ \text{curl } H_n^a &= \alpha_n E_n^a & \text{curl } H_n^b &= \beta_n E_n^b \\ E_n^a &= -a_z \times H_n^c & H_n^b &= -a_z \times E_n^c \\ \beta_n E_n^c &= \text{grad}(a_z \cdot E_n^b) & \alpha_n H_n^c &= \text{grad}(a_z \cdot H_n^a) \end{aligned} \quad (46)$$

We are now ready to expand the various quantities appearing in Eqs. 42(a, b) in terms of the appropriate modes. For E and H we have

$$E = \sum_n (e_n^a E_n^a + e_n^b E_n^b + e_n^c E_n^c) \quad (47a)$$

and

$$H = \sum_n (h_n^a H_n^a + h_n^b H_n^b + h_n^c H_n^c) \quad (47b)$$

The various expansion coefficients can be written, as a result of the orthonormality of the modes, as

$$e_n^a = \int E \cdot E_n^a dS, \quad e_n^b = \int E \cdot E_n^b dS, \quad h_n^a = \int H \cdot H_n^a dS$$

and so on, where the integration is over the cross section of the guide. The other expansions are

$$\begin{aligned}
\text{curl } \mathbf{E} &= \sum_{\mathbf{n}} (\epsilon_{\mathbf{n}}^a \alpha_{\mathbf{n}} \mathbf{H}_{\mathbf{n}}^a + \epsilon_{\mathbf{n}}^b \beta_{\mathbf{n}} \mathbf{H}_{\mathbf{n}}^b) \\
\text{curl } \mathbf{H} &= \sum_{\mathbf{n}} (h_{\mathbf{n}}^a \alpha_{\mathbf{n}} \mathbf{E}_{\mathbf{n}}^a + h_{\mathbf{n}}^b \beta_{\mathbf{n}} \mathbf{E}_{\mathbf{n}}^b) \\
\mathbf{a}_z \times \mathbf{E} &= \sum_{\mathbf{n}} (\epsilon_{\mathbf{n}}^a \mathbf{H}_{\mathbf{n}}^c - \epsilon_{\mathbf{n}}^c \mathbf{H}_{\mathbf{n}}^b) \\
\mathbf{a}_z \times \mathbf{H} &= \sum_{\mathbf{n}} (h_{\mathbf{n}}^b \mathbf{E}_{\mathbf{n}}^c - h_{\mathbf{n}}^c \mathbf{E}_{\mathbf{n}}^a)
\end{aligned} \tag{48}$$

Finally, we have for the electric current

$$\mathbf{J}_e = \sum_{\mathbf{n}} \left[\left(\int \mathbf{J}_e \cdot \mathbf{E}_{\mathbf{n}}^a dS \right) \mathbf{E}_{\mathbf{n}}^a + \left(\int \mathbf{J}_e \cdot \mathbf{E}_{\mathbf{n}}^b dS \right) \mathbf{E}_{\mathbf{n}}^b + \left(\int \mathbf{J}_e \cdot \mathbf{E}_{\mathbf{n}}^c dS \right) \mathbf{E}_{\mathbf{n}}^c \right] \tag{49}$$

and for the magnetic current

$$\mathbf{J}_m = \sum_{\mathbf{n}} \left[\left(\int \mathbf{J}_m \cdot \mathbf{H}_{\mathbf{n}}^a dS \right) \mathbf{H}_{\mathbf{n}}^a + \left(\int \mathbf{J}_m \cdot \mathbf{H}_{\mathbf{n}}^b dS \right) \mathbf{H}_{\mathbf{n}}^b + \left(\int \mathbf{J}_m \cdot \mathbf{H}_{\mathbf{n}}^c dS \right) \mathbf{H}_{\mathbf{n}}^c \right] \tag{50}$$

It has been tacitly assumed that the modes defined by Eqs. 43 and 45 are real. This is generally true except in cases such as the circular waveguide where complex rotating modes offer an advantage. Whenever this is true, all orthogonality relations will be taken in the hermitian sense; for example, $\int \mathbf{E}_{\mathbf{n}}^a \cdot (\mathbf{E}_{\mathbf{m}}^a)^* dS = \delta_{\mathbf{mn}}$.

Substituting the preceding expansions in Eqs. 42(a, b) and equating coefficients, we obtain the basic set

$$\alpha_{\mathbf{n}} \epsilon_{\mathbf{n}}^a + j\omega \mu_0 h_{\mathbf{n}}^a = \int \mathbf{J}_m \cdot \mathbf{H}_{\mathbf{n}}^a dS \tag{51a}$$

$$-j\gamma \epsilon_{\mathbf{n}}^a + j\omega \mu_0 h_{\mathbf{n}}^c = \int \mathbf{J}_m \cdot \mathbf{H}_{\mathbf{n}}^c dS \tag{51b}$$

$$-j\omega \epsilon_0 \epsilon_{\mathbf{n}}^a + \alpha_{\mathbf{n}} h_{\mathbf{n}}^a + j\gamma h_{\mathbf{n}}^c = \int \mathbf{J}_e \cdot \mathbf{E}_{\mathbf{n}}^a dS \tag{51c}$$

and

$$\beta_{\mathbf{n}} \epsilon_{\mathbf{n}}^b + j\gamma \epsilon_{\mathbf{n}}^c + j\omega \mu_0 h_{\mathbf{n}}^b = \int \mathbf{J}_m \cdot \mathbf{H}_{\mathbf{n}}^b dS \tag{52a}$$

$$-j\omega \epsilon_0 \epsilon_{\mathbf{n}}^b + \beta_{\mathbf{n}} h_{\mathbf{n}}^b = \int \mathbf{J}_e \cdot \mathbf{E}_{\mathbf{n}}^b dS \tag{52b}$$

$$-j\omega \epsilon_0 \epsilon_{\mathbf{n}}^c - j\gamma h_{\mathbf{n}}^b = \int \mathbf{J}_e \cdot \mathbf{E}_{\mathbf{n}}^c dS \tag{52c}$$

The grouping of the preceding equations is deliberate. If we set the electric and magnetic currents equal to zero, Eqs. 51 and 52 reduce to two independent sets of equations. Physically, this should correspond to the case of an empty waveguide and the usual TE-, TM-waves. That this is indeed the case may be easily verified. The group in Eq. 51, for example, becomes

$$\begin{aligned} a_n e_n^a + j\omega\mu_0 h_n^a &= 0 \\ -j\gamma e_n^a + j\omega\mu_0 h_n^c &= 0 \\ -j\omega\epsilon_0 e_n^a + a_n h_n^a + j\gamma h_n^c &= 0 \end{aligned}$$

A nonvanishing solution will exist only for such values of γ which render the determinant of this system zero. These are

$$\gamma_n^2 = \omega^2 \mu_0 \epsilon_0 - a_n^2 \quad (53)$$

as we should expect. Substituting this expression back and arbitrarily setting e_n^a equal to unity, we obtain

$$\begin{aligned} E &= -\frac{1}{a_n} a_z \times \text{grad } \psi_n \\ H &= -a_z \frac{a_n}{j\omega\mu_0} \psi_n + \frac{\gamma_n}{\omega\mu_0} \frac{1}{a_n} \text{grad } \psi_n \end{aligned} \quad (54)$$

These, when multiplied by the factor $\exp(-j\gamma_n z)$, will be recognized as the field vectors of the TE-set of modes in an empty waveguide. Similarly, from the expressions in Eq. 52 we obtain

$$\gamma_n^2 = \omega^2 \mu_0 \epsilon_0 - \beta_n^2 \quad (55)$$

and

$$\begin{aligned} H &= -\frac{1}{\beta_n} a_z \times \text{grad } \phi_n \\ E &= a_z \frac{\beta_n}{j\omega\epsilon_0} \phi_n - \frac{\gamma_n}{\omega\epsilon_0} \frac{1}{\beta_n} \text{grad } \phi_n \end{aligned} \quad (56)$$

which correspond to the usual TM-modes.

It has been tacitly assumed from the beginning of this section that the cross section of the empty waveguide is singly connected, such as those of rectangular or circular cylindrical waveguides. All that has been said until now is perfectly valid for doubly connected cross sections, like the cross sections of the ordinary coaxial waveguide, provided we add to Eqs. 43 and 45 these two modes:

$$\mathbf{E}_0^c = \text{grad } \phi \tag{57}$$

$$\mathbf{H}_0^b = -\mathbf{a}_z \times \text{grad } \phi$$

where ϕ satisfies Laplace's equation in the cross section and assumes constant values at the bounding surfaces. Physically, these expressions, when multiplied by $\exp(-jk_0 z)$, represent the usual TEM-wave. Mathematically, their origin is in the fact that when the cross section is doubly or multiply connected, the equation

$$\nabla^2 \phi_n + \beta_n^2 \phi_n = 0$$

admits a solution for $\beta_n = 0$, if ϕ_n takes different constant values on the two boundaries of the cross section. There is, in other words, a solution corresponding to a zero eigenvalue. It can be easily shown that the last two modes are orthogonal to each member of the sets given in Eq. 43 and Eq. 45, respectively.

Before we discuss the application of Eqs. 51 and 52, a few remarks are in order. The integrals on the right-hand side represent the coupling between the various modes. J_e is either an actual current density or a polarization current density, while J_m is always a polarization current density. In most of the practical cases J_e and J_m are simply related to the electric and magnetic fields so that Eqs. 51 and 52 become essentially a homogeneous set. The values of γ that, for a given ω , allow a solution are the propagation constants of the composite structure, that is, the empty waveguide plus the electric and magnetic currents. The formal and exact solution will, in general, require the evaluation of the expansion coefficients of all the modes, a whole infinity of them! Thus, the evaluation of the propagation constants will involve infinite determinants for which the engineer and the physicist have a natural dislike. Although there are instances where a great many modes are indeed necessary if the expansion is to bear any similarity to the actual field, quite frequently we encounter practical cases which may fall into one of two categories: we may find that the actual field can be reasonably well approximated by a small number of modes, in which case we have only a few unknowns with a corresponding number of equations; or, we might expect, on physical grounds, the actual field to be essentially that of an empty-waveguide mode plus a first-order correction term which we can evaluate by an approximate treatment of the infinite determinant. (See, for example, ref. 9.) All the standard techniques of the well-known perturbation calculations in quantum mechanics can, as a matter of fact, be used in connection with Eqs. 51 and 52. We shall now illustrate the preceding method of analysis and perhaps clarify it by working out several examples.

B. APPLICATION

1. **Rectangular Waveguide with a Dielectric Slab at the Center.** Consider a rectangular waveguide of width a , as shown in Fig. 3, partly and symmetrically filled with a dielectric of susceptibility χ_e . Suppose we wish to find the propagation constant of the fundamental mode. An exact solution involving the solution of a transcendental equation is possible in this case (10). Let

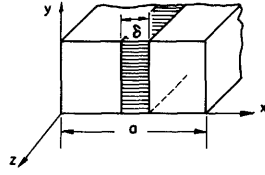


Fig. 3. Rectangular waveguide with a symmetrically placed dielectric slab.

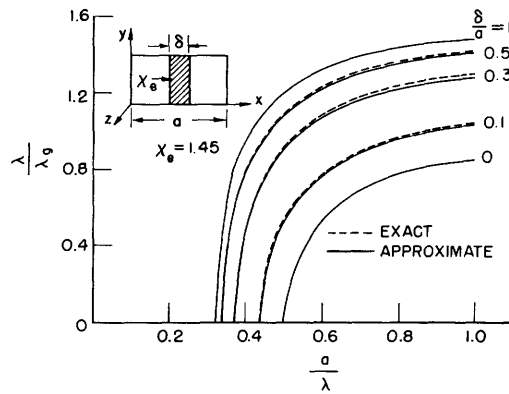


Fig. 4. Variation of propagation constant versus frequency for slabs of various widths in a rectangular waveguide.

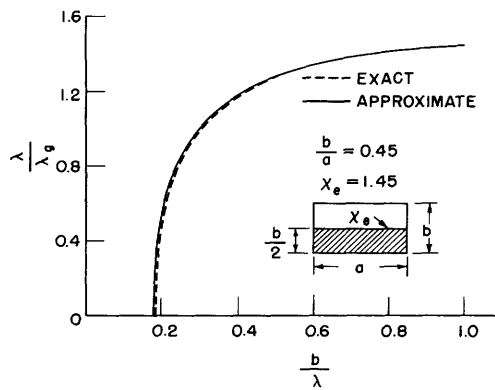


Fig. 5. Waveguide with a horizontal dielectric slab.

us see, however, if we can obtain an approximate but simple solution by applying the general principles of the previous section.

With the slab absent we know that the fundamental mode is the TE_{10} -mode, which has an electric field in the y -direction varying as $\sin \pi x/a$ with x . If the dielectric constant of the slab is not very large, we should expect the field to be a somewhat perturbed version of the TE_{10} -mode of the empty waveguide. Now present in this latter mode are E_{10}^a , H_{10}^a , H_{10}^c , and these only, as we can easily verify. Hence, we write for the E- and H-fields

$$E = e_{10}^a E_{10}^a$$

$$H = h_{10}^a H_{10}^a + h_{10}^c H_{10}^c$$

These are approximate versions of the general expansion, Eq. 47. Correspondingly, we have this simple system of equations to solve

$$\begin{aligned} \alpha_{10} e_{10}^a + j\omega\mu_0 h_{10}^a &= 0 \\ -j\gamma e_{10}^a + j\omega\mu_0 h_{10}^c &= 0 \\ -j\omega\epsilon_0 e_{10}^a + \alpha_{10} h_{10}^a + j\gamma h_{10}^c &= \int J_e \cdot E_{10}^a dS \end{aligned} \tag{58}$$

Now, for a rectangular waveguide, we have

$$\alpha_{10} = \frac{\pi}{a}$$

$$E_{10}^a = a_y \left(\frac{2}{a}\right)^{1/2} \sin \frac{\pi x}{a}$$

$$H_{10}^a = a_z \left(\frac{2}{a}\right)^{1/2} \cos \frac{\pi x}{a}$$

$$H_{10}^c = a_x \left(\frac{2}{a}\right)^{1/2} \sin \frac{\pi x}{a}$$

We assume that the dielectric has the permeability of free space. Consequently, J_m in Eq. 58 has been set equal to zero. J_e is, of course, a polarization current density and is given by

$$J_e = j\omega\epsilon_0 \chi_e e_{10}^a E_{10}^a$$

The system of Eq. 58 will have a solution for the following values of γ

$$\gamma^2 = k_0^2 - \left(\frac{\pi}{a}\right)^2 + k_0^2 \chi_e \left(\frac{\delta}{a} + \frac{1}{\pi} \sin \frac{\pi\delta}{a}\right)$$

If we introduce the free-space and guide wavelengths λ and λ_g , respectively, the preceding equation becomes

$$\left(\frac{\lambda}{\lambda_g}\right)^2 = 1 + \chi_e \left(\frac{\delta}{a} + \frac{1}{\pi} \sin \frac{\pi\delta}{a}\right) - \frac{\lambda^2}{4a^2} \quad (59)$$

Curves of λ/λ_g versus a/λ , shown in solid lines, are compared in Fig. 4 with curves, shown in broken lines, obtained from the exact calculations given in reference 10 (p. 386). The agreement is seen to be good except at high frequencies. This is not a serious restriction, however, since the frequency of operation is seldom raised above the cut-off of the next higher mode. For the two cases $\delta = 0$ and $\delta = a$, the agreement with the exact solution is perfect.

2. Rectangular Waveguide with a Dielectric Slab on a Side. Let us consider solving for the propagation constant of the fundamental mode of a rectangular waveguide with a nonsymmetrically placed dielectric slab such as the ferrite slab in Fig. 9. We shall, incidentally, need the result of this problem in treating example 6 of Section III, part D. We confine our attention to the case in which the slab is located between the middle of the guide and one of the side walls but not quite in the neighborhood of either of these limits. Consideration of the electric field configuration to be expected leads to the assumption

$$E = e_{10}^a E_{10}^a + e_{20}^a E_{20}^a$$

E_{20}^a is the transverse component of the electric field in the TE_{20} -mode of the empty waveguide and varies as $\sin 2\pi x/a$. Taking into account the appropriate magnetic modes, H_{10}^a , H_{20}^a , H_{10}^c , H_{20}^c , and setting the determinant of the six simultaneous equations equal to zero, we obtain

$$2\left(\frac{\lambda}{\lambda_g}\right)^2 = m + (m^2 - n)^{1/2} \quad (60)$$

where m and n are given by

$$m = 2 + \chi_e (A + B) - \frac{5}{4} \frac{\lambda^2}{a^2}$$

and

$$n = 4 \left[1 + \chi_e A - \frac{\lambda^2}{4a^2} \right] \left[1 + \chi_e B - \frac{\lambda^2}{a^2} \right] - 4\chi_e^2 C^2$$

with A , B , and C defined as

$$A = \int_d^{d+\delta} (E_{10}^a)^2 dx$$

$$B = \int_d^{d+\delta} (E_{20}^a)^2 dx$$

$$C = \int_d^{d+\delta} E_{20}^a \cdot E_{10}^a dx$$

3. Rectangular Waveguide with a Dielectric Slab Normal to the Electric Lines of Force.

Next we obtain an approximate expression for the propagation constant of the fundamental mode of a waveguide whose cross section is shown in Fig. 5. This is a good example to show how a purely academic-looking method can be utilized in a practical case by proper use of the known physical aspect of a problem. A detailed amount of the reasoning is difficult to give, since there is a good deal of guessing involved. However, broadly speaking, the argument is as follows. The normal component of the electric field at the interface between air and dielectric must be discontinuous. Hence, the component of the electric field in the dielectric part of the waveguide must be less than that in the part filled with air. Such a configuration can be produced by assuming the transverse part of the E-field to be a linear combination of E_{10}^a and E_{11}^c . In this case, these modes are

$$E_{10}^a = a_y \left(\frac{2}{a}\right)^{1/2} \sin \frac{\pi x}{a}$$

$$E_{11}^c = \frac{1}{\pi[(1/a^2) + (1/b^2)]^{1/2}} \cdot \frac{2}{(ab)^{1/2}} \left(a_x \frac{\pi}{a} \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} + a_y \frac{\pi}{b} \sin \frac{\pi x}{a} \cos \frac{\pi y}{b} \right)$$

so that the y-component of the E_{11}^c -mode adds to the E_{10}^a -mode in half of the cross section and subtracts in the other. Further, it is generally known that the solution we are seeking is neither TE nor TM. The z-components of the E- and H-fields can most simply be accounted for by assuming them proportional to E_{11}^b and H_{10}^a , respectively. We thus have for the field expansions

$$E = e_{10}^a E_{10}^a + e_{11}^b E_{11}^b + e_{11}^c E_{11}^c \quad (61)$$

$$H = h_{10}^a H_{10}^a + h_{10}^c H_{10}^c + h_{11}^b H_{11}^b$$

Consequently we have the following set of six simultaneous equations:

$$a_{10} e_{10}^a + j\omega\mu_0 h_{10}^a = 0$$

$$-j\gamma e_{10}^a + j\omega\mu_0 h_{10}^c = 0$$

$$-j\omega\epsilon_0 e_{10}^a + a_{10} h_{10}^a + j\gamma h_{10}^c = \int J_e \cdot E_{10}^a dS \quad (62)$$

$$\beta_{11} e_{11}^b + j\gamma e_{11}^c + j\omega\mu_0 h_{11}^b = 0$$

$$-j\omega\epsilon_0 e_{11}^b + \beta_{11} h_{11}^b = \int J_e \cdot E_{11}^b dS$$

$$-j\omega\epsilon_0 e_{11}^c - j\gamma h_{11}^b = \int J_e \cdot E_{11}^c dS$$

Again J_e equals $j\omega\epsilon_0 \chi_e E$, where E is given in Eq. 61. Following the usual procedure, we find that the values of the propagation constant are the roots of

$$\gamma^4 + (a_{10}^2 + \beta_{11}^2 - 2k_0^2\xi)\gamma^2 + (a_{10}^2 + k_0^2 \frac{\eta^2 - \xi^2}{\xi}) (\beta_{11}^2 - k_0^2\xi) = 0 \quad (63)$$

where a_{10} has already been defined

$$\beta_{11}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$

$$\xi = 1 + \frac{\chi_e}{2}$$

and

$$\eta = \frac{(2)^{1/2}a}{b^2\beta_{11}} \chi_e$$

We choose the pair of solutions with the smaller absolute value. The other pair corresponds to a higher perturbed mode and is usually in large error. Figure 5 shows a plot of λ/λ_g , the ratio of free-space wavelength to guide wavelength, versus b/λ , compared with the exact solution. (This solution, which involves considerable labor, is given in ref. 10, p. 389.) For high frequencies, the approximate solution breaks down. However, this is of little practical importance, since the frequency of operation is usually kept below any higher mode.

It would be interesting to examine the behavior of the propagation constant, as obtained from Eq. 63, as a function of the thickness of the dielectric layer when the frequency is kept constant. We know that for thicknesses of the dielectric that are widely different from $b/2$, our reasoning and assumptions about the field configuration are in grave error. Let us do it nevertheless. The result is shown in Fig. 6 where the ratio of free-space wavelength to guide wavelength is plotted as a function of the ratio of dielectric thickness to waveguide height. It is compared to the exact solution as given in reference 10 (p. 389.)

4. Circular Waveguide with Coaxial Dielectric Core. We next treat the circular analog of the problem discussed in example 3. Our problem is to find the propagation constant of the fundamental mode of a circular waveguide of diameter 'a' with a concentric dielectric rod of diameter d. Following a line of reasoning similar to the one used in the previous example, we write

$$E = e_{11}^a E_{11}^a + e_{11}^b E_{11}^b + e_{11}^c E_{11}^c$$

$$H = h_{11}^a H_{11}^a + h_{11}^b H_{11}^b + h_{11}^c H_{11}^c$$

In terms of the usual TE-, TM-modes of the empty waveguide, this means that we are assuming the solution to consist essentially of the elements present in the TE_{11} - and TM_{11} -modes. After going through the usual procedure of setting the determinant of the six equations equal to zero, we obtain the relation

$$2Y\left(\frac{\lambda}{\lambda_g}\right)^2 = P + (P^2 - Q)^{1/2} \quad (64)$$

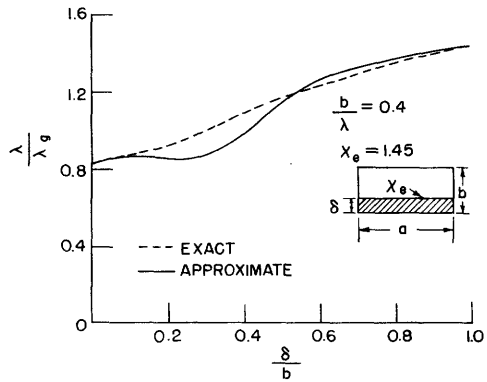


Fig. 6. Propagation constant versus thickness of a horizontal dielectric slab in a rectangular ferrite.

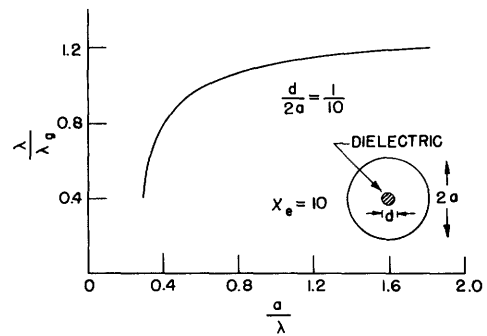


Fig. 7. Propagation constant versus frequency in a circular waveguide with a concentric dielectric rod.

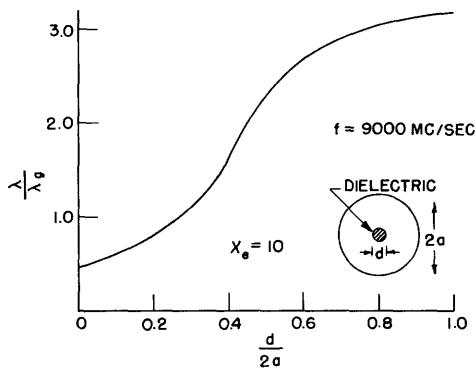


Fig. 8. Propagation constant versus diameter of a dielectric rod in a circular waveguide.

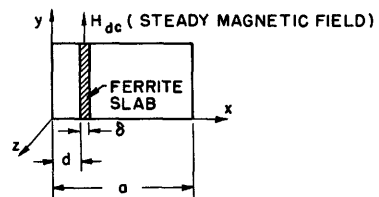


Fig. 9. Rectangular waveguide with a ferrite slab.

where we have

$$Y = 1 + \chi_e \int E_{11}^b \cdot \dot{E}_{11}^b dS$$

$$4\pi^2 P = -\lambda^2 (\alpha_{11}^2 - k_o^2 X) Y - \lambda^2 (\beta_{11}^2 - k_o^2 Y) Z$$

$$4\pi^4 Q = \lambda^4 Y \left[(\alpha_{11}^2 - k_o^2 X) Z + k_o^2 V^2 \right] \left[\beta_{11}^2 - k_o^2 Y \right]$$

and also

$$X = 1 + \chi_e \int E_{11}^a \cdot \dot{E}_{11}^a dS$$

$$Z = 1 + \chi_e \int E_{11}^c \cdot \dot{E}_{11}^c dS$$

$$V^2 = \chi_e^2 \left| \int E_{11}^a \cdot \dot{E}_{11}^c dS \right|^2$$

All the integrals are over the cross section of the dielectric rod. Explicit expressions for E_{11}^a , and the like, are given in Appendix V. Two curves have been obtained from Eq. 64. Figure 7 shows the behavior of λ/λ_g as a function of a/λ for the specific instance when $d = 0.1a$ and $\chi_e = 10$. Figure 8 shows a plot of λ/λ_g versus d/a when the frequency is fixed. Both curves should be expected to deviate considerably from the correct solution in those regions where our initial assumption about the field configuration is not valid. An exact solution of this problem, to the best of our knowledge, has not been given.*

5. Circular Waveguide with a Thin Dielectric Coaxial Sliver. When, in the preceding example, the dielectric constant of the rod does not greatly differ from unity and d is very small as compared to a , we can simplify matters considerably by assuming the solution to be a slightly perturbed form of the empty waveguide TE_{11} -mode. We shall give only the result, since the procedure is the same as in the preceding examples. We have

$$\gamma = \gamma_o \left(1 + \chi_e \frac{k_o^2}{\gamma_o^2} \frac{sC}{S} \right) \quad (65)$$

where γ_o is the propagation constant of the unperturbed TE_{11} -mode, s the cross section of the dielectric, S that of the waveguide, and C a numerical factor approximately equal to 1.04. Equation 65 is applicable to the case of a rectangular waveguide with a square cross section if we put $C = 1$.

* Since the writing of this report, the author has been informed of a paper by R.E. Beam and H.M. Wachowski, *Trans. A.I.E.E.* 70, 874-880 (1951), in which an exact solution is given for the case in which the dielectric is polystyrene. Curves obtained from the exact solution (Fig. 8 of the reference) were compared to those calculated from Eq. 64 at Hughes Research Laboratories, Culver City, California, by L. Kleinman and the author. The error associated with the approximate solution did not exceed a few percent.

6. Circular Waveguide with a Coaxial Ferrite Core. If we substitute for the dielectric rod of the preceding example one which has an 'effective' magnetic tensor susceptibility with components, $\chi_{mxx} = \chi_{myy} = \chi$, $\chi_{mxy} = -j\kappa$, $\chi_{myx} = j\kappa$, we obtain

$$\gamma = \gamma_0 \left[1 + \left(\chi_e \frac{k_0^2}{\gamma_0^2} + \chi \pm \kappa \right) \frac{s}{S} C \right] \quad (66)$$

By 'effective' susceptibility we mean the tensor that operates on the unperturbed external rf field to give the rf magnetization. C is again equal to unity for the rectangular waveguide and is approximately 1.04 for the circular waveguide. Such a solution corresponds to a waveguide with a concentric sliver of ferrite and with the steady magnetic field along the axis of propagation. Note, however, that Eq. 66, like Eq. 65, is nothing but a first-order perturbation expression. Consequently, it is meaningful only when the actual field configuration is similar to the unperturbed wave. Now, typical values of χ_e , χ , and κ are 10, 3, and 2, respectively. These constitute more than perturbing factors. Hence, Eq. 66 should be considered with caution when applied to a ferrite case. Note also that the two signs in front of κ in this equation correspond to the two opposite directions of propagation and express the nonreciprocal nature of the waveguide.

7. Ferrite Slab in a Rectangular Waveguide. Let us consider a ferrite slab of thickness δ in a rectangular waveguide of width a with a transverse steady magnetic field H_{dc} , as shown in Fig. 9. The 'effective' magnetic tensor susceptibility components are, in this case, $\chi_{mxx} = \chi_{mzz} = \chi$; $\chi_{mxz} = j\kappa$; $\chi_{mzx} = -j\kappa$, with all the others equal to zero. If we assume the field to be essentially the TE_{10} -mode of the empty waveguide, we obtain, after going through the usual steps, for the differential propagation constant (the difference of the propagation constants in the two opposite directions)

$$\gamma_+ - \gamma_- = 2 \frac{\pi}{a} \frac{\delta}{a} \kappa \sin \frac{2\pi d}{a} \quad (67)$$

Again, this expression is accurate only when δ/a , κ , and χ are small as compared to unity. Otherwise, it is in grave error. (See example 6 of Section III for a variational treatment of the problem.)

8. Rectangular Waveguide Completely Filled with Ferrite. Next we take up a waveguide completely filled with ferrite and with the steady magnetic field along the axis of propagation. We derive a formal solution of this problem in terms of infinite sums, which, as such, is of academic interest only. We then obtain from this general solution approximate ones for waveguides with circular or square cross sections.

As a first step, consider the basic equations of Eq. 42 and note that the permittivity and permeability appearing on the left side of these equations are those of free space (or air). Suppose, however, that the guide is filled uniformly with a dielectric of susceptibility χ_e . Then, $J_e = j\omega\epsilon_0\chi_e E$ can be transferred to the left side and combined with $j\omega\epsilon_0 E$ to yield $j\omega\epsilon E$, where ϵ is the permittivity of the dielectric filling the guide uniformly. Thus, in this case, we can set

J_e equal to zero and replace ϵ_0 by the permittivity of the ferrite in Eqs. 51 and 52. Next, we note that the magnetic susceptibility matrix of the ferrite is, in the case under consideration, of the form (7)

$$\vec{\chi}_m = \begin{bmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{bmatrix} = \begin{bmatrix} \chi & -j\kappa & 0 \\ j\kappa & \chi & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (68)$$

We can split this

$$\vec{\chi}_m = \begin{bmatrix} 0 & -j\kappa & 0 \\ j\kappa & 0 & 0 \\ 0 & 0 & -\chi \end{bmatrix} + \begin{bmatrix} \chi & 0 & 0 \\ 0 & \chi & 0 \\ 0 & 0 & \chi \end{bmatrix} = \vec{\chi}'_m + \chi$$

and write*

$$J_m = -j\omega\mu_0\chi H - j\omega\mu_0\vec{\chi}'_m \cdot H \quad (69)$$

Now the first term in Eq. 69 can be transferred from the right-hand side of Eq. 52(a, b, c) to the left and combined with $j\omega\mu_0 H$ to form $j\omega\mu' H$, where μ' is a fictitious permeability. Thus we can substitute, in Eqs. 51 and 52, $j\omega\mu_0\vec{\chi}'_m \cdot H$ for J_m and μ' for μ . Taking these facts into account, substituting for H appearing in J_m the series given by Eq. 47(b), and combining Eqs. 51 and 52, we obtain the simultaneous infinite sets of equations

$$\frac{(\gamma_n^2 - \gamma^2) + a_n^2\chi}{\gamma_n^2 + \omega^2\mu'\epsilon\chi} h_n^c + \sum_{\nu} h_{\nu}^b (b_{\nu} | c_n^*) + \sum_{\nu \neq n} h_{\nu}^c (c_{\nu} | c_n^*) = 0 \quad (70a)$$

$$\frac{(\gamma_n^2 - \gamma^2)h_n^b}{\omega^2\mu'\epsilon} + \sum_{\nu \neq n} h_{\nu}^b (b_{\nu} | b_n^*) + \sum_{\nu} h_{\nu}^c (c_{\nu} | b_n^*) = 0 \quad (70b)$$

The abbreviations that have been used are

$$\int H_{\nu}^b \cdot \chi'_m \cdot \dot{H}_n^c dS = (b_{\nu} | c_n^*)$$

$$\int H_{\nu}^c \cdot \chi'_m \cdot \dot{H}_n^c dS = (c_{\nu} | c_n^*)$$

* Several investigators have proposed this artifice. We are directly indebted to Dr. B. Lax of the Lincoln Laboratory, M.I.T., for this suggestion.

with similar meanings for $(b_\nu | b_n^*)$ and $(c_\nu | b_n^*)$.

Also,

$$\frac{\gamma_n^2}{a} = \omega^2 \mu' \epsilon - \alpha_n^2$$

$$\frac{\gamma_n^2}{\beta} = \omega^2 \mu' \epsilon - \beta_n^2$$

The usual perturbation techniques can be used in connection with these equations although we do not employ them here. Instead, we merely obtain the solution for degenerate modes such as the two TE₁₁-modes in a circular waveguide. Thus, writing Eq. 70(a) for the two degenerate modes, indicated by the subscripts n and m , and neglecting all others, we obtain

$$\left(\frac{\gamma_n^2}{a} - \gamma^2 + \alpha_n^2 \chi\right) h_n^c + (c_m | c_n^*) \left(\frac{\gamma_n^2}{a} + \omega^2 \mu' \epsilon \chi\right) h_m^c = 0$$

$$\left(\frac{\gamma_m^2}{a} + \omega^2 \mu' \epsilon \chi\right) (c_n | c_m^*) h_n^c + \left(\frac{\gamma_m^2}{a} \pm \gamma^2 + \alpha_m^2 \chi\right) h_m^c = 0$$

Setting the determinant equal to zero yields

$$\gamma^2 = (\omega^2 \mu_o \epsilon_o - \alpha^2)(1 + \chi \pm C\kappa) \quad (71)$$

where $\alpha = \alpha_n = \alpha_m$ (since the modes are degenerate) and C is a numerical factor having approximately the value 0.81 for a square waveguide, and 0.84 for a circular one. Note in Eq. 71, that the first factor on the right-hand side is an expression for the propagation constant of the same waveguide filled with a dielectric of dielectric susceptibility equal to that of the ferrite but with a permeability equal to that of free space. Calling γ_o this propagation constant, we can approximately rewrite Eq. 71 as

$$\gamma = \gamma_o \left(1 + \frac{1}{2} \chi \pm \frac{1}{2} C\kappa\right)$$

provided γ differs only slightly from γ_o . The two signs in front of κ express the nonreciprocal character of such a transmission line. The reader is again reminded that χ and κ are components of the 'effective' susceptibility.

C. GENERAL PERTURBATION FORMULAS FOR WAVEGUIDES

Equations 51 and 52 may be formally solved for the six expansion coefficients. The results are

$$e_n^a \left(\frac{\gamma_n^2}{a} - \gamma^2\right) = j\omega\mu_o \int J_e \cdot E_n^a dS - \alpha_n \int J_m \cdot H_n^a dS - j\gamma \int J_m \cdot H_n^c dS \quad (72)$$

$$e_n^b \left(\gamma_n^2 - \gamma^2 \right) = -\frac{\gamma_n}{\omega \epsilon_o} \int J_e \cdot E_n^c dS + \frac{\gamma^2 - k_o^2}{j\omega \epsilon_o} \int J_e \cdot E_n^b dS - \beta_n \int J_m \cdot H_n^b dS \quad (73)$$

$$e_n^c \left(\gamma_n^2 - \gamma^2 \right) = \frac{\gamma_n}{\omega \epsilon_o} \int J_e \cdot E_n^b dS - \frac{\gamma_n^2}{j\omega \epsilon_o} \int J_e \cdot E_n^c dS + j\gamma \int J_m \cdot H_n^b dS \quad (74)$$

$$h_n^a \left(\gamma_n^2 - \gamma^2 \right) = -a_n \int J_e \cdot E_n^a dS + \frac{k_o^2 - \gamma^2}{j\omega \mu_o} \int J_m \cdot H_n^a dS + \frac{\gamma a_n}{\omega \mu_o} \int J_m \cdot H_n^c dS \quad (75)$$

$$h_n^b \left(\gamma_n^2 - \gamma^2 \right) = -j\omega \epsilon_o \int J_m \cdot H_n^b dS - \beta_n \int J_e \cdot E_n^b dS - j\gamma \int J_e \cdot E_n^c dS \quad (76)$$

$$h_n^c \left(\gamma_n^2 - \gamma^2 \right) = \frac{\gamma_n^2}{j\omega \mu_o} \int J_m \cdot H_n^c dS - \frac{\gamma a_n}{\omega \mu_o} \int J_m \cdot H_n^a dS + j\gamma \int J_e \cdot E_n^a dS \quad (77)$$

These relations, especially Eq. 72 and Eq. 76, are in a suitable form for perturbation calculations. If the solution we are seeking is a slightly perturbed TE-mode of the empty waveguide, we write Eq. 72 with $n = \nu$ and arbitrarily set $e_\nu^a = 1$, obtaining

$$\gamma_\nu^2 - \gamma^2 = -k_o^2 \int E \cdot \vec{\chi}_e \cdot E_\nu^a dS + j\omega \mu_o a_n \int H \cdot \vec{\chi}_m \cdot H_\nu^a dS - \gamma \omega \mu_o \int H \cdot \vec{\chi}_m \cdot H_\nu^c dS \quad (78)$$

By hypothesis, however, γ is almost equal to γ_ν , and the actual fields E and H are nearly those of the unperturbed TE_ν -mode, given by Eq. 54. If we make these approximations, Eq. 78 becomes

$$\begin{aligned} \gamma - \gamma_\nu &\cong \frac{k_o^2}{2\gamma_\nu} \int E_\nu^a \cdot \vec{\chi}_e \cdot E_\nu^a dS + \frac{a_\nu^2}{2\gamma_\nu} \int H_\nu^a \cdot \vec{\chi}_m \cdot H_\nu^a dS - j \frac{a_\nu}{2} \int H_\nu^c \cdot \vec{\chi}_m \cdot H_\nu^a dS \\ &\quad + j \frac{a_\nu}{2} \int H_\nu^a \cdot \chi_m \cdot H_\nu^c dS + \frac{\gamma_\nu}{2} \int H_\nu^c \cdot \chi_m \cdot H_\nu^c dS \end{aligned} \quad (79)$$

For a waveguide containing isotropic media only, the preceding expression becomes

$$\gamma - \gamma_\nu \cong \frac{k_o^2}{2\gamma_\nu} \int \chi_e (E_\nu^a)^2 dS + \frac{a_\nu^2}{2\gamma_\nu} \int \chi_m |H_\nu^a|^2 dS + \frac{\gamma_\nu}{2} \int \chi_m |H_\nu^c|^2 dS \quad (80)$$

Equation 80 has a simple physical interpretation. It shows that the presence of a dielectric or magnetic material tends to increase the propagation constant, and that the increase is proportional to the amount of excess energy stored per unit length of the waveguide because of the presence of the dielectric or magnetic substance. When the perturbed mode of the empty waveguide is TM, we start with Eq. 76. The results are similar to those of the perturbed TE-mode case and may be obtained from the preceding three equations by substituting ϵ , μ , a , b , a , β , E , H , m , e for μ , ϵ , b ,

$\alpha, \beta, \alpha, H, E, e, m$, respectively. Some of the examples we have already worked out may be treated by direct application of these formulas.

D. INTEGRAL-EQUATION TREATMENT

The problem of anisotropic and cross-sectionally inhomogeneous waveguides can be treated by an integral-equation technique as follows: Equations 42(a, b) may be combined to yield

$$\nabla^2 H + (k_0^2 - \gamma^2)H = f(J_e, J_m) \quad (81)$$

and

$$\nabla^2 E + (k_0^2 - \gamma^2)E = g(J_e, J_m) \quad (82)$$

where f is given by

$$f = -j\omega\epsilon_0 J_m - \text{curl } J_e + j\gamma a_z \times J_e - \frac{j}{\omega\mu_0} \text{grad div } J_m - \frac{\gamma}{\omega\mu_0} a_z \text{ div } J_m \\ - \frac{\gamma}{\omega\mu_0} \text{div } (a_z \cdot J_m) + j \frac{\gamma^2}{\omega\mu_0} a_z (a_z \cdot J_m)$$

A similar expression can be given for g .

The inhomogeneous equations, Eqs. 81 and 82, can be formally solved by introducing the magnetic and electric Green's tensor functions which are defined as

$$\nabla^2 G_H + (k_0^2 - \gamma^2)G_H = \delta I \quad (83)$$

$$\nabla^2 G_E + (k_0^2 - \gamma^2)G_E = \delta I \quad (84)$$

I is the idem factor, and δ is the Dirac delta-function. G_H satisfies the same boundary conditions as the magnetic field; G_E those of the electric field. These Green's functions, as defined, have no simple physical meaning. They are used here for mathematical expediency. We form the scalar product of Eq. 81 with G_H , subtract it from the result of dot multiplying Eq. 83 by H , and obtain, after utilizing Green's vector identity

$$H(r) = \int G_H(r|r') \cdot f(r') dS' \quad (85)$$

where the integral is over the cross section of the waveguide, r' is the coordinate of the source point, and r is that of the field point. It has been tacitly assumed that the boundary of the guide has infinite conductivity. Similarly, we have

$$E(r) = \int G_E(r|r') \cdot g(r') dS' \quad (86)$$

Equations 85 and 86 are formal solutions of the problem, provided G_H and G_E are known. However, G_E and G_H are seldom known in closed form, but they can be expanded in terms of normal modes. The H-like modes may be obtained by setting f equal to zero in Eq. 81; the E-like modes

by putting g equal to zero in Eq. 82. It will be recalled that these modes are those defined by Eqs. 43 and 45. The expansion of G_H and G_E can now be obtained by following the usual procedure of substituting in Eqs. 83 and 84 the expansions of G_H and G_E in terms of the normal modes and determining the coefficients of expansion from the orthonormality property. The result is

$$G_E(r | r') = \sum_n \left[\frac{E_n^a(r)E_n^a(r')}{\gamma^2 - \gamma_n^2/a} + \frac{E_n^b(r)E_n^b(r')}{\gamma^2 - \gamma_n^2/\beta} + \frac{E_n^c(r)E_n^c(r')}{\gamma^2 - \gamma_n^2/\beta} \right] \quad (87)$$

$$G_H(r | r') = \sum_n \left[\frac{H_n^a(r)H_n^a(r')}{\gamma^2 - \gamma_n^2/a} + \frac{H_n^b(r)H_n^b(r')}{\gamma^2 - \gamma_n^2/\beta} + \frac{H_n^c(r)H_n^c(r')}{\gamma^2 - \gamma_n^2/a} \right] \quad (88)$$

In practice, f and g are almost always functions of E and H because J_e and J_m are seldom, if at all, arbitrarily impressed currents. Hence, Eqs. 85 and 86 are actually integral equations.

All the techniques for the formal or approximate solutions of such equations can be, of course, applied. The result of applying them to the solution of the examples treated in part B is the same as that obtained by the mode-expansion method. More generally, we have obtained Eqs. 72 through 77, and the perturbation formulas thereafter by substituting the expansions, Eq. 87 and Eq. 88, in Eqs. 85 and 86.

III. SOME VARIATIONAL PRINCIPLES FOR CAVITIES AND WAVEGUIDES

Known variational expressions for the resonance frequencies of a resonator are restricted to the special case in which the electromagnetic field can be derived from a single scalar function that satisfies the Helmholtz equation (11). When the substance within a cavity is inhomogeneous or anisotropic, such a scalar formulation is inadequate. The need for vector variational principles is thus apparent. In the first part of this section, we present such variational formulas for resonant frequencies directly in terms of the field vectors. Some of these formulas can be obtained as special cases of the abstract operator forms discussed in reference 11(pp.1108-1111). Here, however, we shall obtain these variational expressions (and others that cannot be derived from the operator equations of ref. 11) directly from the equations satisfied by the field vectors. In problems of propagation through anisotropic or inhomogeneous media, vector variational expressions for the propagation constant are also of interest. The second part of this section is concerned with such expressions. The variational formulas appearing in this section are to be distinguished from variational expressions for reflection coefficients, scattering amplitudes, or impedance matrices derived by other authors (11, 12). The section concludes with some illustrative examples and with the derivation of certain perturbation expressions. The main purpose of these examples is to illustrate the methods developed in this section rather than to present accurate solutions of heretofore unsolved problems.

A. VARIATIONAL PRINCIPLES FOR RESONANT (AND CUTOFF) FREQUENCIES

1. E-Field Formulation. Consider a resonator with perfectly conducting walls which enclose a medium of permittivity ϵ and permeability μ . Both permittivity and permeability may be tensors and functions of position.* Now let a resonant angular frequency be ω and let the corresponding electromagnetic field be characterized by the vectors E and H . The following is then asserted to be a variational expression for ω , provided ϵ and μ are hermitian, that is, provided no losses are present

$$\omega^2 = \frac{\int (\text{curl } E^*) \cdot \mu^{-1} \cdot (\text{curl } E) dV}{\int E^* \cdot \epsilon \cdot E dV} \quad (89)$$

The integrals are over the volume of the resonator, μ^{-1} is the inverse of μ , and E^* is the complex conjugate of E . To prove this assertion, we must show that those field configurations E and E^* that render ω^2 stationary are solutions of

$$\text{curl} (\mu^{-1} \cdot \text{curl } E) - \omega^2 \epsilon \cdot E = 0 \quad (90)$$

and of its complex conjugate, and have vanishing tangential components at the boundary. (Eq. 90

* To avoid extremely cumbersome notation, the double-arrow superscript used to denote tensor quantities in previous sections will be omitted.

is the result of eliminating the magnetic field from Maxwell's equations.) This is indeed the case. On varying E and E^* in Eq. 89 we obtain, after utilizing the hermitian character of ϵ and μ , the following expression for the variation of ω^2 :

$$\begin{aligned} \left(\int E^* \cdot \epsilon \cdot E \, dV \right) \delta \omega^2 = & \int \delta E^* \cdot [\text{curl} (\mu^{-1} \cdot \text{curl} E) - \omega^2 \epsilon \cdot E] \, dV - \oint \delta E^* \cdot (\mathbf{n} \times \mu^{-1} \cdot \text{curl} E) \, dS \\ & + \int \delta E \cdot [\text{curl} (\mu^{-1*} \cdot \text{curl} E^*) - \omega^2 \epsilon^* \cdot E^*] \, dV - \oint \delta E \cdot (\mathbf{n} \times \mu^{-1*} \cdot \text{curl} E^*) \, dS \end{aligned} \quad (91)$$

The second and fourth integrals are over the boundary of the cavity. Their appearance is the result of using the vector identity

$$\int A \cdot \text{curl} B \, dS = \int B \cdot \text{curl} A \, dV + \oint \mathbf{n} \cdot (B \times A) \, dS \quad (92)$$

where \mathbf{n} is the outward normal unit vector. The variation of ω^2 will vanish, provided E satisfies Eq. 90, E^* satisfies the complex conjugate of Eq. 90, and the surface integrals in Eq. 91 vanish. The latter condition can be satisfied only if $\mathbf{n} \times \delta E$ and its complex conjugate vanish over the boundary, since $\mathbf{n} \times (\mu^{-1} \cdot \text{curl} E)$, being proportional to the tangential component of the magnetic field, cannot vanish over the complete boundary. Equation 89 is thus a variational formulation of the problem defined by Eq. 90 and the boundary condition $\mathbf{n} \times E = 0$. Admissible trial fields must have vanishing tangential components at the boundary, must be continuous together with their first derivatives, and must possess finite second derivatives everywhere in the cavity except at surfaces where ϵ and μ are discontinuous. At such surfaces $\mathbf{n} \times E$ and $\mathbf{n} \times (\mu^{-1} \cdot \text{curl} E)$ must be continuous.

Equation 89 can be modified so that trial vectors E are not required to satisfy the boundary condition $\mathbf{n} \times E = 0$ at the wall of the resonator. This can be achieved by following a known general method (see ref. 13 and ref. 11, pp. 1131-33): we add appropriate terms to the numerator of Eq. 89

$$\omega^2 = \frac{\int (\text{curl} E^*) \cdot \mu^{-1} \cdot (\text{curl} E) \, dV - \oint \mathbf{n} \cdot [E \times (\mu^{-1*} \cdot \text{curl} E^*)] \, dS - \oint \mathbf{n} \cdot [E^* \times (\mu^{-1} \cdot \text{curl} E)] \, dS}{\int E^* \cdot \epsilon \cdot E \, dV} \quad (93)$$

or, on combining the first and third terms of the numerator we obtain

$$\omega^2 = \frac{\int E^* \cdot \text{curl} (\mu^{-1} \cdot \text{curl} E) \, dV - \oint \mathbf{n} \cdot [E \times (\mu^{-1*} \cdot \text{curl} E^*)] \, dS}{\int E^* \cdot \epsilon \cdot E \, dV} \quad (94)$$

When the distribution of matter within the cavity is discontinuous, Eq. 89 can be further modified so that trial fields will not be required to have continuous tangential components of E and of $(\mu^{-1} \cdot \text{curl } E)$. The modification consists of adding to the numerator of Eq. 89 the terms

$$- \int \mathbf{n} \cdot [\mathbf{E}_+^* \times (\mu_+^{-1} \cdot \text{curl } \mathbf{E}_+) - \mathbf{E}_-^* \times (\mu_-^{-1} \cdot \text{curl } \mathbf{E}_-)] dS - \text{complex conjugate}$$

where the subscripts $+$ and $-$ refer to values on opposite sides of the surface of discontinuity and the integrals are over such a surface. The passage from Eq. 89 to Eq. 94 enables one to expand the class of admissible trial functions.

Finally, Eq. 89 can be modified to apply when the boundary condition at the walls is

$$\mathbf{n} \times (\mu^{-1} \cdot \text{curl } \mathbf{E}) = -j\omega \mathbf{Y} \cdot \mathbf{E}_t \quad (95)$$

where $(j\omega \mathbf{Y})$ is a hermitian admittance dyadic and \mathbf{E}_t is the tangential component of \mathbf{E} . The term to be added to the numerator of Eq. 89 is, in this case, $\oint \mathbf{E}_t^* \cdot (j\omega \mathbf{Y}) \cdot \mathbf{E}_t dS$. An example illustrating the application of Eq. 95 is the junction of a cavity and a tuning stub.

2. H-Field Formulation. A variational expression that is similar to Eq. 89 but is in terms of the magnetic field vector can be obtained by interchanging ϵ and μ and by replacing \mathbf{E} with \mathbf{H} :

$$\omega^2 = \frac{\int (\text{curl } \mathbf{H}^*) \cdot \epsilon^{-1} \cdot (\text{curl } \mathbf{H}) dV}{\int \mathbf{H}^* \cdot \mu \cdot \mathbf{H} dV} \quad (96)$$

Here, trial vectors \mathbf{H} are not required to satisfy the proper boundary condition, $\mathbf{n} \times (\epsilon^{-1} \cdot \text{curl } \mathbf{H}) = 0$, at the wall. However, the differentiability and continuity conditions are the same as those for \mathbf{E} in connection with Eq. 89.

Equations 89 and 96 reduce properly to the scalar variational principle for the Helmholtz equation (see ref. 11, p. 1112) when the electromagnetic problem can be described in terms of a single scalar field.

3. Mixed-Field Formulation. In contrast to the preceding formulas, which are in terms of either the electric vector or the magnetic vector, the following variational expression is in terms of both field vectors:

$$\omega = j \frac{\int \mathbf{H}^* \cdot \text{curl } \mathbf{E} dV - \int \mathbf{E}^* \cdot \text{curl } \mathbf{H} dV}{\int \mathbf{E}^* \cdot \epsilon \cdot \mathbf{E} dV + \int \mathbf{H}^* \cdot \mu \cdot \mathbf{H} dV} \quad (97)$$

where ϵ and μ are again assumed to be hermitian. That Eq. 97 is indeed a variational expression can readily be verified by evaluating the variation of ω and observing that the latter vanishes, provided E and H satisfy Maxwell's equations, E^* and H^* satisfy the complex conjugate of Maxwell's equations, and the trial electric vectors have vanishing tangential components at the boundary. Admissible trial E - vectors must be continuous; they must possess first derivatives; and their tangential components at the boundary must vanish. Admissible trial H -vectors are subject to the same continuity and differentiability conditions as the E -vectors, but they are not required to satisfy any particular boundary condition. If matter is discontinuously distributed within the cavity, trial vectors E and H must have continuous tangential components at the surfaces of discontinuity. The latter restriction and the restriction of vanishing trial tangential E at the walls of the cavity can be eliminated by the addition of appropriate terms to the numerator of Eq. 97. For example, in the variational expression

$$\omega = j \frac{\int H^* \cdot \text{curl } E \, dV - \int E^* \cdot \text{curl } H \, dV - \oint n \cdot (E \times H^*) \, dS}{\int E^* \cdot \epsilon \cdot E \, dV + \int H^* \cdot \mu \cdot H \, dV} \quad (98)$$

both trial vectors are unrestricted at the wall.

4. Stationary Nature of the Preceding Expressions. Because of the positive definite nature of both numerator and denominator in Eqs. 89 and 96, the lowest 'correct' ω is an absolute minimum. Hence, trial values of E yield approximate values of ω that are always larger than the correct one. No such statement can be made for the other variational expressions.

5. Cutoff Frequencies of Waveguides. The cutoff frequencies of a waveguide are the resonant frequencies of a two-dimensional cavity formed by the cross section of the waveguide. Hence, the preceding discussion applies directly to cutoff frequencies if we make the following correspondence:

Cavity	Waveguide
Resonant frequency	Cutoff frequency
Integrals over the volume	Integrals over the cross section
Integrals over the surface	Integrals along the perimeter of the cross section

B. VARIATIONAL FORMULAS FOR PROPAGATION CONSTANTS

1. Mixed-Field Formulation. Consider a waveguide with perfectly conducting walls, possibly enclosing anisotropic matter whose distribution may be a function of the transverse coordinates but not of the coordinate along the direction of propagation. If z is this coordinate, the field vectors may be expressed as $E(x, y) \exp(-j\gamma z)$, $H(x, y) \exp(-j\gamma z)$, where γ is the propagation constant. E and H are three-dimensional vectors but depend only on x and y . They satisfy the following relations obtained by substituting the field vectors in Maxwell's equations

$$\text{curl } \mathbf{E} + j\omega\boldsymbol{\mu} \cdot \mathbf{H} = \gamma j\mathbf{a}_z \times \mathbf{E} \quad (99)$$

$$\text{curl } \mathbf{H} - j\omega\boldsymbol{\epsilon} \cdot \mathbf{E} = \gamma j\mathbf{a}_z \times \mathbf{H} \quad (100)$$

where \mathbf{a}_z is the unit vector in the z-direction. Premultiplying Eq. 99 by \mathbf{H}^* , Eq. 100 by \mathbf{E}^* , integrating over the cross section of the waveguide, and subtracting, we obtain

$$\gamma = \frac{\omega \int \mathbf{E}^* \cdot \boldsymbol{\epsilon} \cdot \mathbf{E} \, dS + \omega \int \mathbf{H}^* \cdot \boldsymbol{\mu} \cdot \mathbf{H} \, dS + j \int \mathbf{E}^* \cdot \text{curl } \mathbf{H} \, dS - j \int \mathbf{H}^* \cdot \text{curl } \mathbf{E} \, dS}{\int \mathbf{H}^* \cdot \mathbf{a}_z \times \mathbf{E} \, dS - \int \mathbf{E}^* \cdot \mathbf{a}_z \times \mathbf{H} \, dS} \quad (101)$$

That Eq. 101 is indeed a variational expression can be shown by evaluating the variation of γ and observing that the latter vanishes if \mathbf{E} and \mathbf{H} satisfy Eqs. 99 and 100 and the tangential component of \mathbf{E} vanishes at the walls of the waveguide. Thus, trial fields \mathbf{E} , \mathbf{H} must be continuous and differentiable throughout the cavity. At the boundary, the tangential component of \mathbf{E} must vanish, but \mathbf{H} is arbitrary. When discontinuities are present in the distribution of matter within the cavity, the tangential components of both \mathbf{E} and \mathbf{H} must be continuous at the surfaces of discontinuity.

In the following modified form of Eq. 101 \mathbf{H} , as well as \mathbf{E} , is arbitrary at the boundary

$$\gamma = \frac{\omega \int \mathbf{E}^* \cdot \boldsymbol{\epsilon} \cdot \mathbf{E} \, dS + \omega \int \mathbf{H}^* \cdot \boldsymbol{\mu} \cdot \mathbf{H} \, dS + j \int \mathbf{E}^* \cdot \text{curl } \mathbf{H} \, dS - j \int \mathbf{H}^* \cdot \text{curl } \mathbf{E} \, dS - j \oint \mathbf{n} \cdot (\mathbf{E} \times \mathbf{H}^*) \, d\ell}{\int \mathbf{H}^* \cdot \mathbf{a}_z \times \mathbf{E} \, dS - \int \mathbf{E}^* \cdot \mathbf{a}_z \times \mathbf{H} \, dS} \quad (102)$$

The last integral in the numerator of Eq. 102 is over the periphery of the cross section of the waveguide.

2. E-Field Formulation. If \mathbf{H} is eliminated between Eq. 99 and Eq. 100 the result is

$$\begin{aligned} \gamma^2 \mathbf{a}_z \times (\boldsymbol{\mu}^{-1} \cdot \mathbf{a}_z \times \mathbf{E}) + j\gamma [\text{curl } (\boldsymbol{\mu}^{-1} \cdot \mathbf{a}_z \times \mathbf{E}) + \mathbf{a}_z \times (\boldsymbol{\mu}^{-1} \cdot \text{curl } \mathbf{E})] - \text{curl } (\boldsymbol{\mu}^{-1} \cdot \text{curl } \mathbf{E}) \\ + \omega^2 \boldsymbol{\epsilon} \cdot \mathbf{E} = 0 \end{aligned} \quad (103)$$

Premultiplying by \mathbf{E}^* , integrating over the cross section, and rearranging, the following equation is obtained:

$$\begin{aligned} \gamma^2 \int (\mathbf{a}_z \times \mathbf{E}^*) \cdot \boldsymbol{\mu}^{-1} \cdot (\mathbf{a}_z \times \mathbf{E}) \, dS - j\gamma \int [(\text{curl } \mathbf{E}^*) \cdot \boldsymbol{\mu}^{-1} \cdot (\mathbf{a}_z \times \mathbf{E}) - (\mathbf{a}_z \times \mathbf{E}^*) \cdot \boldsymbol{\mu}^{-1} \cdot (\text{curl } \mathbf{E})] \, dS \\ + \int (\text{curl } \mathbf{E}^*) \cdot \boldsymbol{\mu}^{-1} \cdot (\text{curl } \mathbf{E}) \, dS - \omega^2 \int \mathbf{E}^* \cdot \boldsymbol{\epsilon} \cdot \mathbf{E} \, dS = 0 \end{aligned} \quad (104)$$

This is a variational principle for γ . Indeed, if E , E^* , and γ are varied, γ remains stationary if E satisfies Eq. 103 and E^* satisfies the complex conjugate of Eq. 103. Admissible trial vectors must have vanishing tangential components at the walls, must be continuous together with their first derivatives, and must possess second derivatives except at surfaces of discontinuity of the medium enclosed within the waveguide. At such surfaces, E and $(\mu^{-1} \cdot \text{curl } E - j\gamma\mu^{-1} \cdot \mathbf{a}_z \times E)$ must have continuous tangential components.

3. H-Field Formulation. If in Eq. 104 μ and ϵ are interchanged and E is replaced by H , a new variational equation results. Admissible trial vectors must satisfy the same continuity and differentiability conditions as E in connection with Eq. 104, but they are arbitrary at the boundary of the waveguide.

C. ILLUSTRATIVE EXAMPLES AND PERTURBATION FORMULAS FOR RESONATORS

We have thus far presented several variational expressions for both the resonant frequencies of a cavity and the propagation constants of a waveguide. We now illustrate them by a number of examples. For a particular problem the choice of a certain variational formula in preference to others largely depends on physical considerations. For example, if the configuration of the electric field can be guessed more readily than that of the magnetic field, it is sensible to use a formula in terms of the electric vector only.

The successful choice of trial fields depends upon our familiarity with the physical aspects of the problem at hand. It is important to remember that the conditions of admissibility of trial functions are minimum requirements and that we should make every effort to select trial fields that satisfy as many of the known features of the solution as possible. In particular, we should attempt to devise trial fields which at surfaces of discontinuity have not only continuous tangential components but continuous normal components of $(\epsilon \cdot E)$ and $(\mu \cdot H)$ as well. This is because the last set of boundary conditions does not follow from the first set unless the trial fields satisfy Maxwell's equations.

1. Resonant Frequency of a Rectangular Cavity. We shall start with an almost trivial example which is meant to clarify the basic principles involved and the procedure to be followed in conjunction with the variational expression, Eq. 97.

Consider a rectangular cavity of sides a , b , and c along the x -, y -, and z -directions. From physical considerations we know that a possible solution is with the E -field along one of the axes, let us say y , and the H -field in the x - z plane. Taking into account the boundary conditions, we can write, as trial fields, the expressions

$$E = a_y \sin \frac{\pi x}{a} \sin \frac{\pi z}{c}$$

$$H_x = A \sin \frac{\pi x}{a} \cos \frac{\pi z}{c}$$

$$H_z = B \cos \frac{\pi x}{a} \sin \frac{\pi z}{c}$$

where A, B are variational parameters to be determined. There is no need for an additional parameter for E , since it can be easily shown that the variational principles derived are independent of the absolute intensity of the electromagnetic field. Substituting these quantities in Eq. 97 and performing the integrations involved, we get

$$\omega = j \frac{(\pi/c)(A - A^*) - (\pi/a)(B - B^*)}{\mu A \cdot A^* + \mu B \cdot B^* + \epsilon}$$

Taking the partial derivative of ω with respect to A^* and setting it equal to zero, we obtain

$$A = -j \frac{1}{\omega \mu} \cdot \frac{\pi}{c}$$

Similarly, taking the partial derivative with respect to B^* and setting it equal to zero, we have

$$B = j \frac{1}{\omega \mu} \cdot \frac{\pi}{a}$$

Substituting the trial fields thus determined back in the variational expression for ω , we find

$$\omega^2 = \frac{1}{\mu \epsilon} \left[\left(\frac{\pi}{a} \right)^2 + \left(\frac{\pi}{c} \right)^2 \right]$$

This will be readily recognized as the exact formula for ω . As a matter of fact, the fields just determined are also exact. The reason for this is that the trial fields E and H happened to belong to the proper class of admissible fields.

2. Cutoff Frequency of a Rectangular Guide with a Dielectric Slab. For our next example we shall choose a more practical case. Suppose that we wish to find the cutoff wavelength of the fundamental mode of a rectangular waveguide partly filled with a dielectric of dielectric susceptibility χ_e , as shown in Fig. 10. If χ_e were zero, we know that an exact solution for the E -field would be

$$E = a_y \sin \frac{\pi x}{a} \tag{105}$$

With the dielectric present, the E -field is certainly going to be different, but to a first approximation we can assume that it is given by Eq. 105. We again utilize Eq. 97 but this time subject to the constraint

$$\text{curl } E = -j\omega \mu_0 H$$

We then have

$$E = a_y \sin \frac{\pi x}{a}; \text{ curl } E = a_z \frac{\pi}{a} \cos \frac{\pi x}{a}$$

$$H = a_z \frac{\pi/a}{-j\omega \mu_0} \cos \frac{\pi x}{a}; \text{ curl } H = a_y \frac{(\pi/a)^2}{-j\omega \mu_0} \sin \frac{\pi x}{a}$$

Substituting these in Eq. 97 and introducing the cutoff wavelength λ_c , we obtain

$$\left(\frac{\lambda_c}{a}\right)^2 = 4 \left[1 + \chi_e \left(\frac{\delta}{a} - \frac{1}{2\pi} \sin \frac{2\pi\delta}{a} \right) \right]$$

Let us test the accuracy of this expression by comparing results obtained through its application to those obtained by exact methods. The latter can be found in reference 10 (p. 387), where numerical results are given for the specific case in which $\chi_e = 1.45$. In the following table, the approximate and exact answers are compared for various widths of the dielectric. The approximate values involve the use of a slide rule. The exact values were read from curves of reference 10.

δ/a	a/λ_c (exact)	a/λ_c (approximate)
0.000	0.50	0.500
0.167	0.48	0.486
0.286	0.45	0.455
0.500	0.38	0.383
0.600	0.35	0.352
1.000	0.32	0.320

The agreement is seen to be excellent – surprisingly so in view of the rather crude assumption about the field. Note that the preceding expression for the cutoff wavelength provides us with a formula in closed form which can be utilized for design purposes; the exact solution, though possible, involves a transcendental equation.

3. Cutoff Frequency of a Rectangular Guide with a Dielectric Slab at the Center. We now consider a case similar to example 2. We wish to find the cutoff wavelength of the fundamental mode of a waveguide with a cross section as shown in Fig. 4. This time let us use the variational expression given by Eq. 89. Making use of the same rather crude trial field given by Eq. 105, we obtain

$$\left(\frac{\lambda_c}{a}\right)^2 = 4 \left[1 + \chi_e \left(\frac{\delta}{a} + \frac{1}{\pi} \sin \frac{\pi\delta}{a} \right) \right]$$

The following is a table of comparison between values computed with a slide rule from the preceding formula and values read from reference 10 (p. 386).

δ/a	a/λ_c (exact)	a/λ_c (approximate)
0.00	0.500	0.500
0.10	0.435	0.438
0.20	0.395	0.397
0.30	0.370	0.372
0.50	0.340	0.338
0.75	0.324	0.318
1.00	0.320	0.319

4. Cavity Surface Perturbation. A resonator originally bounded by conducting surfaces S , S_1 is deformed so that surface S_1 is replaced by S_2 as shown in Fig. 11. If the field configuration and the resonant frequency of the undeformed cavity are characterized by E , H , and ω , an approximate expression for the resonant frequency of the deformed cavity is obtained by substituting E as the trial field in Eq. 94. The result is

$$\omega_1^2 = \omega^2 \left[1 + \frac{\int_{v'} [H^* \cdot \mu \cdot H - E^* \cdot \epsilon \cdot E] dV}{\int_v E^* \cdot \epsilon \cdot E dV} \right] \quad (106)$$

where v' is the volume bounded by S_1 and S_2 . When μ and ϵ are scalars, Eq. 106 is identical with a formula derived by Slater by a different method (see ref. 1, p. 81).

5. Cavity Volume Perturbation. Let a resonator be characterized by ω , E , H , when the permittivity and the permeability of the medium are ϵ , μ . Let ω_1 be the resonant frequency when ϵ , μ change to ϵ_1 , μ_1 , the boundary of the resonator remaining the same. Substituting E , H as trial fields in Eq. 97, we get

$$\omega_1 = \omega \frac{\int H^* \cdot \mu \cdot H dV + \int E^* \cdot \epsilon \cdot E dV}{\int H^* \cdot \mu_1 \cdot H dV + \int E^* \cdot \epsilon_1 \cdot E dV} \quad (107)$$

or, forming the relative frequency shift, we obtain

$$\frac{\Delta\omega}{\omega_1} = \frac{\omega_1 - \omega}{\omega_1} = - \frac{\int H^* \cdot (\mu_1 - \mu) \cdot H dV + \int E^* \cdot (\epsilon_1 - \epsilon) \cdot E dV}{\int H^* \cdot \mu \cdot H dV + \int E^* \cdot \epsilon \cdot E dV} \quad (108)$$

Equation 108 can also be obtained from a well-known formula.*

D. ILLUSTRATIVE EXAMPLES AND PERTURBATION FORMULAS FOR WAVEGUIDES

1. Empty Rectangular Waveguide. As a first example, let us consider a rather trivial problem: the determination of the fundamental mode in a rectangular waveguide. We shall use for this purpose the variational expression given in Eq. 101. On the basis of boundary conditions, an obvious trial field is (see Fig. 12):

* See H. B. G. Casimir, Philips Res. Rep. 6, 162-182 (1951). Essentially the same formula was also derived by H. A. Bethe and J. Schwinger, NDRC, Div. 14, Report D1- 117, M.I.T., March 4, 1943.

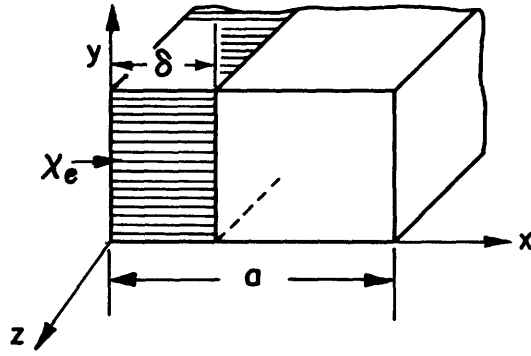


Fig. 10. Rectangular waveguide with dielectric slab.

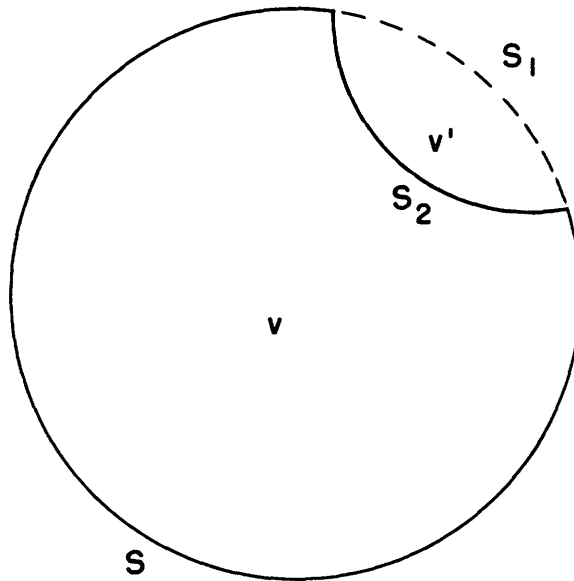


Fig. 11. Cavity with deformed boundary.

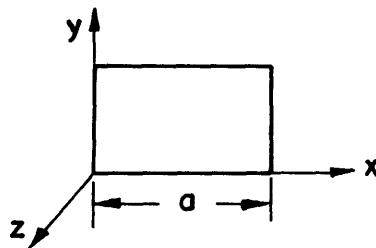


Fig. 12. Coordinates in a rectangular waveguide.

$$E = a_y \sin \frac{\pi x}{a}$$

$$H = a_x A \sin \frac{\pi x}{a} + a_z B \cos \frac{\pi x}{a}$$

so that

$$a_z \times E = -a_x \sin \frac{\pi x}{a}$$

$$a_z \times H = a_y A \sin \frac{\pi x}{a}$$

$$\text{curl } E = a_z \frac{\pi}{a} \cos \frac{\pi x}{a}$$

$$\text{curl } H = a_y \frac{\pi}{a} B \sin \frac{\pi x}{a}$$

where A and B are variational parameters. Substituting in Eq. 101, we obtain

$$\gamma = \frac{\omega\epsilon + \omega\mu A A^* + \omega\mu B B^* + j(\pi/a)(B-B^*)}{-(A^* + A)}$$

Taking the derivative of γ with respect to A^* and setting it equal to zero, we get

$$A = -\frac{\gamma}{\omega\mu}$$

Taking the derivative of γ with respect to B^* and setting it equal to zero, we obtain

$$B = j \frac{\pi/a}{\omega\mu}$$

The field is thus

$$E = a_y \sin \frac{\pi x}{a}$$

$$H = -a_x \frac{\gamma}{\omega\mu} \sin \frac{\pi x}{a} + a_z j \frac{\pi/a}{\omega\mu} \cos \frac{\pi x}{a}$$

Substituting this field in the expression of γ , we obtain

$$\gamma^2 = \omega^2 \mu \epsilon - \left(\frac{\pi}{a}\right)^2$$

This expression, as well as that of the fields, will be readily recognized as the exact solution to our problem. The reason for this is that the assumed class of fields contains the correct solution.

2. Empty Rectangular Waveguide. In the preceding example, we used the mixed-field approach. Let us treat the same example by the E-field formula, Eq. 104. We have, again,

$$E = a_y \sin \frac{\pi x}{a}$$

$$a_z \times E = -a_x \sin \frac{\pi x}{a}$$

$$\text{curl } E = a_z \left(\frac{\pi}{a}\right) \cos \frac{\pi x}{a}$$

Substituting and noting that $(\mu)^{-1} = 1/\mu$ in this case, we get

$$\left(\frac{\pi}{a}\right)^2 + \gamma^2 - \omega^2 \mu \epsilon = 0$$

Thus, as far as evaluating the propagation constant is concerned, this method is simpler and faster because the E-field is reasonably simple to guess. On the other hand, it offers no information about the configuration of the H-field.

3. Empty Rectangular Waveguide. Let us now use the H-field variational equation, described in Section III, part B(3), for the same problem. We have a trial field

$$H = a_x \sin \frac{\pi x}{a} + a_z C \cos \frac{\pi x}{a}$$

with C as a variational parameter. We also have

$$a_z \times H = a_y \sin \frac{\pi x}{a}$$

$$\text{curl } H = a_y \frac{\pi}{a} C \sin \frac{\pi x}{a}$$

Substituting and evaluating C from the condition $\partial\gamma/\partial C^* = 0$, we obtain

$$C = \frac{j\gamma(\pi/a)}{(\pi/a)^2 - \omega^2 \mu \epsilon}$$

Substituting this value back in the variational principle and solving for γ , we get

$$\gamma^2 = \omega^2 \mu \epsilon - \left(\frac{\pi}{a}\right)^2$$

as before. The determination of γ by this method is not as simple as using the variational equation, Eq. 104, since it involves, as an intermediate step, the evaluation of the variational parameter C. It thus illustrates a previous statement to the effect that for a specific case, the choice of a variational principle may be made advantageously by proper use of the physical aspects of the problem.

4. Rectangular Waveguide with a Dielectric Slab at Center. Next, we take up the more practical case already treated by the mode expansion method in example 1, Section II, part B. Our aim this time is to illustrate the use of the variational expressions. We choose for this purpose Eq. 104. This choice is based on the physical fact that the fundamental mode is a TE-mode; the E-field is therefore simpler than the H-field, since it has one less component.

Assuming rather crudely again that the E-field is given by

$$E = a_y \sin \frac{\pi x}{a}$$

we have, after substituting,

$$\frac{1}{\mu_0} \left(\frac{\pi}{a} \right)^2 \frac{a}{2} - j \frac{\gamma}{\mu_0} (0-0) + \gamma^2 \frac{1}{\mu_0} \frac{a}{2} - \omega^2 \epsilon_0 \frac{a}{2} - \omega^2 \epsilon_0 \chi_e \left(\frac{\delta}{2} + \frac{a}{2\pi} \sin \frac{\pi \delta}{a} \right) = 0$$

or, after introducing the free-space and guide wavelengths λ and λ_g , and rearranging,

$$\left(\frac{\lambda}{\lambda_g} \right)^2 = 1 + \chi_e \left(\frac{\delta}{a} + \frac{1}{\pi} \sin \frac{\pi \delta}{a} \right) - \frac{1}{4(a/\lambda)^2} \quad (109)$$

This is the same as Eq. 59 in Section II. The mode-expansion and the variational methods thus yield the same result in this case. Equation 109 has been plotted and compared to the exact solution in Fig. 4.

5. Rectangular Waveguide with Dielectric Slab at Edge. Let us take as our next example the determination of the propagation constant of the fundamental mode of a waveguide with a cross section as shown in Fig. 13. If we try $E = a_y \sin \pi x/a$ as an approximate solution, we find that the error in γ becomes appreciable. This is to be expected, since the assumed form for the electric field clearly violates the nonsymmetrical configuration of the actual field. A better trial field would be

$$E = a_y \left(\sin \frac{\pi x}{a} + A \sin \frac{2\pi x}{a} \right) \quad (110)$$

with A as a variational parameter. Substituting in the variational principle, say Eq. 104, and evaluating A from the condition of $\partial \gamma / \partial A^* = 0$, we obtain

$$A = -R \pm (R^2 + 1)^{1/2} \quad (111)$$

where

$$R = \frac{\frac{3}{2} \left(\frac{\pi}{a} \right)^2 - \frac{\omega^2 \mu_0 \epsilon_0 \chi_e}{2\pi} \sin \frac{2\pi \delta}{a} \sin^2 \frac{\pi \delta}{a}}{\frac{4\omega^2 \mu_0 \epsilon_0 \chi_e}{3\pi} \sin^3 \frac{\pi \delta}{a}}$$

There is little difficulty with the ambiguity of sign in the expression for A. Physically, we know that there is a concentration of the electric field within the dielectric part of the guide. Hence, A must be positive. It follows that 'plus' is the sign to be retained in Eq. 111.

Substituting the electric field thus evaluated in the variational equation and solving for γ or, better, for the ratio of free-space wavelength λ to guide wavelength λ_g , we obtain

$$\left(\frac{\lambda}{\lambda_g}\right)^2 = -\frac{5}{8} \frac{1}{(a/\lambda)^2} + 1 + \chi_e \left(\frac{\delta}{a} - \frac{1}{2\pi} \sin \frac{2\pi\delta}{a} \cos^2 \frac{\pi\delta}{a} \right) + \left\{ \left[\frac{3}{8} \frac{1}{(a/\lambda)^2} - \frac{\chi_e}{2\pi} \right. \right. \\ \left. \left. \times \sin \frac{2\pi\delta}{a} \sin^2 \frac{\pi\delta}{a} \right]^2 + \frac{16}{9} \left(\frac{\chi_e}{\pi} \right)^2 \sin^6 \frac{\pi\delta}{a} \right\}^{1/2} \quad (112)$$

A plot of λ/λ_g versus a/λ , which is essentially a normalized plot of the propagation constant versus frequency, is given for various values of δ/a and for $\chi_e = 1.45$ in Fig. 13. Curves obtained from exact solutions (see ref. 10, p. 387) are also reproduced on the same graph. The agreement is seen to be good. For $\delta = 0$ or $\delta = a$, it is perfect.

Equation 112 has also been obtained as a result of applying the mode-expansion method. Thus, in this case, the two methods of analysis lead to the same approximate answer.

6. Rectangular Waveguide with a Ferrite Slab. A rectangular waveguide with a ferrite slab off-center as shown in Fig. 14 is important as a differential phase shifter. The differential propagation constant is defined as the difference of the propagation constants in the two opposite directions, $\gamma_+ - \gamma_-$. This difference is a consequence of the tensor susceptibility of the ferrite, which for the configuration shown in Fig. 14 is of the form $\chi_{xx} = \chi_{zz} = \chi$; $\chi_{xz} = -j\kappa$, $\chi_{zx} = j\kappa$, $\chi_{xy} = \chi_{yx} = \chi_{yz} = \chi_{zy} = \chi_{yy} = 0$. An important quantity is the displacement of the slab from the wall d for which the differential propagation constant is a maximum. To determine this quantity we must first determine the propagation constant. Let us utilize, for this purpose, Eq. 101 with a trial field selected as follows: For the E-field we let

$$E = a_y \left(\sin \frac{\pi x}{a} + A \sin \frac{2\pi x}{a} \right)$$

where A is a variational parameter. We determine H by substituting this expression of E in Eq. 99 where, for this equation only, μ is assumed to be a scalar and equal to μ_0 , and γ is assumed to equal the propagation constant of the fundamental mode with the ferrite slab replaced by a dielectric one having the same dielectric constant as that of the ferrite, but a permeability equal to that of free space. Substituting E and H in Eq. 101, determining A from the condition $\partial\gamma/\partial A = 0$, and substituting the value of A back into Eq. 101, we obtain γ . Note that this procedure involves several simplifying assumptions and amounts essentially to utilizing the variational principle given by Eq. 101, subject to the constraint of Eq. 99. In Fig. 14 the differential propagation constant plotted as a function of d and for specific values of slab thickness and tensor permeability is shown and compared with the exact solution obtained by Lax, Button, and Roth (14).

Note the agreement in the location of the maximum. The ratio of slab thickness to the width of the guide is 0.044.

For ratios appreciably less than 1 per cent, we can simply substitute for E and H in Eq. 101 the expressions valid for an empty waveguide, taking into account, however, the continuity of the normal component of the magnetic flux density at the surface of the slab. The following result is readily obtained:

$$\gamma_+ - \gamma_- = 2 \frac{\pi}{a} \frac{\delta}{a} \frac{\kappa}{1+\chi} \sin \frac{2\pi d}{a}$$

where δ is the thickness of the slab. This is the expression Lax, Button, and Roth (14) obtained from an expansion of a transcendental equation.

7. Circular Waveguide with Concentric Ferrite Core. We now evaluate the propagation constant of the lowest mode of a circular waveguide with a coaxial ferrite and a steady magnetic field in the axial direction. If we simply substitute the solution of example 4, Section II in the variational expression given in Eq. 101 and plot λ/λ_g versus a/λ , we obtain the curve shown in Fig. 15. It is assumed that the ferrite has a radius of one-tenth of the waveguide radius, that the dielectric constant is 11, and that the components of the susceptibility matrix are $\chi = 2.3$, $\kappa = 3.4$. There are several reasons why the curve on Fig. 15 should be considered with extreme caution. First, the solution of example 4, Section II is a rough approximation itself; second, even if the latter were an exact solution, the result of substituting it in Eq. 101 would be only approximately correct; third, the susceptibility matrix will vary with frequency even when the ferromagnetic resonance is outside the range of the frequencies considered.

8. Waveguide Wall Perturbation. Let E, H, and γ characterize a known solution of a waveguide. Suppose, now, that the cross section is deformed so that s is the cross-sectional area between the original and final cross sections, and let γ_1 be the new propagation constant. Substituting E, H as trial vectors in Eq. 102, we obtain

$$\gamma_1 - \gamma = \omega \frac{\int_s \mathbf{H}^* \cdot \boldsymbol{\mu} \cdot \mathbf{H} dS - \int_s \mathbf{E}^* \cdot \boldsymbol{\epsilon} \cdot \mathbf{E} dS}{\int_S \mathbf{H}^* \cdot \mathbf{a}_z \times \mathbf{E} dS - \int_S \mathbf{E}^* \cdot \mathbf{a}_z \times \mathbf{H} dS} \quad (113)$$

The integrals of the denominator are over the cross-sectional surface of the deformed waveguide.

9. Waveguide Volume Perturbation. Let E, H, and γ characterize a known solution of a waveguide when the permittivity and permeability of the medium are ϵ and μ . Let γ_1 be the propagation constant when ϵ , μ are changed to ϵ_1 , μ_1 , the boundary of the waveguide remaining the same. Substituting E, H as trial fields in Eq. 101, we obtain

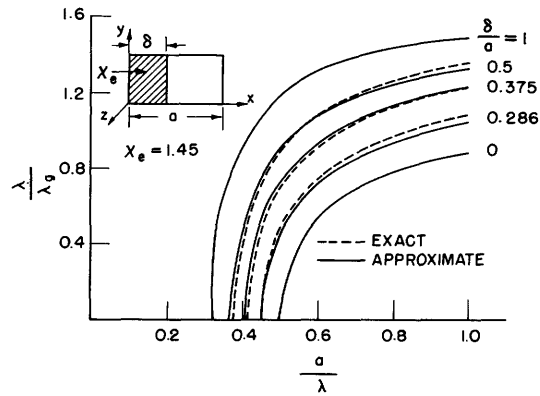


Fig. 13. Propagation constant versus frequency of rectangular waveguide with a dielectric slab.

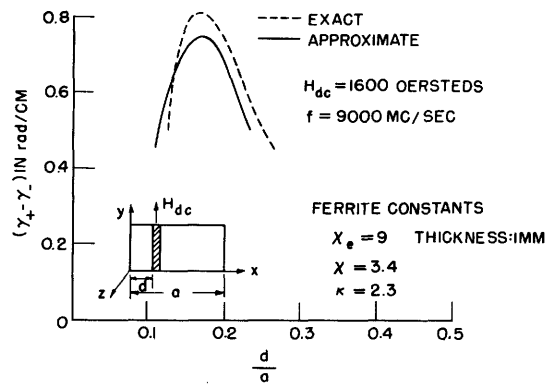


Fig. 14. Differential propagation constant as a function of the location of a ferrite slab in a rectangular waveguide.

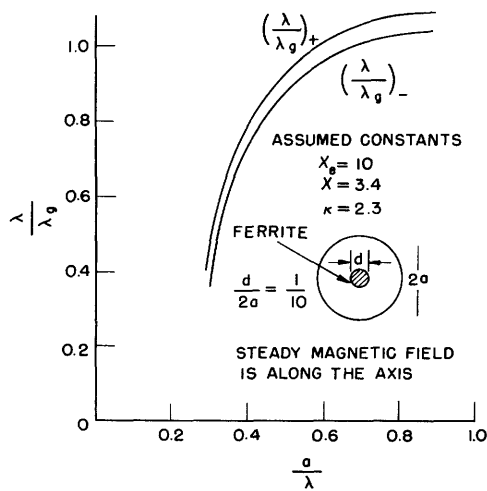


Fig. 15. Circular waveguide with a concentric ferrite rod.

$$\gamma_1 - \gamma = \omega \frac{\int \mathbf{E}^* \cdot (\epsilon_1 - \epsilon) \cdot \mathbf{E} \, dS + \int \mathbf{H}^* \cdot (\mu_1 - \mu) \cdot \mathbf{H} \, dS}{\int \mathbf{H}^* \cdot \mathbf{a}_z \times \mathbf{E} \, dS - \int \mathbf{E}^* \cdot \mathbf{a}_z \times \mathbf{H} \, dS} \quad (114)$$

This formula can also be derived from a formula first derived (unpublished) by B. Lax of Lincoln Laboratory, M.I.T.

Acknowledgment

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APPENDIX I

COMPLETENESS OF CAVITY MODES

We briefly discuss in this appendix the interesting problem of completeness of a set of cavity modes. We do this by giving brief descriptions of Slater's treatment (1), a modified form of Slater's treatment, Teichmann and Wigner's analysis (3), and Schwinger's integral-equation method (4), and by comparing these among themselves and with the approach of part B of Section I.

Method I (Slater's). The problem consists of determining the electromagnetic field in a region containing electric currents and completely enclosed by surfaces S and S' (Fig. I-1) over which

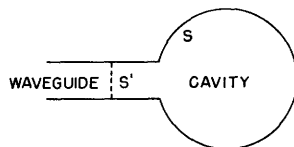


Fig. I-1. Driven cavity.

tangential components of electric and magnetic fields, respectively, are arbitrarily impressed. The fundamental steps of the method are: For the expansion of the divergence-less part of the field, solenoidal electric and magnetic modes are introduced and defined by Eqs. 5(a,b). For the expansion of the irrotational part of the electric field, an irrotational set which is defined by Eq. 6 and the boundary condition of vanishing tangential component on both parts (S and S') of the boundary is introduced. Note that this is different from the boundary condition given in Eq. 7. From the physical absence of magnetic charges, it is concluded that no irrotational modes are needed for the expansion of the magnetic field. If suitable expansions in terms of the preceding modes are substituted for the various quantities in Maxwell's equations, relations (2.6), (2.7), and (2.8) of reference 1 are obtained.

In the less general case in which the tangential electric field is given over the entire surface of the cavity, part S' of the boundary vanishes and the surface integral in Eq. (2.7) of reference 1 disappears. The other two equations remain the same. Note, however, that the various modes satisfy homogeneous 'short-circuit' boundary conditions over the total surface of the cavity.

Returning to Eqs. (2.6), (2.7), and (2.8) of reference 1, we observe that they imply the absence of irrotational magnetic and electric fields when the cavity is empty. The same implication is valid for the case described in the preceding paragraph.

Method II (Modification of Method I). That the last implications are not in general valid will be shown by following essentially the same reasoning as that presented in reference 3. Consider, first, the general case of mixed boundary conditions, that is, where the total boundary of the cavity is formed by S and S' . We shall show that (a) an arbitrary electric field E usually has a component, in function space, along an irrotational mode as defined by Eqs. 6 and 7, (b) it has no

component along an irrotational mode as defined in reference 1, and (c) the modes defined by Eqs. 6 and 7 are orthogonal to the solenoidal electric modes so that they cannot be expanded in terms of the latter. To prove part (a) of the argument, we simply form $\int E \cdot F_b dV$, transform it identically to

$$= \frac{1}{j\omega\epsilon_0} \int_{S'} n \times H \cdot F_b dS \quad (I-1)$$

and note that the last integral may be different from zero. Part (b) is proved by observing that the integral of $E \cdot F_b$, which is again given by Eq. I-1, vanishes, since $n \times F_b$ in reference 1 is zero on S' . To demonstrate part (c) of the argument, we form the volume integral of $E_a \cdot F_b$ which can be transformed to an expression similar to Eq. I-1, where H is replaced by H_a and $j\omega\epsilon_0$ by k_a . This last expression vanishes because of the boundary condition of the magnetic normal modes. By following a similar procedure we can show that the set of irrotational magnetic modes defined by Eq. 8 is needed, in general, to complete the set of solenoidal magnetic modes. The set of Eqs. 10 and 13 in the text is the result of utilizing the modified electric irrotational modes and introducing the set of magnetic irrotational modes.

When the tangential electric field is specified over the total surface of the cavity, it can be shown, by following the general argument just given for the case of mixed boundary conditions, that an irrotational set of magnetic modes that satisfies the vector Helmholtz equation and the boundary condition of vanishing normal component at the boundary is needed.

Method III (Teichmann and Wigner). The main contention in reference 3 is that for an empty cavity driven by an arbitrary distribution of tangential electric field over the total boundary, the solenoidal electric modes are complete but the solenoidal magnetic modes need to be completed with an irrotational magnetic field $H_\beta = \text{grad } u$. The authors demonstrate this contention by showing that H_β is always orthogonal to the solenoidal magnetic modes and that an arbitrary magnetic field has, in general, a component along H_β , in vector function space. Thus, they write for the electromagnetic field:

$$E = \sum_a f_a E_a \quad (I-2)$$

$$H = \sum_a f_a H_a + H_\beta \quad (I-3)$$

Since the divergence of H is undoubtedly zero within the cavity, we must have $\text{div } H_\beta = 0$; consequently, u must satisfy Laplace's equation. From this last condition and from a knowledge of the normal component of the magnetic field at the boundary, u can be found by solving 'the second boundary value problem' of potential theory whose formal solution is given by

$$u = \int_S (n \cdot H) N dS \quad (I-4)$$

where N is the Neumann function of the cavity. Note that the field H may exist only when $n \cdot H$ differs from zero on the surface of the cavity.

It can be shown that the field H_β is identical with the expansion

$$\sum_c \left(\int H \cdot G_c \, dV \right) G_c$$

(with the coefficients determined by Eq. 16 with J_m equal to zero). This can be done easily if we note the equality:

$$\int_S n \times E \cdot G_c \, dS = -\frac{j\omega\mu_0}{k_c} \int_S n \cdot H \psi_c \, dS \quad (I-5)$$

where ψ_c and k_c are defined as

$$k_c G_c = \text{grad } \psi_c$$

$$\nabla^2 \psi_c + k_c^2 \psi_c = 0; \quad \frac{\partial \psi_c}{\partial n} = 0 \text{ on } S$$

Thus, Methods II and III yield essentially the same results.

Method IV (Schwinger). The following steps constitute the essence of this method. First, Maxwell's equations are combined to form

$$\text{curl curl } E - k^2 E = -j\omega\mu_0 J_e - \text{curl } J_m \quad (I-6a)$$

and

$$\text{curl curl } H - k^2 H = -j\omega\epsilon_0 J_m + \text{curl } J_e \quad (I-6b)$$

Then the electric and magnetic Green's dyadics $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are introduced, respectively. These are defined by

$$\text{curl curl } \Gamma^{(1)} - k^2 \Gamma^{(1)} = \delta I \quad (I-7a)$$

$$n \times \Gamma^{(1)} = 0 \text{ on the boundary} \quad (I-7b)$$

and

$$\text{curl curl } \Gamma^{(2)} - k^2 \Gamma^{(2)} = \delta I \quad (I-8a)$$

$$n \times \text{curl } \Gamma^{(2)} = 0 \text{ on the boundary} \quad (I-8b)$$

I is the idem factor; δ the Dirac delta-function. Combining Eqs. I-6(a) and I-7(a) in the usual fashion, making use of the Green's vector identity, and utilizing the boundary condition given in Eq. I-7(b), we obtain for the electric field:

$$\mathbf{E}(\mathbf{r}) = \int_{\mathbf{v}} \Gamma^{(1)}(\mathbf{r}|\mathbf{r}') \cdot \mathbf{f}(\mathbf{r}') dV' - \int_{\mathbf{s}} \mathbf{n} \times \mathbf{E}(\mathbf{r}') \cdot \text{curl } \Gamma^{(1)}(\mathbf{r}|\mathbf{r}') dS' \quad (\text{I-9})$$

The symbol \mathbf{f} is used here to abbreviate the right side of Eq. I-6(a). The influence point is denoted by \mathbf{r}' , the field point by \mathbf{r} .

If the same procedure is followed and Eqs. I-6(b) and I-8(a, b) are used, an expression for the magnetic field is obtained. The result can be derived by replacing \mathbf{E} by \mathbf{H} , $\Gamma^{(1)}$ by $\Gamma^{(2)}$, and \mathbf{f} by the right side of Eq. I-6(b).

The next step consists of expanding the two Green's dyadics in terms of 'short-circuit' normal modes:

$$\Gamma^{(1)} = \sum_{\mathbf{a}} \frac{\mathbf{E}_{\mathbf{a}}(\mathbf{r})\mathbf{E}_{\mathbf{a}}(\mathbf{r}')}{k_{\mathbf{a}}^2 - k^2} - \frac{1}{k^2} \sum_{\mathbf{b}} \mathbf{F}_{\mathbf{b}}(\mathbf{r})\mathbf{F}_{\mathbf{b}}(\mathbf{r}') \quad (\text{I-10})$$

and

$$\Gamma^{(2)} = \sum_{\mathbf{a}} \frac{\mathbf{H}_{\mathbf{a}}(\mathbf{r})\mathbf{H}_{\mathbf{a}}(\mathbf{r}')}{k_{\mathbf{a}}^2 - k^2} - \frac{1}{k^2} \sum_{\mathbf{c}} \mathbf{G}_{\mathbf{c}}(\mathbf{r})\mathbf{G}_{\mathbf{c}}(\mathbf{r}') \quad (\text{I-11})$$

The modes used are identical with those utilized in part B of Section I. By substituting the expansions of the Green's dyadics in Eq. I-9 and its companion for the magnetic field, expansions of the fields in terms of the normal modes are obtained. For a current-free cavity, the fields turn out to be

$$\mathbf{E} = \sum_{\mathbf{a}} \frac{k_{\mathbf{a}} \mathbf{E}_{\mathbf{a}}}{k^2 - k_{\mathbf{a}}^2} \int_{\mathbf{s}} \mathbf{n} \times \mathbf{E} \cdot \mathbf{H}_{\mathbf{a}} dS \quad (\text{I-12})$$

$$\mathbf{H} = \sum_{\mathbf{a}} \frac{j\omega\epsilon_0 \mathbf{H}_{\mathbf{a}}}{k^2 - k_{\mathbf{a}}^2} \int_{\mathbf{s}} \mathbf{n} \times \mathbf{E} \cdot \mathbf{H}_{\mathbf{a}} dS - \sum_{\mathbf{c}} \frac{\mathbf{G}_{\mathbf{c}}}{j\omega\mu_0} \int_{\mathbf{s}} \mathbf{n} \times \mathbf{E} \cdot \mathbf{G}_{\mathbf{c}} dS \quad (\text{I-13})$$

Note that we may have $\text{div } \mathbf{H} = 0$ even though the divergence of the individual irrotational modes is nonvanishing. The preceding equations imply Eqs. 24-27 (for zero current). Therefore, the two methods of analysis, Schwinger's and that presented in part B of Section I, lead to identical results. They differ in two respects, one being a consequence of the other: first, the mathematical formulations of the problem as expressed by the inhomogeneous equations, Eqs. 19 and I-6(b) are different; second, the Green's functions are different.

Schwinger's approach has the advantage over that of part B, Section I, in that it is mathematically less tedious and has a physically more meaningful Green's dyadic. Its disadvantage (if one could call it a disadvantage) is associated with the divergent character of the expansion for the Green's functions. The second series on the right side of Eqs. I-10 and I-11 are indeed divergent, because as \mathbf{b} and \mathbf{c} increase indefinitely the terms of these series remain finite. This

is not a serious disadvantage, since the field expansions themselves are convergent series. In Eq. I-12, for example, $\int \mathbf{n} \times \mathbf{E} \cdot \mathbf{G}_c \, dS$ tends to zero as c increases indefinitely.

The expansions of the Green's functions of part B, Section I do not suffer from this disadvantage because all terms contain factors which tend to zero as b or c increases indefinitely. This kind of Green's dyadics and their expansions are extensively used by Morse and Feshbach (11) in treating problems connected with the vector wave equation.

We shall now resume the discussion of Method IV. Having obtained Eq. I-12, Schwinger reasons as follows. The magnetic field for an empty cavity has no divergence. Hence, it should be expandable in terms of the solenoidal modes only. He then obtains such an expansion by a series of steps, one of which involves taking the curl of the divergent expansion for $\Gamma^{(2)}$. The result is

$$\mathbf{H} = \frac{1}{-j\omega\mu_0} \sum_a \frac{k_a^2}{k^2 - k_a^2} \mathbf{H}_a \int \mathbf{n} \times \mathbf{E} \cdot \mathbf{H}_a \, dS \quad (\text{I-14})$$

Schwinger observes that this expression is exactly what we would get if we substituted the electric field as given by Eq. I-12 in Maxwell's equation and solved for \mathbf{H} .

We believe that this reasoning and Eq. I-14 is in error for the following reasons.

a. A divergence-free field is not necessarily a solenoidal one. Fields derived from potentials satisfying Laplace's equation in a bounded region also have zero divergence. Hence, there is no reason, a priori, why the solenoidal set should be complete.

b. The apparent verification that Eq. I-14 is also the result that we would get if we substituted the electric field solution, Eq. I-12, in Maxwell's equation and solved for the magnetic field is not entirely rigorous; for it assumes that the curl of the infinite sum, Eq. I-12, is equal to the sum of the curls of the individual terms. This is questionable in view of the nonuniformly convergent character of the expansion of the electric field.* We should rather expand the term curl \mathbf{E} directly. To summarize: Eq. I-13 agrees with the solution obtained by other methods, whereas Eq. I-14 is believed to be in error.

Remarks on the Preceding Methods. Methods II, III, and IV (with the solution, in the latter, given by Eqs. I-12 and I-13 but not by Eq. I-14) and that in part B, Section I, although different in approach and mathematical details, are seen to give the same results. We can summarize these in the following way. In a bounded 'empty' region of space excited by a distribution of tangential electric field over the boundary, the electric field is expressible in terms of solenoidal 'short-circuit' electric modes only, whereas the expansion of the magnetic field requires, in addition to the solenoidal magnetic modes, an irrotational term of zero divergence.

In the more general case of mixed boundary conditions, that is, when over part of the

* If we substitute in the coordinates of the boundary, we have $\mathbf{n} \times \mathbf{E}_a = 0$ by definition so that the infinite sum is zero. If, however, we form the infinite sum first and then take the limit as the coordinates approach those of the surface, the sum equals the prescribed electric field at the boundary.

boundary the tangential electric field is specified, and over the rest of the boundary the tangential magnetic field is specified, the expansions of both fields require irrotational terms of zero divergence. Note, however, that in this case the normal modes satisfy different boundary conditions from those of the short-circuit modes.

Remarks on Admittance Calculations. The general method of calculating admittance or impedance matrices of cavities is treated in references 1 and 2 in detail and will not be repeated here. It appears that the admittance, computed with Eqs. I-12 and I-13 as a basis, has, in addition to the usual resonant terms, terms originating from the second sum in Eq. I-13 and having a frequency behavior of $1/\omega$. The latter terms will be absent whenever the normal component of the impressed magnetic field is zero. This is obvious in Method III and can be shown to be true in all the other methods by observing the relation I-5. Hence, in all calculations in which the driving waveguide mode is TM, the irrotational terms vanish. They also vanish when the excitation is through a waveguide with a TEM-mode. In the case of a loop coupling, the zero-frequency behavior (admittance $\rightarrow \infty$ as $\omega \rightarrow 0$) originates from the 'zero-frequency resonant mode' corresponding to $k_a = 0$. Such a mode is possible in a doubly connected region such as we find in a coupling loop and a cavity. Thus, the only instance in which the irrotational terms may exist is when the driving waveguide modes are TE. Whether they are actually excited depends upon the coupling conditions between the waveguide and the cavity.

Similar considerations apply to impedance calculations. When it is a TEM-mode that drives a cavity, no irrotational term is needed; and the zero-frequency behavior of the impedance is caused by the existence of a zero-frequency resonant mode. When the driving waveguide mode is TE, we still have no irrotational modes. The latter may exist only when the excitation of the cavity is through TM modes in the guide.

In view of the preceding discussion, it is important to note that the results given in reference 1 for impedance matrices are complete as long as the driving waveguides are excited in their lowest mode (TEM for coaxial; TE, otherwise). Since this is usually the case, the entire discussion in this appendix, as far as the calculation of impedance of practical cases is concerned, becomes academic.

APPENDIX II

EXPANSION OF MAXWELL'S EQUATIONS IN TERMS OF MODES

The expansions for the field vectors and current densities are straightforward and are based on the orthonormality of the modes. Thus, the coefficient of the E_a mode, for example, is simply given by the integral of $E \cdot E_a$ over the volume of the cavity. The expansions of the curls require a little more consideration. Consider the curl of the electric field, for example. If the series expansion of the electric field were differentiable term-by-term, we could simply take the sum of the curls. The sum representing the electric field is, however, nonuniformly convergent. To see this we may note that the value of the tangential component of the electric field at S is by hypothesis nonvanishing, while its expansion is in terms of cavity modes with vanishing tangential components. Thus, the value of the sum depends upon the order in which we take the limit of summation and the limit of approaching the boundary from the interior of the cavity.

This difficulty can be avoided by following the artifice used in reference 1 (p. 64) and expanding the curls independently. For the curl of the electric field, for example, we first write, formally,

$$\text{curl } E = \sum_a \left(\int \text{curl } E \cdot H_a \, dV \right) H_a + \sum_c \left(\int \text{curl } E \cdot G_c \, dV \right) G_c$$

and then compute the integrals that constitute the coefficients of expansion. The procedure of evaluating the first integral is given in reference 1 (p. 64) and will not be repeated here. To calculate the second integral, we transform it (using the vector identity involving the divergence of a cross product) to the expression:

$$\int \text{curl } E \cdot G_c \, dV = \int_S n \times E \cdot G_c \, dS$$

In obtaining this result, we have utilized the vector identity $\text{curl grad} = 0$ and the boundary condition that the tangential component of G_c at S is zero.

The expansion of the curl of the electric field will thus, in general, require irrotational terms. We may note, incidentally, that although each of these terms has a nonvanishing divergence, it is perfectly possible that the sum of the divergences is zero. Note also that the right side of the preceding equation can be rewritten as

$$= -\frac{j\omega\mu_0}{k_c} \int_S n \cdot H \psi_c \, dS$$

by using the vector identity for the divergence of a product of a scalar and a vector and by introducing the scalar functions ψ_c defined by

$$\nabla^2 \psi_c + k_c^2 \psi_c = 0; \quad \psi_c = 0 \text{ on } S', \quad \frac{\partial \psi_c}{\partial n} = 0 \text{ on } S,$$

and

$$G_c = \frac{1}{k_c} \text{grad } \psi_c$$

Thus, the modes G_c may have nonvanishing coefficients only when $n \cdot H$ is different from zero over the surface S .

APPENDIX III

IMPEDANCE MATRIX OF A CAVITY CONTAINING A FERRITE SPHERE

By hypothesis we have for the electric field in the cavity

$$\vec{E} = \epsilon_a \vec{E}_a + \epsilon_\beta \vec{E}_\beta \quad (\text{III-1})$$

and for the magnetic field

$$\vec{H} = h_a \vec{H}_a + h_\beta \vec{H}_\beta \quad (\text{III-2})$$

The subscripts refer to the two degenerate TE_{111} -modes; ϵ_a , ϵ_β , h_a , and h_β are expansion coefficients. The transverse fields \vec{E}_1, \vec{H}_1 at S'_1 (see Fig. III-1) can be expressed in terms of

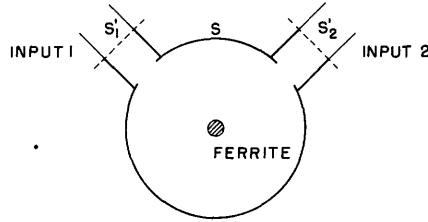


Fig. III-1. A TE_{111} cylindrical cavity of the transmission type containing a small sphere of ferrite.

\vec{E}_{t1} and \vec{H}_{t1} , the fundamental orthonormal transverse electric and magnetic modes of the waveguide of input 1, as

$$\vec{E}_1 = V_1 \vec{E}_{t1} \quad (\text{III-3})$$

$$\vec{H}_1 = i_1 Z_{c1} \vec{H}_{t1} \quad (\text{III-4})$$

where Z_{c1} is the characteristic impedance of the waveguide of input 1; V_1 and i_1 are coefficients indicating the intensity of the transverse electric and magnetic fields. Similarly, we have

$$\vec{H}_2 = i_2 Z_{c2} \vec{H}_{t2} \quad (\text{III-5})$$

and

$$\vec{E}_2 = V_2 \vec{E}_{t2} \quad (\text{III-6})$$

We now substitute Eqs. III-1 and III-2 in Eqs. 10 and 11 and obtain

$$k_a \epsilon_a + [Z_s H_{aa}^s + j\omega\mu_o (1 + J_{aa})] h_a + [Z_s H_{\beta a}^s + j\omega\mu_o J_{\beta a}] h_\beta = 0$$

$$k_{\beta} e_{\beta} + [Z_s H_{a\beta}^s + j\omega\mu_o J_{a\beta}] h_a + [Z_s H_{\beta\beta}^s + j\omega\mu_o (1 + J_{\beta\beta})] h_{\beta} = 0 \quad (\text{III-7})$$

$$j\omega\epsilon_o [(1 + I_{a\alpha}) e_a + I_{\beta\alpha} e_{\beta}] - k_a h_a = L_1$$

$$j\omega\epsilon_o [I_{a\beta} e_a + (1 + I_{\beta\beta}) e_{\beta}] - k_{\beta} h_{\beta} = L_2$$

where $k_a = k_{\beta} = \omega_o(\mu_o\epsilon_o)^{1/2}$, the wave number of each degenerate TE₁₁₁-mode; the remaining symbols are:

$Z_s = (1 + j)(\omega\mu/2\sigma)^{1/2}$, the surface impedance of the metal boundary;

$$J_{pq} = \int \vec{H}_p \cdot \vec{\chi}_m \cdot \vec{H}_q dv \quad ; \quad I_{pq} = \int \chi_e \vec{E}_p \cdot \vec{E}_q dv$$

$$H_{pq}^s = \int_s \vec{H}_p \cdot \vec{H}_q dS \quad ; \quad p, q = a \text{ or } \beta$$

$$L_1 = \int_{s'_1} \vec{n} \times \vec{H} \cdot \vec{E}_a dS \quad ; \quad L_2 = \int_{s'_2} \vec{n} \times \vec{H} \cdot \vec{E}_{\beta} dS$$

$\vec{\chi}_m$ is the magnetic susceptibility tensor

χ_e is the dielectric susceptibility.

From the set of equations in Eq. III-7, we determine the values of e_a , e_{β} , h_a , and h_{β} . Hence the electric field in the cavity, given by Eq. III-1, is known. If \vec{E}_{ta1} is the tangential component of \vec{E}_a at S'_1 , and $\vec{E}_{t\beta1}$ is the tangential component of \vec{E}_{β} at S'_1 , then from Eq. III-1

$$\vec{E}_1 = e_a \vec{E}_{ta1} + e_{\beta} \vec{E}_{t\beta1} \quad (\text{III-8})$$

Similarly,

$$\vec{E}_2 = e_a \vec{E}_{ta2} + e_{\beta} \vec{E}_{t\beta2} \quad (\text{III-9})$$

Because of the nature of the TE₁₁₁-modes, we have $E_{t\beta1} = 0$ and $\vec{E}_{ta2} = 0$. Furthermore, \vec{E}_{ta1} and $\vec{E}_{t\beta2}$ differ from the waveguide modes, \vec{E}_{t1} and \vec{E}_{t2} , simply by the coupling coefficients v_{a1} and $v_{\beta2}$. Hence

$$\vec{E}_1 = e_a v_{a1} \vec{E}_{t1} \quad (\text{III-10})$$

$$\vec{E}_2 = e_{\beta} v_{\beta2} \vec{E}_{t2} \quad (\text{III-11})$$

Also,

$$L_1 = i_1 v_{a1} \quad , \quad L_2 = i_2 v_{\beta2} \quad (\text{III-12})$$

If we compare Eq. III-3 with Eq. III-10 and Eq. III-6 with Eq. III-11, we obtain

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

where the impedance matrix is given by Eqs. 28 and 29.

APPENDIX IV

GENERATION OF A COMPLETE SET OF WAVEGUIDE MODES

A general method of obtaining the two sets, Eqs. 43(a, b, c) and 45(a, b, c), is fairly well known (see, for example, ref. 11, sec. 13.1). It consists of the following steps. We first give the three vector solutions in terms of scalar functions χ

$$E^a = \text{curl} (a_z \chi)$$

$$E^b = \text{curl curl} (a_z \chi)$$

$$E^c = \text{grad} \chi$$

These will satisfy the vector Helmholtz equation if χ satisfies the scalar Helmholtz equation. χ is further specified by the boundary conditions of vanishing tangential electric field at the walls of the waveguide. For the E^a -modes χ turns out to be ψ_n , while for the E^b -, E^c -modes it becomes ϕ_n . The definitions of ψ_n and ϕ_n have been given in Eq. 44(a). The derivation of the magnetic normal modes is similar except that the boundary conditions of χ are derived from the condition of vanishing tangential component of the curl of the modes.

Ordinarily, this method of deriving complete vector sets is followed whenever we wish the irrotational part of the solution to be separate from the solenoidal part, the curl and curl curl terms giving the part of the field without divergence, and the grad term giving the part with no curl. Note, however, that in our case E^a , H^a , and so forth, are really not entire expressions for the field vectors but only the part which is independent of z . Hence, the total fields, $E_n^a \exp(-j\gamma_n z)$, and the like, will, in general, have both solenoidal and irrotational parts. Thus, the preceding technique of deriving a complete set of modes is followed only for mathematical expediency.

The orthogonality of these modes can easily be proved by repeated application of vector identities, transformation from volume integrals to surface integrals, and application of the boundary conditions. Thus, we have

$$\int E_n^p \cdot E_m^q dS = \delta_{nm} \delta_{pq} \quad ; \quad \int H_n^p \cdot H_m^q dS = \delta_{nm} \delta_{pq}$$

where $p, q = a, b, c$.

These modes, as defined in Eqs. 43 and 45 are also normalized (as implied by the preceding expressions), provided ψ_n and ϕ_n are normalized.

APPENDIX V

FIELD EXPRESSIONS OF $E_{11}^{a, b, c}$ -MODES IN A CIRCULAR WAVEGUIDE

We have

$$E_{11}^a = A e^{j\vartheta} \left[a_r \frac{j}{1.84 \frac{r}{a}} J_1 \left(1.84 \frac{r}{a} \right) - a_\vartheta J_1' \left(1.84 \frac{r}{a} \right) \right]$$

$$E_{11}^b = a_z B e^{j\vartheta} J_1 \left(3.83 \frac{r}{a} \right)$$

$$E_{11}^c = C e^{j\vartheta} \left[a_r J_1' \left(3.83 \frac{r}{a} \right) + a_\vartheta \frac{j}{3.83 \frac{r}{a}} J_1 \left(3.83 \frac{r}{a} \right) \right]$$

J_1 is the Bessel function of first order, J_1' is its derivative, and a_r and a_ϑ are the unit vectors of the radial and angular coordinates r and ϑ . A , B , and C are normalization factors.

We also have

$$\alpha_{11} = \frac{1.84}{a} ; \quad \beta_{11} = \frac{3.83}{a}$$

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