CODING AND DECODING FOR TIME-DISCRETE AMPLITUDE-CONTINUOUS MEMORYLESS CHANNELS

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Abstract

In this report we consider some aspects of the general problem of encoding and decoding for time-discrete, amplitude-continuous memoryless channels. The results can be summarized under three main headings.

1. Signal Space Structure: A scheme for constructing a discrete signal space, for which sequential encoding-decoding methods are possible for the general continuous memoryless channel, is described. We consider random code selection from a finite ensemble. The engineering advantage is that each code word is sequentially generated from a small number of basic waveforms. The effects of these signal-space constraints on the average probability of error, for different signal-power constraints, are also discussed.

2. Decoding Schemes: The application of sequential decoding to the continuous asymmetric channel is discussed. A new decoding scheme for convolutional codes, called successive decoding, is introduced. This new decoding scheme yields a bound on the average number of decoding computations for asymmetric channels that is tighter than has yet been obtained for sequential decoding. The corresponding probabilities of error of the two decoding schemes are also discussed.

3. Quantization at the Receiver: We consider the quantization at the receiver, and its effects on probability of error and receiver complexity.

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GLOSSARY

Symbol	Definition
a	Number of input symbols per information digit
$A = \frac{\xi}{\sigma}$	Voltage signal-to-noise ratio
$A_{max} = \frac{\xi_{max}}{\sigma}$	Maximum signal-to-noise ratio
b	Number of branches emerging from each branching point in the convolutional tree code
C	Channel capacity
$D(u, v) = \ln \frac{f(v)}{p(v\|u)}$	The "distance" between u and v
$d(x, y) = \ln \frac{f(y)}{p(y\|x)}$	The "distance" between x and y
d	Dimensionality (number of samples) of each input symbol
$E(R)$	Optimum exponent of the upper bound to the probability of error (achieved through random coding)
$E_{\ell, d}(R)$	Exponent of the upper bound to the probability of error when the continuous input space is replaced by the discrete input set X_ℓ
$f(y)$	A probabilitylike function (Appendix A)
$g(s)$, $g(r, t)$	Moment-generating functions (Appendix A)
i	Number of source information digits per constraint length (code word)
ℓ	Number of input symbols (vectors) in the discrete input space X_ℓ
m	Number of d-dimensional input symbols per constraint length (code word)
n	Number of samples (dimensions) per constraint length (code word)
$\overline{N}$	Average number of computations
P	Signal power
R	Rate of information per sample
R_{crit}	Critical rate above which E(R) is equal to the exponent of the lower bound to the probability of error

GLOSSARY

Symbol	Definition
R_{comp}	Computational cutoff rate (Section III)
U	The set of all possible words of length n samples
u	Transmitted code word
u'	A member of U other than the transmitted message u
V	The set of all possible output sequences
v	The output sequence (a member of V)
X	The set of all possible d-dimensional input symbols
x	A transmitted symbol
x'	A member of X other than x
X_ℓ	The discrete input set that consists of ℓ d-dimensional vectors (symbols)
Y	The set of all possible output symbols
Ξ	The set of all possible input samples
ξ	A sample of the transmitted waveform u
ξ'	A sample of u'
H	The set of all possible output samples
η	A sample of the received sequence v
σ^2	The power of a Gaussian noise

I. INTRODUCTION

We intend to study some aspects of the problem of communication by means of a memoryless channel. A block diagram of a general communication system for such a channel is shown in Fig. 1. The source consists of M equiprobable words of length T seconds each. The channel is of the following type: Once each T/n seconds a real number is chosen at the transmitting point. This number is transmitted to the receiving point but is perturbed by noise, so that the i^{th} real number ξ_i is received as η_i. Both ξ and η are members of continuous sets and therefore the channel is time-discrete but amplitude-continuous.

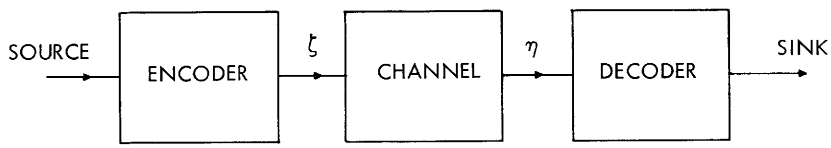

Fig. 1. Communication system for memoryless channels.

The channel is also memoryless in the sense that its statistics are given by a probability density $p(\eta_i|\xi_1, \xi_2, \ldots, \xi_i)$ so that

$$p(\eta_i|\xi_1, \xi_2, \ldots, \xi_i) = p(\eta_i|\xi_i), \tag{1}$$

where

$$p(\eta_i|\xi_i) = p(\eta|\xi); \quad \xi = \xi_i, \eta = \eta_i, \tag{2}$$

and η_i is independent of η_j for $i \neq j$.

A code word, or signal, of length n for such a channel is a sequence of n real numbers $(\xi_1, \ldots, \xi_n)$. This may be thought of geometrically as a point in n-dimensional Euclidean space. The type of channel that we are studying is, of course, closely related to a bandlimited channel (W cycles per seconds wide). For such a bandlimited channel we have n = 2WT.

The encoder maps the M messages into a set of M code words (signals). The decoding system for such a code is a partitioning of the n-dimensional output space into M subsets corresponding to the messages from 1 to M.

For a given coding and decoding system there is a definite probability of error for receiving a message. This is given by

$$P_e = \frac{1}{M} \sum_{i=1}^{M} P_{e_i}, \tag{3}$$

where P_{e_i} is the probability, if message i is sent, that it will be decoded as a message other than i.

The rate of information per sample is given by

$$R = \frac{1}{n} \ln M . \tag{4}$$

We are interested in coding systems that, for a given rate R, minimize the probability of error, P_e.

In 1959, C. E. Shannon[1] studied coding and decoding systems for a time-discrete but amplitude-continuous channel with additive Gaussian noise, subject to the constraint that all code words were required to have exactly the same power. Upper and lower bounds were found for the probability of error when optimal codes and optimal decoding systems were used. The lower bound followed from sphere-packing arguments, and the upper bound was derived by using random coding arguments.

In random coding for such a Gaussian channel one considers the ensemble of codes obtained by placing M points randomly on a surface of a sphere of radius $\sqrt{nP}$ (where nP is the power of each one of M signals, and n = 2WT, with T the time length of each signal, and W the bandwidth of the channel). More precisely, each point is placed independently of all other points with a probability measure proportional to surface area, or equivalently to solid angle. Shannon's upper and lower bounds for the probability of error are very close together for signaling rates from some R_{crit} up to channel capacity C.

R. M. Fano[2] has recently studied the general discrete memoryless channel. The signals are not constrained to have exactly the same power. If random coding is used, the upper and lower bounds for the probability of error are very close together for all rates R above some R_{crit}.

The detection scheme that was used in both of these studies is an optimal one, that is, one that minimizes the probability of error for a given code. Such a scheme requires that the decoder compute an a posteriori probability measure, or a quantity equivalent to it, for each of, say, the M allowable code words.

In Fano's and Shannon's cases it can be shown that a lower bound on the probability of error has the form

$$P_e \geqslant K^* e^{-E^*(R)n}, \tag{5a}$$

where K^* is a constant independent of n. Similarly, when optimum random coding is used, the probability of error is upper-bounded by

$$P_e \leqslant Ke^{-E(R)n}; \quad E(R) = E^*(R) \qquad \text{for } R \geqslant R_{crit}. \tag{5b}$$

In general, construction of a random code involves the selection of messages with some probability density P(u) from the set U of all possible messages. When P(u) is such that E(R) is maximized for the given rate R, the random code is called optimum.

The behavior of $E^*(R)$ and E(R) as a function of R is illustrated in Fig. 2. Fano's upper-bounding technique may be extended to include continuous channels, for all cases in which the integrals involved exist. One such case is the Gaussian channel. However, the lower bound is valid for discrete channels only. Therefore, as far as the continuous channel is concerned, the upper and lower bounds are not necessarily close together for rates $R \geqslant R_{crit}$.

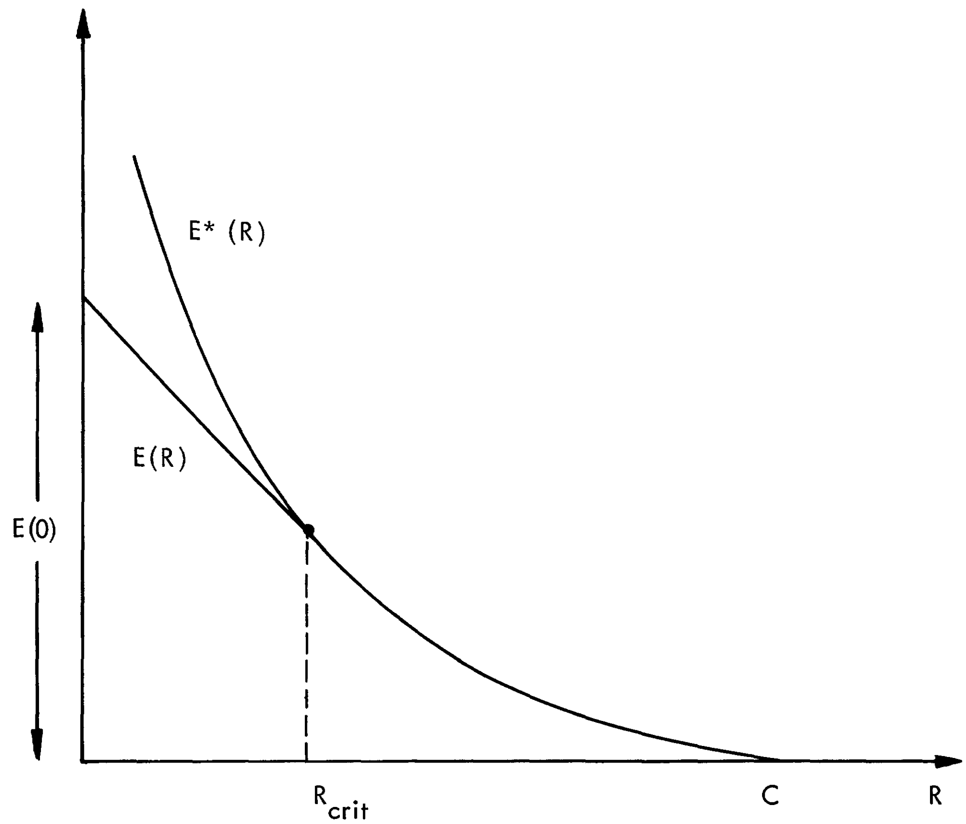

Fig. 2. Behavior of $E^*(R)$ and $E(R)$ as a function of R.

The characteristics of many continuous physical channels, when quantized, are very close to the original ones if the quantization is fine enough. Thus, for such channels, we have $E^*(R) = E(R)$ for $R \geqslant R_{crit}$.

We see from Fig. 2 that the specification of an extremely small probability of error for a given rate R implies, in general, a significantly large value for the number of words M and for the number of decoding computations.

J. L. Kelly[3] has derived a class of codes for continuous channels. These are block codes in which the (exponentially large) set of code words can be computed from a much smaller set of generators by a procedure analogous to group coding for discrete channels. Unfortunately, there seems to be no simple detection procedure for these codes. The receiver must generate each of the possible transmitted combinations and must then compare them with the received signal.

The sequential coding scheme of J. M. Wozencraft,[4] extended by B. Reiffen,[5] is a code that is well suited to the purpose of reducing the number of coding and decoding computations.[6] They have shown that, for a suitable sequential decoding scheme, the average number of decoding computations for channels that are symmetric at their output is bounded by an algebraic function of n for all rates below some R_{comp}. (A channel with transition probability matrix $P(y|x)$ is symmetric at its output if the set of probabilities $P(y|x_1)$, $P(y|x_2)$, ... is the same for all output symbols y.) Thus, the average number of decoding computations is not an exponential function of n as is the case when an optimal detection scheme is used.

In this research, we consider the following aspects of the general problem of encoding and decoding for time-discrete memoryless channels: (a) Signal-space structure, (b) sequential decoding schemes, and (c) the effect of quantization at the receiver. Our results for each aspect are summarized below.

(a) Signal-space structure: A scheme for constructing a discrete signal space, in such a way as to make the application of sequential encoding-decoding possible for the general continuous memoryless channel, is described in Section II. In particular, whereas Shannon's work[1] considered code selection from an infinite ensemble, in this investigation the ensemble is a finite one. The engineering advantage is that each code word can be sequentially generated from a small set of basic waveforms. The effects of these signal-spare constraints on the average probability of error, for different signal power constraints, are also discussed in Section II.

(b) Sequential decoding schemes: In Section III we discuss the application of the sequential decoding scheme of Wozencraft and Reiffen to the continuous asymmetric channel. A lower bound on R_{comp} for such a channel is derived. The Wozencraft-Reiffen scheme provides a bound on the average number of computations which is needed to discard all of the messages of the incorrect subset. No bound on the total number of decoding computations for asymmetric channels has heretofore been derived.

A new systematic decoding scheme for sequentially generated random codes is introduced in Section III. This decoding scheme, when averaged over the ensemble of code words, yields an average total number of computations that is upper-bounded by a quantity proportional to n^2, for all rates below some cutoff rate R_{comp}.

The corresponding probabilities of error of the two decoding schemes are also discussed in Section III.

(c) Quantization at the receiver: The purpose of introducing quantization at the receiver is to curtail the utilization of analogue devices. Because of the large number of computing operations that are carried out at the receiver, and the large flow of information to and from the memory, analogue devices may turn out to be more complicated and expensive than digital devices. In Section IV, the process of quantization at the receiver and its effect on the probability of error and the receiver complexity are discussed.

II. SIGNAL-SPACE STRUCTURE

We shall introduce a structured signal space, and investigate the effect of the particular structure on the probability of error.

2.1 THE BASIC SIGNAL-SPACE STRUCTURE

Let each code word of length n channel samples be constructed as a series of m elements, each of which has the same length d, as shown in Fig. 3. Each one of the m elements is a member of a finite input space X_ℓ which consists of ℓ d-dimensional vectors (d = n/m), as shown in Fig. 3. The advantage of such a structure is that a set of randomly constructed code words may be generated sequentially,[4, 5] as discussed in section 2.4.

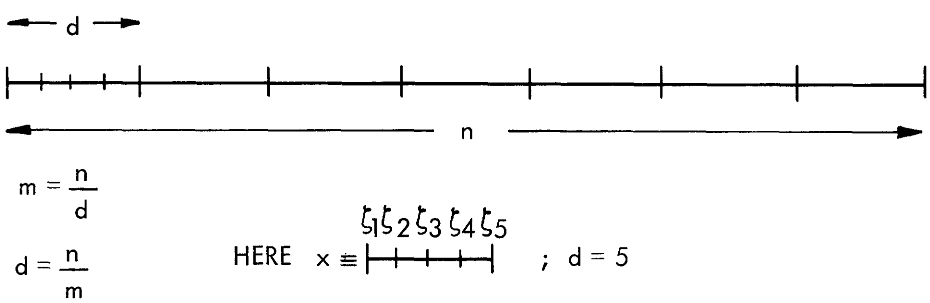

Fig. 3. Construction of a code word as a series of elements.

Two cases will be considered:

Case 1: The power of each of the n samples is less than or equal to P.

Case 2: All code words have exactly the same power nP.

2.2 THE EFFECT OF THE SIGNAL-SPACE STRUCTURE ON THE AVERAGE PROBABILITY OF ERROR – CASE 1

In order to evaluate the effect of a constrained input space on the probability of error, let us first consider the unrestricted channel.

The constant memoryless channel is defined by the set of conditional probability densities $p(\eta|\xi)$, where ξ is the transmitted sample, and η is the corresponding channel output. The output η is considered to be a member of a continuous output ensemble H. From case 1, we have

$$|\xi| \leq \sqrt{P}. \tag{6a}$$

Let us consider the optimal unrestricted random code for which each particular message of length n is constructed by selecting the n samples independently at random with probability density $p(\xi)$ from a continuous ensemble Ξ. Then, following Fano,[2] it can be shown (Appendix A.4) that the average probability of error over the ensemble of codes is bounded by

$$P_e \leqslant \begin{cases} 2e^{-nE(R)}; & R_{crit} \leqslant R < C \\ e^{-nE(R)} = e^{-n[E(0)-R]}; & 0 \leqslant R \leqslant R_{crit} \end{cases} \tag{6b}$$

where $R = 1/n \ln M$ is the rate per sample, and $E(R)$ is the optimum exponent in the sense that it is equal, for large n and for $R \geqslant R_{crit}$, to the exponent of the lower bound to the average probability of error (Fig. 2). For any given rate R, $p(\xi)$ is chosen so as to maximize $E(R)$ [i.e., to minimize P_e].

Let us now constrain each code word to be of the form shown in Fig. 3, with the exception that we let the set X_ℓ be replaced by a continuous ensemble with an infinite, rather than finite, number of members. We shall show that in this case, the exponent $E_d(R)$ of the upper bound to the average probability of error for such an input space can be made equal to the optimum exponent $E(R)$.

THEOREM 1: Let us introduce a random code that is constructed in the following way: Each code word of length n consists of m elements, where each element x is an d-dimensional vector

$$x = \xi_1, \xi_2, \ldots, \xi_d \tag{7}$$

selected independently at random with probability density $p(x)$ from the d-dimensional input ensemble X. Let the output event y that corresponds to x be

$$y = \eta_1, \eta_2, \ldots, \eta_d. \tag{8}$$

Here, y is a member of a d-dimensional output ensemble Y. The channel is defined by the set of conditional probabilities

$$p(y|x) = \prod_{i=1}^{d} p(\eta_i|\xi_i) \tag{9}$$

Also, let

$$p(x) = \prod_{i=1}^{d} p(\xi_i), \tag{10}$$

where $p(\xi_i) \equiv p(\xi)$, for all i, is the one-dimensional probability density that yields the optimum exponent $E(R)$. The average probability of error is then bounded by

$$P_e \geqslant \begin{cases} 2\exp[-nE_d(R)]; & R_{crit} \leqslant R < C \\ \exp[-nE(R)] = \exp[-n[E_d(0)-R]]; & R \leqslant R_{crit} \end{cases}, \tag{11}$$

where

$$E_d(R) \equiv E(R); \quad E_d(0) \equiv E(0). \tag{12}$$

PROOF 1: The condition given by Eq. 10 is statistically equivalent to an independent, random selection of each one of the d samples of each element x. This corresponds to the construction of each code word by selecting each of the n samples independently at random with probability density $p(\xi)$ from the continuous space Ξ, and therefore by Eqs. 10 and 6, yields the optimum exponent E(R). Q.E.D.

The random code given by (9) is therefore an optimal random code, and yields the optimal exponent E(R).

We now proceed to evaluate the effect of replacing the continuous input space x by the discrete d-dimensional input space x, which consists of ℓ vectors. Consider a random code, for which the m elements of each word are picked at random with probability $1/\ell$ from the set X_ℓ of ℓ waveforms (vectors)

$$X_\ell = \{x_k; k=1, \ldots, \ell\}. \tag{13}$$

The length or dimensionality of each x_k is d. Now let the set X_ℓ be generated in the following fashion: Each vector x_k is picked at random with probability density $p(x_k)$ from the continuous ensemble X of all d-dimensional vectors matching the power constraint of Statement 2-1. The probability density $p(x_k)$ is given by

$$p(x_k) \equiv p(x); \quad k = 1, \ldots, \ell \tag{14}$$

where p(x) is given by Eq. 10. Thus, we let $p(x_k)$ be identical with the probability density that was used for the generation of the optimal unrestricted random code. We can then state the following theorem.

THEOREM: Let the general memoryless channel be represented by the set of probability densities $p(y|x)$. Given a set X_ℓ, let $E_{\ell,d}(R)$ be the exponent of the average probability of error over the ensemble of random codes constructed as above. Let $\overline{E_{\ell,d}(R)}$ be the expected value of $E_{\ell,d}(R)$ averaged over all possible sets X_ℓ.

Now define a tilted probability density for the product space XY

$$Q(x,y) = \frac{e^{sD(x,y)}p(x)\,p(y|x)}{\int_Y \int_X e^{sD(x,y)}p(x)\,p(y|x)\,dxdy} = \frac{p(x)\,p(y|x)^{1-s}f(y)^s}{\int_Y \int_X p(x)\,p(y|x)^{1-s}f(y)^s\,dxdy} \tag{15}$$

where

$$f(y) = Q(y) = \frac{\left[\int_X p(x)\,p(y|x)^{1-s}\,dx\right]^{1/1-s}}{\int_Y \left[\int_X p(x)\,p(y|x)\,dx\right]^{1/1-s} dy}; \quad 0 \leqslant s \leqslant \frac{1}{2}$$

$$Q(x|y) = \frac{Q(x,y)}{Q(y)} = \frac{p(x)\,p(y|x)^{1-s}}{\int_X p(x)\,p(y|x)^{1-s}\,dx}; \quad 0 \leq s \leq \frac{1}{2}$$

Then

Part 1. $$\overline{E_{\ell,d}(R)} \geq E(R) - \frac{1}{d}\ln\frac{\exp[F_1(R)] + \ell - 1}{\ell}, \tag{16}$$

s and $F_1(R)$ are related parametrically to the rate R as shown below.

$$0 \leq F_1(R) = \ln \frac{\int_X \int_Y p(x)\,p(y|x)^{2(1-s)}\,Q(y)^{2s-1}\,dxdy}{\int_Y \left[\int_X p(x)\,p(y|x)^{1-s}\,dx\right]^2 Q(y)^{2s-1}\,dy}; \quad 0 \leq s \leq \frac{1}{2} \tag{17}$$

$$R = \frac{1}{d}\int_X \int_Y Q(x,y)\ln\frac{Q(x|y)}{p(x)}\,dxdy \geq R_{crit}$$

$$R_{crit} = [R]_s = \frac{1}{2}.$$

Also, when $R \leq R_{crit}$,

$$F_1(R) = F_1(R_{crit}) = dE(0) = -\ln \int_Y \left[\int_X p(x)\,p(y,x)^{1/2}\,dx\right]^2 dy; \quad s = \frac{1}{2}. \tag{18}$$

Part 2. $$\overline{E_{\ell,d}(R)} \geq E\left(R + \frac{1}{d}\ln\frac{\exp[F_2(R)] + \ell - 1}{\ell}\right), \tag{19}$$

where $F_2(R)$ is related parametrically to the rate R by

$$0 \leq F_2(R) = \ln \frac{\int_X \int_Y p(x)\,p(y|x)^{2(1-s)}\,Q(y)^{2s-1}\,dxdy}{\int_Y \left[\int_X p(x)\,p(y|x)^{1-s}\right]^2 Q(y)^{2s-1}\,dxdy}; \quad 0 \leq s \leq \frac{1}{2} \tag{20a}$$

$$R = \frac{1}{d}\int_X \int_Y Q(x,y)\ln\frac{Q(x|y)}{p(x)}\,dxdy - \frac{1}{d}\ln\frac{\exp[F_2(R)d] + \ell - 1}{\ell}$$

$$\geq R_{crit} - \frac{1}{2}\ln\frac{e^{E(0)d} + \ell - 1}{\ell}. \tag{20b}$$

Also, when $R \leq R_{crit} - \frac{1}{d}\ln\frac{e^{E(0)d} + \ell - 1}{\ell}$,

$$F_2(R) = F_2(R_{crit}) = E(0) = -\frac{1}{d}\ln \int_Y \left[\int_X p(x)\,p(y|x)^{1/2}\,dx\right]^2 dy. \tag{21}$$

PROOF: Given the set X_ℓ, each of the successive elements of a code word is generated by first picking the index k at random with probability $1/\ell$ and then taking x_k to be the element. Under these circumstances, by direct analogy with Appendix A, Eqs. A-46, A-41, and A-26 with the index k replacing the variable x, the average probability of error is bounded by

$$p(e|X_\ell) \leq \exp\left[-nE^{(1)}_{\ell,d}(R)\right] + \exp\left[-nE^{(2)}_{\ell,d}(R)\right], \tag{22}$$

where

$$E^{(1)}_{\ell,d}(R) = -R - \frac{1}{d}\left[\gamma_{\ell,d}(t,r) - r\frac{D_o}{m}\right] \tag{23}$$

$$E^{(2)}_{\ell,d}(R) = -\frac{1}{d}[\gamma_{\ell,d}(s)-sD_o] \tag{24a}$$

$$\gamma_{\ell,d}(t,r) = \ln g_{\ell,d}(t,r) \tag{24b}$$

$$g_{\ell,d}(t,r) = \int_Y \sum_{k=1} \sum_{k=1} p(k)\,p(k')\,p(y|k)\,\exp[(r-t)D(ky)+tD(k',y)]\,dy; \quad r \leq 0; \quad t \leq 0$$

$$= \int_Y \sum_{i=1} \sum_{j=1} \frac{1}{\ell^2}\,p(y|x_i)\,\exp[(r-t)D(x_iy)+tD(x_jy)]\,dy; \quad \text{with } r \leq 0; \quad t \leq 0, \tag{24c}$$

$$D(ky) = D(x_k,y) = n\,\frac{f(y)}{p(y|x_k)} \tag{25a}$$

Here, f(y) is a positive function of y satisfying $\int_Y f(y)\,dy = 1$, and D_o is an arbitrary constant.

$$\gamma_{\ell,d}(s) = \ln g_{\ell,d}(s) \tag{25b}$$

$$g_{\ell,d}(s) = \int_Y \sum_{k=1} p(k)\,p(y|k)\,e^{sD(ky)}\,dy; \quad 0 \leq s$$

$$= \int_Y \sum_{k=1} \frac{1}{2}\,p(y|x_k)\,\exp[sD(x_k,y)]\,dy; \quad 0 \leq s \tag{25c}$$

As in Eq. A-47, let D_o be such that

$$E^{(1)}_{\ell,d}(R) = E^{(2)}_{\ell,d}(R) \tag{26}$$

Inserting Eqs. 23 and 24a into Eq. 26 yields

$$-R - \frac{1}{d}\left[\gamma_{\ell,d}(t,r) - r\frac{D_o}{m}\right] = -\frac{1}{d}\left[\gamma_{\ell,d}(s) - s\frac{D_o}{m}\right]. \tag{27}$$

Thus

$$\frac{1}{d}\frac{D_o}{m} = \left[\frac{1}{d}\gamma_{\ell,d}(s) - \frac{1}{d}\gamma_{\ell,d}(t,r) - R\right]\frac{1}{s-r} \tag{28}$$

Inserting Eq. 28 into Eqs. 23 and 24a yields

$$E_{\ell,d}(R) = E^{(1)}_{\ell,d}(R) = E^{(2)}_{\ell,d}(R)$$

$$= -\frac{1}{s-r}\left[-\frac{r\gamma_{\ell,d}(s)}{d} + \frac{s\gamma_{\ell,d}(r,t)}{d} + sR\right] \tag{29}$$

with $0 \leq s, r \leq 0$; $t \leq 0$. Inserting Eq. 29 into Eq. 22 yields

$$p(e|X_\ell) \leq 2\exp[-nE_{\ell,d}(R))] ,$$

where $E_{\ell,d}(R)$ is given by Eq. 29.

We now proceed to evaluate a bound on the expected value of $E_{\ell,d}(R)$ when averaged over all possible sets X. The average value of $E_{\ell,d}(R)$, by Eq. 29, is

$$\overline{E_{\ell,d}(R)} \geq -\frac{1}{s-r}\left[-\frac{r}{d}\gamma_{\ell,d}(s)+\frac{s}{d}\gamma_{\ell,d}(r,t)+sR\right] \tag{30}$$

with $0 \leq s, r \leq 0, t \leq 0$. Inequality (30) is not an equality, since, in general, the parameters s, r, and t should be chosen so as to maximize $E_{\ell,d}(R)$ of each individual input set X_ℓ, rather than to be the same for all sets. From the convexity of the logarithmic function, we have

$$-\overline{\ln x} \geq -\ln \bar{x} . \tag{31}$$

Inserting Eq. 31 into Eqs. 24b and 25b yields

$$-\overline{\gamma_{\ell,d}(s)} = -\ln \overline{g_{\ell,d}(s)} \geq -\ln \overline{g_{\ell,d}(s)} \tag{32}$$

$$-\overline{\gamma_{\ell,d}(r,t)} \geq -\ln \overline{g_{\ell,d}(r,t)}. \tag{33}$$

Now, since $r \leq 0$, $s \geq 0$, we have

$$\frac{r}{s-r} \leq 0; \quad -\frac{s}{s-r} \leq 0 . \tag{34}$$

Inserting (32), (33), and (34) into (30) yields

$$\overline{E_{\ell,d}(R)} \geq \frac{r}{s-r}\frac{1}{d}\ln \overline{g_{\ell,d}(s)} - \frac{s}{s-r}\frac{1}{d}\ln \overline{g_{\ell,d}(r,t)} - \frac{s}{s-r} R . \tag{35}$$

From Eqs. 25c and 14, we have

$$\overline{g_{\ell,d}(s)} = \int_X p(x_k)\, g_{\ell,d}(s)\, dx_k = \frac{1}{\ell}\sum_{k=1} \int_Y \int_X p(x)\, p(y|x)\, e^{sD(xy)}\, dxdy$$

where the index k has been dropped, since $p(x_k) = p(x)$. Thus

$$\overline{g_{\ell,d}(s)} = \int_Y \int_X p(x)\, p(y|x)\, e^{sD(xy)}\, dxdy . \tag{36}$$

From Eq. 24c, we have

$$\overline{g_{\ell,d}(t,r)} = \int_{X_i} \int_{X_j} p(x_i, x_j)\, g_{\ell,d}(t,r)\, dx_i dx_j . \tag{37}$$

Here, by construction,

$$p(x_i, x_j) \begin{cases} = p(x_i)\, p(x_j); & i \neq j \\ = p(x_i)\, \delta(x_i - x_j); & i = j \end{cases} \tag{38}$$

and, by Eq. 14,

$$p(x) \equiv p(x), \qquad \text{for all } i . \tag{39}$$

Thus, from (24c), (37), (38), and (39), we have

$$\overline{g_{\ell,d}(r,t)} = \underset{i \neq j}{\overline{g_{\ell,d}(r,t)}} + \underset{i = j}{\overline{g_{\ell,d}(r,t)}}, \tag{40}$$

where

$$\underset{i \neq j}{\overline{g_{\ell,d}(r,t)}} = \frac{1}{\ell^2} \sum_{\substack{i=1 \\ i \neq j}} \sum_{j=1} \int_{X_i} \int_{X_j} p(x_i)\, p(x_j) \int_Y p(y|x_i) \exp[(r-t)D(x_i, y) + tD(x_j, y)]\, dy dx_i dx_j$$

$$= \frac{1}{\ell^2} \sum_{i=1} \sum_{j=1} \int_Y \int_X \int_{X'} p(x)\, p(x')\, p(y|x) \exp[(r-t)D(x,y) + tD(x', y)]\, dx dx' dy \tag{41}$$

with $r \leqslant 0$; $t \leqslant 0$, and

$$\underset{i = j}{\overline{g_{\ell,d}(t,r)}} = \frac{1}{\ell^2} \sum_{i=1} \int_Y \int_{X_i} p(x_i)\, p(y|x_i) \exp[rD(x_i, y)]\, dx dy$$

$$= \frac{1}{\ell^2} \sum_{i=1} \int_Y \int_X p(x)\, p(y|x) \exp[rD(x,y)]\, dx dy ; \quad r \leqslant 0 \tag{42}$$

Inserting (25a) into (41) and (42) yields

$$\underset{i \neq j}{\overline{g_{\ell,d}(r,t)}} = \frac{\ell(\ell-1)}{\ell^2} \int_Y \int_X \int_{X'} p(x)\, p(x')\, p(y|x)^{1-r+t}\, p(y|x')^{-t} f(y)^r\, dx dx' dy \tag{43}$$

with $r \leqslant 0$; $t \leqslant 0$;

$$\underset{i = j}{\overline{g_{\ell,d}(r,t)}} = \frac{\ell}{\ell^2} \int_Y \int_X \int_{X'} p(x)\, p(y|x)^{1-r} f(y)^r\, dx dy \tag{44}$$

with $r \leqslant 0$.

In general, $f(y)$ of Eq. 25a should be chosen so as to maximize $E_{\ell,d}(R)$ for each individual input set X_ℓ. However, we let $f(y)$ be the same for all sets X_ℓ and equal to the $f(y)$ that maximizes the exponent $E_d(R)$ for the unrestricted continuous set X. Thus inserting Eqs. A-52 and A-53 into Eqs. 43 and 36 yields

$$\overline{\underset{i \neq j}{g_{\ell,d}(r,t)}} = \frac{\ell - 1}{\ell} \exp[\gamma_d(r,t)] = \frac{\ell - 1}{\ell} g_d(r,t) \tag{45}$$

$$\overline{\underset{i = j}{g_{\ell,d}(r,t)}} = \frac{1}{\ell} \exp[\gamma_d(r)] = \frac{1}{\ell} g_d(r) \tag{46}$$

Thus, by Eq. 40,

$$\overline{g_{\ell,d}(r,t)} = \frac{\ell - 1}{\ell} g_d(r,t) + \frac{1}{\ell} g_d(r). \tag{47}$$

Also, by Eqs. A-52 and 36,

$$\overline{g_{\ell,d}(s)} = g_{\ell,d}(s). \tag{48}$$

Inserting Eqs. 47 and 48 into Eq. 35 yields

$$\overline{E_{\ell,d}(R)} \geq \frac{r}{s-r}\frac{1}{d} \ln g_d(s) - \frac{s}{s-r}\frac{1}{d} \ln\left[\frac{\ell-1}{\ell} g_d(r,t) + \frac{1}{\ell} g_d(r)\right] - \frac{s}{s-r} R$$

$$= \frac{r}{s-r}\frac{1}{d} \ln g_d(s) - \frac{s}{s-r}\frac{1}{d} \ln g_d(r,t) - \frac{s}{s-r}\frac{1}{d} \ln\left[\frac{g_d(r)/g_d(r,t) + \ell - 1}{\ell}\right] - \frac{s}{s-r} R.$$

Thus

$$\overline{E_{\ell,d}(R)} \geq \frac{r}{s-r}\frac{1}{d}\gamma_d(s) - \frac{s}{s-r}\frac{1}{d}\gamma_d(r,t) - \frac{s}{s-r}\frac{1}{d} \ln\left[\frac{\exp[\gamma_d(r) - \gamma_d(r,t)] + \ell - 1}{\ell}\right]. \tag{49}$$

Now, the exponent $E_d(R)$ that corresponds to the unconstrained d-dimensional continuous space X is given by Eq. A-49:

$$E_d(R) = -\frac{1}{d}\left[\gamma_d(s) - s\frac{D_o}{m}\right] = -R + \frac{1}{d}\left[\gamma_d(r,t) - r\frac{D_o}{m}\right]$$

Eliminating D_o yields

$$E_d(R) = \frac{r}{s-r}\frac{1}{d}\gamma_d(s) - \frac{s}{s-r}\frac{1}{d}\gamma_d(r,t) - \frac{r}{s-r} R. \tag{50}$$

Furthermore, $E_d(R)$ is maximized, as shown in Eqs. A-50, A-51, A-54, A-55 and A-56, by letting

$$f(y) = \frac{\left[\int_X p(x)\,p(y|x)^{1-s}\,dx\right]^{1/1-s}}{\int_Y \left[\int_X p(x)\,p(y|x)^{1-s}\,dx\right]^{1/1-s} dy} \tag{51a}$$

$$r = 2s - 1; \quad t = s - 1 \tag{51b}$$

For $R \geq R_{crit}$, s is such that

$$R = \frac{1}{d}\left[(s-1)\gamma_{d'}(s) - \gamma_d(s)\right]; \quad 0 \leqslant s \leqslant \frac{1}{2}, \tag{51c}$$

where

$$R_{crit} = [R]_{s=1/2}.$$

If we let the parameters r, s, and t of (49) be equal to those of (50), we have

$$\overline{E_{\ell,d}(R)} \geqslant E_d(R) - \frac{s}{s-r}\frac{1}{d}\ln\left[\frac{\exp[\gamma_d(r)-\gamma_d(r,t)] + \ell - 1}{\ell}\right]. \tag{52}$$

The insertion of Eq. 51b yields

$$\overline{E_{\ell,d}(R)} \geqslant E_d(R) - \frac{s}{1-s}\frac{1}{d}\ln\left[\frac{\exp[F_1(R)] + \ell - 1}{\ell}\right], \tag{53}$$

where

$$F_1(R) = \gamma_d(2s-1) - \gamma_d(2s-1;s-1). \tag{54}$$

Inserting Eqs. A-52 and A-53 into Eq. 54 yields

$$F_1(R) = \ln \int_Y \int_X p(x)\,p(y|x)^{2-2s}[f(y)]^{2s-1}\,dxdy$$

$$- \ln \int_Y \int_{X'} \int_X p(x)\,p(x')\,p(y|x)^{1-s}\,p(y|x')^{1-s}\,f(y)^{2s-1}\,dxdx'dy.$$

Thus

$$F_1(R) = \ln \frac{\int_Y \int_X p(x)\,p(y|x)^{2(1-s)}\,f(y)^{2s-1}\,dxdy}{\int_Y \left[\int_X p(x)\,p(y|x)^{1-s}\,dx\right]^2 f(y)^{2s-1}\,dxdy}, \tag{55}$$

where s and $F_1(R)$ are related parametrically to the rate R by Eq. A-60c, for all rates above $R_{crit} = [R]_{s=1/2}$.

As for rates below R_{crit}, we let

$$s = \frac{1}{2}; \quad t = -\frac{1}{2}; \quad r = 0. \tag{56}$$

Inserting Eq. 56 into Eqs. 54 and 55, with the help of Eqs. A-69 and A-71, yields

$$[F_1(R)]_{s=1/2} = -\ln \int_Y \left[\int_X p(x)\,p(y|x)^{1/2}\,dx\right]^2 = dE_d(0) \tag{57}$$

where

$$E_d(0) = [E_d(R)]_{R=0}$$

$$\overline{E_{\ell,d}(R)} \geq E(R) - \frac{1}{d}\ln\left(\frac{\exp\left[F_1(R)\Big|_{s=1/2}\right] + \ell - 1}{\ell}\right)$$

$$= E(R) - \frac{1}{d}\ln\left(\frac{\exp[dE_d(0)] + \ell - 1}{\ell}\right)$$

for $R \leq R_{crit}$. From Eqs. 14 and 12 we also have $E_d(R) \equiv E(R)$ for all rates, by construction. The proof of the first part of the theorem has therefore been completed (a simplified proof for the region $R \leq R_{crit}$ is given elsewhere[7]). Q.E.D.

In order to prove the second part, let us rewrite (49) as

$$\overline{E_{\ell,d}(R)} \geq \frac{r}{s-r}\frac{1}{d}\gamma_d(s) - \frac{s}{s-r}\frac{1}{d}\gamma_d(r,t) - \frac{s}{s-r}\left[R + \frac{1}{d}\ln\frac{\exp[\gamma_d(r)-\gamma_d(r,t)+\ell-1]}{\ell}\right]$$

$$= \frac{r}{s-r}\frac{1}{d}\gamma_d(s) - \frac{s}{s-r}\frac{1}{d}\gamma_d(r,t) - \frac{s}{s-r}[R'], \tag{58}$$

where

$$R' = R + \frac{1}{d}\ln\frac{\exp[\gamma_d(r)-\gamma_d(r,t)] + \ell - 1}{\ell} \tag{59}$$

Comparing (59) with (50) yields

$$\overline{E_{\ell,d}(R)} \geq E_d(R') = E_d\left(R + \frac{1}{d}\ln\frac{\exp[F_2(R)] + \ell - 1}{\ell}\right). \tag{60}$$

By Eqs. 51a, 55, A-57, A-58, A-59, A-60c, and A-60b, $F_2(R)$ is related parametrically to the rate R by

$$F_2(R) = \ln\frac{\int_Y\int_X p(x)\,p(y|x)^{2(1-s)}\,Q(y)^{2s-1}\,dxdy}{\int_Y\left[\int_X p(x)\,p(y|x)^{1-s}\,dx\right]^2 Q(y)^{2s-1}\,dxdy}; \quad 0 \leq s \leq \frac{1}{2} \tag{61}$$

$$R' = R + \frac{1}{d}\ln\frac{\exp[F_2(R)] + \ell - 1}{\ell} \equiv \frac{1}{d}\int_X\int_Y Q(x,y)\ln\frac{Q(x|y)}{p(x)}; \quad 0 \leq s \leq \frac{1}{2} \tag{62}$$

for all

$$R' \geq R'_{crit} = [R']_{s=1/2} \tag{63}$$

Inequality (63) can be rewritten

$$R \geq R_{crit} - \frac{1}{d}\ln\frac{\exp\left[F_2(R)\Big|_{s=1/2}\right] + \ell - 1}{\ell}$$

$$= R_{crit} - \frac{1}{d}\ln\frac{e^{dE(0)} + \ell - 1}{\ell} \tag{64}$$

As for rates below $R_{crit} - \frac{1}{d} \ln \frac{e^{dE(0)} + \ell - 1}{\ell}$, we let

$$s = \frac{1}{2}, \quad t = -\frac{1}{2}, \quad r = 0. \tag{65}$$

Inserting (65) into (60) and (61) yields

$$\overline{E_{\ell, d}(R)} \geq E_d\left(R + \frac{1}{d} \ln \frac{e^{E(0)d} + \ell - 1}{\ell}\right); \quad R \leq R_{crit} \tag{66}$$

From Eqs. 14 and 12, by construction, $E_d(R) \equiv E(R)$ for all rates. Thus, the proof of the second part of the theorem has been completed. Q. E. D.

Discussion: We proceed now to discuss the bounds that were derived in the theorem above. We shall consider particularly the region $0 \leq R \leq R_{crit}$, which, as we shall see in Section III, is of special interest. From Eqs. 16 and 18,

$$\overline{E_{\ell, d}(R)} \geq E(R) - \frac{1}{d} \ln \left[\frac{e^{dE(0)} + \ell - 1}{\ell}\right] \qquad \text{for } R \leq R_{crit}. \tag{67}$$

From Eq. 6, we have

$$E(R) = E(0) - R \qquad \text{for } R \leq R_{crit}. \tag{68}$$

Inserting (68) into (66) yields

$$\overline{E_{\ell, d}(R)} \geq E(0) - R - \frac{1}{d} \ln \left[\frac{e^{dE(0)} + \ell - 1}{\ell}\right] \qquad \text{for } R \leq R_{crit}. \tag{69}$$

Now, whenever $dE(0) \ll 1$, we have

$$\begin{aligned} \overline{E_{\ell, d}(R)} &\gtrapprox E(0) - R - \frac{1}{d} \ln \left(\frac{1+dE(0)+\ell-1}{\ell}\right) \\ &\cong E(0) - R - \frac{1}{d} \ln \left(1 + \frac{dE(0)}{\ell}\right) \\ &\cong E(0) - R - \frac{E(0)}{\ell}; \qquad \text{for } R \leq R_{crit}. \end{aligned}$$

Thus

$$\overline{E_{\ell, d}(R)} \gtrapprox E(0)\left[1 - \frac{1}{\ell}\right] - R, \qquad \text{for } R \leq R_{crit}, \text{ and} \tag{70}$$

$$dE(0) \ll 1. \tag{71}$$

Comparing (71) with (68), we see that $\overline{E_{\ell, d}(0)}$ can be made practically equal to E(R) by using quite a small number ℓ, of input symbols, whenever $E(0)d \ll 1$. In cases in which $|\xi|_{max} \to 0$ and $\left.\frac{dp(\eta | \xi)}{d\xi}\right|_{\xi=0} \neq 0$ so that $p(\eta | \xi)$ can be replaced by the first two terms of the Taylor series expansion, $p(\eta | \xi) = p(\eta | 0) + \left.\frac{dp(\eta | \xi)}{d\xi}\right|_{\xi=0}$, it can be shown (by insertion into Eqs. A-74a-d) that the optimum input space consists of two oppositely directed vectors, ξ_{max} and $-\xi_{max}$, for all rates $0 \leq R < C$, and that

$$E(0) = \frac{1}{2} C = \frac{1}{4} \xi_{max}^2 \int \frac{\left[\left. \frac{dp(\eta \mid \xi)}{d\xi} \right|_{\xi=0} \right]^2}{p(\eta \mid 0)} d\eta ,$$

where C is the channel capacity.

Inequality (69) may be bounded by

$$\overline{E_{\ell,d}(R)} \geq E(0) - R - \frac{1}{d} \ln \left[\frac{e^{E(0)d} + \ell}{\ell} \right] \qquad R \leq R_{crit}$$

$$= E(0) - R - \frac{1}{d} \ln \left[e^{E(0)d - \ln \ell} + 1 \right] \tag{72}$$

Thus, whenever $dE(0) \gg 1$, we have from (72)

$$\overline{E_{\ell,d}(R)} \gtrapprox E(0) - R - \frac{1}{d} \ln \left[e^{E(0)d - \ln \ell} \right] \cong \frac{1}{d} \ln \ell - R \tag{73}$$

when

$$\frac{1}{d} \ln \ell \ll E(0)$$

and

$$\overline{E_{\ell,d}(R)} \gtrapprox E(0) - R - \frac{1}{d} \ln \left[e^{E(0)d - \ln \ell} + 1 \right]$$

$$\cong E(0) - \frac{1}{d} e^{-[\ln \ell - E(0)d]} - R \tag{74}$$

when

$$\frac{1}{d} \ln \ell \gg E(0)$$

Comparing Eqs. 73 with 74 yields

$$E(R) \geq \overline{E_{\ell,d}(R)} \cong E(R); \quad R \leq R_{crit}, dE(0) \gg 1 \tag{75}$$

or

$$E_{\ell,d}(R) \cong E(R) \tag{76}$$

if

$$\frac{1}{d} \ln \ell \geq E(0) \qquad dE(0) \gg 1 . \tag{77}$$

In the following section we shall discuss the construction of a semioptimum finite input set X_ℓ for the Gaussian channel. [A semioptimum input space is one that yields an exponent $E_{\ell,d}(R)$ which is practically equal to $E(R)$.] We shall show that the number of input vectors ℓ needed is approximately the same as that indicated by Eqs. 75 and 71. This demonstrates the effectiveness of the bounds derived in this section.

2.3 SEMIOPTIMUM INPUT SPACES FOR THE WHITE GAUSSIAN CHANNEL – CASE 1

The white Gaussian channel is defined by the transition probability density

$$p(\eta \mid \xi) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(\eta-\xi)^2}{2\sigma^2}\right) \tag{78}$$

in which, by inequality (6a),

$$|\xi| \leq |\xi|_{max} = \sqrt{P}. \tag{79}$$

Let us define the voltage signal-to-noise ratio A, as

$$A = \frac{\xi}{\sigma}. \tag{80}$$

Inserting (79) into (80) yields

$$A \leq A_{max} = \frac{|\xi|_{max}}{\sigma} = \frac{\sqrt{P}}{\sigma}. \tag{81}$$

We shall first discuss the case in which

$$dA^2_{max} \ll 1 \tag{82}$$

and proceed with the proof of the following theorem.

THEOREM: Consider a white Gaussian channel whose statistics are given by Eq. 78. Let the input signal power be constrained by (79) and by (82). Let the input space consist of two d-dimensional oppositely directed vectors. Then the exponent of the upper bound to the probability of error, $E_{2,d}(R)$ is asymptotically equal to the optimum exponent E(R).

PROOF: From Eq. 7 we have

$$x = \xi_1, \xi_2, \ldots, \xi_d.$$

The input set X_2 consists of two oppositely directed vectors. Let those two vectors be

$$x_1 = \xi_1^1, \xi_2^1, \ldots, \xi_d^1, \tag{83}$$

where $\xi_1^1 = \xi_2^1 = \ldots = \xi_d^1 = \xi_{max}$, and

$$x_2 = \xi_1^2, \xi_2^2, \ldots, \xi_d^2 \tag{84}$$

where $\xi_1^2 = \xi_2^2 = \ldots = \xi_d^2 = -\xi_{max}$.

From Eqs. 8 and 9 we have

$$y = \eta_1, \eta_2, \ldots, \eta_d$$

and

$$p(y \mid x) = \prod_{i=1}^{d} p(\eta_i \mid \xi_i).$$

Inserting Eqs. 78, 83, and 84 into Eq. 9 yields

$$p(y|x_1) = \prod_{i=1}^{d} \frac{1}{(2\pi)^{d/2}\sigma^d} \exp\left(-\frac{(\eta_i-\xi_{max})^2}{2\sigma^2}\right) \tag{85a}$$

$$p(y|x_2) = \prod_{i=1}^{d} \frac{1}{(2\pi)^{d/2}\sigma^d} \exp\left(-\frac{(\eta_i+\xi_{max})^2}{2\sigma^2}\right) \tag{85b}$$

From Eqs. 29 and 30 we have

$$p(e|X_2) \leq 2\exp[-nE_{2,d}(R)] \tag{86}$$

where

$$E_{2,d}(R) = \frac{r}{s-r}\frac{1}{d}\gamma_{2,d}(s) - \frac{s}{s-r}\frac{1}{d}\gamma_{2,d}(r,t) - \frac{s}{s-r}R. \tag{87}$$

Let

$$r = 2t+1; \quad s = 1+t; \quad 0 \leq s \leq \frac{1}{2} \tag{88}$$

and let

$$f(y) = p(y|0) = \prod_{i=1}^{d} \frac{1}{(2\pi)^{d/2}\sigma^d} \exp\left(-\frac{\eta_i^2}{2\sigma^2}\right) \tag{89}$$

and

$$p(x_1) = p(x_2) = \frac{1}{2}. \tag{90}$$

Inserting Eqs. 88, 89, and 90 into Eqs. 24b, 25, and 87 yields

$$E_{2,d}(R) = -\frac{1-2s}{1-s}\frac{1}{d}\gamma_{2,d}(s) - \frac{s}{1-s}\frac{1}{d}\gamma_{2,d}(2s-1,s-1) - \frac{s}{1-s}R \tag{91}$$

$$\gamma_{2,d}(s) = \ln \sum_{i=1}^{2} \int_Y \frac{1}{2} p(y|x_i)^{1-s}\, p(y|0)^s\, dy \tag{92}$$

$$\gamma_{2,d}(2s-1,s-1) = \ln \sum_{i=1}^{2} \int_Y \frac{1}{4} p(y|x_i)^{1-s}\, p(y|x_j)^{1-s}\, p(y|0)^{2s-1} \tag{93}$$

Inserting Eqs. 85 and 89 into Eqs. 92 and 93 yields

$$\gamma_{2,d}(s) = \ln\frac{1}{2}\left\{\left[\int_\eta \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(1-s)(\eta-\xi_{max})^2+\eta^2 s}{2\sigma^2}\right)d\eta\right]^d + \left[\int_\eta \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(1-s)(\eta+\xi_{max})^2+\eta^2 s}{2\sigma^2}\right)d\eta\right]^d\right\}$$

$$= \ln\frac{1}{2}\left\{2\exp\left(-\frac{\xi_{max}^2(1-s)\,sd}{2\sigma^2}\right)\right\} = -\frac{\xi_{max}^2 ds(1-s)}{2\sigma^2}; \quad 0 \leq s \leq \frac{1}{2} \tag{94}$$

$$\gamma_{2,d}(2s-1, s-1) = \ln \frac{1}{4} \left\{ \left[\int_\eta \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(- \frac{2(1-s)(\eta - \xi_{max})^2 + (2s-1)\,\eta^2}{2\sigma^2} \right) d\eta \right]^d \right.$$

$$+ \left[\int_\eta \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(- \frac{2(1-s)(\eta + \xi_{max})^2 + (2s-1)\,\eta^2}{2\sigma^2} \right) d\eta \right]^d$$

$$\left. + 2 \left[\int_\eta \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(- \frac{(1-s)(\eta - \xi_{max})^2 - (1-s)(\eta + \xi_{max}) - (2s-1)\eta^2}{2\sigma^2} \right) d\eta \right]^d \right\}$$

$$= \ln \frac{1}{4} \left\{ 2 \exp\left(- \frac{\xi_{max}^2 d[2(1-s) - 4(1-s)^2]}{2\sigma^2} \right) + 2 \exp\left(- \frac{2(1-s)\,\xi_{max}^2 d}{2\sigma^2} \right) \right\} ;$$

$$0 \leq s \leq \frac{1}{2} \qquad\qquad 0 \leq s \leq \frac{1}{2} \qquad (95)$$

Now, since by (82) $dA_{max}^2 = \frac{\xi_{max}^2}{\sigma^2} \ll 1$, we have

$$\gamma_{2,d}(2s-1, s-1) \cong \ln \frac{1}{4} \left[4 - \frac{2\xi_{max}^2}{2\sigma^2} d[2(1-s) - r(1-s)^2] - \frac{2\xi_{max}^2}{2\sigma^2} d\, 2(1-s) \right]$$

$$= \ln \left[1 - \frac{\xi_{max}^2}{2\sigma^2} 2ds(1-s) \right]$$

$$\cong \frac{\xi_{max}^2}{2\sigma^2} 2ds(1-s) = 2\gamma_{2,d}(s) . \qquad (96)$$

Inserting Eqs. 94 and 96 into Eq. 91 yields

$$E_{2,d}(R) = + \left[\frac{1-2s}{1-s} + \frac{2s}{1-s} \right] s(1-s) \frac{\xi_{max}^2}{2\sigma^2} - \frac{s}{1-s} R$$

$$= s \frac{\xi_{max}^2}{2\sigma^2} - \frac{s}{1-s} R ; \quad 0 \leq s \leq \frac{1}{2}$$

$$= s \frac{A_{max}^2}{2} - \frac{s}{1-s} R ; \quad 0 \leq s \leq \frac{1}{2} . \qquad (97)$$

Maximizing $E_{2,d}(R)$ with respect to s yields

$$s = \frac{1}{2} \quad \text{for } R \leq R_{crit} = \frac{1}{8} A_{max}^2$$

$$s = 1 - \frac{\sqrt{2R}}{A} \quad \text{for } R \geq R_{crit} = \frac{1}{8} A_{max}^2 .$$

Thus

$$E_{2,d}(R) \cong \frac{1}{4} A_{max}^2 - R; \quad R \leq \frac{1}{8} A_{max}^2 \tag{98a}$$

$$E_{2,d}(R) \cong \frac{1}{2} A^2 - 2A\sqrt{\frac{R}{2}} + R; \quad \frac{1}{8} A_{max}^2 \leq R \leq \frac{1}{2} A_{max}^2 \tag{98b}$$

Comparing Eqs. 98 with the results given by Shannon yields

$$E(R) \geq E_{2,d}(R) \geq E(R) \tag{99}$$

Shannon's results[1] are derived for the power constraint of Case 2, and are valid also in cases where the average power is constrained to be equal to P. The set of signals satisfying Case 1 is included in the set of signals satisfying the average power constraint above. Thus, Shannon's exponent of the upper bound to the probability of error is larger than or equal to the optimum exponent E(R) which corresponds to the constraint of Case 1. Thus, from (99),

$$E_{2,d}(R) \cong E(R) \tag{100}$$

for $A_{max}^2 d \ll 1$. Q.E.D.

We shall now discuss cases in which the condition of (82) is no longer valid. The first step will be the evaluation of $E_{\ell,d}(0)$ for the white Gaussian channel. From Eqs. A-69 and A-71 we have

$$E_{\ell,d}(0) = -\frac{1}{d} \ln \sum_{X'_\ell} \sum_{X_\ell} p(x)\, p(x') \int_Y p(y|x)^{1/2}\, p(y|x')^{1/2}\, dy. \tag{101}$$

Inserting Eqs. 9 and 78 into (101) yields

$$E_{\ell,d}(0) = -\frac{1}{d} \ln \sum_X \sum_{X'} p(x)\, p(x') \prod_{i=1}^{d} \int_\eta \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\eta_i-\xi_i)^2 - (\eta_i-\xi_i')^2}{4\sigma^2}\right] d \tag{102}$$

where $x = \xi_1, \xi_2, \ldots, \xi_d$, and $x' = \xi_1', \xi_2', \ldots, \xi_d'$. Thus

$$E_{\ell,d}(0) = -\frac{1}{d} \ln \sum_X \sum_{X'} p(x)\, p(x') \prod_{i=1}^{d} \exp\left[-\frac{(\xi_i - \xi_i')^2}{8\sigma^2}\right]$$

$$= -\frac{1}{d} \ln \sum_X \sum_{X'} p(x)\, p(x') \exp\left[-\sum_{i=1}^{d} \frac{(\xi_i - \xi_i')^2}{8\sigma^2}\right]. \tag{103}$$

Let D be the geometrical distance between the two vectors x and x', given by

$$D^2 = [x-x']^2 = \sum_{i=1}^{d} (\xi_i - \xi_j')^2. \tag{104}$$

Then, inserting (104) into (103) yields

$$E_{\ell,d}(0) = -\frac{1}{d}\ln \sum_X \sum_{X'} p(x)p(x') \exp\left(-\frac{|x-x'|^2}{8\sigma^2}\right) \tag{105a}$$

or

$$E_{\ell,d}(0) = -\frac{1}{d}\ln \sum_X \sum_{X'} p(D) \exp\left(-\frac{D^2}{8\sigma^2}\right). \tag{105b}$$

Here, p(D) can be found from p(x) and p(x').

When the input set X_2 consists of two oppositely directed vectors given by Eqs. 83, 84, and 90, from Eq. 103, we obtain

$$E_{2,d}(0) = -\frac{1}{d}\ln\frac{1}{2}\left(1+\exp\left(-\frac{\xi_{max}^2 d}{2\sigma^2}\right)\right)$$

$$= -\frac{1}{d}\ln\frac{1}{2}\left(1+\exp\left(-\frac{A_{max}^2 d}{2}\right)\right). \tag{106}$$

For $A_{max}^2 d/2 \ll 1$, we have $E_{2,d}(0) \cong A_{max}^2/4$ as in (98a). For higher values of peak signal-to-noise ratio let d = 1. Then, from (106),

$$E_{2,1}(0) = -\ln\frac{1}{2}\left(1+\exp\left(-\frac{A_{max}^2}{2}\right)\right). \tag{107}$$

In Table I, $E_{2,1}(0)$, together with $C_{2,1}$, the rate for which $E_{2,1}(R) = 0$, is given. Also given in the same table are the channel capacity C and the zero-rate exponent E(0), that correspond to the power constraint of Case 2. (The channel capacity has been

Table 1. Tabulation of $E_{2,1}(0)$ and $C_{2,1}$ vs A_{max}.

A_{max}	$E_{2,1}(0)$	$E(0)$	$\frac{E_{2,1}(0)}{E(0)}$	$C_{2,1}$	C	$C_{2,1}/C$
1	0.216	0.22	0.99	0.343	0.346	0.99
2	0.571	0.63	0.905	0.62	0.804	0.77
3	0.683	0.95	0.72	0.69	1.151	0.60
4	0.69	1.20	0.57	0.69	1.4	0.43

computed by F. J. Bloom et al.,[9] and E(0) is given by Fano[2] and is also computed in Appendix B.) The channel capacity C and the zero-rate exponent E(0) for Case 1 are upper bounded by the C and E(0) that correspond to the power constraint of Case 2. From Table I we see that the replacing of the continuous input set by the discrete input set X_2, consisting of two oppositely directed vectors, has a negligible effect on the exponent of the probability of error because $A^2_{max} \lessapprox 1$.

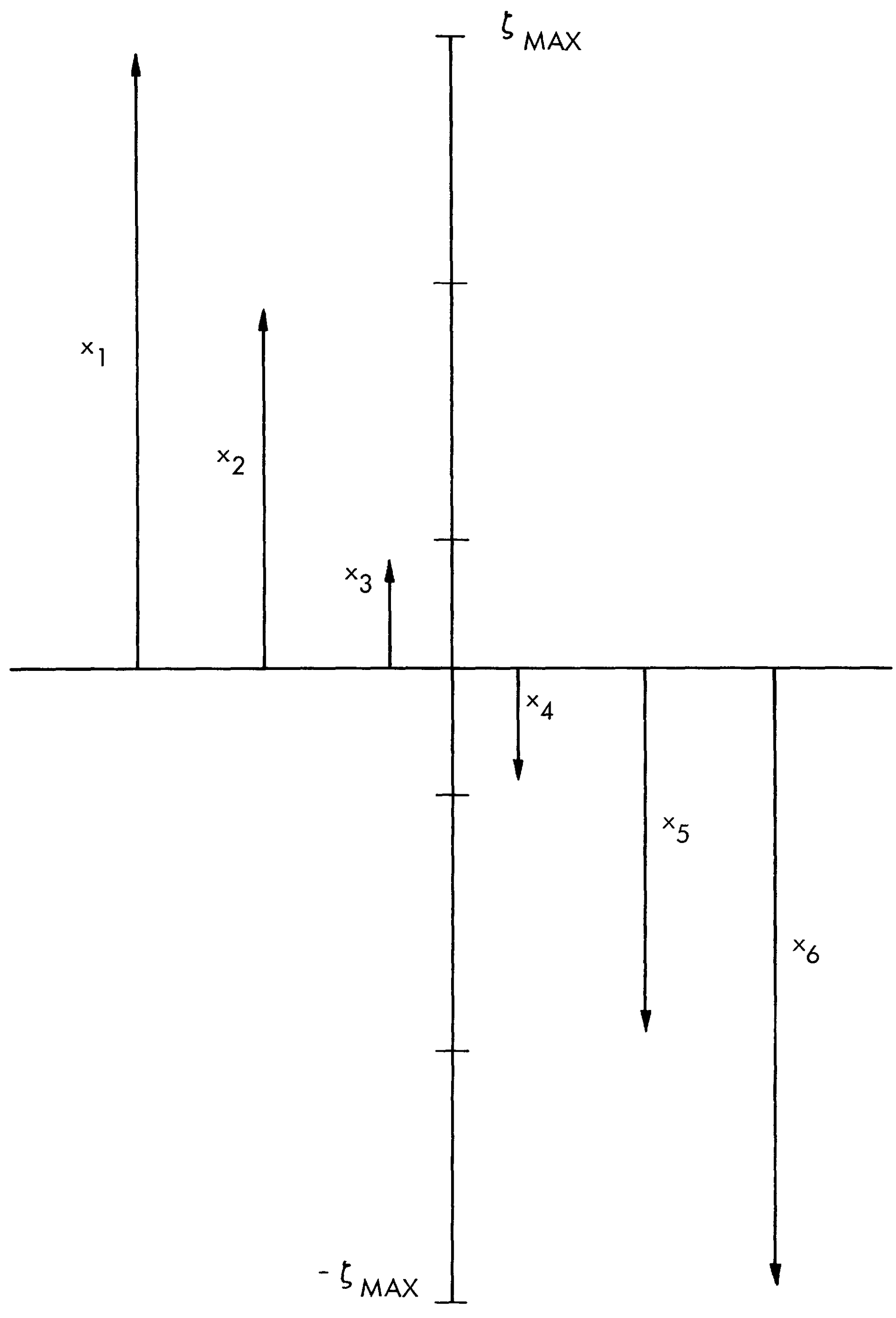

Fig. 4. Semioptimum input space for the Gaussian channel.

Let us consider, next, the case in which the input set X_ℓ consists of ℓ one-dimensional vectors as shown in Fig. 4. The distance between each two adjacent vectors is

$$D_{min} = \frac{2\xi_{max}}{\ell - 1}. \tag{108}$$

Let

$$p(x_i) = \frac{1}{\ell}; \quad i = 1, \ldots, \ell. \tag{109}$$

Inserting (108) and (109) into (105) yields

$$E_{2,1}(0) = -\ln \frac{1}{\ell^2}\left[\ell + 2\sum_{k=1}^{\ell} (\ell-k) \exp\left(-\frac{(kD_{min})^2}{8\sigma^2}\right)\right]. \tag{110}$$

Thus, since $4k < k^2$; $k \geq 2$, we have

$$E_{2,1}(0) \geq -\ln \frac{1}{\ell^2}\left[\ell + 2(\ell-1)\exp\left(-\frac{D^2_{min}}{8\sigma^2}\right) + 2(\ell-2)\sum_{k=1}^{\ell}\exp\left(-\frac{4kD^2_{min}}{8\sigma^2}\right)\right]$$

$$\geq -\ln \frac{1}{\ell^2}\left[\ell + 2(\ell-1)\exp\left(-\frac{D^2_{min}}{8\sigma^2}\right) - 2(\ell-2)\frac{\exp\left(-4kD^2_{min}/8\sigma^2\right)}{\exp\left(-4D^2_{min}/8\sigma^2\right) - 1}\right]$$

$$\geq -\ln \frac{1}{\ell}\left\{1 + \left[2\exp\left(-\frac{D^2_{min}}{8\sigma^2}\right) + 2\,\frac{1}{\exp\left(4D^2_{min}/8\sigma^2\right) - 1}\right]\right\}. \tag{111}$$

Now define K as

$$\ell - 1 = \frac{A_{max}}{K}; \quad 0 \leq K. \tag{112}$$

Inserting (112) and (81) into (108) yields

$$D_{min} = 2\sigma K. \tag{113}$$

Inserting (113) into (111) then yields

$$E_{2,1}(0) \geq \ln\ (1+KA_{max}) - \ln\left\{1 + 2\exp\left(-\frac{K^2}{2}\right) + \frac{2}{e^{+2K^2} - 1}\right\}. \tag{114a}$$

If we choose ℓ so that $K \cong 1$, we have

$$E_{2,1}(0) = \ln\ (1+A_{max}) - \ln 2,52. \tag{114b}$$

From Eqs. 114b and 112, for $A_{max} \gg 1$, we have

$$E_{2,1}(0) \cong \ln A_{max} \tag{115a}$$

$$\ln \ell \cong \ln A_{max} = E_{2,1}(0). \tag{115b}$$

On the other hand, it can be shown (Appendix B) that

$$E(0) \lessapprox \ln A_{max}; \quad A_{max} \gg 1. \tag{116}$$

Thus, by (116) and (115), we have

$$E_{2,1}(R) \cong E(R); \quad R < R_{crit}, A_{max} \gg 1. \tag{117a}$$

For

$$\ln \ell = E(0); \quad d = 1. \tag{117b}$$

Comparing Eqs. 71 and 75-77 with Eqs. 100 and 117, respectively, gives the result that the lower bound on $E_{\ell,d}(R)$ previously derived is indeed a useful one.

2.4 THE EFFECT OF THE SIGNAL-SPACE STRUCTURE ON THE AVERAGE PROBABILITY OF ERROR – CASE 2

For a power constraint such as that of Case 2, we consider the ensemble of codes obtained by placing M points on the surface of a sphere of radius $\sqrt{nP}$.

The requirement of Case 2 can be met by making each of the m elements in our signal space have the same power dP (see Fig. 3). (The power of each word is $mdP = nP$ and therefore Case 2 is satisfied.) This additional constraint produces an additional reduction in the value of $E_{\ell,d}(R)$ as compared with E(R). Even if we let the d-dimensional input space X_ℓ be an infinite set ($\ell = \infty$), the corresponding exponent $E_d(R)$, in general, will be

$$E_d(R) \leq E(R). \tag{118}$$

The discussion in this section will be limited to the white Gaussian channel and to rates below R_{crit}. Thus

$$E_d(R) = E_d(0) - R; \quad R \leq R_{crit}. \tag{119}$$

Let

$$E_d(0) = E(0) - k_d(\overline{A^2})\, E(0), \tag{120a}$$

where

$$\overline{A^2} = \frac{P}{\sigma^2}. \tag{120b}$$

Then, from (119) and (120), we have

$$E_d(R) = E(0) - k_d(\overline{A^2})\, E(0) - R; \quad R \leq R_{crit}. \tag{121}$$

We shall now proceed to evaluate $k_d(\overline{A^2})$ as a function of $\overline{A^2}$ for different values of d.

The input space X is, by construction, a set of points on the surface of a d-dimensional sphere of radius $\sqrt{dP}$.

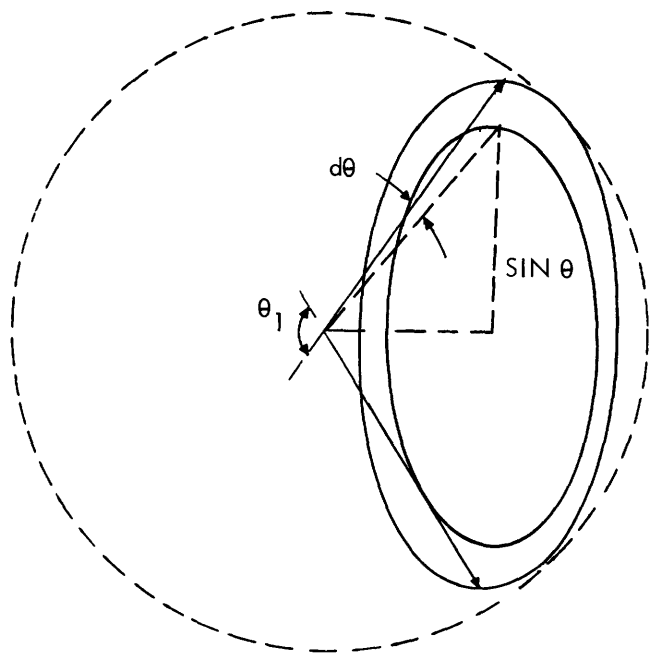

Fig. 5. Cap cut out by a cone on the unit sphere.

Let each point of the set X be placed at random and independently of all others with probability measure proportional to surface area or, equivalently, to solid angle. The probability $Pr(0 \leq \theta \leq \theta_1)$ that an angle between two vectors of the space X is less than or equal to θ_1, is therefore proportional to the solid angle of a cone in d-dimensions with half-angle θ_1. This is obtained by summing the contributions that are due to ring-shaped elements of area (spherical surfaces in d-1 dimensions of radius $\sin\theta$ and incremental width $d\theta$ as shown in Fig. 5). Thus the solid angle of the cone, as given by Shannon,[1] is

$$\Omega(\theta_1) = \frac{(d-1)\,\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \int_0^{\theta_1} (\sin\theta)^{d-2}\, d\theta. \qquad (122)$$

Here we have used the formula for the surface $s_d(r)$ of a sphere of radius r in d-dimensions, $s_d(r) = \pi^{d/2} r^{d-1}/\Gamma(d/2+1)$.

From (122), we have

$$\Pr(0\leq\theta\leq\theta_1) = \frac{\Omega(\theta_1)}{\Omega(\pi)} = \frac{(d-1)\,\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \frac{\int_0^{\theta_1} (\sin\theta)^{d-2}\, d\theta}{d\pi^{d/2}/\Gamma\left(\frac{d+1}{2}\right)}$$

$$= \frac{d-1}{d\sqrt{\pi}} \frac{\left(\Gamma\ \frac{d}{2}+1\right)}{\left(\Gamma\ \frac{d+1}{2}\right)} \int_0^{\theta_1} (\sin\theta)^{d-2}\, d\theta$$

$$= \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^{\theta_1} (\sin\theta)^{d-2}\, d\theta. \tag{123}$$

The probability density $p(\theta)$ is therefore given by

$$p(\theta) = \frac{d\Pr(0\leq\theta\leq\theta_1)}{d\theta_1} = \frac{1}{\sqrt{\pi}} \frac{\left(\Gamma\ \frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} (\sin\theta)^{d-2}. \tag{124}$$

Now, by (104), the geometrical distance between two vectors with an angle θ between them (see Fig. 5) is

$$D^2 = 4\left(dP\sin^2\frac{\theta}{2}\right). \tag{125}$$

Inserting (125) and (124) into (105b) yields

$$E_d(0) = -\frac{1}{d}\ln \int_0^{\pi} p(\theta)\exp\left(-\frac{dP}{2\sigma^2}\sin^2\frac{\theta}{2}\right) d\theta$$

$$= -\frac{1}{d}\ln\left\{\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^{\pi} \exp\left(-\frac{dP}{2\sigma^2}\sin^2\frac{\theta}{2}\right)(\sin\theta)^{d-2}\, d\theta\right\}. \tag{126}$$

Inserting (120b) into (126) for $d \geq 2$ yields

$$E_d(0) = -\frac{1}{d}\ln\left\{\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \int_0^{\pi} \exp\left(-\frac{d\overline{A^2}}{2}\sin^2\frac{\theta}{2}\right)(\sin\theta)^{d-2}\, d\theta\right\} \tag{127a}$$

or

$$E_d(0) = -\frac{1}{d}\ln\left\{\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \exp\left(-\frac{d\overline{A^2}}{4}\right) \int_0^{\pi} \exp\left(\frac{d\overline{A^2}}{4}\cos\theta\right)\sin\theta^{d-2}\, d\theta\right\}. \tag{127b}$$

Equation 127 is valid for all $d \geq 2$. As for $d = 1$, it is clear that in order to satisfy the power constraint of Case 2 the input space **X** must consist of two oppositely directed vectors with an amplitude of $\sqrt{P}$. Thus

$$A^2_{max} = \overline{A^2} = P . \tag{128}$$

Inserting (128) into (106) yields

$$E_1(0) = - \ln \left(\frac{1 + e^{-\overline{A^2}/2}}{2} \right) \tag{129}$$

In Appendix B we show that for all d

$$E_d(0) \cong \frac{1}{4} A^2 = E(0) ; \quad A^2 d \ll 1 ; \quad d \geqslant 2 \tag{130a}$$

$$E_d(0) \cong \frac{d-1}{d} \frac{1}{2} \ln A^2 ; \quad A^2 \gg 1 ; \quad d \geqslant 2 \tag{130b}$$

Thus

$$E_d(0) \cong \frac{d-1}{d} E(0) \tag{130c}$$

Inserting Eqs. 129 and 130 into Eq. 120 yields, for <u>any</u> d,

$$E_d(0) = E(0) - k_d(\overline{A^2}) E(0) , \tag{131}$$

where

$$k_d(\overline{A^2}) = 0 ; \quad \overline{A^2} d \ll 1$$

$$k_d(\overline{A2}) \cong \frac{1}{d} ; \quad \overline{A^2} \gg 1 .$$

The qualitative behaviour of $k_d(\overline{A^2})$ as a function of $\overline{A^2}$ with d as a parameter is illustrated in Fig. 6. From (127a) it is clear that $E_d(0)$ is a monotonic increasing

Table 2. Tabulation of $k_1(A^2)$ and $k_3(A^2)$ vs A^2.

A^2	$k_1(\overline{A^2})$	$k_3(\overline{A^2})$
1	0.01	
4	0.095	0.046
9	0.28	0.095
16	0.43	0.135
100	0.7	0.2
10^4	0.9	0.28

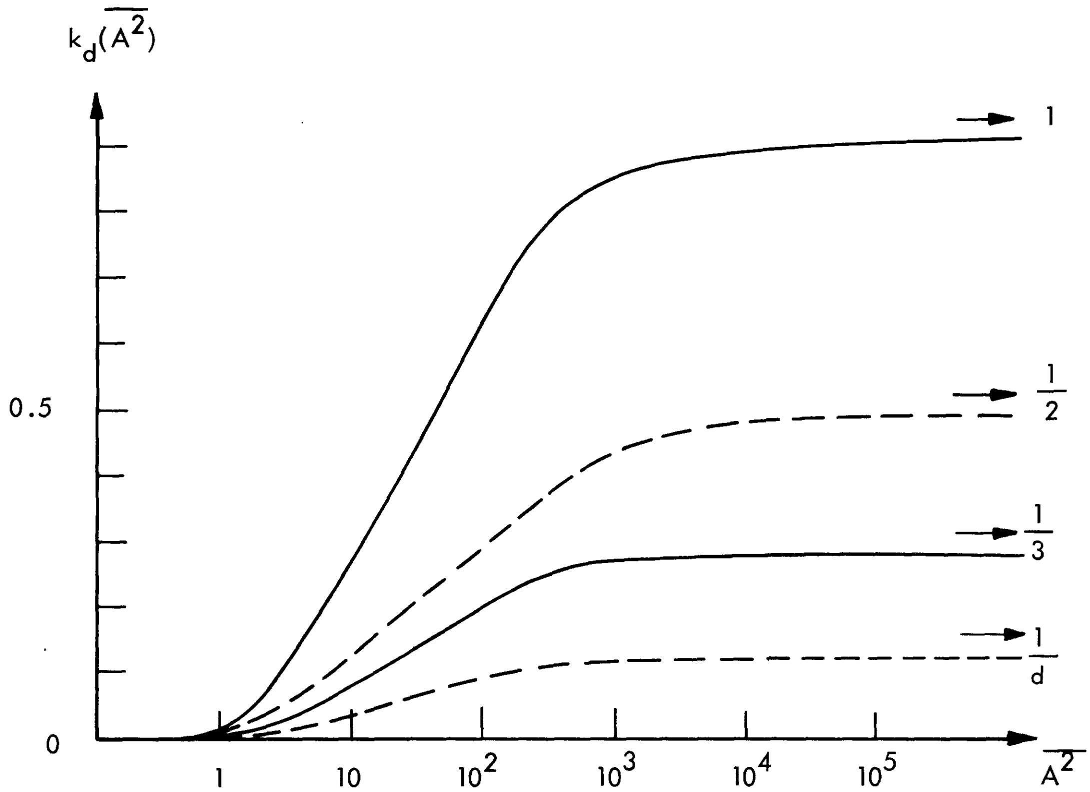

Fig. 6. Plot of $k_d(\overline{A^2})$ vs $\overline{A^2}$.

function of $\overline{A^2}$. The functions $k_1(\overline{A^2})$ and $k_3(\overline{A^2})$ are tabulated in Table II.

We shall now evaluate the effect of replacing the continuous d-dimensional input space X by a discrete d-dimensional input space X_ℓ, which consists of ℓ vectors.

Let each of the m elements be picked at random with probability $1/\ell$ from the set X_ℓ of ℓ vectors (waveforms), $X_\ell \equiv \{x_k : k=1, \dots, \ell\}^\ell$.

Let the set X_ℓ be generated in the following fashion: Each vector x_k is picked at random with probability $p(x_k)$ from the ensemble X of all d-dimensional vectors matching the power constraint of Case 2. The probability $p(x_k)$ is given by

$$p(x_k) = p(x)\Big|_{x=x_k}; \quad k = 1, \dots, \ell,$$

where $p(x_k)$ is the same probability distribution that is used to generate $E_d(0)$. The following theorem can then be stated.

THEOREM 2: Let $E_{\ell,d}(0)$ be the zero-rate exponent of the average probability of error for random codes constructed as those above. Let $\overline{E_{\ell,d}(0)}$ be the expected value of $E_{\ell,d}(0)$ averaged over all possible sets X_ℓ. Then

$$\overline{E_{\ell,d}(0)} \geq E_d(0) - \frac{1}{d}\ln\left(\frac{\exp[dE_d(0)] + \ell - 1}{\ell}\right). \tag{132}$$

The proof is identical with that of Theorem 1. Inserting (131) into (132) yields

$$\overline{E_{\ell,d}(0)} \geq E(0) - k_d(\overline{A^2})\,E(0) - \frac{1}{d}\ln\left(\frac{\exp[dE_d(0)] + \ell - 1}{\ell}\right). \tag{133}$$

Thus there is a combined loss that is due to two independent constraints:

1. Constraining the power of each of the input vectors to be equal to dP; the resulting loss is equal to $k_d(\overline{A^2})\,E(0)$.

2. Constraining the input space to consist of ℓ vectors only; the resulting loss is equal to $\frac{1}{d}\ln\frac{\exp[dE_d(0)] + \ell - 1}{\ell}$.

We now evaluate the effect of these constraints at high (and low) values of $\overline{A^2}$. From (132) we have

$$\overline{E_{\ell,d}(0)} \geq E_d(0) - \frac{1}{d}\ln\left(\frac{\exp[dE_d(0)] + \ell}{\ell}\right)$$

$$= E_d(0) - \frac{1}{d}\ln\left(\exp(dE_d(0) - \ln\ell) + 1\right). \tag{134}$$

Thus, for $E(0) \cong \ln A^2 \gg 1$, we have

$$\overline{E_{\ell,d}(0)} \geq \frac{1}{d}\ln\ell; \quad \frac{1}{d}\ln\ell \ll E_d(0) \cong \frac{d-1}{d}\,\frac{\ln\overline{A^2}}{2} \tag{135a}$$

$$\overline{E_{\ell,d}(0)} \cong E_d(0); \quad \frac{1}{d}\ln\ell \gg E_d(0) \cong \frac{d-1}{d}\,\frac{\ln\overline{A^2}}{2} \tag{135b}$$

On the other hand we always have $E_{\ell,d}(0) \leq E_d(0)$, and inserting it into (135b) yields

$$E_{\ell,d}(0) \cong E_d(0) = \frac{d-1}{d}\,\frac{1}{2}\ln\overline{A^2} \tag{136}$$

for $\frac{1}{d}\ln\ell \gg E_d(0)$.

Whenever $\overline{A^2}d \ll 1$, an input space X_2 that consists of two oppositely directed vectors with an amplitude of $\sqrt{dP}$ yields the optimum exponent E(R) for all rates $0 \leq R < C$, as shown in section 2.3.

2.5 CONVOLUTIONAL ENCODING

We have established a discrete signal space generated from a d-dimensional input space which consists of ℓ input symbols. We have shown that a proper selection of ℓ and d yields an exponent $E_{\ell,d}(R)$ which is arbitrarily close to the optimum exponent E(R).

We proceed to describe an encoding scheme for mapping output sequences from an independent letter source into sequences of channel input symbols that are all members of the input set X_ℓ. We do this encoding in a sequential manner so that sequential or other systematic decoding may be attempted at the receiver. By sequential encoding we mean that the channel symbol to be transmitted at any time is uniquely determined by the sequence of the output letters from the message source up to that time. Decoding schemes for sequential codes will be discussed in Section III.

Let us consider a randomly selected code with a constraint length n, for which the size of M(w), the set of allowable messages at the length w input symbols, is an exponential function of the variable w.

$$M(w) \leq A_1 e^{wRd}; \quad 1 \leq w \leq m, \tag{137}$$

where A_1 is some small constant ≥ 1.

A code structure that is consistent with Eq. 137 is a tree, as shown in Fig. 7. There is one branch point for each information digit. Each information digit consists of "a" channel input symbols. All the input symbols are randomly selected from a d-dimensional input space X_ℓ which consists of ℓ vectors. From each branch point there diverge b branches. The constraint length is n samples and thus equal to m input symbols or i information digits where $i = m/a$.

The upper bound on the probability of error that was used in the previous sections and which is discussed in Appendix A, is based on random block codes, not on tree codes, to which we wish to apply them. The important feature of random block codes, as far as the average probability of error is concerned, is the fact that the M code words are statistically independent of each other, and that there is a choice of input symbol a priori probabilities which maximize the exponent in the upper bound expression.

In the case of a tree structure we shall seek in decoding to make a decision only about the first information digit. This digit divides the entire tree set M into two subsets: M′ is the subset of all messages which start with the same digit as that of the transmitted message, and M″ is the subset of messages other than those of M′. It is clear that the messages in the set M′ cannot be made to be statistically independent. However, each member of the incorrect subset M″ can be made to be statistically independent of the transmitted sequence which is a member of M′.

Reiffen[5] has described a way of generating such randomly selected tree codes when the messages of the incorrect subset M″ are statistically independent of the messages in the correct subset M′.

Thus, the probability of incorrect detection of the first information digit in a tree code is bounded by the same expressions as the probability of incorrect detection of a message encoded by a random block code.

Furthermore, these trees can be made infinite so that the above-mentioned statistical characteristics are common to all information digits, which are supposed to be emitted from the information source in a continuous stream and at a constant rate. These codes can be generated by a shift register,[5] and the encoding complexity per information digit is proportional to m, where $m = n/d$ is the number of channel input symbols per constraint length.

Clearly, the encoding complexity is also a monotonically increasing function of ℓ (the number of symbols in the input space X). Thus, let M_e be an encoding complexity measure, defined as

$$M_e = \ell m = n\frac{\ell}{d}. \tag{138}$$

The decoding complexity for the two decoding schemes that will be discussed in Section III is shown to be proportional to m^{α}, $1 \le \alpha \le 2$, for all rates below some computational cutoff rate R_{comp}.

Clearly, the decoding complexity must be a monotonically increasing function of ℓ. Thus, let M_d be the decoding complexity measure defined as

$$M_d = \ell m^2 = n^2\frac{\ell}{d^2}. \tag{139}$$

We shall discuss the problem of minimizing M_e and M_d with respect to ℓ and d, for a given rate R, a given constraint length n, and a suitably defined loss in the value of the exponent of the probability of error.

2.6 OPTIMIZATION OF ℓ AND d

This discussion will be limited to rates below R_{crit}, and to cases for which the power constraint of Case 1 is valid. Let L be a loss factor, defined as

$$L = \frac{E(0) - \overline{E_{\ell,d}(0)}}{E(0)}. \tag{140}$$

Now, for rates below R_{crit}, from Eq. A-70, we have

$$E_{\ell,d}(R) = E_{\ell,d}(0) - R; \quad R \le R_{crit}.$$

Thus

$$\overline{E_{\ell,d}(R)} = E(0)(1-L) - R; \quad R \le R_{crit}. \tag{141}$$

Therefore specifying an acceptable $E_{\ell,d}(R)$ for any rate $R \le R_{crit}$, is equivalent to the specification of a proper loss factor L.

We proceed to discuss the minimization of M_e and M_d with respect to ℓ and d, for a given acceptable loss factor L, and a given constraint length n.

For $dE(0) \ll 1$, from Eq. 71, we have

$$\frac{\overline{E_{\ell,d}(0)}}{E(0)} \ge \left(1 - \frac{1}{\ell}\right); \quad dE(0) \ll 1, \quad R \le R_{crit}. \tag{142}$$

Inserting (140) into (142) yields

$$\ell \le \frac{1}{L}; \quad E(0)d \ll 1; \quad R \le R_{crit}.$$

Thus, by Eqs. 138 and 139, we have

$$M_e \leq \frac{n}{Ld}; \quad E(0)d \ll 1 \tag{143a}$$

$$M_d \leq \frac{n^2}{Ld^2}; \quad E(0)d \ll 1\,. \tag{143b}$$

The lower bounds to M_e and M_d decrease when d increases.

Thus, d should be chosen as large as possible and the value of d that minimizes M_e and M_d is therefore outside the region of d for which $E(0)d \ll 1$. The choice of ℓ should be such as to yield the desired loss factor L. Also, by Eq. 72,

$$\overline{E_{\ell,d}(0)} \geq E(0) - \frac{1}{d}\ln\left(\frac{e^{E(0)d}}{\ell} + 1\right); \quad R \leq R_{crit}\,. \tag{144}$$

This bound is effective whenever $\ell \gg 1$. This corresponds to the region $E(0)d \gg 1$. (In order to get a reasonably small L, ℓ should be much larger than unity if $E(0)d \gg 1$.) Inserting (144) into (140) yields

$$L = \frac{1}{dE(0)} \ln\left(\frac{e^{E(0)d}}{\ell} + 1\right).$$

Thus

$$\ell = \frac{e^{E(0)d}}{e^{LE(0)d} - 1}\,. \tag{145}$$

Inserting (145) into (138) and (139) yields

$$M_e \leq \frac{n}{e} \frac{e^{E(0)d}}{e^{LE(0)d} - 1} \tag{146a}$$

$$M_d \leq \frac{n^2}{d^2} \frac{e^{E(0)d}}{e^{LE(0)d} - 1}\,. \tag{146b}$$

From (146a), the bound to M_e has an extremum point at

$$\frac{E(0)d - 1}{E(0)\,d(1-L) - 1} = e^{LE(0)d}\,. \tag{147}$$

if a solution exists. Thus, for $E(0)d \gg 1$,

$$\frac{1}{1 - L} = e^{LE(0)d},$$

or

$$dE(0) = \frac{1}{L} \ln \frac{1}{1 - L}\,.$$

Now, for reasonably small variables of the loss factor L, we have

$$dE(0) \cong 1\,. \tag{148}$$

This point is outside the region of d for which $dE(0) \gg 1$.

From (146b), the bound to M_d has an extremum point at

$$\frac{E(0)d - 2}{E(0)d(1-L) - 2} = e^{LE(0)d}, \qquad (149)$$

if a solution exists. Thus, for $E(0)d \gg 1$,

$$\frac{1}{1 - L} = e^{LE(0)d},$$

or

$$dE(0) = \frac{1}{L} \ln \frac{1}{1 - L}.$$

For reasonably small variables of the loss factor L, therefore, we have $dE(0) \cong 1$. This point is outside the region of d for which $dE(0) \gg 1$.

We may conclude that the lower bounds to M_e and M_d are monotonically decreasing functions of d in the region $dE(0) \ll 1$, and are monotonically increasing functions of d in the region $dE(0) \gg 1$.

Both M_e and M_d are therefore minimized if

$$E(0)d \cong 1; \quad E(0) \leqslant 1, \quad R \leqslant R_{crit} \qquad (150a)$$

And since $d \geqslant 1$, if

$$d = 1; \quad E(0) \gtreqqless 1, \quad R \leqslant R_{crit}. \qquad (150b)$$

The number ℓ is chosen to yield the desired loss factor L.

III. DECODING SCHEMES FOR CONVOLUTIONAL CODES

3.1 INTRODUCTION

Sequential decoding implies that we decode one information digit at a time. The symbol s_i is to be decoded immediately after s_{i-1}. Thus the receiver has the decoded set $(\ldots, s_{-1}, s_o)$ when it is about to decode s_1. We assume that these symbols have been decoded without an error. This assumption, although crucial to the decoding procedure, is not as restrictive as it may appear.

We shall restrict our attention to those s_t that are consistent with the previously decoded symbols.

3.2 SEQUENTIAL DECODING (AFTER WOZENCRAFT AND REIFFEN)

Let u_w be the sequence that consists of the first w input symbols of the transmitted sequence that diverges from the last information digit to be detected. Let u_w' be a member of the incorrect set M''. Therefore u_w' starts with an information digit other than that of the sequence u_w. Let v_w be the sequence that consists of the w output symbols that correspond to the transmitted segment u_w. Let

$$D_w(u,v) = \ln \frac{p(v_w)}{p(v_w|u_w)}. \tag{151}$$

We call this the distance between u_w and v_w, where

$$p(v_w) = \prod_{i=1}^{w} p(y_i)$$

$$p(v_w|u_w) = \prod_{i=1}^{w} p(y_i|x_i)$$

Let us define a constant D_w^j given by

$$P\left(D_w(u,v) \geqslant D_w^j\right) \leqslant e^{-k_j}, \tag{152a}$$

where k_j is some arbitrary positive constant that we call "probability criterion" and is a member of an ordered set

$$K = \{k: k_j = k_{j-1} + \Delta; \quad k_{j_{max}} = E(R)n\}, \tag{152b}$$

where $\Delta \geqslant 0$ is a constant.

Let us now consider the sequential decoding scheme in accordance with the following rules:

1. The decoding computer starts out to generate sequentially the entire tree set M (section 2.5). As the computer proceeds, it discards any sequence u_w' of length w symbols $(1 \leqslant w \leqslant m)$ for which the distance $D_w(u',v) \geqslant D_w^1$. ($D_w^1$ is for

the smallest "probability criterion" k_1).

2. As soon as the computer discovers any sequence M that is retained through length m, it prints out the corresponding first information digit.

3. If the complete set M is discarded, the computer adopts the next larger criterion k_2, and its corresponding distance D_w^2; $(1 \leq w \leq m)$.

4. The computer continues this procedure until <u>some</u> sequence in M is retained through length m. It then prints the corresponding first information digit.

When these rules are adopted, the decoder never uses a criterion K_j unless the correct subset M′ (and hence the correct sequence u_w) is discarded for k_{j-1}. The probability that u_w is discarded depends on the channel noise only. By averaging both over all noise sequences and over the ensemble of all tree codes, we can bound the required average number of computations, $\overline{N}$, to eliminate the incorrect subset M″.

3.3 DETERMINATION OF A LOWER BOUND TO R_{COMP} OF THE WOZENCRAFT-REIFFEN[5,6] DECODING SCHEME

Let N(w) be the number of computations required to extend the search from w to w + 1. Using bars to denote averages, we have

$$\overline{N} = \sum_w \overline{N(w)}.$$

$\overline{N(w)}$ may be upper-bounded in the following way: The number of incorrect messages of length w, M(w), is given by Eq. 143.

$$M(w) \leq A_1 e^{dRw}.$$

The probability that an incorrect message is retained through length w + 1 when the criterion k_j is used is given by

$$\Pr\left[D_w(u', v) \leq D_w^j \middle| j\right]. \tag{153}$$

The criterion k_j is used whenever all sequences are discarded at some length λw $(1/w \leq \lambda \leq m/w)$ with the criterion k_{j-1}.

Thus the probability Pr(j) of such an event is upper-bounded by the probability that the correct sequence u is discarded at some length λw. Therefore

$$p(j) \leq \sum_\lambda \Pr\left(D_{\lambda w}(u, v) \geq D_{\lambda w}^{j-1}\right). \tag{154}$$

Thus, by Eqs. 143, 153, and 154,

$$\overline{N(w)} \leq A_1 e^{dwR} \sum_j \Pr\left(D_w(u', v) \leq D_w^j \middle| j\right) \Pr(j). \tag{155}$$

Inserting (154) into (155) yields

$$\overline{N(w)} \leq A_1 e^{dwR} \sum_{j, \lambda} \Pr\left(D_w(u', v) \leq D_w^j; d_{\lambda w}(u, v) \geq D_{\lambda w}^{j-1}\right). \tag{156}$$

Inserting (156) into (152) yields

$$\overline{N} \leq \sum_{w\lambda, j} A_1 e^{wdR} \Pr\left[D_w(u', v) \leq D_w^j; D_{\lambda w}(u, v) \geq D_{\lambda w}^{j-1}\right]. \tag{157}$$

We would like to obtain an upper bound on

$$\Pr\left[D_w(u', v) \leq D_w^j; D_{\lambda w}(u, v) \geq D_{\lambda w}^{j-1}\right]$$

of the form

$$\Pr\left[D_w(u', v) \leq D_w^j; D_{\lambda w}(u, v) \geq D_{\lambda w}^{j-1}\right] \leq Be^{-R^* dw}, \tag{158}$$

where B is a constant that is independent of w and λ, and R^* is any positive number which is such that (158) is true. Inserting (158) into (157) yields

$$\overline{N} \leq \sum_{w, j, \lambda} Ke^{(R-R^*)wd}, \tag{159}$$

where $k = BA_1$.

The minimum value of R^*, over all w, λ, and j is called "R_{comp}." Thus

$$R_{comp} = \min_{\lambda, w} \{R^*\}. \tag{160}$$

Inserting (160) into (159) yields

$$\overline{N} \leq \sum_{w, j, \lambda} K \exp[-(R_{comp}-R)wd].$$

For $R < R_{comp}$, the summation on w is a geometric series that may be upper-bounded by a quantity independent of the constraint length m. The summation on λ contains m identical terms. The summation on j will contain a number of terms proportional to m. This follows because the largest criterion, $k_{j_{max}}$, has been made equal to E(R)n = E(R)md. Thus for rates $R < R_{comp}$, $\overline{N}$ may be upper-bounded by a quantity proportional to m^2. Reiffen[6] obtained an upper bound

$$R_{comp} \leq E(0).$$

It has been shown[5] that $R_{comp} = E(0)$ whenever the channel is symmetrical.

We proceed to evaluate a lower bound on R_{comp}. From Eq. 151, we have

$$D_{\lambda w}(u, v) = \sum_{i=1}^{\lambda w} d(x, y) = \sum_{i=1}^{\lambda w} \ln \frac{p(y_i)}{p(y_i|x_i)} \tag{161a}$$

$$D_w(u',v) = \sum_{i=1}^{w} d(x',y) = \sum_{i=1}^{w} \ln \frac{p(y_i)}{p(y_i|x_i')} . \tag{161b}$$

Thus, by the use of Chernoff bounds (Appendix A),

$$\Pr\left(D_{\lambda w}(u,v) \geq D_{\lambda w}^{j}\right) \leq e^{\lambda w(\gamma(s)-s\gamma'(s))} . \tag{162}$$

By Eqs. A-27, A-29, and 161a,

$$\gamma(s) = \ln \sum_{X_\ell} \int_Y P(x)\, p(y|x)\, e^{sd(x,y)}\, dy$$

$$= \ln \sum_{X_\ell} \int_Y P(x)\, p(y|x)^{1-s}\, p(y)^s\, dy ; \quad s \geq 0 \tag{163}$$

and

$$\gamma'(s) = \frac{D_{\lambda w}^{j}}{\lambda w} . \tag{164}$$

Thus, by (152) and (162),

$$\Pr\left[D_{\lambda w}(u,v) \geq D_{\lambda w}^{j}\right] \leq e^{\lambda w(\gamma(s)-s\gamma'(s))} = e^{-kj} \qquad \text{for all } \lambda \tag{165a}$$

and

$$D_{\lambda w}^{j} = \lambda w \gamma'(s) . \tag{165b}$$

Here, s is determined as the solution of

$$\frac{k_j}{\lambda w} = s\gamma'(s) - \gamma(s) .$$

In the same way,

$$\Pr\left[D_w(u',v) \leq D_w^{j}\right] \leq e^{w(\mu(t)-t\mu'(t))} . \tag{166a}$$

By the results of section A.3 and Eq. 161b,

$$\mu(t) = \ln \sum_{X_\ell'} \int_Y P(x')\, p(y)\, e^{td(x',y)}\, dy$$

$$= \ln \sum_{X_\ell'} \int_Y P(x')\, p(y)^{1+t}\, p(y,x')^{-t}\, dy dx' ; \quad t \leq 0 \tag{166b}$$

and

$$\mu'(t) = \frac{D_{\lambda w}^{j}}{\lambda w} . \tag{166c}$$

Returning to (158), we find that

$$\Pr\left[D_w(u',v) \leq D_w^j; D_{\lambda w}(y,v) \geq D_{\lambda w}^{j-1}\right] \leq \Pr\left[D_w(u',v) \leq D_w^j\right]. \tag{167a}$$

Also,

$$\Pr\left[D_w(u',v) \leq D_w^j; D_{\lambda w}(u,v) \geq D_{\lambda w}^{j-1}\right] \leq \Pr\left[D_{\lambda w}(u,v) \geq D_{\lambda w}^{j-1}\right]. \tag{167b}$$

Thus, by (167), (165), and (166),

$$\Pr\left[D_w(u',v) \leq D_w^j; D_{\lambda w}(u,v) \geq D_{\lambda w}^{j-1}\right]$$

$$\leq \min\left\{\Pr\left[D_w(u',v) \leq D_w^j\right]; \Pr\left[D_{\lambda w}(u',v) \geq D_{\lambda w}^{j-1}\right]\right\}$$

$$\min\left\{e^{\mu(t)-t\mu'(t)}; \exp[-k_{j-1}]\right\} \tag{168}$$

Now (see 152b) $k_j = k_{j-1} + \Delta;\ \Delta \geq 0$. Thus by (165a)

$$\exp[-k_{j-1}] = e^{\Delta}\exp[-k_j] = e \quad e^{(\gamma(s)-s\gamma'(s))w}, \tag{169a}$$

where

$$\gamma'(s) = \frac{D_w^j}{w} \tag{169b}$$

and

$$\gamma(s) - s\gamma'(s) = -\frac{k^j}{w}. \tag{169c}$$

Therefore, inserting Eqs. 169 into (168) yields

$$\Pr\left[D_w(u',v) \leq D_w^j; D_{\lambda w}(u,v) \geq D_{\lambda w}^{j-1}\right]$$

$$\leq e^{\Delta} \min\left\{e^{w(\gamma(s)-s\gamma'(s))}; e^{w(\mu(t)-t\mu'(t))}\right\}. \tag{170}$$

Thus, if we choose

$$R^* = \max\left\{-\gamma(s)+s\gamma'(s); -\mu(t)+t\mu'(t)\right\}$$

or

$$R^* \geq \frac{1}{2}\left\{-\gamma(s)+s\gamma'(s)-\mu(t)+t\mu'(t)\right\}, \tag{171}$$

then Eq. 158 is valid. Inserting (171) into (160) yields

$$R_{comp} \geq \min \frac{1}{2}\left\{-\gamma(s)+s\gamma'(s)-\mu(t)+t\mu'(t)\right\}. \tag{172}$$

Now, by Eqs. 164 and 166,

$$\gamma'(s) = \frac{\sum\limits_{X_\ell} \int_Y P(x)\, p(y|x)^{1-s}\, p(y)^s \ln \frac{p(y)}{p(y|x)}\, dy}{\sum\limits_{X_\ell} \int_Y P(x)\, p(y|x)^{1-s}\, p(y)^s\, dy} \tag{173a}$$

$$\mu'(t) = \frac{\sum\limits_{X_\ell} \int_Y P(x)\, p(y)^{1+t}\, p(y|x)^{-t} \ln \frac{p(y)}{p(y|x)}\, dy}{\sum\limits_{X_\ell} \int_Y P(x)\, p(y)^{1+t}\, p(y|x)^{-t}\, dy}. \tag{173b}$$

If we let $t = s - 1$, we have

$$\mu'(t) = \gamma'(s); \quad \mu(t) = \gamma(s). \tag{174}$$

Hence

$$R_{comp} \geq \min \frac{1}{2}[(2s-1)\gamma'(s)-2(s)] \; . \tag{175}$$

The minimum occurs at that s, for which

$$[(2s-1)\gamma'(s)-2\gamma(s)]' = 0, \tag{176}$$

which corresponds to $s = 1/2$. Also, $[(1-2s)\gamma'(s)-2\gamma(s)]'' = 2\gamma''(1/2) \geq 0$, since $\gamma''(1/2)$ is the variance[2] of a random variable. Thus, $s = 1/2$ is indeed a minimum point.

Inserting (176) into (175) yields

$$R_{comp} \geq -\gamma\left(\frac{1}{2}\right). \tag{177}$$

Now, by Eq. 163,

$$\gamma\left(\frac{1}{2}\right) = \ln \int_Y \sum_{X_\ell} P(x)\, p(y|x)^{1/2}\, p(y)^{1/2}\, dy, \tag{178}$$

where $p(y) = \sum\limits_X P(x)\, p(y|x)$. Therefore

$$2\gamma\left(\frac{1}{2}\right) = \ln \left\{\int_Y \sum_{X_\ell} p(x)\, p(y|x)^{1/2}\, p(y)^{1/2}\, dy\right\}^2$$

$$= \ln \left\{\int_Y g(y)^{1/2}\, p(y)^{1/2}\, dy\right\}^2, \tag{179}$$

where

$$g(y) = \left\{\sum_{X_\ell} p(x)\, p(y|x)^{1/2}\right\}^2.$$

By the Schwarz inequality,

$$\left\{\int_Y g(y)^{1/2}\, p(y)^{1/2}\, dy\right\}^2 \leq \int_Y g(y)dy \int_Y p(y)dy = \int_Y g(y)dy \tag{180}$$

Inserting (180) into (179) yields

$$2\gamma\left(\frac{1}{2}\right) \leq \ln \int_Y \left[\sum_{X_\ell} p(x)\, p(y|x)^{1/2}\right]^2 dy . \tag{181}$$

Inserting (33) into (177) yields

$$R_{comp} \geq -\frac{1}{2} \ln \int_Y \left[\sum_{X_\ell} p(x)\, p(y|x)^{1/2}\right]^2 dy . \tag{182}$$

Now, by Eqs. A-69 and A-71, we have

$$-\ln \int_Y \left[\sum_{X_\ell} p(x)\, p(y|x)^{1/2}\right]^2 dy$$

which is equal to the zero-rate exponent $E_{\ell,d}(0)$ for the given channel. By a proper selection of p(x) and the number of input symbols, $E_{\ell,d}(0)$ can be made arbitrarily close to the optimum zero-rate exponent E(0). Thus

$$R_{comp} \geq \frac{1}{2} E_{\ell,d}(0) \tag{183}$$

and for semioptimum input spaces

$$R_{comp} \geq \frac{1}{2} E(0) . \tag{184}$$

We have been able to meaningfully bound the average number of computations for discarding the incorrect subset. The harder problem of bounding the computation on the correct subset has not been discussed. A modification of the decoding procedure above, adapted from a suggestion by Gallager[12] for binary symmetric channels, yields a bound on the total number of computations for any symmetric channel. However, no such bound for asymmetric channels has been discovered.

3.4 UPPER BOUND TO THE PROBABILITY OF ERROR FOR THE SEQUENTIAL DECODING SCHEME

Let us suppose that we (conservatively) count as a decoding error the occurrence of either one of the following events.

1. The transmitted sequence μ and the received sequence v are such that they fail to meet the largest criterion $k_{j_{max}}$. The probability of this event, over the ensemble, is less than $m \exp\left[-k_{j_{max}}\right]$.

2. Any element μ' of the incorrect subset M'', together with the received v, satisfies some $k_j < k_{j_{max}}$ when the j^{th} criterion is used.

An element of M'' picked at random, together with the received v_n has a probability of satisfying some k_j equal to

$$\sum_j \Pr\left[D_m(u', v) \leq D_m^j;\ k_j \text{ is used}\right].$$

Since the probability of a union of events is upper-bounded by the sum of the probabilities of the individual events, the probability that any element of M″ together with the received signal v satisfies k_j is less than

$$e^{nR} \sum_j \Pr\left[D_m(u', v) \leq D_m^j;\ k_j \text{ is used}\right].$$

The two events stated above are not in general independent. However, the probability of their union is upper-bounded by the sum of their probabilities. Thus the probability of error p_e may be bounded by

$$p_e \leq m \exp\left[-k_{j_{max}}\right] + e^{mdR} \sum_j \Pr\left[D_m(u', v) \geq D_m^j;\ k_j \text{ is used}\right] \tag{185}$$

It has been shown by Reiffen[5] that for channels that are symmetric at their output, the probability of error is bounded by $p_e \leq m \exp(-E_{\ell, d}(R)n)$, where $E_{\ell, d}(R)$ is the optimum exponent for the given channel and the given input space. (See Appendix A.) We proceed to evaluate (185) for the general asymmetric memoryless channel. The event that k_j is used is included in the event that u′, together with v, will not satisfy the criterion k_{j-1}, or $D_m(u', v) \geq D_m^{j-1}$. Thus

$$\Pr\left[D_m(u', v) \leq D_m^j; k_j \text{ is used}\right] \leq \Pr\left[D_m^j \geq D_m(u', v) \geq D_m^{j-1}\right] \tag{186}$$

Inserting (186) into (185) yields

$$P_e \leq m \exp\left[-k_{j_{max}}\right] + e^{dmR} \Pr\left[D_m(u', v) \leq D_m^{j_{max}}\right]. \tag{187}$$

Now, by (152), D_m^j is chosen so as to make

$$\Pr\left[D_m(u, v) \geq D_m^{j_{max}}\right] \leq \exp\left[-k_{j_{max}}\right].$$

Also, by (162),

$$\Pr\left[D_m(u, v) \geq D_m^{j_{max}}\right] \leq e^{m[\gamma(s) - s\gamma'(s)]}; \quad s \geq 0,$$

where

$$\gamma'(s) = \frac{D_m^{j_{max}}}{m}.$$

Thus, we let $-k_j = m[\gamma(s) - \gamma'(s)]$; therefore

$$\exp[-k_j] = e^{m[\gamma(s) - s\gamma'(s)]}; \quad s \geq 0, \tag{188}$$

where

$$\gamma'(s) = \frac{D_m^{j_{max}}}{m}.$$

From (166), we have

$$\Pr\left[D_m(u', v) \leq d_m^{j_{max}}\right] \leq e^{m(\mu(t)-t\mu'(t))}; \quad t \leq 0, \tag{189}$$

where

$$\mu'(t) = \frac{D_m^{j_{max}}}{m}$$

Inserting (188) and (189) into (187) yields

$$P_e \leq m[e^{m(\gamma(s)-s\gamma'(s))} + e^{m(dR+\mu(t)-t\mu'(t))}], \tag{190a}$$

$$\text{where } \mu'(t) = \gamma'(s) = D_m^{j_{max}}. \tag{190b}$$

By (174), we find that (190b) is satisfied if we let $t = s - 1$. Thus, by Eq. 174,

$$P_e \leq m\left[e^{m(\gamma(s)-s\gamma'(s))} + e^{m(dR+\gamma(s)-(s-1)\gamma'(s))}\right]. \tag{191}$$

Making $\gamma(s) - s\gamma'(s) = dR + \gamma(s) - (s-1)\gamma'(s)$, we obtain

$$P_e \leq 2me^{m(\gamma(s)-s\gamma'(s))} = 2m \exp[-nE_{sq}(R)], \tag{192a}$$

$$\text{where } -\frac{1}{d}\gamma'(s) = R, \text{ and} \tag{192b}$$

$$E_{sq}(R) = \frac{1}{d}(\gamma(s)-s\gamma'(s)). \tag{192c}$$

The rate that makes $E_{sq}(R) = 0$ is the one that corresponds to $s = 0$, since $\{\gamma(s)-s\gamma'(s)\}\big|_{s=0} = 0$. By Eq. 192b,

$$[R]_{s=0} = -\frac{1}{2}\gamma'(s)\Big|_{s=0}.$$

Also, by Eq. 173a,

$$-\frac{1}{d}\gamma'(s)\Big|_{s=0} = -\frac{1}{d}\sum_{X_\ell}\int_Y P(x)\,p(y|x)\ln\frac{p(y)}{p(y|x)}\,dy.$$

Thus

$$E_{sq}(R) \geq 0; \quad R \leq [R]_{s=0}, \tag{193a}$$

where

$$[R]_{s=0} = \frac{1}{d} \sum_{X_\ell} \int_Y P(x)\, p(y|x) \ln \frac{p(y|x)}{p(y)} \, dy . \tag{193b}$$

Comparing Eq. 193 with Eq. A-57 yields $E_{sq}(R) \geq 0$ for the same region of rates as $E_{\ell, d}(R)$. Thus, if the input space X is semioptimal, one can get an arbitrarily small probability of error for rates below the channel capacity C.

The zero-rate exponent $E_{sq}(0)$ is given by

$$E_{sq}(0) = -\gamma(s) + s\gamma'(s) = -\gamma(s) + (s-1)\gamma'(s) , \tag{194}$$

where s is the solution of $\gamma'(s) = 0$. Thus

$$E_{sq}(0) \geq \min \frac{1}{2}\{-2\gamma(s)+(2s-1)\gamma'(s)\} . \tag{195}$$

Following Eqs. 175-184 and substituting for R_{comp} $E_{sq}(0)$, we get

$$E_{sq}(0) \geq \frac{1}{2} E_{\ell, d}(0) , \tag{196}$$

and for semioptimum input spaces

$$E_{sq}(0) \geq \frac{1}{2} E(0) . \tag{197}$$

3.5 NEW SUCCESSIVE DECODING SCHEME FOR MEMORYLESS CHANNELS

A new sequential decoding scheme for random convolutional codes will now be described. The average number of computations does not grow exponentially with n; for rates below some R^*_{comp}, the average number of computations is upper-bounded by a quantity proportional to

$$m^{(1+R/R^*_{comp})} \leq m^2 .$$

The computational cutoff rate R^*_{comp} of the new systematic decoding scheme is equal to the lower bound on R_{comp} for sequential decoding with asymmetric channels (see section 3.3).

However, for sequential decoding, R_{comp} is valid only for the incorrect subset of code words: the existence of R_{comp} for the correct subset has not yet been proved for asymmetric channels. The successive decoding scheme, which is different from other effective decoding schemes such as sequential decoding and low-density parity-check codes[9] yields a bound on the total average number of computations.

A convolutional tree code is shown in Fig. 7 and has been discussed in section 2.5.

Let us now consider the decoding procedure that consists of the following successive operations.

<u>Step 1</u>: Consider the set of b^{k_1} paths of k_1 information digits that diverge from the the first node (branch point). Each path consists, therefore, of $k_1 a$ input symbols.

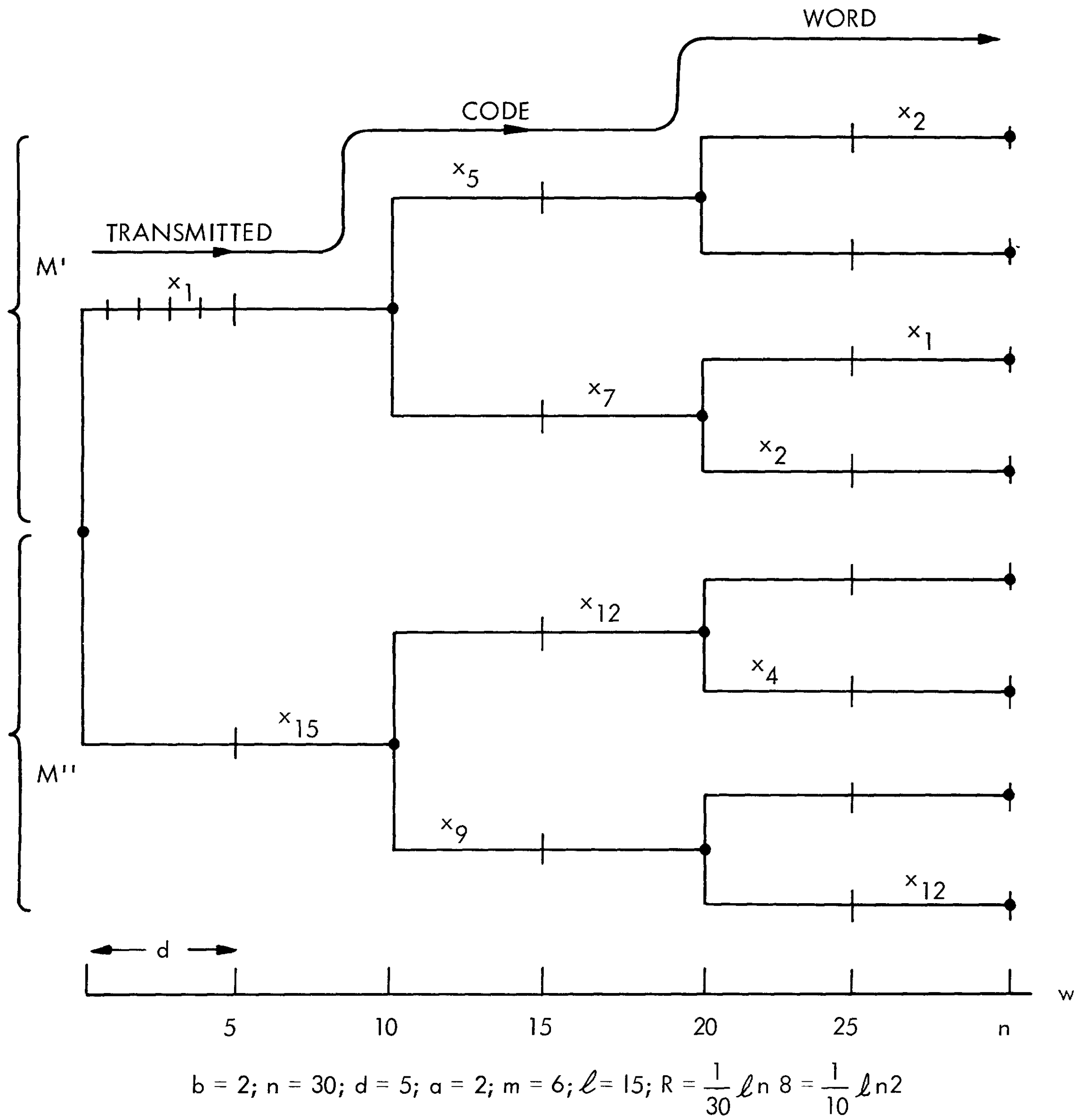

Fig. 7. Convolutional tree code.

The a posteriori probability of each one of the b^{k_1} paths, given the corresponding segment of v, is computed. The first branch of the path of length k_1a which, given v, yields the largest a posteriori probability is tentatively chosen to represent the corresponding first transmitted digit (see Fig. 8).

Let us next consider the set of b^{k_1} paths of length k_1a symbols which diverges from the tentative decision of the previous step. The a posteriori probability of each one of these b^{k_1} paths, given the corresponding segment of the sequence v, is computed. The first branch of the link of length k_1a which, given v, yields the largest a posteriori

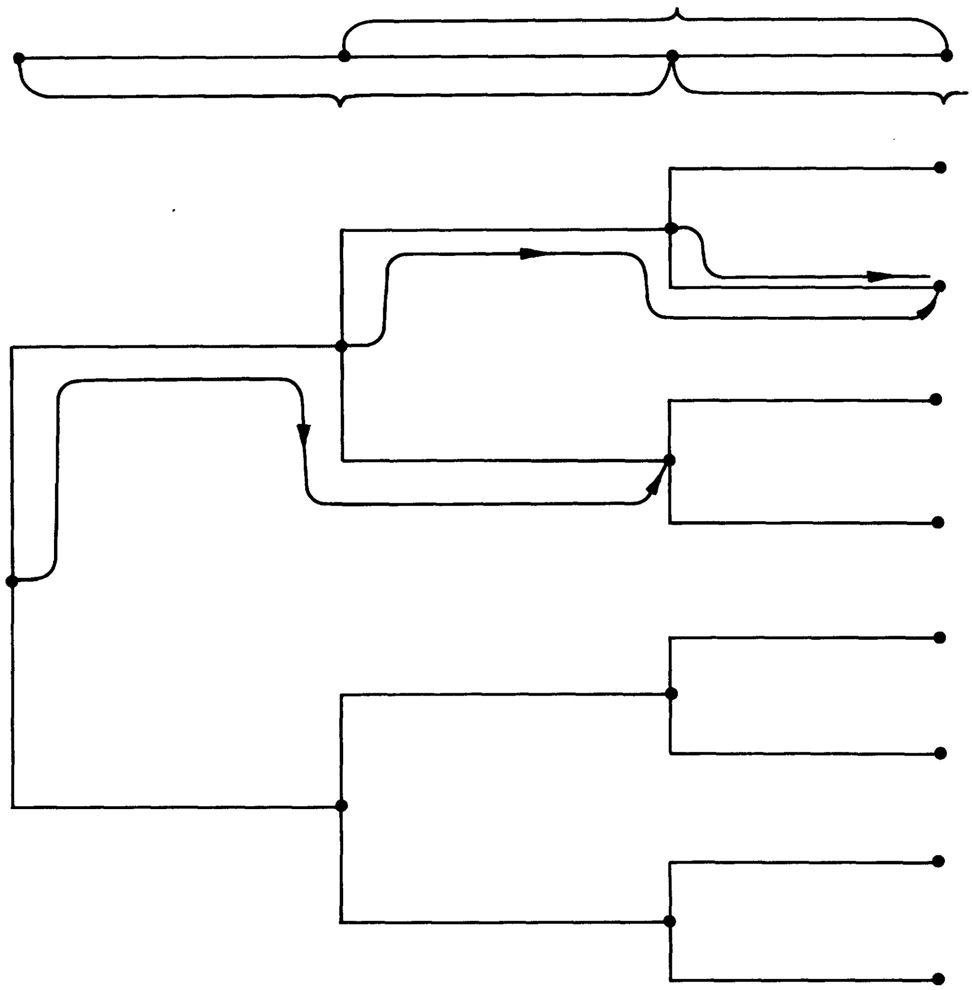

Fig. 8. Successive decoding procedure; $k_i = 2$.

probability is tentatively chosen to represent the second transmitted digit.

This procedure is continued until $i = m/a$ information digits have been tentatively detected.

The distance $D(u_1, v) = \ln \dfrac{p(v)}{p(v|u_1)}$ is then computed for the complete word u_1 of length m input symbols thus obtained.

If $D(u_1, v)$ is smaller than some preset threshold D_0, a firm decision is made that the first digit of u_1 represents the first encoded information digit. If, however, $D(u_1, v) \geq D_0$, the computation procedure is to proceed to Step 2.

Step 2: The decoding procedure of Step 2 is identical with that of Step 1, with the exception that the length k_1 (information digits for $k_1 a$ channel symbols) is replaced by

$$k_2 = k_1 + \Delta; \quad \Delta \text{ a positive integer}. \tag{198}$$

Let u_2 be the detected word of Step 2. If $D(u_2, v) = \ln \frac{p(v)}{p(v|u_2)} < D_0$, a final decision is made, and the detection of the first information digit is completed. If $D(u_2, v) \geq D_0$, no termination occurs and the computation procedure is to then go to Step 3, and so on.

In general, for the j^{th} step, we have

$$k_j = k_{j-1} + \Delta; \quad \Delta \text{ a positive integer} \tag{199}$$

and the detected word is u_j.

Following the detection of the first information digit, the whole procedure is repeated for the next information digit, and so forth.

3.6 THE AVERAGE NUMBER OF COMPUTATIONS PER INFORMATION DIGIT FOR THE SUCCESSIVE DECODING SCHEME

Let us count as a computation the generation of one branch of a random tree code at the receiver. The number of computations that are involved in step j is bounded by

$$N_j \leq mb^{k_j}. \tag{200}$$

Let C_j be the condition that no termination occurs at step j. Step j will be used only if there are no terminations on all the j-1 previous steps. Thus the probability of step j being used is

$$P(j) = \Pr(C_1, C_2, C_3, \ldots, C_{j-1}). \tag{201}$$

The average number of computations is given by

$$\overline{N} = N_1P(1) + N_2P(2) + \ldots + N_jP(j) + \ldots + N_{j_{max}}P(j_{max}) \leq \sum_{j=1}^{\infty} N_jP(j), \tag{202}$$

where $P(1) \equiv 1$, and P(j) may be bounded by

$$P(j) = \Pr(C_1, C_2, C_3, \ldots, C_{j-1}) \leq \Pr(C_{j-1}). \tag{203}$$

Inserting (203) and (200) into (202) yields

$$\overline{N} \leq N_1 + \sum_{j=2}^{\infty} N_j \Pr(C_{j-1}) \leq mb^{k_1} + m \sum_{j=1}^{\infty} b^{k_j} \Pr(C_{j-1}). \tag{204}$$

Now let u_j be the code word detected at step j, and let u be the transmitted code word. Then

$$\begin{aligned} \Pr(C_j) &= \Pr(D(u_j, v) \geq D_0) \\ &= \Pr[D(u_j, v) \geq D_0; u_j = u] + \Pr[D(u_j, v) \geq D_0; u_j \neq u] \\ &= \Pr[D(u, v) \geq D_0; u_j = u] + \Pr[D(u_j, v) \geq D_0; u_j \neq u] \\ &\quad \Pr[D(u, v) \geq D_0] + \Pr[u_j \neq u]. \end{aligned} \tag{205}$$

We are free to choose the threshold D_0 so as to satisfy

$$\Pr[D(u,v) \geq D_0] \leq \exp\left(-\frac{1}{2}E_{\ell,d}(0)n\right) . \tag{206}$$

Now, let e_{jr} be the event that the r^{th} information digit of u_j is not the same as the corresponding digit of the transmitted sequence u. Then

$$\Pr(u_j \neq u) = \Pr\left[\bigcup_{r=1}^{i} \{e_{jr}\}\right] . \tag{207}$$

The probability of a union of events is upper-bounded by the sum of the probabilities of the individual events. Thus

$$\Pr(u_j \neq u) \leq \sum_{r=1}^{i} \Pr(e_{jr}) . \tag{208}$$

There are $\frac{b-1}{b} b^{k_j}$ paths of length k_j information digits that diverge from the $(r-1)^{th}$ node of u, and do not include the r^{th} information digit of u. Over the ensemble of random codes these $\frac{b-1}{b} b^{k_j}$ are statistically independent of the corresponding segment of the transmitted sequence u (see section 2.5). The event e_{jr} occurs whenever the a posteriori probability of one of these $\frac{b-1}{b} b^{k_j}$ paths yields, given v, an a posteriori probability that is larger than that of the corresponding segment of u. Thus, $\Pr(e_{jr})$ is identical with the probability of error for randomly constructed block codes of length $k_j a$ channel input symbols. (All input symbols are members of the d-dimensional input space X_ℓ which consists of ℓ vectors.) Bounds to the probability of error for such block codes are given in Appendix A. Thus

$$\Pr(e_{jr}) \leq \exp(-E_{\ell,d}(R)k_j ad) , \tag{209a}$$

where

$$R = \frac{1}{n} \ln b^{m/a} = \frac{1}{n} \ln b^{n/ad} = \frac{1}{ad} \ln b \tag{209b}$$

$$E_{\ell,d}(R) = E_{\ell,d}(0) - R ; \quad R \leq R_{crit} \tag{209c}$$

$$E_{\ell,d}(R) \geq E_{\ell,d}(0) - R ; \quad R \geq R_{crit} . \tag{209d}$$

Inserting (209) into (207) yields

$$\Pr(u_j \neq u) \leq m \exp(-E_{\ell,d}(R)k_j ad) \tag{210}$$

Inserting (206) and (210) into (205) yields

$$\Pr(C_j) \leq \exp\left(-\frac{1}{2}E_{\ell,d}(0)n\right) + 2m \exp(-E_{\ell,d}(R)k_j ad) . \tag{211}$$

Now, by (209), we have

$$\tfrac{1}{2}E_{\ell,d}(0) \leq E_{\ell,d}(R); \quad R \leq \tfrac{1}{2}E_{\ell,d}(0). \tag{212}$$

Also

$$n = md = iad \geq k_j ad. \tag{213}$$

Inserting (211) and (212) into (208) yields

$$\Pr(C_j) \leq 2m \exp\left(-\tfrac{1}{2}E_{\ell,d}(0)k_j ad\right); \quad R \leq \tfrac{1}{2}E_{\ell,d}(0). \tag{214}$$

Inserting (214) into (204) yields

$$\overline{N} \leq mb^{k_1} + 2m^2 \sum_{j=2} b^{k_j} \exp\left(-\tfrac{1}{2}E_{\ell,d}(0)k_{j-1}ad\right); \quad R \leq \tfrac{1}{2}E_{\ell,d}(0) \tag{215}$$

Inserting (209b) into (215) yields

$$\overline{N} \leq 2\left\{m \exp(Rk_1 ad) + m^2 \sum_{j=2} \exp\left[Rk_j - \tfrac{1}{2}E_{\ell,d}(0)k_{j-1}\right] ad\right\}; \quad R \leq \tfrac{1}{2}E_{\ell,d}(0). \tag{216}$$

By Eq. 199, we have

$$\overline{N} \leq 2\left\{m \exp(Rk_1 ad) + m^2 e^{\Delta Rad} \sum_{j=1} \exp\left[R - \tfrac{1}{2}E_{\ell,d}(0)\right][k_1 + j\Delta]\, ad\right\}; \quad R \leq \tfrac{1}{2}E_{\ell,d}(0) \tag{217}$$

Let R^*_{comp} be defined as

$$R^*_{comp} = \tfrac{1}{2}E_{\ell,d}(0). \tag{218}$$

Then, for all rates below R^*_{comp}, $R - \frac{1}{2}E_{\ell,d}(0) < 0$, and therefore

$$\overline{N} \leq 2\left\{m \exp[Rk_1 ad] + m^2 \frac{\exp[R-R^*_{comp}]\, k_1 ad}{1 - \exp[R-R^*_{comp}]\, \Delta ad} e^{\Delta Rad}\right\}; \quad R \leq R^*_{comp} = \tfrac{1}{2}E_{\ell,d}(0). \tag{219}$$

The bound on the average number of computations given by (219) is minimized if we let

$$\Delta = \frac{1}{ad}\left(\frac{\ln R/R^*_{comp}}{R/R^*_{comp} - 1}\right)\frac{1}{R^*_{comp}} \tag{220a}$$

$$k_1 = \Delta + \frac{1}{ad}(\ln m)\frac{1}{R^*_{comp}}. \tag{220b}$$

Equation 220 can be satisfied only if both Eqs. 220a and 220b yield positive integers.

Inserting (220) yields

$$\overline{N} \leq \frac{B^{B/1-B}}{1-B} m^{1+B}; \quad R \leq R^*_{comp} = \frac{1}{2} E_{\ell, d}(0), \tag{221}$$

where $B = R/R^*_{comp} \leq 1$.

3.7 THE AVERAGE PROBABILITY OF ERROR OF THE SUCCESSIVE DECODING SCHEME

Let u be the transmitted sequence of length n samples. Let u′ be a member of the incorrect subset M″. (The set M″ consists of M″ members.) As we have shown, u′ is statistically independent of u. The probability of error is then bounded by

$$P_e \leq \Pr[D(u,v) \geq D_0] + M''\Pr[D(u',v) < D_0] \leq \Pr[D(u,v) \geq D_0] + e^{mdR}\Pr[D(u',v) < D_0]. \tag{222}$$

Now

$$D(u,v) = \ln \frac{p(v)}{p(v|u)},$$

where

$$p(v) = \prod_{i=1}^{m} p(y_i)$$

and

$$p(v|u) = \prod_{i=1}^{m} p(y_i|x_i).$$

Thus, by the use of Chernoff Bounds (Appendix A),

$$\Pr[D(u,v) \geq D_0] \leq e^{m[\gamma(s)-s\gamma'(s)]}, \tag{223}$$

where

$$\gamma(s) = \ln \sum_{X_\ell} \int_Y P(x)\, p(y|x)^{1-s}\, p(y)^s\, dy; \quad s \geq 0$$

$$\gamma'(s) = D_0.$$

Also

$$\Pr[D(u',v) < D_0] \leq e^{m[\mu(t)-t\mu'(t)]} \tag{224}$$

$$\mu(t) = \ln \sum_{X_\ell} \int_Y P(x)\, p(y|x)^{-t}\, p(y)^{1+t}\, dy$$

$$\mu'(t) = D_0.$$

If we let $t = s - 1$, by Eq. 174, we have

$$\gamma(s) = \mu(t); \quad \gamma'(s) = \mu'(t).$$

Inserting (223), (224), and (174) into (222) yields

$$P_e \leqslant e^{m[\gamma(s)-s\gamma'(s)]} + e^{m[dR+\gamma(s)-(s-1)\gamma'(s)]}, \tag{225}$$

where $\gamma'(s) = D_0$.

Now, comparing (223) with (206) yields

$$-\gamma(s) + s\gamma'(s) = \frac{1}{2} dE_{\ell,d}(0). \tag{226}$$

On the other hand, we have by (175) through (183)

$$-2\gamma(s) + (2s-1)\gamma'(s) \geqslant dE_{\ell,d}(0). \tag{227}$$

Thus, inserting (226) into (227) yields

$$-\gamma(s) + (s-1)\gamma'(s) \geqslant \frac{1}{2} dE_{\ell,d}(0). \tag{228}$$

Inserting (226) and (228) into (225) yields

$$\begin{aligned} P_e &\leqslant \exp\left(-\frac{1}{2}nE_{\ell,d}(0)\right) + \exp\left(-n\left[\frac{1}{2}E_{\ell,d}(0)-R\right]\right) \\ &\leqslant 2\exp\left(-[\frac{1}{2}E_{\ell,d}(0)-R]\right) = 2\exp\left(-n[R^*_{comp}-R]\right); \quad R \leqslant R^*_{comp} \end{aligned} \tag{229}$$

If the input space is semioptimal, we have $E_{\ell,d}(0) \cong E(0)$. Thus

$$P_e \leqslant 2e^{(-[1/2\,E(0)-R]n)}; \quad R \leqslant R^*_{comp} = \frac{1}{2} E(0). \tag{230}$$

If, instead of setting D_0 as we did in (206), we set it so as to make $\gamma(s) - s\gamma'(s) = dR + \gamma(s) - (s-1)\gamma'(s)$, where $\gamma'(s) = D_0$, we have by (225)

$$P_e \leqslant 2\exp[-nE_s(R)],$$

where by (191)-(193) we have (for semioptimal input spaces)

$$E_s(0) \geqslant \frac{1}{2} E(0)$$

$$E_s(R) > 0; \quad R < C.$$

However, following Eqs. 206-218, it can be shown that the new setting of D_0 yields $R^*_{comp} \geqslant \frac{1}{4} E_{\ell,d}(0)$.

The fact that the successive decoding scheme yields a positive exponent for rates above R^*_{comp} does not imply that this scheme should be used for such rates, since the number of computations for $R \geqslant R^*_{comp}$ grows exponentially with m.

IV. QUANTIZATION AT THE RECEIVER

4.1 INTRODUCTION

The purpose of introducing quantization at the receiver is to avoid the utilization of analogue devices. Because of the large number of computing operations that are carried out at the receiver, and the large flow of information to and from the memory, analogue devices may turn out to be more complicated and expensive than digital devices.

Our discussion is limited to the Gaussian channel and to rates below R_{crit}. The effect of quantization on the zero-rate exponent of the probability of error will be discussed for three cases. (See Fig. 9.)

Case I. The quantizer is connected to the output terminals of the channel.

Case II. The logarithm of the a posteriori probability per input letter (that is, $p(y|x_i)$; $i = 1, \ldots, \ell$) is computed and then quantized.

Case III. The logarithm of the a posteriori probability per p input letters (i.e., $p(y^p|x_j^p)$; $j = 1, \ldots, \ell^p$) is computed and then quantized. x_j^p is the vector sum of p successive input-letters of one of the M code words; y^p is the vector sum of the p received outputs.

It was shown in section 2.3 that whenever semioptimum input spaces are used with white Gaussian channels, $E_{\ell,d}(0)$ is a function of A_{max}^2, the maximum signal-to-noise ratio. In this section, the effect of quantization is expressed in terms of "quantization loss" L_q in the signal-to-noise ratio of the unquantized channel.

Let $E_{\ell,d}^q(0)$ be the zero-rate exponent of the quantized channel. Then, by Eq. A-70,

$$E_{\ell,d}^q(R) = E_{\ell,d}^q(0) - R; \quad R \leq R_{crit}.$$

Therefore specifying an acceptable $E_{\ell,d}^q(R)$ for any rate $R \leq R_{crit}$, is the same as specifying a proper loss factor L_q.

Let M_q be the number of quantization levels that are to be stored in the memory of the "Decision Computer" per one transmitted symbol.

Under the assumption that one of the two decoding schemes discussed in Section III is used, it is then convenient to define the total decoding complexity measure (including the quantizer).

$$M = M_d M_q \tag{231}$$

with M_d given by Eq. 139. We shall minimize M with respect to ℓ and d for a fixed n and a given quantization loss, L_q. In Section II we discussed the ways of minimizing M_d with respect to ℓ and d.

We shall show that if semioptimal input spaces are used with a white Gaussian channel, M_q of the quantization scheme of Case III (Fig. 9) is always larger than that of Case II and therefore the quantization scheme of Case III should not be used.

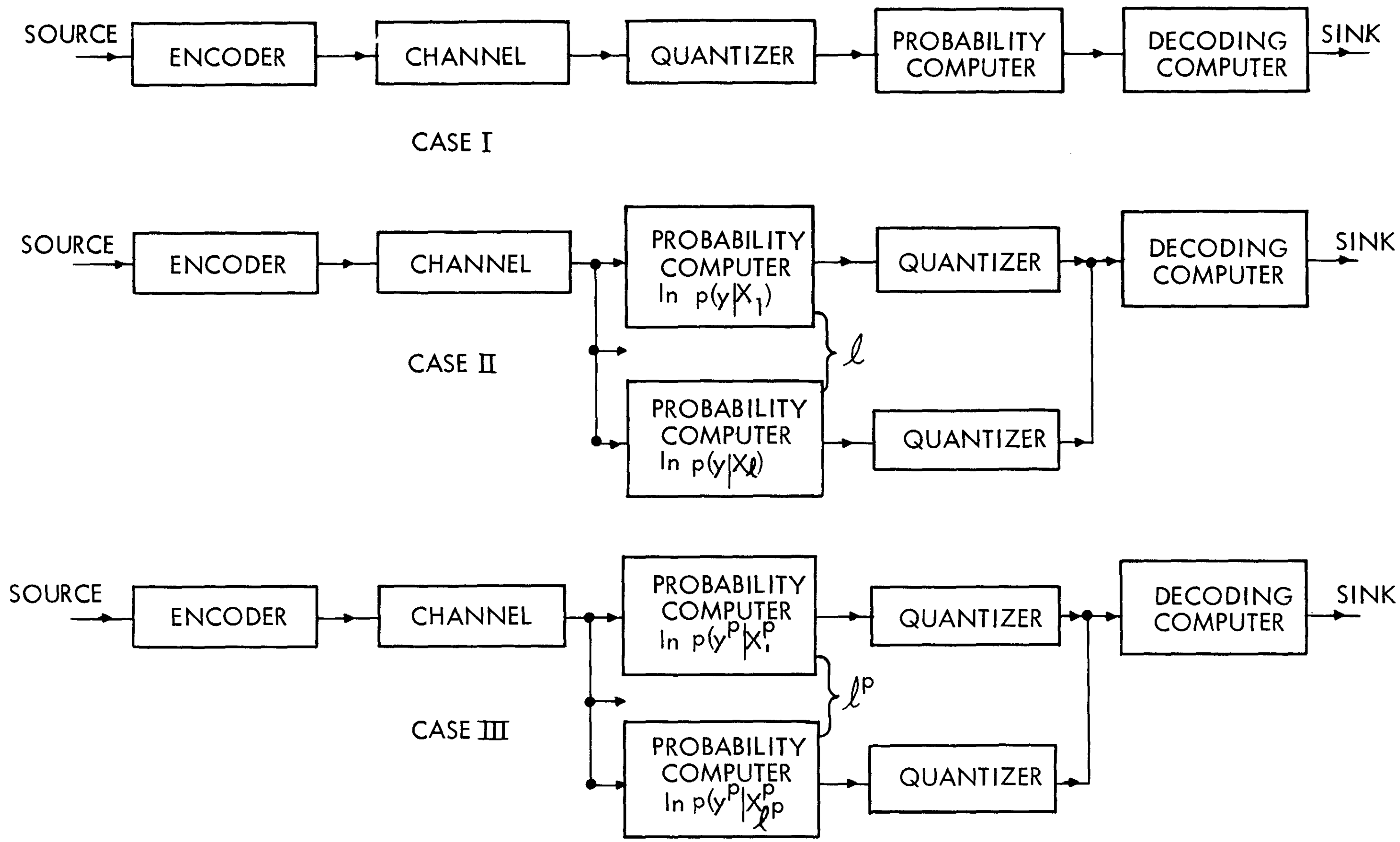

Fig. 9. Quantization schemes.

Also, whenever $E(0) \approx \frac{1}{2} \ln A_{max}^2 \gg 1$, M_q of the quantization scheme of Case I (Fig. 9) is smaller than that of Case II, and therefore the quantization scheme of Case I should be used in such cases. On the other hand, whenever $E(0) \ll 1$ (or $A_{max}^2 \ll 1$), M_q of Case II is smaller than that of Case I.

Furthermore, it will be shown that M, like M_d, is minimized if we let $d \approx \frac{1}{E(0)}$; $E(0) \ll 1$.

The results mentioned above are derived for the quantizer shown in Fig. 10a which is equivalent to that of Fig. 10b.

The interval Q (Fig. 10b) is assumed to be large enough so that the limiter effect can be neglected as far as the effect on the exponent of the probability of error is concerned.

Thus, the quantizer of Fig. 10b can be replaced by the one shown in Fig. 10c. However, the actual number of quantization levels is not infinite as in Fig. 10c, but is equal to $k = Q/q$ as in Fig. 10b.

4.2 QUANTIZATION SCHEME OF CASE I (FIG. 9)

The quantized zero-rate exponent $E_{\ell,d}^q(0)$ of Case I can be lower-bounded by the zero-rate exponent of the following detection scheme. The distance

$$d^q(x, y) = \frac{-2y^q x + x^2}{2\sigma^2} \tag{232}$$

is computed for each letter x_i of the tested code word. Here y^q is the quantized vector of the channel output y:

$$y^q = \eta_1^q, \eta_2^q, \dots, \eta_d^q . \tag{233}$$

The distance

$$D^q(u, v) = \sum_{i=1}^{n} d_i^q(y_i, x_i)$$

is then computed.

The one code word that yields the smallest distance is chosen to represent the transmitted word. This detection procedure is optimal for the unquantized Gaussian variable y. However, y^q is not a Gaussian random variable and therefore $d^q(x, y)$ is not necessarily the best distance.

Thus, this detection scheme will yield an exponent $E^*(0)$, which will be a lower bound on $E_{\ell,d}^q(0)$.

$$E^*(0) \leq E_{\ell,d}^q(0) \leq E_{\ell,d}(0) . \tag{234}$$

The probability of error is bounded by

$$P_e \leq (M-1) \Pr[D(v^q, u') \leq D(v^q, u)] \tag{235a}$$

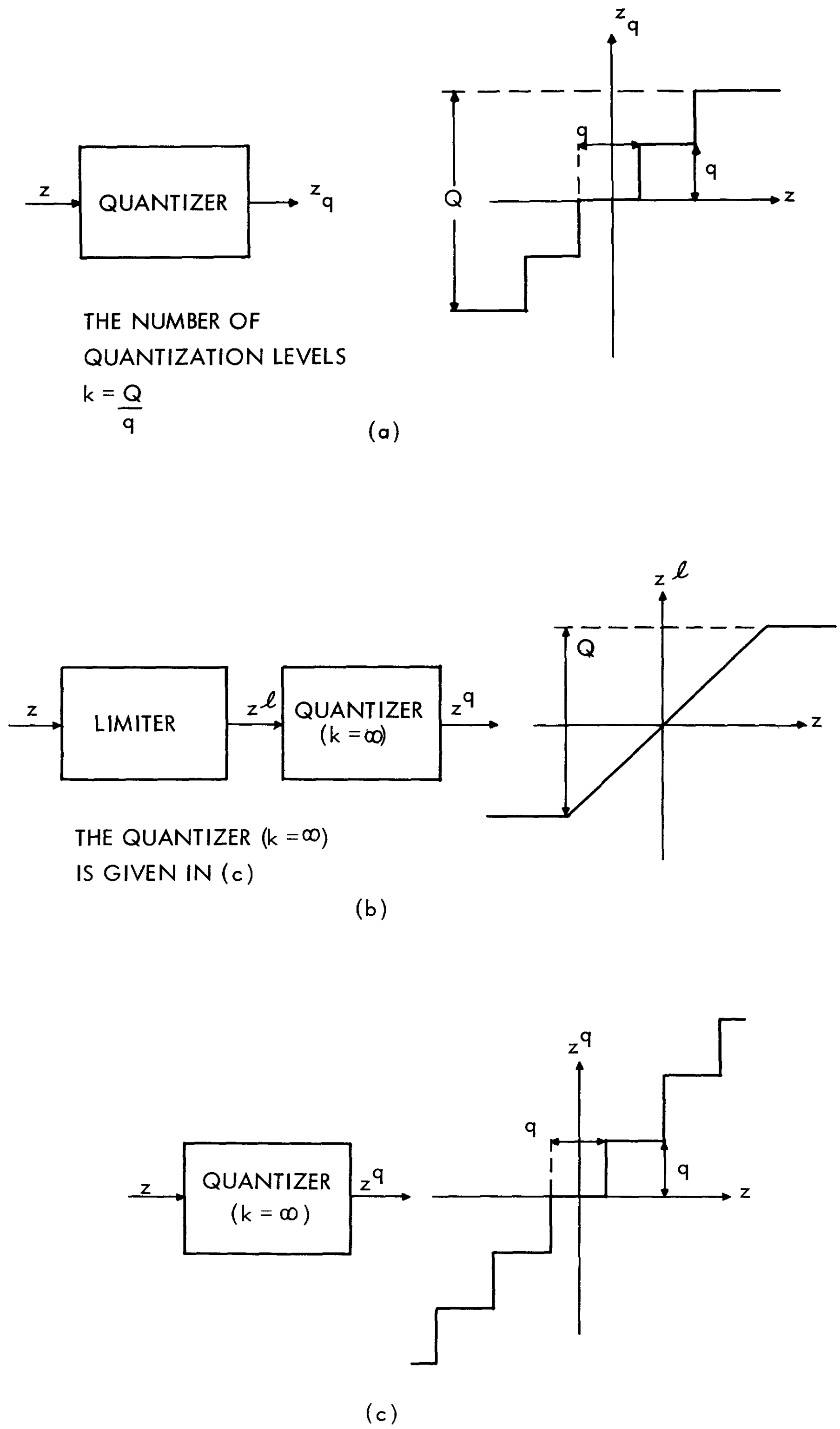

Fig. 10. Quantizers and their transfer characteristics.

or

$$P_e \leq M \Pr[D(v^q, u) - D(v^q, u') \geq 0] , \tag{235b}$$

where u represents the transmitted word, and u' some other code word.

$D^q(u, v)$, as well as $D^q(u', v)$, are sums of n independent random variables:

$$D^q(u, v) = \sum_{i=1}^{n} d_i^q(y_i, x_i) \tag{236a}$$

$$D^q(u', v) = \sum_{i=1}^{n} d_i^q(y_i, x_i') , \tag{236b}$$

where x_i is the i^{th} transmitted letter, and x_i' is the i^{th} letter of some other code word.

By the use of the Chernoff bounds (Appendix A.3) it can be shown that

$$P_e \leq (M-1)\, e^{-E^*(0)n} \leq e^{-n[E^*(0)-R]} \tag{237}$$

and, by Eq. A-65,

$$-E^*(0) = \mu^*(s) = \frac{1}{d} \ln \sum_{Y^q} \sum_{X_\ell} \sum_{X_\ell'} P(x)\, P(x')\, p(y^q|x)\, e^{s[d^q(x,y) - d^q(x',y)]} \tag{238}$$

$$s \geq 0. \tag{239}$$

Now, let s = 1/2. Then, by Eqs. 102a and 237,

$$E^*(0) = -\frac{1}{d} \ln \sum_{X_\ell} \sum_{X_\ell'} P(x')\, P(x) \exp\left(\frac{|x|^2 - |x'|^2}{4\sigma^2}\right) \sum_{y^q} p(y^2|x) \exp\left(\frac{y^q(x'-x)}{2\sigma^2}\right), \tag{240}$$

where

$$y^q x = \eta_1^q \xi_i + \eta_2^q \xi_2 + \ldots + \eta_d^q \xi_d . \tag{241}$$

Thus, by Eqs. 240, 234, and 102,

$$E^q_{\ell,d}(0) \geq -\frac{1}{d} \ln \sum_{X_\ell} \sum_{X_\ell'} P(x)\, P(x') \exp\left(\frac{|x|^2 - |x'|^2}{4\sigma^2}\right) \sum_{y^q} p(y^q|x) \exp\left(\frac{y^q(x'-x)}{2\sigma^2}\right) \tag{242a}$$

$$E^q_{\ell,d}(0) \leq -\frac{1}{d} \ln \sum_{X_\ell} \sum_{X_\ell'} P(x)\, P(x') \exp\left(\frac{|x|^2 - |x'|^2}{4\sigma^2}\right) \int_Y p(y|x) \exp\left(\frac{y(x'-x)}{2\sigma^2}\right) dy = E_{\ell,d}(0) \tag{242b}$$

The complete information about the quantization effects is therefore carried by the term

$$g^q(x, x') = \sum_{Y^q} p(y^q|x) \exp\left(\frac{y^q(x'-x)}{2\sigma^2}\right) \tag{243a}$$

when compared with the unquantized term

$$g(x, x') = \int_Y p(y|x) \exp\left(\frac{y(x'-x)}{2\sigma^2}\right) dy \tag{243b}$$

The quantizer is a memoryless device; therefore, since the channel is memoryless as well, by Eq. 9, we have

$$p(y^q|x) = p\left(\eta_1^q \middle| \xi_1\right) p\left(\eta_2^q \middle| \xi_2\right) \dots p\left(\eta_d^q \middle| \xi_d\right) .$$

Thus

$$g^q(x, x') = \prod_{i=1}^{d} \left\{ \sum_{\eta_i^q} p\left(\eta_i^q \middle| \xi_i\right) \exp\left(\frac{\eta_i^q(\xi_i'-\xi_i)}{2\sigma^2}\right) \right\} \tag{244}$$

Two important signal-to-noise ratio conditions will be discussed.

<u>Condition 1.</u> $\frac{\xi_{max}}{\sigma} = A_{max} \leq 1$. At the same time $q \leq 2\sigma$. (245)

<u>Condition 2.</u> $\frac{\xi_{max}}{\sigma} = A_{max} > 1$. (246)

a. Condition 1. $A_{max} \leq 1$

We have

$$\left|\frac{(\xi_i-\xi_i')}{2\sigma}\right| \leq 1 ; \qquad \text{for all } \xi \text{ and } \xi' \tag{247}$$

It is shown in Appendix C that whenever the quantizer of Fig. 10c is used and the input to the quantizer is a Gaussian random variable with a probability density such as that in Eq. 78, we have

$$\sum_{\eta_i^q} p\left(\eta_i^q \middle| \xi_i\right) \exp\left(\frac{\eta_i^q(\xi_i'-\xi_i)}{2\sigma^2}\right)$$

$$= \left[\int_{\eta_i} p(\eta_i|\xi_i) \exp\left(\frac{\eta_i(\xi_i'-\xi_i)}{2\sigma^2}\right) d\eta_i\right] \frac{\operatorname{sh} \frac{(\xi_i'-\xi_i)\, q}{4\sigma^2}}{\frac{(\xi_i'-\xi_i)\, q}{4\sigma^2}}; \qquad \text{for } q < 2\sigma. \tag{248}$$

Inserting Eq. 248 into Eq. 244 yields

$$g^q(x, x') = \left[\int_Y p(y|x) \exp\left(\frac{y(x'-x)}{2\sigma^2}\right) dy\right] \prod_{i=1}^{d} \left[\frac{\operatorname{sh} \frac{(\xi_i'-\xi_i) q}{4\sigma^2}}{\frac{(\xi_i'-\xi_i) q}{4\sigma^2}}\right]. \tag{249}$$

Thus, by Eqs. 249 and 243a,

$$g^q(x', x) = g(x', x) \prod_{i=1}^{d} \left[\frac{\operatorname{sh} \frac{(\xi_i'-\xi_i) q}{4\sigma^2}}{\frac{(\xi_i'-\xi_i) q}{4\sigma^2}}\right]. \tag{250}$$

Now

$$\frac{\operatorname{sh} x}{x} \leqslant \exp\left(\frac{x^2}{6}\right). \tag{251a}$$

Also, for $x < 1$,

$$\frac{\operatorname{sh} x}{x} \approx \exp\left(\frac{x^2}{6}\right). \tag{251b}$$

Thus, by Eqs. 251a and 247,

$$\frac{\operatorname{sh} \frac{(\xi_i'-\xi_i) q}{4\sigma^2}}{\frac{(\xi_i'-\xi_i) q}{4\sigma^2}} \lessapprox \exp\left(\frac{(\xi_i'-\xi_i)^2 q^2}{96\sigma^4}\right) \tag{251c}$$

Inserting Eq. 251c into Eq. 250 yields

$$g^q(x', x) \lessapprox g(x', x) \prod_{i=1}^{d} \exp\left(\frac{(\xi_i'-\xi_i)^2}{96\sigma^4}\right)$$

or

$$g^q(x', x) \lessapprox g(x', x) \exp\left(\sum_{i=1}^{d} (\xi_i'-\xi_i)^2 \frac{q^2}{96\sigma^4}\right). \tag{252}$$

Therefore, by Eq. 7,

$$g^q(x, x') \lessapprox g(x, x') \exp\left(\frac{q^2 |x-x'|^2}{96\sigma^4}\right). \tag{253}$$

Replacing Eq. 243b with Eq. 253 and inserting Eq. 253 into Eq. 242a yields

$$E^q_{\ell,d}(0) \geqslant -\frac{1}{d} \ln \sum_{X_\ell} \sum_{X_\ell'} P(x) p(x') \exp\left(\frac{|x|^2 - |x'|^2}{4\sigma^2}\right) g(x, x') \exp\left(\frac{q^2 |x-x'|^2}{96\sigma^4}\right). \tag{254}$$

Inserting Eq. 243b into Eq. 254 yields

$$E^q_{\ell,d}(0) \geq -\frac{1}{d} \ln \sum_{X_\ell} \sum_{X'_\ell} P(x) P(x') \exp\left(\frac{q^2 |x-x'|^2}{96\sigma^4}\right) \exp\left(\frac{|x|^2 - |x'|^2}{4\sigma^2}\right) \int_y p(y\ x) \exp\left(\frac{y(x'-x)}{2\sigma^2}\right) dy. \tag{255}$$

Inserting Eq. 78 into Eq. 255 yields

$$E^q_{\ell,d}(0) \geq -\frac{1}{d} \ln \sum_{X_\ell} \sum_{X'_\ell} P(x) P(x') \exp\left(\frac{q^2 |x-x'|^2}{96\sigma^4}\right) \exp\left(-\frac{|x-x'|^2}{8\sigma^2}\right) \tag{256}$$

$$E^q_{\ell,d}(0) \geq -\frac{1}{d} \ln \sum_{X_\ell} \sum_{X'_\ell} P(x') P(x) \exp\left(-\frac{|x-x'|^2(1-q^2/12\sigma^2)}{8\sigma^2}\right). \tag{257}$$

Then, by Eq. 105a, we have

$$E_{\ell,d}(0) = -\frac{1}{d} \ln \sum_{X_\ell} \sum_{X'_\ell} P(x') P(x) \exp\left(-\frac{|x-x'|^2}{8\sigma^2}\right). \tag{258}$$

Comparing Eq. 256 with Eq. 258 shows that whenever the channel is in Condition 1 (Eq. 245), the zero-rate exponent $E^q_{\ell,d}(0)$ of the quantized channel is lower-bounded by the zero-rate exponent of an unquantized Gaussian channel with an average poise power of

$$\sigma_q^2 = \frac{\sigma^2}{1 - \frac{q^2}{12\sigma^2}} \qquad \text{for } q < 2\sigma. \tag{259}$$

This result does not depend on the kind of input space that is used nor on its dimensionality, d. The effective signal-to-noise ratio of the quantized channel is given by

$$A_q^2 = \frac{\xi^2_{max}}{\sigma_q^2} = \frac{\xi^2_{max}}{\sigma^2}\left(1 - \frac{q^2}{12\sigma^2}\right).$$

Thus

$$A_q^2 = A^2_{max}\left(1 - \frac{q^2}{12\sigma^2}\right). \tag{260}$$

Therefore, for a given quantization loss in the signal-to-noise ratio, let

$$q = \sqrt{12}\, L_q \sigma \qquad q < 2\sigma, \tag{261}$$

where L_q, the "quantization loss," is a constant that is determined by the acceptable loss in signal-to-noise ratio,

$$\frac{A_q^2}{A_{max}^2} = 1 - L_q^2. \tag{262}$$

The number of quantization levels, as shown in Fig. 10b, is equal to

$$k = \frac{Q}{q}. \tag{263}$$

It is quite clear from the nature of the Gaussian probability density that if we let

$$\frac{Q}{2} = |\xi|_{max} + B\sigma, \tag{264}$$

where B is a constant, then the effect of the limiter on $E_{\ell,d}(0)$ (shown in Fig. 10b) becomes negligible if B is large enough (approximately 3).

Thus, inserting Eq. 264 into Eq. 263 yields

$$k = \frac{2\xi_{max} + 2B\sigma}{12\ L_q\sigma}. \tag{265}$$

Now, if $\frac{\xi_{max}}{\sigma} = A \ll 1$,

$$k = \frac{2B}{L_q\sqrt{12}}. \tag{266}$$

The number of quantization levels for a given effective loss in signal-to-noise ratio is therefore independent of A_{max}, for $A_{max} \ll 1$. In the following section, the effective loss in signal-to-noise ratio for higher values of A, and the corresponding number of quantization levels k, are discussed.

b. Condition 2. $A_{max} > 1$

In this condition we have $A_{max} > 1$, and therefore

$$\left|\frac{\xi_i - \xi_i'}{2\sigma}\right| > 1$$

for some ξ and ξ'. Now, if $q \leq 2\sigma$, Eqs. 251c-262 are valid.

The number of quantization levels is given by

$$k = \frac{2\xi_{max} + 2B\sigma}{\sqrt{12}\ L\sigma}$$

or by

$$k = \frac{2A_{max} + 2B}{\sqrt{12}\ L}. \tag{267a}$$

Thus, for $A_{max} \gg 1$,

$$k = \frac{2A_{max}}{\sqrt{12}\, L}; \quad q < 2\sigma. \tag{267b}$$

In this case, again, k does not depend on the kind of input space that is used. There are many cases, however, in which the assumption that $q < 2\sigma$ is unrealistic, since much larger quantization grain q can be used and still yield the acceptable loss L_q^2.

The effects of quantization in these cases depend heavily on the kind of input set which is used. This fact will be demonstrated by the following typical input sets.

Set 1. This input set consists of two equiprobable oppositely directed vectors

$$x_1 = x; \quad x_2 = -x, \tag{268a}$$

where

$$P(x_1) = P(x_2) = \frac{1}{2}. \tag{268b}$$

As shown in section 2.3, this input set is not optimal for $A_{max} > 1$. A semioptimal input space for $A_{max} > 1$, as shown in section 2.3, is Set 2.

Set 2. This input set consists of ℓ equiprobable one-dimensional vectors. The distance between two adjacent vectors is A_{max}/ℓ, as shown in Fig. 4.

When the input set consists of two oppositely directed vectors, by Eq. 106, we have

$$E_{2,d}(0) = -\frac{1}{d} \ln \left[\frac{1}{2} + \frac{1}{2} \exp\left(-\frac{A_{max}^2 d}{2}\right)\right].$$

Also, by Eqs. 242a, 243a, and 244,

$$E_{2,d}^q(0) \geq -\frac{1}{d} \ln \sum_{X_\ell} \sum_{X'_\ell} P(x) P(x') g^q(x, x') \exp\left(\frac{|x|^2 - |x'|^2}{2\sigma^2}\right). \tag{269}$$

In this case,

$$g^q(x, x') = \left[\sum_{\eta^q} p(\eta^q|\xi) \exp\left(\frac{\eta^q(\xi'-\xi)}{2\sigma^2}\right)\right]^d$$

since by Eq. 268, $\xi_i = \xi_j = \xi$ and $\xi'_i = \xi'_j = \pm\xi$, for all $i = 1, \ldots, d$ and $j = 1, \ldots, d$.

Thus, by Eq. 269

$$g^q(x, x') = 1; \quad x' = x \tag{270a}$$

$$g^q(x, x') = \left[\sum_{\eta^q} p(\eta^q|\xi) \exp\left(\frac{\eta^q(\xi'-\xi)}{2\sigma^2}\right)\right]^d; \quad x \neq x' = -x \tag{270b}$$

Now for $x' \neq x$,

$$\frac{|\xi'-\xi|}{2\sigma} = \frac{|2\xi|_{max}}{2\sigma} > 1,$$

since $A_{max} > 1$. It is shown in Appendix C that in such a case

$$\sum_{\eta^q} p(\eta^q|\xi) \exp\left(\frac{\eta^q(\xi'-\xi)}{2\sigma^2}\right) \leq \left[\sum_{\eta} p(\eta|\xi) \exp\left(\frac{(\xi'-\xi)\eta}{2\sigma^2}\right)\right] \exp\left(\frac{|q(\xi'-\xi)|}{4\sigma^2}\right). \tag{271}$$

Thus, inserting Eq. 271 into Eq. 270 yields

$$g^q(x,x') \leq \left[\sum_{\eta} p(\eta|\xi) \exp\left(\frac{(\xi'-\xi)\eta}{2\sigma^2}\right)\right]^d \exp\left(\frac{|q(\xi'-\xi)|d}{4\sigma^2}\right)$$

or

$$g^q(x,x') \leq \int_y p(y|x) \exp\left(\frac{y(x'-x)}{2\sigma^2}\right) dy \, \exp\left(\frac{|q(\xi'-\xi)|d}{4\sigma^2}\right). \tag{272}$$

Thus, by Eqs. 272 and 243b, we have

$$g^q(x,x') \leq g(x,x') \exp\left(\frac{|q(\xi'-\xi)|d}{4\sigma^2}\right).$$

Thus

$$g^q(x,x') = g(x',x) = 1; \quad x' = x \tag{273a}$$

and

$$g^q(x,x') = g(x,x') \exp\left(\frac{qAd}{2\sigma}\right); \quad x' \neq x. \tag{273b}$$

Thus, inserting Eqs. 273 and 243b into Eq. 242a yields, together with Eq. 268,

$$E^q_{2,d}(0) \geq -\frac{1}{d} \ln\left(\frac{1}{2} + \frac{1}{2}\exp\left(-\frac{A^2d}{2}\right)\exp\left(\frac{qAd}{2\sigma}\right)\right). \tag{274}$$

The zero-rate exponent of the unquantized channel is given by Eq. 126. Let

$$A_q^2 = A^2 - \frac{q}{\sigma}A. \tag{275}$$

Inserting Eq. 275 into Eq. 274 yields

$$E^q_{2,d}(0) \geq -\frac{1}{d} \ln\left(\frac{1}{2} + \frac{1}{2}\exp\left(-\frac{A_q^2 d}{2}\right)\right). \tag{276}$$

Thus, by comparing Eq. 276 with Eq. 126, we find that the zero-rate exponent $E^q_{2,d}(0)$ of the quantized channel may be lower-bounded by the zero-rate exponent of the unquantized channel (with the same input set) if the original signal-to-noise ratio A^2 is replaced with A_q^2, which is given by Eq. 275.

Let

$$q = L_q^2 \xi_{max} = L_q^2 \xi ; \quad \xi_{max} = \xi , \tag{277}$$

where L_q is the "quantization loss" factor determined by the acceptable loss in the effective signal-to-noise ratio.

$$\frac{A_q^2}{A^2} = \frac{-(L_q^2 \xi/\sigma A) + A^2}{A^2} = 1 - L_q^2 ; \quad A = \frac{\xi}{\sigma} \tag{278}$$

Inserting Eqs. 277 and 264 into Eq. 263 yields

$$k = \frac{2\xi + 2B\sigma}{L_q^2 \xi} = \frac{2 + 2B/A}{L_q^2} .$$

Thus, for $A \gg 1$,

$$k = \frac{2}{L_q^2} . \tag{279}$$

The number of quantization levels, for a given quantization loss in signal-to-noise ratio, is therefore independent of A_{max} for $A_{max} \gg 1$. Comparing Eq. 279 with Eq. 267b shows that for reasonably small L_q, the number of quantization levels needed for a given loss in signal-to-noise ratio is higher for $A_{max} \gg 1$ than it is for $A_{max} \ll 1$.

The binary input set does not yield the optimum zero-rate exponent because more than two letters are needed for $A_{max} > 1$. It was shown in section 2.3 that an input set that consists of ℓ one-dimensional vectors, yields a zero-rate exponent that is very close to the optimum one if the distance between two adjacent vectors is

$$\frac{2\xi_{max}}{\ell} = 2\sigma \quad \text{or} \quad A_{max} = \ell .$$

The zero-rate exponent of this input set is given by Eq. 110.

$$E_{\ell,1}(0) = -\ln\left(\frac{1}{\ell} + 2\frac{\ell-1}{\ell^2}\exp\left(-\frac{4A_{max}^2}{8\ell^2}\right) + 2\frac{\ell-2}{\ell^2}\exp\left(-\frac{16A_{max}^2}{8\ell^2}\right) + \dots\right) .$$

Since $\frac{A_{max}}{\ell} = 1$, from Eq. 111, we obtain

$$E_{\ell,1}(0) \cong -\ln\left(\frac{1}{\ell} + 2\frac{\ell-1}{\ell^2}\exp\left(-\frac{4A_{max}^2}{8\ell^2}\right)\right)$$

In other words, only adjacent vectors with a distance $|\xi'-\xi| = 2\sigma$ are considered.

For all such vectors we have

$$\frac{|\xi'-\xi|}{2\sigma} = 1 \tag{280}$$

Following Eqs. 247 through 265 gives the result that the number of quantization levels is

$$k = \frac{2\xi_{max} + 2B\sigma}{\sqrt{12}\, L_q \sigma}$$

or

$$k = \frac{2A_{max} + 2B}{\sqrt{12}\, L_q} \tag{281}$$

Thus

$$k \approx \frac{2A_{max}}{\sqrt{12}\, L_q}; \quad A_{max} \gg 1. \tag{282}$$

The number of quantization levels in this case therefore increases with the signal-to-noise ratio.

The zero-rate exponent $E^q_{\ell,d}(0)$ of the quantized channel of Case I (Fig. 9) may be lower-bounded by the zero-rate exponent of the unquantized channel with the same input space if the signal-to-noise ratio, A^2, is replaced with A_q^2, with

$$\frac{A_q^2}{A^2} = 1 - L_q^2.$$

The quantization loss L^2 is a function of the number of quantization levels, k. The number of quantization levels for a given loss L_q^2 is constant for all $A \ll 1$, for all input sets. However, the number of quantization levels does depend on the input space whenever $A_{max} > 1$. Two typical input sets have been introduced. The first input set consisted of two letters only, while the second input set was large enough to yield an $E_{\ell,d}(0)$ that is close to the optimum exponent E(0).

It has been shown that for both input spaces discussed in this section the number of quantization levels for a given loss L_q is higher for $A_{max} \ll 1$ than it is for $A_{max} \gg 1$. In the case of the semioptimal input space shown in Fig. 4, the number of quantization levels increases linearly with A_{max} (for $A_{max} \gg 1$). The results are summarized in Table 3.

4.3 QUANTIZATION SCHEME OF CASE II (FIG. 9)

The logarithm of the a posteriori probability per input letter is computed and then quantized. The a posteriori probability per input letter, by Eq. 78, is

$$p(y|x) = \frac{1}{(2\pi)^{d/2}\sigma^d} \exp\left(-\frac{y^2 - 2xy + x^2}{2\sigma^2}\right), \tag{283}$$

where

$$xy = \xi_1\eta_1 + \xi_2\eta_2 + \dots + \xi_d\eta_d.$$

Thus

$$\ln p(y|x) = \frac{2xy - x^2}{2\sigma^2} - \frac{y^2}{2\sigma^2} + \ln (2\pi)^{d/2}(\sigma^2)^{d/2} . \tag{284}$$

The only part of Eq. 284 that carries information about x is

$$d(x, y) = \frac{|x|^2 - 2yx}{2\sigma^2} . \tag{285}$$

Thus, the computation of $\ln p(y|x)$ may be replaced by the somewhat simpler computation of $d(x, y)$ with no increase in the probability of error. The decoding scheme for

Table 3. Quantization scheme of Case I — Results.

Input Space	A^2 Signal-to-noise Ratio	k No. of Quantization Levels	q Quantization Grain
All	$A \ll 1$	$\frac{2B}{L_q\sqrt{12}}$	$q = \sqrt{12}\, L_q \sigma$
Binary	$A \gg 1$	$\frac{2}{L_q^2}$ $\frac{2A}{L_q\sqrt{12}}$ for $q < 2\sigma$	$q = L_q^2\, \xi_{max}$ $q = 12\, L\sigma$
Optimal	$A_{max} \gg 1$	$\frac{2A_{max}}{L_q\sqrt{12}}$	$q = \sqrt{12}\, L_q \sigma$

the unquantized channel is discussed in Appendix A.2, with $d(x, y)$ of Eq. 285 replacing $d(x, y)$ of Eq. A-18. The corresponding probability of error is bounded in section A.3:

$$P_e \leq e^{-[E_{\ell,d}(0)-R]n} ; \quad R \leq R_{crit}, \tag{286}$$

where $E_{\ell,d}(0)$ is given by

$$E_{\ell,d}(0) = -\frac{1}{d}\ln \sum_{X_\ell} \sum_{X'_\ell} P(x)\ P(x') \int_y p(y|x)^{1/2} \exp[d(x,y)-d(x',y)]\ dy$$

or

$$E_{\ell,d}(0) = -\frac{1}{d}\ln \sum_{X_\ell} \sum_{X'_\ell} P(x)\ P(x')\ g(x,x') \tag{287}$$

where

$$g(x',x) = \int_y p(y|x) \exp\left\{\frac{1}{2}[d(x,y)-d(x',y)]\right\} dy$$

Now, comparing Eqs. 287 and 285 with Eq. 242b yields

$$g(x',x) = \exp\left(-\frac{(x-x')^2}{8\sigma^2}\right). \tag{288}$$

Let the input to the quantizer be $d(x,y)$ given by Eq. 285 and let the output be $d^q(x,y)$. (According to Eq. 285 the quantity $x^2/2\sigma^2$ should be added to $-2yx/2\sigma^2$ at the input to the quantizer rather than at its output. If each $\frac{1}{2}x_i^2/\sigma^2$ is equal to one of the k quantization levels exactly, one can add the quantity $\frac{1}{2}x_i^2/\sigma^2$ at the output to the quantizer, and the bounds will still be the same as those derived below.) The zero-rate exponent $E^q_{\ell,d}(0)$ of the quantized channel in Case II can be lower-bounded by the zero-rate exponent $E^*(0)$ of the following detection scheme: The distance $d_i(x_i,y_i)$, given by Eq. 285, is computed for each letter x_i of the tested code word and then quantized to yield $d_i^q(x,y)$, the quantized version of $d_i(x,y)$. The distance

$$D^q(u,v) = \sum_{i=1}^{m} d_i^q(x_i,y_i) \tag{289}$$

is then computed. The one code word that yields the smallest distance $D^q(u,v)$ is then chosen to represent the transmitted code word. Thus

$$E^*(0) \le E^q_{\ell,d}(0) \le E_{\ell,d}(0). \tag{290}$$

Following Eqs. A-65 and A-70, we have

$$E^*(0) = -\frac{1}{d}\sum_{X_\ell} \sum_{X'_\ell} P(x)\ P(x') \int_Y p(y|x)\ e^{[d^q(x',y)-d^q(x,y)]}\ dy; \qquad t \le 0 \tag{291}$$

and if we let $t = -\frac{1}{2}$,

$$E^*(0) = -\frac{1}{d}\ln \sum_{X_\ell} \sum_{X'_\ell} P(x)\ p(x')\ g^q(x',x), \tag{292}$$

where

$$g^q(x, x') = \int_Y p(y|x) \exp\{\tfrac{1}{2}[d^q(x, y)-d^q(x', y)]\} dy. \tag{293}$$

Now,

$$d(x, y) = \frac{|x|^2 - 2yx}{2\sigma^2},$$

where y is a d-dimensional Gaussian vector that, for a given x, consists of d independent Gaussian variables, each of which has a mean power of σ^2. Thus, d(x, y) is a Gaussian variable with an average variance

$$\sigma_{d(x, y)} = \sqrt{\overline{(d(x, y))^2} - (\overline{d(x, y)})^2} = \sqrt{\frac{4\sigma^2 |x|^2}{4\sigma^4}} = \frac{|x|}{\sigma}. \tag{294}$$

Let

$$\left.\begin{aligned} \frac{d(x, y)}{\sigma\, d(x, y)} &= \frac{d(x, y)}{\frac{|x|}{\sigma}} = z \\ \frac{d(x', y)}{\sigma\, d(x', y)} &= \frac{d(x', y)}{\frac{|x'|}{\sigma}} = z' \end{aligned}\right\} \tag{295}$$

Thus, by Eqs. 294 and 295, z and z' are normalized Gaussian variables with a unit variance.

Inserting Eq. 295 into Eq. 293 yields

$$g^q(x, x') = \int_Z \int_{Z'} p(z, z'|x, x') \exp\left\{\frac{1}{2}\left[\left(\frac{z\,|x|}{\sigma}\right)^q - \left(\frac{z'\,|x'|}{\sigma}\right)^q\right]\right\} dz\, dz', \tag{296}$$

since the product space Z Z' is identical with the space y for given x and x'. Then

$$\left(\frac{z|x|}{\sigma}\right)^q = d^q(x, y) = d(x, y) + n_q,$$

where n_q is the "quantization noise." Thus

$$\left(\frac{|x|}{\sigma} z\right)^q = \frac{z|x|}{\sigma} + \left(\frac{n_q}{\frac{x}{\sigma}}\right)\frac{|x|}{\sigma}$$

or

$$\left(\frac{|x|}{\sigma} z\right)^q = \frac{|x|}{\sigma}(z^q) = \frac{|x|}{\sigma}\left[z + \frac{n_q}{\frac{|x|}{\sigma}}\right], \tag{297}$$

where

$$z^q = z + \frac{n_q}{\frac{|x|}{\sigma}} = \frac{d^q(x, y)}{\frac{x}{\sigma}}.$$

Thus, z^q is equivalent to the output of a quantizer with z as an input and with a quantization grain that is equal to

$$q_z = \frac{q}{\frac{|x|}{\sigma}}. \tag{298}$$

Here, q is the quantization grain of the quantizer d(x, y). Inserting Eq. 297 into Eq. 296 yields

$$g^q(x, x') = \int_{Z'^q} \int_{Z^q} p(z^q, z'^q | x, x') \exp\left\{\frac{1}{2}\left[\frac{|x|}{\sigma} z^p - \frac{|x'|}{\sigma} z'^q\right]\right\} dz^q dz'^q. \tag{299}$$

Both z and z' are Gaussian random variables, governed by the joint probability density

$$p(z, z' | x, x') = \frac{1}{2\pi(1-\zeta^2)^{1/2}} \exp\left[\frac{-(z-\bar{z})^2 + 2\zeta(z-\bar{z})(z'-\bar{z}') - (z'-z')^2}{2(1-\zeta^2)}\right] \tag{300}$$

where

$$\zeta = \overline{(z-\bar{z})(z'-\bar{z}')}. \tag{301}$$

It is shown in Appendix C that for such a joint probability density such as that in Eq. 301, we have the following situations.

1. When $|\zeta| = 1$ $(x=ax')$:

$$g^p(x, x') = [g(x, x')] \frac{\operatorname{sh}\left(\frac{|x| - |x'|}{4\sigma}\right) q_z}{\frac{(|x| - |x'|)}{4\sigma} q_z}; \quad \text{for } q_z < 2. \tag{302}$$

By Eq. 296,

$$g(x, x') = \int_Z \int_{Z'} p(z, z' | x, x') \exp\left\{\frac{1}{2}\left[\frac{|x|}{\sigma} z - \frac{|x'|}{\sigma} z'\right]\right\} dz dz' = \int_Y p(y|x) \exp\left\{\frac{1}{2}\left[d(x, y) - d(x', y)\right]\right\} dy \tag{303}$$

Also, it is assumed that

$$q^z = \frac{q}{\frac{|x|}{\sigma}} = \frac{q'}{\frac{|x'|}{\sigma}} = q'_z,$$

where q is the quantization grain of the quantizer of d(x, y), and q' is the quantization grain of the quantizer $d(x',y)$. In other words, it is assumed that both $d(x,y)$ and $d(x',y)$ have the same normalized quantization grain. (The quantization grain of each of one of the ℓ quantizers of Case II (Fig. 9) is assumed to be proportional to the variance of the Gaussian variable $d_i(s,y)$ fed into that quantizer. The ℓ quantizers are therefore not identical.

2. When $|\zeta| < 1$:

$$g^q(x,x') = [g(x,x')]\left(\frac{\frac{\mathrm{sh}\,|x|q_z}{4\sigma}}{\frac{xq_z}{4\sigma}}\right)\left(\frac{\frac{\mathrm{sh}\,|x'|q_z'}{4\sigma}}{\frac{x'q_z}{4\sigma}}\right) \text{ for } q_z < 2(1-\zeta^2). \tag{305}$$

3. When $|\zeta| = 1$ and $q_z > 2$:

$$g^q(x,x') \leqslant g(x,x') \exp\left(\frac{(|x|-|x'|)q_z}{4\sigma}\right) \text{ for } q_z = q_z'. \tag{306}$$

4. When $|\zeta| < 1$ and $q_z > 2(1-\zeta^2)$:

$$g_q(x,x') \leqslant g(x',x) \exp\left(\frac{|x|q_z+|x'|q_z}{4\sigma}\right). \tag{307}$$

Study of Eqs. 205-307 shows that the effects of quantizations depend on the kind of input space that is used. The effect of quantization for three important input spaces will now be discussed.

a. Binary Input Space

The binary input space consists of two oppositely directed vectors

$$x_1 = x; \qquad x_2 = x, \tag{308a}$$

where

$$P(x_1) = P(x_2) = \frac{1}{2}. \tag{308b}$$

This corresponds to $x = -1$.

The first signal-to-noise condition to be considered is

$$\sqrt{A^2 d} = \frac{|x|}{\sigma} = 1; \qquad q < 2. \tag{309}$$

By Eq. 302, we have

$$g^q(x,x') = [g(x,x')] \frac{\frac{\mathrm{sh}\,|x|q_z}{2\sigma}}{\frac{|x|q_z}{2\sigma}} \tag{310a}$$

and

$$g^q(x,x') = g(x,x') = 1; \qquad x = x'. \tag{310b}$$

Now, inserting Eqs. 301 and 293 into Eq. 310 yields

$$E^q_{\ell,d}(0) \geq -\frac{1}{d}\ln\left[\frac{1}{2}+\frac{1}{2}\exp\left(-\frac{x^2}{2\sigma^1}\right)\frac{\dfrac{sh|x|q_z}{2\sigma}}{\dfrac{|x|q_z}{2\sigma}}\right]; \qquad q < 2. \tag{311}$$

Inserting Eq. 304 into Eq. 311 yields

$$E^q_{\ell,d}(0) \geq -\frac{1}{d}\ln\left[\frac{1}{2}+\frac{1}{2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\frac{\dfrac{shq}{2}}{\dfrac{q}{2}}\right]; \qquad q < 2. \tag{312}$$

Inserting Eq. 251a into Eq. 312 yields

$$E^q_{\ell,d}(0) \geq -\frac{1}{d}\ln\left[\frac{1}{2}+\frac{1}{2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\exp\left(-\frac{q^2}{24}\right)\right]; \qquad q < 2. \tag{313}$$

Let

$$q = 12\, L_q \frac{|x|}{\sigma} \tag{314}$$

Inserting Eq. 314 into Eq. 313 yields

$$E^q_{\ell,d}(0) \geq -\frac{1}{d}\ln\left[\frac{1}{2}+\frac{1}{2}\left(-\frac{A^2 d}{2}\left(1-L_q^2\right)\right)\right]. \tag{315}$$

Comparing Eq. 315 with Eq. 106, we have

$$E^q_{\ell,d}(0) = -\frac{1}{d}\ln\left[\frac{1}{2}+\frac{1}{2}\exp\left(-\frac{A^2 d}{2}\right)\right]. \tag{316}$$

Thus, the zero-rate exponent $E^q_{\ell,d}(0)$ of the quantized channel is lower-bounded by the zero-rate exponent of an unquantized channel with an effective signal-to-noise ratio $A_q^2/A^2 = L_q^2$, where L_q is the "quantization-loss" factor.

Now the mean value of the Gaussian variable d(x,y) is, in general, different from zero. Thus, Eq. 264, which was derived for the Gaussian variable y that has a zero mean, is replaced by

$$Q = \overline{d(x',y)}_{max} - \overline{d(x',y)}_{min} + 2B\sqrt{\overline{d^2(x'y)} - \left(\overline{d(x',y)}\right)^2}. \tag{317}$$

Now $y = \phi + x$, where ϕ is a Gaussian vector that consists of d independent Gaussian variables with a zero mean and a variance σ. Thus, by Eq. 285

$$\overline{d(x',y}_{max} = \frac{|x|^2 + 2|x||x|}{2\sigma^2} = \frac{3|x|^2}{2\sigma^2} \tag{318a}$$

$$\overline{d(x',y)}_{min} = \frac{|x|^2 - 2|x||x|}{2\sigma^2} = \frac{|x|^2}{2\sigma^2} \tag{318b}$$

Inserting Eqs. 318 and 294 into Eq. 317 yields

$$Q = \frac{2|x|^2}{\sigma^2} + 2B\frac{|x|}{\sigma} \tag{319}$$

Inserting Eqs. 319 and 314 into Eq. 263 yields

$$k = \frac{2\frac{|x|}{\sigma} + 2B}{\sqrt{12}\,L_q}. \tag{320}$$

Thus, for $A\sqrt{d} = \frac{|x|}{\sigma} \ll 1$,

$$k = \frac{2B}{\sqrt{12}\,L_q}. \tag{321}$$

Equation 315 is valid also for cases in which $\sqrt{A^2 d} = \frac{|x|}{\sigma} > 1$, as long as $q_z < 2\left(\text{or } q < \frac{2|x|}{\sigma}\right)$. Thus, by Eq. 320,

$$k = \frac{2\frac{|x|}{\sigma}}{\sqrt{12}\,L} = \frac{2A\sqrt{d}}{\sqrt{12}\,L}; \qquad A\sqrt{d} = \frac{|x|}{\sigma} \gg 1,\ q < \frac{2|x|}{\sigma}. \tag{322}$$

However, there are cases in which much larger grain may be used. In such cases, where $q > \frac{2|x|}{\sqrt{\sigma}}$ and $\frac{|x|}{\sqrt{\sigma}} \gg 1$, Eq. 306 should be used. Therefore, by Eq. 306,

$$g^q(x,x') = 1; \qquad x' = x \tag{323a}$$

and

$$g^q(x,x') = g(x,x')\exp\left(\frac{|x|q_z}{2\sigma}\right); \qquad x' = -x. \tag{323b}$$

Inserting Eq. 298 into Eqs. 323 yields

$$g^q(x,x') = 1; \qquad x = x' \tag{324a}$$

and

$$g^q(x, x') = g(x, x') \, e^{q/2}; \qquad x' = -x. \tag{324b}$$

Inserting Eqs. 324 and 268b into Eq. 292 yields

$$E^q_{\ell, d}(0) \geq -\frac{1}{d} \ln \left(\frac{1}{2} + \frac{1}{2} \exp\left(-\frac{|x|^2}{2\sigma} \right) e^{q/2} \right). \tag{325}$$

Let

$$q = L_q^2 \frac{|x|^2}{\sigma^2} = L_q^2 A^2 d. \tag{326}$$

Then

$$E^q_{\ell, d}(0) \geq -\frac{1}{d} \ln \left[\frac{1}{2} + \frac{1}{2} \exp\left(-\frac{A^2 d}{2\sigma^2} \left(1 - L_q^2 \right) \right) \right]. \tag{327}$$

Comparing Eq. 327 with Eq. 316 yields

$$E_{\ell, d}(0) \geq -\left[\frac{1}{d} \ln \frac{1}{2} + \frac{1}{2} \exp\left(-\frac{A^2 d}{2\sigma^2} \right) \right]. \tag{328}$$

The number of quantization levels, for a given loss of signal-to-noise ratio, is determined by inserting Eqs. 326 and 319 into Eq. 213. Thus

$$k = \frac{2 \frac{|x|^2}{\sigma^2} + 2B \frac{|x|}{\sigma}}{L_q^2 \frac{|x|^2}{\sigma^2}}$$

For $A \; d = \frac{|x|}{\sigma} \gg 1$, we have

$$k = \frac{2}{L_q^2}. \tag{329}$$

b. Orthogonal Input Set

The binary input space is an optimal one, for $A < 1$, as shown in section 2.3. Another optimal input space for $A \ll \frac{1}{d}$ is the orthogonal input space. In this case

$$x_i \, x_j = 0; \qquad i \neq j \tag{330a}$$

$$x_i \, x_i = x_j \, x_j = |x|^2; \qquad i = j \tag{330b}$$

for all $i = 1, \ldots, \ell$; $\quad j = 1, \ldots, \ell$.

Inserting Eqs. 330 into Eq. 105a yields

$$E_{\ell,d}(0) = -\frac{1}{d}\ln\left(\frac{1}{\ell} + \frac{\ell-1}{\ell}\exp\left(-\frac{|x|^2}{4\sigma^2}\right)\right) . \tag{331}$$

Now, since the input signals are orthogonal, it can be shown that $\zeta = 0$. Following Eqs. 309-330, with (305) replacing (310), (331) replacing (106), and with Eq. 318 replaced by

$$\overline{d(x',y)}_{max} = \frac{3|x|^2}{2\sigma^2}; \qquad \overline{d(x',y)}_{min} = -\frac{|x|^2}{2\sigma^2}, \tag{332}$$

it can be shown that the number of quantization levels is

$$k = \frac{2B}{1a\ L_q}; \qquad A\sqrt{d} = \frac{|x|}{\sigma} \ll 1 \tag{333a}$$

$$k = \frac{2A\sqrt{d}}{12\ L_q}; \qquad A\sqrt{d} = \frac{|x|}{\sigma} \gg 1; \qquad q < \frac{2|x|}{\sigma} \tag{333b}$$

$$k = \frac{2}{L_q^2}; \qquad A\sqrt{d} = \frac{|x|}{\sigma} \gg 1; \qquad q > \frac{2|x|}{\sigma} \tag{333c}$$

c. Optimal Input Space

Both the binary and the orthogonal input spaces are nonoptimal for $A \gg 1$. An input set that is semioptimal for $A \gg 1$ is shown in Fig. 4. Now, if $d = 1$, it can be shown that $E^q_{\ell,1}(0)$ of the quantization scheme of Case II is equal to that of Case I.

The results of this section are summarized in Table 4. From Table 4 we conclude that in Case II, as in Case I, the number of quantization levels for a given "quantization loss" increases with the signal-to-noise ratio, which in this case is equal to A^2d/σ^2.

4.4 QUANTIZATION SCHEME OF CASE III (Fig. 9)

In this case the logarithm of the a posteriori probability per p input letters is computed and then quantized.

Let x^p be the vector sum of p input symbols. One can regard the vector sum x^p as a member of a new input space with "dp"dimensions. Equations 283-307 are therefore valid in Case III, once x is replaced by x^p.

It has been demonstrated that, in Case I and Case II, the number of quantization levels increases with the signal-to-noise ratio. If the signal-to-noise ratio in Case II is A^2d, the signal-to-noise ratio in Case III is then A^2dp.

Thus, given a quantization loss L_q and given an input space X_ℓ,

$$k_{\text{Case II}} \leq k_{\text{Case III}}. \tag{334}$$

Table 4. Quantization scheme of Case II – Results.

Input Space	$A^2 = \frac{\lvert x\rvert^2}{d\sigma^2}$ Signal-to-noise ratio	k No. of Quantization Levels	q Quantization Grain
Binary	$A^2d << 1$	$\frac{2B}{L\sqrt{12}}$	$\sqrt{12}\,A\sqrt{d}$
Binary	$A^2d >> 1$	$\frac{2}{L^2}$; $q > 2A\sqrt{d}$ $\frac{2A\sqrt{d}}{\sqrt{12}\,L}$; $q < 2A\sqrt{d}$	L^2A^2d $\sqrt{12}\,A\sqrt{d}$
Orthogonal	$A^2d << 1$	$\frac{2B}{L\sqrt{12}}$	$\sqrt{12}\,A\sqrt{d}$
Orthogonal	$A^2d >> 1$	$\frac{2}{L^2}$; $q > 2A\sqrt{d}$ $\frac{2A\sqrt{d}}{\sqrt{12}\,L}$; $q < 2A\sqrt{d}$	$\frac{L^2A^2d}{2}$ $\sqrt{12}\,A\sqrt{d}$
$d = 1$	See Table 10.1		

4.5 CONCLUSIONS

Let M_q be the number of digits to be stored in the memory of the decision computer per each transmitted symbol.

Let M_{qI} and k_I be M_q and k of Case I.

Let M_{qII} and k_{II} be M_q and k of Case II.

Let M_{qIII} and k_{III} be M_q and k of Case III.

Therefore

$$M_{qI} = k_I d \tag{335a}$$

$M_{qII} = k_{II}$ for a binary input space (only one "matched filter" should be used for both signals) (335b)

$M_{qII} = k_{II}\ell$ (for any input space other than binary)

$$M_{qIII} = \frac{1}{p} k_{III}\,\ell^p . \tag{335d}$$

Inserting (334) into (335d) yields

$$M_{qIII} \geq \frac{1}{p} k_{II} \ell^{p}. \tag{336}$$

Then $\frac{1}{p}\ell^{p} \geq \ell$; $\ell \geq 2$. Thus

$$M_{qIII} \geq k_{II}\ell; \qquad \ell \geq 2 \tag{337}$$

Comparing (337) with (335d) and (335c) yields

$$M_{qIII} \geq M_{qII}. \tag{338}$$

Thus, we conclude that the quantization scheme of Case III should not be used.

Comparing Table 3 with Table 4, we find that $k_I = k_{II}$; $d = 1$. Thus, by Eqs. 335,

$$M_{qI} \leq M_{qII}; \qquad d = 1. \tag{339}$$

We therefore conclude that the quantization scheme of Case I should be used whenever $d = 1$.

Tables 3 and 4 show that for the binary input space, $k_I = k_{II}$ for $A^2 d \leq 1$. Thus, by Eqs. 335, we have

$$M_{qI} = k_I d \qquad k_{II} = M_{qII}$$

or

$$M_{qI} > M_{qII}; \qquad \text{(binary input space; } A^2 d \leq 1). \tag{340}$$

We therefore conclude that whenever the signal-to-noise ratio is low enough ($A^2 d \ll 1$), the quantization scheme of Case II should be used.

As shown in Table 4, the number of quantization levels for a given L_q is not a function of d (as long as $A^2 d \ll 1$). Thus, the complexity measure, M, defined in section 4.1, minimized by letting

$$d \approx \frac{1}{E(0)}; \qquad E(0) \approx \frac{1}{4}A^2 \ll \frac{1}{d}.$$

From the results of sections 2.3 and 2.5 it is clear that the binary input space is the best semioptimal input space (for $A^2 d \ll 1$), since it yields the optimum exponent, and the number of input vectors is kept as small as possible (that is, $\ell = 2$).

If $E(0) \approx \ell n\, A_{max} \gg 1$, by section 2.2, $\frac{1}{d}\ell n \ell \approx E(0) \gg 1$. Thus

$$\ell > \ell n \ell \gg d. \tag{341}$$

On the other hand, one should expect k_{II} to be larger than k_I because the signal-to-noise ratio $A^2 d$ of Case II is larger than that of Case I (which is A^2) when $d > 1$. Thus

$$k_I \leq k_{II}; \qquad d > 1. \tag{342}$$

Inserting (341) and (342) into (335a) and (335c) yields

$$M_{qI} < M_{qII}; \qquad d > 1; \qquad E(0) \gg 1.$$

We conclude that whenever $E(0) \approx \ell n\, A_{max} \gg 1$ and $d > 1$, the quantization scheme of Case I should be used.

If an orthogonal set is used, and at the same time $A^2 d \ll 1$, we find from Tables 3 and 4 that $k_I = k_{II}$. Thus, by Eqs. 335,

$$M_{qI} > M_{qII}; \qquad A^2 d \ll 1; \qquad \ell < d.$$

4.6 EVALUATION OF $E^q_{\ell, d}(0)$ FOR THE GAUSSIAN CHANNEL WITH A BINARY INPUT SPACE (ℓ=2).

Methods have been derived to lower-bound $E^q_{\ell, d}(0)$. In this section the exact value of $E^q_{\ell, d}(0)$ is evaluated for a binary input space (see Eq. 83). Let us first discuss the case for which d = 1 and the output of the channel is quantized as follows: (Case I; k=2).

$$\left.\begin{array}{ll} \text{For all } y \geq 0; & y^q = 1 \\ \text{For all } y < 0; & y^q = -1 \end{array}\right\} \tag{343}$$

Here, y^q is the output of the quantizer. The channel is converted into a binary symmetric channel, described by the following probabilities:

$$P(x_1) = P(x_2) = \frac{1}{2}; \qquad x_1 = \xi_{max},\ x_2 = -\xi_{max} \tag{344a}$$

$$P(1|x_1) = P(-1|x_2) = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y-x_1)}{2\sigma^2}\right) dy \tag{344b}$$

$$P(1|x_2) = P(2|x_1) = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y+x_1)^2}{2\sigma^2}\right) dy. \tag{344c}$$

By Eqs. A-71 and A-69,

$$E^q_{2,1}(0) = -\ln \sum_{Y^q} \sum_{X_2} \sum_{X_2'} P(x)\, P(x')\, P\left(y^q|x\right)^{1/2} P\left(y^q|x'\right)^{1/2} \tag{345}$$

Inserting (344) into (345) yields

$$E^q_{2,1}(0) = -\ln\left\{\frac{1}{2} + \frac{1}{2}\left[\int_0^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y-x_1)^2}{2\sigma^2}\right) dy \int_0^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y+x_1)^2}{2\sigma^2}\right) dy\right]^{1/2}\right. \tag{346}$$

Now

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) dy \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}\,\sigma} x; \qquad \frac{x}{\sigma} \ll 1. \tag{347}$$

Inserting (347) into (346) yields

$$E^q_{2,1}(0) \approx \ln\left\{\frac{1}{2} + \frac{1}{2}\left(1 - \frac{2}{\pi}\frac{x_1^2}{2\sigma^2}\right)\right. \approx \frac{2}{\pi}\frac{\zeta 1^2}{4\sigma^2} = \frac{2}{\pi}\frac{A^2_{max}}{4}; \qquad A_{max} \ll 1; \quad k = 2 \tag{348}$$

Thus, by Eq. 98a, we have

$$\frac{E^q_{2,1}(0)}{E_{2,1}(0)} = \frac{2}{\pi}; \qquad A_{max} \ll 1; \quad k = 2 \tag{349a}$$

and

$$L^2_q = 1 - \frac{2}{\pi}; \qquad A_{max} \ll 1; \; k = 2. \tag{349b}$$

Also,

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) dy \approx 1; \qquad \frac{|x|}{\sigma} \gg 1, \; x > 0 \tag{350a}$$

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) dy \approx \exp\left(-\frac{x^2}{2\sigma^2}\right); \qquad \frac{|x|^2}{\sigma} \gg 1, \; x > 0 \tag{350b}$$

Inserting (350) into (346) yields

$$E^q_{2,1}(0) \approx -\ln\left(\frac{1 + \exp\left(-\frac{x_1^2}{4\sigma^2}\right)}{2}\right)$$

$$= -\ln\left(\frac{1 + \exp\left(-\frac{A^2_{max}}{4}\right)}{2}\right); \qquad k = 2, \; A_{max} \gg 1. \tag{351}$$

Comparing Eq. 107 with Eq. 351 yields

$$L^2_q = \frac{1}{2}; \qquad A_{max} \gg 1; \; k = 2 \tag{352}$$

If three quantization levels are used, (k=3), it can be shown that

$$\frac{E^q_{2,1}(0)}{E^q_{2,1}(0)} = 0.81; \qquad A_{max} \ll 1,\ k = 3 \tag{353a}$$

$$L^2_q = 0.19; \qquad A_{max} \ll 1, k = 3 \tag{353b}$$

If four quantization levels are used, (k=4), it can be shown that

$$\frac{E^q_{2,1}(0)}{E_{2,1}(0)} = 0.8\sim; \qquad A_{max} \ll 1;\ k = 4 \tag{354a}$$

$$L^2_q = 0.14; \qquad A_{max} \ll 1,\ k = 4. \tag{354b}$$

Eqs. 348, 349, 352, 353, and 354 are valid also for the quantization scheme of Case II, if A^2_{max} is replaced by $A^2_{max}d$.

V. CONCLUDING REMARKS

The important features of this research are

1. It presents a method of sending data over a time-discrete, amplitude-continuous memoryless channel with a probability of error which, for $R > R_{crit}$, has an exponent that can be made arbitrarily close to the optimum exponent $E(R)$. This is achieved by using a discrete input space.

2. It presents a decoding scheme with a probability of error no greater than a quantity proportional to $\exp\left[-n\left(\frac{1}{2}E(0)-R\right)\right]$ and an average number of computations no greater than a quantity proportional to m^2. The number of channel input symbols is roughly equal to $\ln E(0)$ when $E(0) \gg 1$, and is very small when $E(0) \ll 1$ (for the Gaussian channel we have $\ell = 2$). The dimensionality of each input symbol is $d = 1$, when $E(0) \gg 1$ and is equal to $d \cong \frac{1}{E(0)}$ whenever $E(0) \ll 1$.

3. It presents a method of estimating the effects of quantization at the receiver, for the white Gaussian channel. It has been shown that the quantization scheme of Case I is to be used whenever $A_{max}^2 \gg 1$. The quantization scheme of Case II is the one to be used whenever $A_{max}^2 \ll 1$.

Suggestions for Future Research

A method has been suggested by Elias[11] for adapting coding and decoding schemes for memoryless channels to channels with memory converted into memoryless channels by "scrambling" the transmitted messages. Extension of the results of this report to channels with memory, using scrambling or more sophisticated methods, would be of great interest.

Another very important and attractive extension would be the investigation of communication systems with a feedback channel. One should expect a further decrease in the decoding complexity and, probably, a smaller probability of error if feedback is used.

APPENDIX A

BOUNDS ON THE AVERAGE PROBABILITY OF ERROR-SUMMARY

A.1 DEFINITIONS

Following Fano,[2] we shall discuss here a general technique for evaluating bounds on the probability of decoding error when a set of M equiprobable messages are encoded into sequences of m channel input events.

Let us consider a memoryless channel that is defined by a set of conditional probability densities $p(\eta|\xi)$, where ξ is the transmitted sample, and η is the corresponding channel output ($p(\eta|\xi)$ is a probability distribution if η is discrete). We consider the case in which each input event x is a d-dimensional vector, and is a member of the (continuous) input space X. The vector x is given by $x = \xi_1, \xi_2, \ldots, \xi_d$.

The corresponding d-dimensional output vector y is a member of the d-dimensional continuous space Y, with $y = \eta_1, \eta_2, \ldots, \eta_d$. The number of dimensions d is given by $d = n/m$, where n is the number of samples per message. The channel statistics are therefore given by

$$p(y|x) = \prod_{i=1}^{d} p(\eta_i|\xi_i), \quad \text{where } p(\eta_i|\xi_i) = p(\eta|\xi);\ \xi_i = \xi,\ \eta = \eta_i.$$

The m^{th} power of this channel is defined as a channel with input space U consisting of all possible sequences u of m events belonging to X, and with output space V consisting of all possible sequences of m events belonging to Y. The i^{th} event of the sequence u will be indicated by y^i. Thus

$$u = x^1, x^2, x^3, \ldots, x^m, \qquad v = y^1, y^2, y^3, \ldots, y^m, \tag{A-1}$$

where x^i may be any point of the input space X, and y^i may be any point of the output space Y.

Since the channel is constant and memoryless, the conditional probability density $p(v|u)$ for the m^{th} power channel is given by

$$p(v|u) = \prod_{i=1}^{m} p(y^i|x^i), \tag{A-2}$$

where

$$p(y^i|x^i) = p(y|x); \quad y^i = y,\ x^i = x. \tag{A-3}$$

We shall assume in the following discussion that the message space consists of M equiprobable messages $m_1, m_2, \ldots, m_M$.

A.2 RANDOM ENCODING FOR MEMORYLESS CHANNELS

For random encoding we consider the case in which the input sequences assigned to messages are selected independently at random with probability density p(u), if U is a continuous space, or with probability distribution p(u) if U is discrete. The average probability of error corresponding to any such random assignment of input sequences to messages depends, of course, on the probability density p(u). We set

$$p(u) = \prod_{i=1}^{m} p(x^i), \tag{A-4}$$

where

$$p(x^i) = p(x); \quad x^i = x. \tag{A-5}$$

Here, p(x) is an arbitrary probability density whenever X is continuous, and is an arbitrary probability distribution whenever X is discrete. Eq. A-4 is equivalent to the statement that the input sequence corresponding to each particular message is constructed by selecting its component events independently at random with probability (density) p(x).

We shall assume, unless it is mentioned otherwise, that the channel output is decoded according to the maximum likelihood criterion; that is, that any particular output sequence v is decoded into the message m_i that maximizes the conditional probability (density) $p(v|m_i)$. Since messages are, by assumption, equiprobable, this decoding criterion is equivalent to maximizing the a posteriori probability $p(m_i|v)$, which, in turn, results in the minimization of the probability of error.

Let us assume that a particular message has been transmitted, and indicate by u the corresponding input sequence, and by v the resulting output sequence. According to the specified decoding criterion, an error can occur only if one of the other M-1 messages is represented by an input sequence u' for which

$$p(v|u') \geq p(v|u). \tag{A-6}$$

Let F(v) be an arbitrary positive function of v satisfying the condition

$$\int F(v)\, dv = 1 \tag{A-7a}$$

or

$$\sum F(v) = 1 \tag{A-7b}$$

if v is discrete.

Also, define

$$D(u, v) = \ln \frac{F(v)}{p(v|u)} \tag{A-8}$$

as the "distance" between u and v. In terms of this measure of distance the condition expressed by Eq. A-6 becomes

$$D(u', v) \leqslant D(u, v). \tag{A-9}$$

For any arbitrary constant D_o, the average probability of error then satisfies the inequality

$$P_e \leqslant MP_1 + P_2, \tag{A-10}$$

where

$$P_1 = \Pr[D(u, v) \leqslant D_o,\ D(u', v) \leqslant D(u, v)] \tag{A-11}$$

and

$$P_2 = \Pr[D(u, v) > D_o]. \tag{A-12}$$

The bound of Eq. A-10 corresponds to the following decoding scheme: $D(u, v)$ of Eq. A-8 is computed for each one of the M sequences of the input space U and the one given output sequence v. The only distances $D(u, v)$ that are taken into further consideration are those for which $D(u, v) \leqslant D_o$, where D_o is an arbitrary constant. The one sequence u, out of all of the sequences for which $D(u, v) \leqslant D_o$, which yields the smallest distance $D(u, v)$ is chosen to represent the transmitted signal. If no such sequence u exists, an error will occur.

If the decoding procedure above is carried out with an arbitrary distance function of u and v, $D^q(u, v)$, other than the $D(u, v)$ of Eq. A-8, then the average probability of error satisfies the inequality

$$P_e \leqslant MP_1 + P_2, \tag{A-13}$$

where

$$P_1 = \Pr[D^q(u, v) \leqslant D_o;\ D^q(u', v) \leqslant D_q(u, v)] \tag{A-14}$$

$$P_2 = \Pr[D^q(u, v) > D_o]. \tag{A-15}$$

However, one would expect the bound of (A-13) to be larger than that of (A-10), if $D_q(u, v)$ is not a monotonic function of the a posteriori probability $p(u|v)$.

A.3 UPPER BOUNDS ON P_1 AND P_2 BY MEANS OF CHERNOFF BOUNDS

The m events constituting the sequence u assigned to a particular message are selected independently at random with the same probability $p(x)$. If we let

$$F(v) = \prod_{i=1}^{m} f(y^i), \tag{A-16}$$

where

$$[f(y^i)]_{y^i = y} \equiv f(y); \quad \int f(y)\, dy = 1,$$

or

$$\sum f(y) = 1, \tag{A-17}$$

when y is discrete, it then follows from Eqs. A-2, A-3, and A-16 that the random variable D(u, v) defined by Eq. A-8 is the sum of m statistically independent, equally distributed, random variables:

$$D(u, v) = \sum_{i=1}^{m} d(x_i, y_i), \tag{A-18a}$$

where

$$d(x^i, y^i) \equiv d(x, y) = \ln \frac{f(y)}{p(y|x)}; \quad x^i = x;\ y^i = y. \tag{A-18b}$$

In cases for which an arbitrary distance $D^q(uv)$ other than D(uv) of Eq. A-8 is used, the discussion will be limited to such distances $D^q(uv)$ that may be represented as a sum of m statistically independent, equally distributed, random variables.

$$D^q(u, v) = \sum_{i=1}^{m} d^q(x^i, y^i), \tag{A-19}$$

where

$$d^q(x^i, y^i) = d^q(x, y); \quad x^i = x,\ y^i = y. \tag{A-20}$$

The moment-generating function of the random variable D(u, v), is

$$G(s) = \int_{D(u,v)} p[D(u, v)]\, e^{sD(u, v)}\, dD(u, v), \tag{A-21}$$

where p[D(u, v)] is the probability density of D(u, v). Thus

$$G(s) = \int_u \int_v p(u)\, p(v|u)\, e^{sD(u, v)}\, dudv. \tag{A-22}$$

From Eqs. A-18, A-4, A-5, A-2, and A-3, we get

$$G(s) = \prod_{i=1}^{m} \int_x \int_y p(x)\, p(y|x)\, e^{sd(x, y)}\, dxdy = [g_d(s)]^m, \tag{A-23}$$

where

$$g_d(s) = \int_x \int_y p(x)\, p(y|x)\, e^{sd(xy)}\, dxdy. \tag{A-24}$$

For the P_2 of Eq. A-12 we are interested in the probability that D(u, v) is greater than some value D_o. For all values of D(uv) for which $D(u, v) \geq D_o$,

$$e^{sD(uv)} \geq e^{sD_o}, \qquad \text{for } s \geq 0.$$

Using this fact, we may rewrite Eq. A-21 as

$$G(s) \geq e^{sD_o} \int_{D(u,v) > D_o} p[D(u, v)] \, dD(u, v). \qquad \text{(A-25)}$$

Using Eq. A-23, we have

$$\Pr[D(u, v) > D_o] \leq \exp(m\gamma_d(s) - sD_o), \qquad \text{(A-26)}$$

where

$$\gamma_d(s) = \ln g_d(s) = \ln \int_X \int_Y p(x) \, p(y|x) \, e^{sd(xy)} \, dxdy. \qquad \text{(A-27)}$$

Equation A-26 is valid for all $s \geq 0$. We may choose s that is such that the exponent is minimized. Differentiation with respect to s and setting the result equal to zero yields

$$\Pr[D(u, v) > D_o] \leq \exp\{m[\gamma_d(s) - s\gamma_{d'}(s)]\}; \qquad s \geq 0, \qquad \text{(A-28)}$$

where s is the solution to

$$\gamma'_d(s) = \frac{d\gamma_d(s)}{ds} = \frac{D_o}{m}. \qquad \text{(A-29)}$$

In the same way,

$$\Pr[D^q(u, v) > D_o] \leq \exp(m\gamma_d^q(s) - sD_o), \qquad \text{(A-30)}$$

where

$$\gamma_d^q(s) = \ln g_d^q(s) = \ln \int_X \int_Y p(x) \, p(y|x) \exp[sD^q(x, y)] \, dxdy. \qquad \text{(A-31)}$$

The exponent of Eq. A-30 is minimized if we choose s that is such that

$$\gamma_d^{q'}(s) = \frac{d\gamma_d^q(s)}{ds} = \frac{D_o}{m}. \qquad \text{(A-32)}$$

Thus

$$\Pr[D^q(u, v) > D_o] \leq \exp\{m[\gamma_d^q(s) - s\gamma_d^{q'}(s)]\}. \qquad \text{(A-33)}$$

For P_1 we desire an upper bound to the probability (A-11)

$$P_1 = \Pr[D(u,v) \leq D_o,\ D(u',v) \leq D(u,v)].$$

For this purpose, let us identify the point uu'v of the product space UU'V with the point a of a space A, the probability (density) $p(uu'v) = p(u)\, p(v')\, p(v|u)$ with the probability (density) $p(a)$, the random variable $D(u,v)$ with $\phi(a)$, and the random variable $D(u',v) - D(u,v)$ with the random variable $\theta(a)$. Inserting $\theta(a)$ and $\phi(a)$ into Eq. A-11 yields

$$p_1 = \Pr[\phi(a) \leq D_o,\ \theta(a) \leq 0]. \tag{A-34}$$

Let us form the moment-generating function of the pair $(\phi(a), \theta(a))$.

$$G(r,t) = \int p(a)\, e^{r\phi(a)+t\theta(a)}\, da. \tag{A-35}$$

Now, for all values of $\{a:\ \phi(a) \leq D_o;\ \theta(a) \leq 0\}$,

$$e^{r\phi(a)+t\theta(a)} \geq e^{rD_o} \qquad \text{for } r \leq 0;\ t \leq 0.$$

By using this fact, Eq. A-35 may be rewritten as

$$G(r,t) \geq e^{rD_o} \int_{\{a:\ \phi(a) \leq D_o;\ \theta(a) \leq 0\}} p(a)\, da$$

or

$$G(r,t) \geq e^{rD_o} \Pr[\phi(a) \leq D_o,\ \theta(a) \leq 0]. \tag{A-36}$$

Thus

$$p_1 = \Pr[\phi(a) \leq D_o,\ \theta(a) \leq 0] \leq G(r,t)\, e^{-rD_o}, \qquad r \leq 0;\ t \leq 0. \tag{A-37}$$

Now

$$\phi(a) = D(u,v), \quad \theta(a) = D(u',v) - D(u,v),$$

and

$$p(a) = p(uu'v) = p(u)\, p(u')\, p(v|u).$$

Thus, from Eqs. A-18, A-4, A-5, A-2, A-3, and A-35, we get

$$G(r,t) = [g_d(r,t)]^m, \tag{A-38}$$

where

$$g_d(r,t) = \int_Y \int_{X'} \int_X p(x)\, p(x')\, p(y|x)\, e^{(r-t)\, d(x,y)+td(x',y)}\, dxdx'dy. \tag{A-39}$$

Inserting Eq. A-38 into Eq. A-36 yields

$$p_1 = \Pr[D(u,v) \leq D_o,\ D(u',v) \leq D(u,v)] \leq \exp(m\gamma_d(r,t) - rD_o);$$

$$r \leq 0;\ t \leq 0, \qquad \text{(A-40)}$$

where

$$\gamma_d(r,t) = \ln g_d(r,t)$$

$$= \ln \int_Y \int_X \int_{X'} p(x)\, p(x')\, p(y|x)\, e^{(r-t)\, d(xy) + td(x',y)}\, dx dx' dy. \qquad \text{(A-41)}$$

We may choose r and t in such a way that the exponent of the right-hand side of (A-40) is minimized. Differentiating with respect to r and setting the result equal to zero and then repeating the same procedure with respect to t, we obtain

$$p_1 = \Pr[D(u,v) \leq D_o,\ D(u',v) \leq D(u,v)] \leq \exp\{m[\gamma_d(r,t) - r\gamma'_{d_r}(r,t)]\}, \qquad \text{(A-42)}$$

where

$$\gamma'_{d_r}(r,t) = \frac{\partial \gamma_d(r,t)}{\partial r} = \frac{D_o}{m} \qquad \text{(A-43)}$$

and

$$\gamma'_{d_t}(r,t) = \frac{\partial \gamma(r,t)}{\partial t} = 0. \qquad \text{(A-44)}$$

In the same way,

$$\Pr[D^q(uv) \leq D_o,\ D^q(u'v) \leq D_q(uv)] \leq \exp\{m[\gamma_d{}^q(r,t) - rD_o/m]\}, \qquad \text{(A-45a)}$$

where

$$\gamma_d^q(r,t) = \ln g_d^q(r,t) = \ln \int_Y \int_X p(x)\, p(y|x)\, \exp\{(r-t)\, d_q(xy) + td_q(x'y)\}\, dxdy. \qquad \text{(A-45b)}$$

Inserting (A-40) and (A-30) into (A-13) yields

$$P_e \leq \exp\{m[\gamma_d(s) - sD_o/m]\} + \exp\{m[n/m\ R + \gamma_d(r,t) - rD_o/m]\}, \qquad \text{(A-46a)}$$

where R, the rate of information per sample, is given by

$$R = \frac{1}{n} \ln M. \qquad \text{(A-46b)}$$

From Eqs. A-11 and A-12, the two probabilities, p_1 and p_2, vary monotonically with D_o in the opposite directions. Thus, the right-hand side of (A-45) is approximately minimized by the value of D_o for which the two exponents are equal to each other. Therefore, let D_o be such that

$$\gamma_d(s) - s\frac{D_o}{m} = \frac{n}{m}R + \gamma_d(r,t) - r\frac{D_o}{m}. \quad \text{(A-47)}$$

The insertion of (A-47), (A-44), and (A-43) into (A-46a) yields

$$P_e \leqslant 2\exp\{m[\gamma_d(s) - sD_o/m]\} = 2\exp[-nE_d(R)], \quad \text{(A-48)}$$

where

1. $E_d(R) = -\frac{m}{n}\left[\gamma_d(s) - s\frac{D_o}{m}\right] = -R + \frac{m}{n}\left(\gamma_d(r,t) - r\frac{D_o}{m}\right).$ (A-49)

2. $\gamma'_{d_s}(s) = \gamma'_{d_r}(r,t) = \frac{D_o}{m}; \quad s \geqslant 0, \ t \leqslant 0, \ r \leqslant 0.$ (A-50)

3. $\gamma'_{d_t}(r,t) = 0; \quad r \leqslant 0; \ t \leqslant 0.$ (A-51)

Now, from (A-25) and (A-18) we have

$$\gamma_d(s) = \ln \int_Y \int_X p(x)\, p(y|x)\, e^{sd(xy)}\, dxdy$$

$$= \ln \int_Y \int_X p(x)\, p(y|x)^{1-s}\, f(y)^s\, dxdy. \quad \text{(A-52)}$$

Also, from (A-39) and (A-18) we have

$$\gamma_d(r,t) = \ln \int_y \int_{x'} \int_x p(x)\, p(x')\, p(y|x)^{1-r\ t}\, p(y|x')^{-t}\, f(y)^r. \quad \text{(A-53)}$$

It can be shown[14] that

1. Eq. A-44 is satisfied if we let

   $1 - r + t = -t; \quad r \leqslant 0, \ t \leqslant 0$

or

$$r = 1 + 2t; \quad t \leqslant -\frac{1}{2}. \quad \text{(A-54)}$$

2. Eq. A-50 is satisfied by letting

$$f(y) = \frac{\left[\int_X p(x)\, p(y|x)^{1-s}\, dx\right]^{1/1-s}}{\int_Y \left[\int_X p(x)\, p(y|x)^{1-s}\, dx\right]^{1/1-s} dy} \quad \text{(A-55a)}$$

and

$$s = 1 + t; \quad 0 \leq s \leq \frac{1}{2}. \tag{A-55b}$$

3. Equation A-49 is then satisfied if we let

$$R = \frac{1}{d}[(s-1)\ \gamma_d'(s) - \gamma_d(s)]; \quad 0 \leq s \leq \frac{1}{2}. \tag{A-56a}$$

We should notice, however, that Eqs. A-44, A-50, and A-49 are satisfied if, and only if, R is such as to make $0 \leq s \leq \frac{1}{2}$. It can be shown[14] that this is for the region

$$R_{crit} \leq R \leq I, \tag{A-56b}$$

where

$$I = \frac{1}{d} \int_Y \int_X p(x)\ p(y|x) \ln \frac{p(y|x)}{p(y)} dxdy = [R]_{s=0} \tag{A-56c}$$

and

$$R_{crit} = [R]_{s=1/2}. \tag{A-56d}$$

Let us now define the tilted probability (density) for the product space XY.

$$Q(x, y) = \frac{e^{sD(x,y)}\ p(x)\ p(y|x)}{\int_Y \int_X e^{sD(x,y)}\ p(x)\ p(y|x)\ dxdy}$$

$$= \frac{p(x)\ p(y|x)^{1-s}\ f^s(y)}{\int_Y \int_X p(x)\ p(y|x)^{1-s}\ f^s(y)\ dxdy} \tag{A-57}$$

where

$$Q(y) = f(y) = \frac{\left[\int_X p(x)\ p(y|x)^{1-s}\ dx\right]^{1/1-s}}{\int_Y \left[\int_X p(x)\ p(y|x)^{1-s}\ dx\right]^{1/1-s} dy}; \quad 0 \leq s \leq \frac{1}{2}, \tag{A-58}$$

$$Q(x|y) = \frac{Q(x,y)}{Q(y)} \frac{p(x)\ p(y|x)^{1-s}}{\int_X p(x)\ p(y|x)^{1-s}\ dx}; \quad 0 \leq s \leq \frac{1}{2}. \tag{A-59}$$

Using Eqs. A-52, A-53, A-54, A-56, A-57, A-59 yields

$$P_e \leq 2e^{-nE(R)}; \qquad R_{crit} \leq R \leq I. \tag{A-60a}$$

Here, the exponent E(R) is related parametrically to the transmission rate per sample R, for $R_c \leq R \leq I$, by

$$0 \leq E(R) = \frac{1}{d} \int_Y \int_X Q(x,y) \ln \frac{Q(x,y)}{p(x)\, p(y|x)} dxdy \tag{A-60b}$$

$$I \leq R \leq \frac{1}{d} \int_X \int_Y Q(x,y) \ln \frac{Q(x|y)}{P(x)} \geq R_{crit}; \quad 0 \leq s \leq \frac{1}{2} \tag{A-60b}$$

$$R_{crit} = [R]_{s=1/2}; \quad I = [R]_{s=0} = \frac{1}{d} \int_X \int_Y p(x)\, p(y|x) \ln \frac{p(y|x)}{p(y)}. \tag{A-60d}$$

Whenever $R < R_{crit}$, there does not exist a D_o that simultaneously satisfies Eqs. A-49, A-50, and A-51. However, the average probability of error may always, for any rate, be bounded by

$$P_e \leq M Pr[D(u'v) \leq D(uv)]. \tag{A-61}$$

This is equivalent to setting $D_o = D(u,v)$ in Eqs. A-11 and A-12. Thus

$$P_1 = Pr[D(u'v) \leq D(uv)]; \quad P_2 = 0. \tag{A-62}$$

In the same way,

$$P_e \leq MP_1 = MPr[D_q(u'v) \leq D_q(uv)]. \tag{A-63}$$

The evaluation of P_1 under these conditions proceeds as before, except for setting r = 0 in (A-42) and (A-45a). Therefore

$$P_e \leq \exp\{m[n/m\ R + \gamma_d(0,t)]\}; \qquad t \leq 0, \tag{A-64a}$$

where

$$\gamma_d(0,t) = \gamma(t)$$

$$= \ln \int_Y \int_X \int_{X'} p(x')\, p(x)\, p(y|x)\, e^{t[d(x'y)-d(xy)]}\, dx'dxdy; \qquad t \leq 0 \tag{A-64b}$$

and

$$P_e \leq \exp\{m[n/m\ R + \gamma_d^q(0,t)]\}; \qquad t \leq 0, \tag{A-65}$$

where

$$\gamma_d^q(0,t) = \gamma_d^q(t)$$

$$= \ln \int_Y \int_X \int_{X'} p(x)\, p(x')\, p(y|x)\, e^{t[d^q(x'y)-d^q(xy)]}\, dx'dxdy; \qquad t \leq 0. \tag{A-65a}$$

Thus

$$\gamma_d(0,t) = \ln \int_X \int_Y p(x)\, p(y|x)^{1-t}\, p(y|x')^t; \qquad t \leqslant 0. \tag{A-66}$$

Here, $\gamma_d(0,t)$ may be minimized by choosing a proper t. Differentiation with respect to t and setting the result equal to zero yields

$$t = -\tfrac{1}{2}. \tag{A-67}$$

$$\gamma_d\left(0,-\tfrac{1}{2}\right) = \gamma_d\left(\tfrac{1}{2}\right) = \ln \int_Y \int_X \int_{X'} p(x)\, p(x')\, p(y|x)^{1/2}\, p(y|x')^{1/2} \tag{A-68}$$

or

$$\gamma_d\left(0,\tfrac{1}{2}\right) = \ln \int_Y \left[\int p(x)\, p(y|x)^{1/2}\, dx\right]^2. \tag{A-69}$$

The insertion of Eq. A-67 into Eq. A-64 yields

$$P_e \leqslant \exp\{-n[E_d(0)-R]\}, \tag{A-70}$$

where

$$E_d(0) = -\tfrac{1}{d}\gamma_d\left(0,\tfrac{1}{2}\right). \tag{A-71}$$

From Eq. A-60 for $R = R_{crit}$, we have $s = \frac{1}{2}$, $t = -\frac{1}{2}$, and $r = 0$. Thus, by Eq. A-49,

$$E_d(R)\Big|_{R_{crit}} = -R_{crit} + \tfrac{1}{d}\gamma_d\left(0,\tfrac{1}{2}\right) = E_d(0) - R_{crit} \tag{A-72}$$

and the exponentials of (A-70) and (A-49) are indeed identical for $R = R_{crit}$.

It can also be shown that $dE_d(R)/dR\big|_{R_{crit}} = -1$, so that the derivatives of the two exponents with respect to R are also the same at $R = R_{crit}$.

The average probability of error can therefore be bounded by

$$P_e \leqslant \begin{cases} e^{-n[E_d(0)-R]}; & R \leqslant R_{crit} \\ 2e^{-n[E_d(R)]}; & R_{crit} \leqslant R \leqslant I \end{cases} \tag{A-73a}$$

where

$$E_d(0) - R_{crit} = E_d(R)\Big|_{R_{crit}} \tag{A-73b}$$

and

$$\left.\frac{d[E_d(0)-R]}{dR}\right|_{R_{crit}} = \left.\frac{dE_d(R)}{dR}\right|_{R_{crit}} = -1. \tag{A-73c}$$

A.4 OPTIMUM UPPER BOUNDS FOR THE AVERAGE PROBABILITY OF ERROR

The upper bound of Eq. A-73 may be optimized by choosing p(x) that is such that, for a given rate R, the exponent of the bound is minimized.

Let d = 1 and m = n. x^i is then identical with ξ^i, where ξ^i is the i^{th} input sample that is a member of the (continuous) one-dimensional space Ξ. y^i is identical with η^i, where η^i is the i^{th} output sample that is a member of the (continuous) one-dimensional space H.

It can be shown[15] that there exists an optimum probability (density) $p(x) \equiv p(\xi)$ defined on Ξ that minimizes the upper bound to the average probability of error, so that, for large n and for $R \geq R_{crit}$, it becomes exponentially equal to the <u>lower</u> bound on the probability of error.

The characteristics of many continuous physical channels, when quantized and thus converted into a discrete channel, are very close to the original ones if the quantization is fine enough. Thus, for such continuous channels there exists one random code with an optimum probability density $p(x) = p(\xi)$ which yields an exponent E(R) that is equal to the exponent of the lower bound of the average probability of error, for n very large and for $R \geq R_{crit}$.

APPENDIX B

EVALUATION OF $E_d(0)$ IN SECTION 2.3, FOR CASES IN WHICH EITHER $A^2d \ll 1$ OR $A^2 > 1$

We shall evaluate lower bounds to the exponent $E_d(0)$ which is given by Eq. 127.

$$E_d(0) = -\frac{1}{n}\ln\left\{\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^\pi \exp\left(-\frac{dA^2}{2}\sin^2\frac{\theta}{2}\right)\sin\theta^{d-2}d\theta\right\}.$$

When d = 2 we have

$$E_2(0) = -\frac{1}{2}\ln\left\{\frac{1}{\pi}\exp\left(-\frac{A^2}{2}\right)\int_0^\pi \exp\left(\frac{A^2}{2}\cos\theta\right)d\theta\right\}$$

$$= -\frac{1}{2}\ln\left\{\exp\left(-\frac{A^2}{2}\right)\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{A^2}{2}\cos\theta\right)d\theta\right\}.$$

Thus

$$E_2(0) = -\frac{1}{2}\ln\left\{\exp\left(-\frac{A^2}{2}\right)I_0\left(\frac{A^2}{2}\right)\right\}. \qquad \text{(B-1)}$$

For $\frac{A^2}{2} \ll 1$, we have

$$E_2(0) \approx -\frac{1}{2}\ln\exp\left(-\frac{A^2}{2}\right) \cong \frac{A^2}{4}. \qquad \text{(B-2)}$$

For $\frac{A^2}{2} \gg 1$, we have

$$E_2(0) \approx -\frac{1}{2}\ln\left\{\exp\left(-\frac{A^2}{2}\right)\frac{1}{\sqrt{2\pi\frac{A^2}{2}}}\exp\left(+\frac{A^2}{2}\right)\right\}$$

$$\approx \frac{1}{4}\ln A^2 + \frac{1}{4}\ln\pi \approx \frac{1}{2}\ln\left(A^2\right)^{1/2}. \qquad \text{(B-3)}$$

Now, for d = 3 we have, from Eq. 127,

$$E_3(0) = -\frac{1}{3}\ln\left\{\frac{1}{2}\exp\left(-\frac{3}{4}A^2\right)\int_0^\pi \exp\left(\frac{3}{4}A^2\cos\theta\right)\sin\theta\,d\theta\right\}$$

$$= -\frac{1}{3}\ln\left\{\frac{1}{2}\exp\left(-\frac{3}{4}A^2\right)\frac{4}{3A^2}\int_\pi^0 \frac{d}{d\theta}\exp\left(\frac{3}{4}A^2\cos\theta\right)d\theta\right. .$$

Thus

$$E_3(0) = -\frac{1}{3} \ln \left\{ \exp\left(-\frac{3}{4} A^2\right) \frac{\operatorname{sh} \frac{3}{4} A^2}{\frac{3}{4} A^2} \right\}. \tag{B-4}$$

For $\frac{3A^2}{4} \ll 1$, we have

$$E_3(0) \approx -\frac{1}{3} \ln \exp\left(-\frac{3}{4} A^2\right) = \frac{A^2}{4}. \tag{B-5}$$

For $\frac{3}{4} A^2 \gg 1$, we have

$$E_3(0) \approx \frac{1}{3} \ln \frac{3}{4} A^2 = \frac{1}{3} \ln A^2 + \frac{1}{3} \ln \frac{3}{4}$$

$$\approx \frac{1}{3} \ln A^2 = \frac{1}{2} \ln \left(A^2\right)^{2/3} . \tag{B-6}$$

In general, for an $d \geqslant 3$, we have

$$E_d(0) = \frac{1}{4} A^2 ; \quad A^2 d \ll 1 \tag{B-7}$$

$$E_d(0) \approx \frac{1}{2} \ln \left(A^2\right)^{(d-1)/d} = \frac{d-1}{d} E(0) ; \quad A^2 \gg 1 , \tag{B-8a}$$

where

$$E(0) = E_d(0)\Big|_{d=n \gg 1} \approx \frac{1}{2} \ln \left(A^2\right) ; \quad A^2 \gg 1 . \tag{B-8b}$$

PROOF: From Eq. 127 we have, for $A^2 d \ll 1$,

$$E_d(0) \approx -\frac{1}{d} \ln \left\{ \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \left(1 - \frac{dA^2}{4}\right) \int_0^{2\pi} \left(1 - \frac{dA^2}{4}\right) \cos \theta \sin \theta^{d-2} \, d\theta \right\}$$

$$= -\frac{1}{d} \ln \left(1 - \frac{dA^2}{4}\right) \approx \frac{A^2}{4} .$$

Thus

$$E_d(0) \approx \frac{A^2}{4} ; \quad A^2 d \ll 1 . \qquad \text{Q.E.D.}$$

We now proceed to prove Eqs. 130b and 130c. Let $x = \sin^2 \frac{\theta}{2}$ and insert x into Eq. 127. We have, then,

$$E_d(0) = -\frac{1}{d} \ln \left\{ \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} 2^{d-3} I \right\} , \tag{B-9a}$$

where

$$I = \int_0^1 \exp\left(-\frac{dA^2}{2} x\right) x^{(d-3)/2}(1-x)^{(d-3)/2}\, dx . \qquad \text{(B-9b)}$$

Now

$$1 - x \leq e^{-x} . \qquad \text{(B-10)}$$

Inserting (B-10) into (B-9b) yields

$$I \leq \int_0^1 \exp\left(-\left[\frac{A^2 d}{d-3} + 1\right] \frac{d-3}{2} x\right) x^{(d-3)/2}\, dx$$

$$= \frac{\Gamma\left(\frac{d-1}{2}\right)}{\left[\frac{d-3}{2}\left(\frac{A^2 d}{d-3} + 1\right)\right]^{(d-1)/2}} . \qquad \text{(B-11)}$$

Inserting (B-11) into (B-9a) yields

$$E_d(0) \geq -\frac{1}{d} \ln \left\{ \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}\right)}{\left(\frac{d}{2}\right)^{(d-1)/2}} 2^{d-3} + \frac{1}{d} \ln \left[\frac{d-3}{d} + A^2\right]^{(d-1)/2} \right\} .$$

The first term on the right-hand side of inequality (134) is bounded by

$$-\frac{1}{d} \ln \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{d}{2}\right)}{\left(\frac{d}{2}\right)^{(d-1)/2}} 2^{d-3} \leq -\frac{1}{d} \ln 2^{d-3}$$

$$= -\frac{d-3}{d} \ln 2 .$$

Thus, for $A^2 \gg 1$, we have

$$E_d(0) \gtrsim \frac{1}{d} \ln \left(A^2\right)^{(d-1)/2} = \frac{d-1}{d} \frac{1}{2} \ln A^2 . \qquad \text{(B-12)}$$

Now, let

$$E(0) = E_{d=n}(0)\Big|_{n \gg 1} . \qquad \text{(B-13)}$$

Inserting $d = n \gg 1$ into (B-12) yields

$$E(0) \gtrsim \frac{1}{2} \ln A^2 .$$

From the convexity of the exponent $E(R)$, when plotted as a function of R, we have

$$E(0) \leq C , \qquad \text{(B-14)}$$

where C is the channel capacity, given by Shannon,[1]

$$C \approx \frac{1}{2} \ln A^2 ; \quad A^2 \gg 1 .$$

Thus, by (B-13) and (B-14),

$$E(0) \approx C \approx \frac{1}{2} \ln A^2 ; \quad A^2 \gg 1 . \tag{B-15}$$

Inserting (B-15) into (B-12) yields

$$E_d(0) \approx \frac{d-1}{d} \frac{1}{2} \ln A^2 \approx \frac{d-1}{d} E(0) .$$

APPENDIX C

MOMENT GENERATING FUNCTIONS OF QUANTIZED GAUSSIAN VARIABLES (after WIDROW[9])

A quantizer is defined as a nonlinear operator having the input-output relation shown in Fig. 10c. An input lying somewhere within a quantization "box" of width q will yield an output corresponding to the center of the box (i.e., the output is rounded off to the center of the box).

Let the input z be a random variable. The probability density distribution of z, p(z), is given.

The moment-generating function (m.g.f.) of the input signal is therefore

$$g(s) = \int_Z p(z)\, e^{-sz}\, dz\,. \tag{C-1}$$

Our attention is devoted to the m.g.f. of the quantized signal z^q, given by

$$g^q(s) = \int_{Z^q} p(z^q)\, e^{-sz^q}\, dz^q, \tag{C-2}$$

where $p(z^q)$ is the probability density of the output of the quantizer, z^q, and consists of a series of impulses. Each impulse must have an area equal to the area under the probability density p(z) within the bound of the "box" of width q, in which the impulse is centered. Thus the probability density $p(z^q)$ of the quantizer output consists of "area samples" of the input probability density p(z). The quantizer may be thought of as an area sampler acting upon the "signal," the probability density p(z).

Thus, $p(z^q)$ may be constructed by sampling the difference $\phi\left(z+\frac{q}{2}\right) - \phi\left(z-\frac{q}{2}\right)$, where $\phi(z)$ is the input probability distribution given by

$$\phi(z) = \int_{-\infty}^{z} p(z)\, dz\,. \tag{C-3}$$

This operation is equivalent to, first, modifying p(z) by a linear "filter" whose transfer function is

$$\frac{e^{(sq)/2} - e^{-(sq)/2}}{s} = q\,\frac{\operatorname{sh}\frac{qs}{2}}{\frac{qs}{2}} \tag{C-4}$$

and then impulse-modulating it to give $p(z^q)$.

Using "Δ" notation to indicate sampling, we get

$$g^q(s) = \left[g(s)\,\frac{\operatorname{sh}\frac{qs}{2}}{\frac{qs}{2}}\right]^{\Delta} = F^{\Delta}(s)\,, \tag{C-5a}$$

where

$$F(s) = g(s)\, q \frac{\operatorname{sh} \frac{qs}{2}}{q \frac{s}{2}} \tag{C-5b}$$

Now, let the function F(s) be the transform of a function f(z).

$$F(s) = \int_Z f(z)\, e^{-sz}\, dz \,. \tag{C-6}$$

Then

$$F^{\Delta}(s) = \int_Z f^{\Delta}(z)\, e^{-sz}\, dz \,, \tag{C-7}$$

where $f^{\Delta}(z)$ is the sampled version of f(z).

Thus

$$f^{\Delta}(z) = f(z)\, c(z) \,, \tag{C-8}$$

where c(z) is a train of impulses, q amplitude units apart. A Fourier analysis may be made of the impulse train c(z). The form of the exponential Fourier series will be

$$c(z) = \frac{1}{q} \sum_{k=-\infty}^{\infty} e^{ik\Omega z} \,; \quad \Omega = \frac{2\pi}{q} \tag{C-9}$$

Inserting (C-9) into (C-7) yields

$$F^{\Delta}(s) = \frac{1}{q} \sum_{k=-\infty}^{\infty} F(s - ik\Omega) \,. \tag{C-10}$$

Inserting (C-10) into (C-5) yields

$$g^{q}(s) = \sum_{k=-\infty}^{\infty} g(s-ik\Omega) \frac{\operatorname{sh}\left[q \frac{(s-ik\Omega)}{2}\right]}{q \frac{(s-ik\Omega)}{2}} \,. \tag{C-11}$$

Now, if the input is a Gaussian variable governed by the probability density

$$p(z|x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(z-x)^2}{2\sigma^2}\right) \,, \tag{C-12}$$

it can be shown that

$$g^{q}(s) \approx g(s) \frac{\operatorname{sh} \frac{qs}{2}}{\frac{qs}{2}} \,; \quad q < 2\sigma \,, \tag{C-13}$$

where

$$g(s) = \exp\left(\frac{s^2 \sigma^2}{2} + xs\right) \,. \tag{C-14}$$

Let z and z' be two random Gaussian variables governed by the probability density

$$p(z,z') = \frac{1}{2\pi(1-\varsigma^2)^{1/2}} \exp\left[\frac{-(z-\bar{z})^2 + 2\varsigma(z-\bar{z})(z'-\bar{z}) - (z'-\bar{z}')^2}{2(1-\varsigma^2)}\right], \tag{C-15a}$$

where

$$\varsigma = (z-\bar{z})(z'-\bar{z}') . \tag{C-15b}$$

Let the corresponding m.g.f. be given by

$$g(r,t) = \int_{Z'} \int_{Z} p(z,z')\, e^{rz+tz'}\, dzdz' . \tag{C-16}$$

Now let z and z' be quantized by the quantizer of Fig. 10c, to yield z^q and z'^q. Thus

$$g^q(r,t) = \int_{Z^q} \int_{Z'^q} p(z^q, z'^q)\, e^{-rz^q - tz'^q}\, dz^q dz'^q . \tag{C-17}$$

It can then be shown (as was shown in the derivation of (C-13)) that

$$g^q(r,t) = g(r,t) \frac{\operatorname{sh}\frac{qr}{2}}{\frac{qr}{2}} \frac{\operatorname{sh}\frac{qt}{2}}{\frac{qt}{2}}; \quad q < 2(1-\varsigma^2) . \tag{C-18}$$

Also, if z = z' ($\varsigma = 1$), we have

$$g^q(r,t) = g(r,t) \frac{\operatorname{sh}\frac{q}{2}(r+t)}{\frac{q}{2}(r+t)}; \quad q < 2 . \tag{C-19}$$

Now, if z = −z' ($\varsigma = -1$), we have

$$g^q(r,t) = g(r,t) \frac{\operatorname{sh}\frac{q}{2}(r-t)}{\frac{q}{2}(r-t)}; \quad q < 2 . \tag{C-20}$$

We now proceed to derive upper bounds to g(s) and g(r, t) to be used whenever the quantization grain q is large, so that (C-13), (C-17), (C-18), and (C-19) are not valid any more.

Let $z^q = z + n_q(z)$, where $n_q(z)$ is the "quantization noise." Thus, by Eq. C-2,

$$g^q(s) = \int_Z p(z) \exp[-s(z+n_q(z))]\, dz .$$

Now, $|n_q(z)| \leq \frac{q}{2}$. Therefore

$$g^q(s) \leq \int_Z p(z)\, e^{-sy}\, dz \exp\left|s\frac{q}{2}\right|$$

$$= g(s) \exp\left|s\frac{q}{2}\right| . \tag{C-21}$$

In the same way, let

$$z^q = z + n_q(z); \quad z'^q = z' + n_q(z').$$

Thus, by (C-17),

$$g^q(r,t) = \int_Z \int_{Z'} p(z,z') \exp[-r(z+n_q(z))-t(z'+n_q(z'))] \, dz'dz .$$

Now

$$|n_q(z)| \leq \frac{q}{2}; \quad |n_q(z')| \leq \frac{q}{2}.$$

Thus

$$g^q(r,t) \leq \int_Z \int_{Z'} p(z,z') \exp(-rz-tz') \, dz'dz \ \exp\left|r \frac{q}{2}\right| + \left|t \frac{q}{2}\right|$$

$$= g(r,t) \exp\left|r \frac{q}{2}\right| + \left|t \frac{q}{2}\right|. \tag{C-22}$$

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ERRATA

(added in press)

Page 1, line 25. Change to read "and η_i is independent of ξ_i for $i \neq j$."

Page 6, lines 30, 31. $e^{sd(x,y)}$ should read $e^{sD(x,y)}$

Page 9, line 15. D(k,y) should read d(k,y)
D(k',y) should read d(k',y)

line 18. $D(x_i,y)$ should read $d(x_i,y)$
$D(x_j,y)$ should read $d(x_j,y)$

line 21. Should read $d(k,y) = d(x_{k,y}) = \ln \frac{f(s)}{p(y|x_k)}$

line 27. D(k,y) should read d(k,y)

line 30. $D(x_k,y)$ should read $d(x_k,y)$

Page 10, lines 35 and 38. D(x,y) should read d(x,y)

Page 11, line 16. $D(x_i, y)$ should read $d(x_i, y)$

line 19. $D(x_j,y)$ should read $d(x_j,y)$

line 24. $D(x_i,y)$ should read $d(x_i,y)$

line 27. D(x,y) should read d(x,y)

Page 13, line 1. Add prime. $R = \frac{1}{d}\left[(s-1)\,\gamma_d'(s) - \gamma_d(s)\right]$

Page 16, line 17. The inequality should be $\geqslant$.

line 25. Replace with

$$\overline{E_{\ell,d}(R)} \cong E(R); \quad R \leqslant R_{crit}, \quad dE(0) \gg 1 \tag{75}$$

line 28. Replace with

$$E(R) \geqslant E_{\ell,d}(R) \cong E(R) \tag{76}$$

line 29. Should read "with probability one if"

Page 20, line 7. Replace with

$$E_{2,d}(R) \cong E^*(R) \quad (E^*(R) \text{ is Shannon's upper-bound exponent}) \tag{99}$$

Page 22, line 5. Add "exponent of the probability of error as long as $A^2_{max} \leqslant 1$."

Page 23, line 12. Change $E_{2,1}(0)$ to $E_{\ell,1}(0)$

line 14. Should read "Thus, since $4(k-1) \leqslant k^2$; $k \geqslant 2$, we have"

line 16. Change $E_{2,1}(0)$ to $E_{\ell,1}(0)$

line 20. Should read

$$\geqslant -\ln \frac{1}{\ell^2}\left[\ell + 2(\ell-1)\exp\left(\frac{-D^2_{min}}{8\sigma^2}\right) + 2(\ell-2)\,\frac{\exp\left(-4kD^2_{min}/8\sigma^2\right)}{\exp\left(-4kD^2_{min}/8\sigma^2\right) - 1}\right]$$

lines 33 and 34. Change $E_{2,1}(0)$ to $E_{\ell,1}(0)$

Page 24, line 1. Change $E_{2,1}(0)$ to $E_{\ell,1}(0)$

line 11. Should read "Comparing Eqs. 72 and 75-77. . . "

Page 26, line 15. $\left(\Gamma\frac{d}{2}\right)$ should read $\Gamma\left(\frac{d}{2}\right)$

Page 27, line 13. Should read

$$E_d(0) \cong \frac{d-1}{d} E(0); \quad A^2 \gg 1 \tag{130c}$$

ERRATA

(added in press)

Page 29, line 15. Should read "Thus, for $A^2 \gg 1$, we have"

Page 31, line 22. Should read "Now, from Eq. A-70, we have"

line 23. Should read

$$\overline{E_{\ell,d}(R)} \geq \overline{E_{\ell,d}(0)} - R.$$

line 24. Should read "Thus, by Eqs. 69 and 140, we have"

line 25. In Eq. 141 change equality sign to $\geq$

lines 34 and 38. Delete "$R \leq R_{crit}$"

Page 32, line 39. Change "variables" to values

Page 33, line 20. Should read "And, since $d \geq 1$ always, if"

Page 35, line 18. Add equation number (153)

line 21. Change "143" to 137.

line 27. Change equation number to (154)

line 33. Change equation number to (155)

line 35. Should read "Thus, by Eqs. 137, 153, and 154,"

line 37. Change equation number to (156)

Page 36, line 1. Change equation number to (156a)

line 3. Should read "Inserting (156a) into (152) yields"

line 4. $\sum_{w,j,\lambda}$ should read $\sum_{w,\lambda,j}$

line 18. Change lower-case k to capital K

Page 40, line 2. Should read

$$2\gamma\left(\frac{1}{2}\right) \ln \int_Y \left[\sum_{X_\ell} p(x)\, p(y|x)^{1/2}\right]^2 dy$$

Page 41, lines 18-21. Replace lines 18-21 with "Summation on j will contain a number of terms proportional to m. Thus

$$\sum_j \Pr\left[D_m(u',v) \geq D_m^j;\ k_j \text{ is used}\right]$$

$$\leq Km \Pr\left[D_m(u',v) \leq D_m^{j_{max}}\right] \tag{186}$$

line 24. Change to

$$P_e \leq Km\left\{\exp[-k_{j_{max}}] + e^{dmR} \Pr\left[D_m(u',v) \leq D_m^{j_{max}}\right]\right\}$$

line 26. Change D_m^j to $D_m^{j_{max}}$

Page 42, line 6. Change $d_m^{j_{max}}$ to $D_m^{j_{max}}$

lines 13 and 19. Should read "$P_e \leq Km[\ldots$"

line 22. Should read "$P_e \leq 2Kme^m \ldots$"

Page 73, Table 4. Last row should read "See Table 3"