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# AUSTRALIAN ATOMIC ENERGY COMMISSION

## RESEARCH ESTABLISHMENT

## LUCAS HEIGHTS

# MULTIPARTICLE COLLISIONS

# PART 2. APPLICATION OF UNITARITY

by

J.L. COOK



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## II. APPLICATION OF UNITARITY

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ABSTRACT

The application of unitarity to multiparticle production processes is studied and relationships between production and scattering amplitudes are derived.

CONTENTS

## 1. INTRODUCTION

In Part I of this work (Cook 1966) the possible structures of many particle partial wave amplitudes were reviewed. In this report, the application of unitary properties of the scattering matrix is investigated to ascertain how much information is provided by the principle of unitarity. From standard tests such as Blatt and Weisskopf (1952), as applied by Kibble (1960), the unitary condition may be written:

$$S^+S = I \quad .$$

from which one gets:

$$\frac{1}{2i}\left(A_{fi} - A^*_{fi}\right) = \sum_n \int d\Omega_{\underset{\sim}{n}} \, A^*_{fn} \, A_{ni} \qquad \ldots\ldots(1)$$

where $A_{fi} = \langle f/T/i \rangle$ is the transition amplitude between states of i and f particles respectively,

$T$ = the transition matrix

= $1/2i\,(I-S)$, where

$S$ = the scattering matrix,

$d\Omega_{\underset{\sim}{n}}$ = the volume element of all degrees of freedom in intermediate states of n particles.

The theorem of reciprocity further states that:

$$T_{fi} = T_{-i-f} \quad . \qquad \ldots\ldots(2)$$

Only cases where the interacting particles have no spin are dealt with, and then $A_{fi}$ is simply a scalar complex number, and both sides of Equation 1 are real.

## 2. INTEGRATION OVER INTERMEDIATE STATES

If the partial wave expansion of the general vertex describing transitions from a state with i particles to a state with f particles were known, it would prove possible to carry out the integrations on the right-hand side of Equation 1 and so obtain algebraic relationships between the partial wave amplitudes for production and scattering. As in the previous paper, we write:

$$A_{fi} = \sum_L \frac{(2L+1)}{\sqrt{2\pi}} \, \Psi_f\,(LM',\underset{\sim}{\Omega}') \; a_{fi}(L,W) \; D^L_{M'M}\,(W_{fi}) \; \Psi_i(LM,\underset{\sim}{\Omega}), \qquad \ldots\ldots(3)$$

where $\Psi_f$, $\Psi_i$ are multiparticle states of orbital angular momentum L and z component M for f and i particles respectively,

$D^L_{M'M}\,(W_{fi})$ is the rotation group operator,

$A_{fi}$ is the partial wave amplitude,

$\underset{\sim}{\Omega}', \underset{\sim}{\Omega}$ are the additional degrees of freedom required to specify final and initial configurations respectively.

The integrated product in (1) is evaluated as follows. After elimination of the kinematical constraints we obtain:

$$\int d\underset{\sim}{\Omega}_n \, A^*_{fn} A_{ni} = \int d\underset{\sim}{\Omega}''_n \sum_{L'} \sum_{L''} \psi^*_f (L'M';\Omega) \, a^*_{fn}(L';W) \times \psi_n(L'M'';\underset{\sim}{\Omega}'')$$

$$D^{L'}_{M'M''} (W_{fn}) \, D^{L}_{M'M} (W_{ni}) \, \psi_n (LM;\underset{\sim}{\Omega}'') a_{ni}(L;W) \times \psi_i(LM;\underset{\sim}{\Omega}) \; . \qquad .....(4)$$

The phase space factors $J_n$ incorporated into the $\psi_n$ and $d\Omega''_n$ will cancel, and following from the orthogonality of the $\psi_n$, that is,

$$\int d\Omega''_n \, \psi_n(L'M';\underset{\sim}{\Omega}'') \, \psi_n(LM;\underset{\sim}{\Omega}'') = \delta(L',L)\delta(M',M) \quad ,$$

as well as the addition theorem for the rotation group operators (Edmunds 1957),

$$\sum_{M''} D^{L}_{M'M''} (W_{fn}) \, D^{L}_{M''M} (W_{ni}) = D^{L}_{M'M} (W_{fi}) \quad ,$$

one obtains:

$$\int d\underset{\sim}{\Omega}_n \, A^*_{fn} \, A_{ni} = \sum_{L\,M'M} \psi_f \, (LM';\underset{\sim}{\Omega}') \, a^*_{fn} \, (L,W) \, a_{ni} \, (L,W) \, D^{L}_{M'M} \, (W_{fi})$$

$$\times \; \psi_i \, (LM;\underset{\sim}{\Omega}) \; . \qquad .....(5)$$

Now we select co-ordinates in initial and final states such that the z-axis lies in a plane perpendicular to L in each case, hence M' = M = 0. The rotation group operators obey the property (Rose 1957),

$$\int dw \, D^{L_1}_{M_1M_1'} (w) \; D^{L_2}_{M_2M_2'} (w) = \frac{8\pi^2}{2L_1 + 1} \, \delta(M_1,M_2) \, \delta(M_1',M_2') \, \delta(L_1,L_2), \qquad .....(6)$$

while it is assumed that $\psi_n(L0;\underset{\sim}{\Omega})$ is a real function. Using these rules, we can project out the M = M' = 0 states in (5), and the left-hand side of (1), to obtain:

$$\frac{1}{2i} \left( a_{fi}(L,W) - a^*_{fi} \, (L,W) \right) = \sum_n a^*_{fn} \, (L,W) \, a_{ni} \, (L,W) \; . \qquad .....(7)$$

Any scalar amplitude may be written:

$$a_{fi} = \rho_{fi} \, e^{i\delta_{fi}} \quad ,$$



and substituting this form into (7), we get:

$$\mathrm{Im}\, a_{fi}(L,W) = \sum_n \rho_{fn}(L,W) \, \rho_{ni}(L,W) \, e^{i[\delta_{ni}(L,W) - \delta_{fn}(L,W)]} \qquad .....(8)$$

The left-hand side of Equation 8 is a real number, and the right-hand side must therefore satisfy:

$$\sum_n \rho_{fn} \, \rho_{ni} \; \sin \, (\delta_{ni} - \delta_{fn}) = 0 \quad , \qquad .....(9)$$

where subscripts (L,W) have been dropped for convenience.

One simple way to satisfy the stringent condition (9) is to introduce an equiphase principle. We assume:

$$\delta_{fn} = \delta_{ni} \qquad \text{for all } (f,n,i) \quad , \qquad .....(10)$$

and explore the consequences.

## 3. <u>PARTIAL WAVE AMPLITUDES IN THE EQUIPHASE ASSUMPTION</u>

If $\rho_{fi}$ is regarded as an n × n matrix, where up to n initial, final, or intermediate particles are kinematically possible, then in the equiphase assumption, Equation 8 states:

$$\underline{\rho} . \underline{\rho} = \underline{\rho} \sin \delta \; . \qquad .....(11)$$

The partial wave projections from the T matrix elements can be written

$$\mathrm{I} = \underline{\rho} \; e^{i\delta} \; . \qquad .....(12)$$

Since $\underline{\rho}$ is a real symmetric matrix, it follows from (11) that:

$$\underline{\rho}^n = (\sin\delta)^{n-1} \underline{\rho} = \sin\delta \; \underline{\rho}^{n-1} .$$

When the determinant of both sides of (11) is taken, one finds:

$$(\det \underline{\rho})^2 = (\sin\delta)^n \det \underline{\rho} \; ,$$

that is, either det $\underline{\rho}$ = 0 or $(\sin\delta)^n$.

The second result corresponds to the trivial solution:

$$\underline{\rho} = \sin\delta . \, \underline{\mathrm{I}} \quad ,$$

but for the first solution, the physically interesting one, $\underline{\rho}$ is a singular matrix. Its characteristic equation:

$$\det |\underline{\rho} - \lambda \mathrm{I}| = 0 \quad ,$$

has n-1 zero roots with one root equal to sinδ. Therefore $\underline{\rho}$ is of unit rank and all principal minors of order greater than unity vanish. It follows from the expansion of the characteristic equation and the Cayley-Hamilton theorem (Mirsky 1955) that:

$$(-\sin\delta)^n + (-\sin\delta)^{n-1} \text{ trace } \underline{\rho} = 0, \text{ so}$$

$$\text{trace } \underline{\rho} = \sin\delta$$

$$\text{trace } \underline{T} = \sin\delta \; e^{i\delta}, \text{ and}$$

$$\det \underline{T} = 0. \qquad .....(13)$$

Thus $\rho$ and $\underline{T}$ possess no inverse: on the other hand, the matrix:

$$\underline{S} = \underline{I} - 2i\,\underline{\rho}\, e^{i\delta} ,$$

fulfils the conditions $\underline{S}^+\underline{S} = \underline{I}$ ,

$$\det \underline{S} = e^{2i\delta}$$

$$\det \underline{S} \det \underline{S}^+ = 1$$

$$S^+ = S^{-1} .$$

$\underline{\rho}$ also has the curious property that

$$(\text{trace } \underline{\rho})^n = \text{trace } \underline{\rho}^n .$$

4. BRANCHING RATIOS

The condition that every minor of $\underline{\rho}$ of order greater than unity should vanish leads to the condition that all 2 × 2 minors should vanish. Hence:

$$\rho_{fi}\, \rho_{kl} = \rho_{fl}\, \rho_{ki} . \qquad .....(14)$$

For example $\rho_{23}^2 = \rho_{22}\, \rho_{33}$ .

We define branching ratios $\Gamma_{fi}$ by:

$$\Gamma_{fi} = \Gamma\rho_{fi}/\sin\delta , \qquad .....(15)$$

and from (13) we find:

$$\Gamma = \sum_i \Gamma_{ii} . \qquad .....(16)$$

Also, Equation 14 relates all off-diagonal elements to diagonal ones by the relation:

$$\Gamma_{fi} = \pm\sqrt{\Gamma_{ff}\,\Gamma_{ii}} . \qquad .....(17)$$

Since $\Gamma_{fi}$ appears in total cross sections as a factor of proportionality, we



chose the positive roots of (17). The matrix:

$$\text{(i)} \quad B_{fi} = \Gamma_{fi}/\Gamma = \rho_{fi}/\sin\delta$$

satisfies (ii) $\underline{B}^2 = \underline{B}$

(iii) trace $\underline{B} = 1$

(iv) det $\underline{B} = 0$ . .....(18)

5. CROSS SECTIONS AND PHASE SHIFTS

The total cross section for a particular reaction is defined by

$$\sigma_{fi}(W,\underset{\sim}{\Omega}) = \int d\underset{\sim}{\Omega}_f \; | A_{fi} |^2 . \qquad .....(19)$$

This expression can be evaluated by substituting (3) into (19) to obtain

$$\sigma_{fi}(W,\underset{\sim}{\Omega}) = 4\pi \sum_L (2L+1)\, \rho_{fi}^{\,2}(L,W)\, |\psi_i(L0;\underset{\sim}{\Omega})|^2 . \qquad .....(20)$$

With the equiphase assumption we get:

$$\sigma_{fi}(W,\underset{\sim}{\Omega}) = 4\pi \sum_L (2L+1) \frac{\Gamma_{fi}^{\,2}(L,W)}{\Gamma^2(L,W)} \sin^2\delta(L,W)\, |\psi_i(L0;\underset{\sim}{\Omega})|^2 ,$$

$$= 4\pi \sum_L (2L+1) \frac{\Gamma_{ff}(L,W)\,\Gamma_{ii}(L,W)}{\Gamma^2(L,W)} \sin^2\delta(L,W)\, |\psi_i(L0;\underset{\sim}{\Omega})|^2 . \qquad .....(21)$$

For a two-particle initial state:

$$\psi_2\,(L0;\underset{\sim}{\Omega}) = 1/\sqrt{J_2} , \qquad .....(22)$$

where $J_2$ is the phase space factor for the state. The inelastic cross sections obtained from a two-particle state are found by substituting (22) into (21) to obtain:

$$\sigma_{f2}(W) = \frac{4\pi}{J_2} \sum_L (2L+1) \frac{\Gamma_{ff}(L,W)\,\Gamma_{22}(L,W)}{\Gamma^2(L,W)} \sin^2\delta(L,W) . \qquad .....(23)$$

Now the scattering amplitude is usually represented in terms of a complex phase shift $(\alpha + i\beta)$ such that in each eigenstate of L:

$$a_n(L,W) = \sin(\alpha + i\beta)\, e^{i(\alpha + i\beta)} . \qquad .....(24)$$

By equating (24) to the polar form, one finds:

$$\text{(i)} \quad \rho_{22} = \tfrac{1}{2}(1 + e^{-4\beta} - 2\cos\alpha\, e^{-2\beta})^{\frac{1}{2}}$$

$$\text{and (ii)} \quad \delta = \tan^{-1}\left(\frac{1 - e^{-2\beta}\cos 2\alpha}{e^{-\beta}\,\sin 2\alpha}\right) . \qquad .....(25)$$

The inverse transformations are

$$\text{(i)} \quad \alpha = \frac{1}{2} \tan^{-1} \left\{ \frac{2\rho_{22} \cos\delta}{1 - 2\rho_{22} \sin\delta} \right\},$$

$$\text{and (ii)} \quad \beta = -\frac{1}{4} \ln \left\{ 1 + 4\rho_{22} (\rho_{22} - \sin\delta) \right\}. \qquad \ldots\ldots(26)$$

The absorption coefficient:

$$\eta(L,W) = e^{-2\beta(L,W)},$$

is such that in each eigenstate:

$$\rho_{22}(\sin\delta - \rho_{22}) = \frac{1}{4}(1 - e^{-4\beta}),$$

that is,

$$\frac{\Gamma_{22}}{\Gamma}\left(1 - \frac{\Gamma_{22}}{\Gamma}\right) \sin^2\delta = \frac{1}{4}(1 - \eta^2).$$

The total production cross section from an initial state of 2 particles becomes:

$$\begin{aligned}
\sigma_{prod,2}(W) &= \sum_{f=3}^{n} \sigma_{f2}(W) = \frac{4\pi}{J_2} \sum_{f=3}^{n} \sum_{L} (2L+1)\, \rho_{f2}^{\,2}(L,W) \\
&= \frac{4\pi}{J_2} \sum_{f=3}^{n} \sum_{L} (2L+1)\, \rho_{22}(L,W) \left[\rho_{33}(L,W) + \rho_{44}(L,W) + \ldots\right] \\
&= \frac{4\pi}{J_2} \sum_{L} (2L+1) \frac{\Gamma_{22}(L,W)}{\Gamma(L,W)} \left(1 - \frac{\Gamma_{22}(L,W)}{\Gamma(L,W)} \sin^2\delta(L,W)\right) \\
&= \frac{\pi}{J_2} \sum_{L} (2L+1) [1 - \eta^2(L,W)]. \qquad \ldots\ldots(27)
\end{aligned}$$

For single level approximations, one may put:

$$\cot\delta = 2(E_r - E)/\Gamma,$$

where $E_r$ = the energy at resonance,

$E$ = the initial particles' total energy,

which yields the Breit-Wigner (19) form in (23) for the partial cross sections.

## 6. A SIMPLE COUPLING SCHEME

In the matrix $\underline{B}$ of the branching ratios, given by Equation 18 (i), the diagonal elements are unrelated. The reciprocity theorem (2) leads to the result:

$$\rho_{fi} = \rho_{if}.$$



If we assume that this assumption may be generalized in such a way that:

$$\rho_{fi} = \rho_{f-r,i+r} = \rho_{f+r,i-r}, \quad |i-r|, |f-r| \geq 2, \qquad \ldots\ldots(28)$$

the diagonal elements become related. The consequence of the postulate (28) is that the total cross sections (21) integrated over the initial configuration, become invariant under the complex Lorentz transformations which change a particle from an initial state incoming to a final state outgoing configuration. That is:

$$\text{if} \quad \sigma_{fi}(W) = \int d\underset{\sim}{\Omega}_i \; \sigma_{fi}(W,\Omega) = 4\pi \sum_{L} (2L+1)\, \rho_{fi}^{\,2}(L,W),$$

$$\text{then} \quad \sigma_{fi}(W) = \sigma_{f+r,\, i-r}(W). \qquad \ldots\ldots(29)$$

The diagonal elements of $\underline{B}$ become related by

$$\xi = \Gamma_{33}/\Gamma_{22} = \Gamma_{44}/\Gamma_{33} = \Gamma_{ii}/\Gamma_{i-1,i-1}, \qquad \ldots\ldots(30)$$

in which case

$$\underline{B} = \frac{\Gamma_{22}}{\Gamma} \begin{pmatrix} 1 & \xi & --- \\ \xi & \xi^2 & - \end{pmatrix}$$

$$\text{trace } \underline{B} = \frac{\Gamma_{22}}{\Gamma} \; \frac{1 - \xi^{2(n-1)}}{1 - \xi^2} = 1 \qquad \ldots\ldots(31)$$

$$\text{and} \quad \rho_{fi} = (\xi)^{i+f-4} \rho_{22}. \qquad \ldots\ldots(32)$$

In this way all production amplitudes are related to scattering and the entire set of $n^2$ reactions are specified by two parameters such as $(\rho_{22},\delta)$ or $(\Gamma_{22},\delta)$ per eigenstate of L.

Using the above theory, similar results are derived for different types of $(2 \to 2)$ or $(i \to f)$ reactions. In these cases we simply subdivide $\Gamma_{22}$ into subsets:

$$\Gamma_{22} = \Gamma_{22}(a) + \Gamma_{22}(b) + --$$

$$\Gamma_{fi} = \Gamma_{fi}(a) + \Gamma_{fi}(b) + --$$

to obtain branching ratios for reactions (a), (b) etc. respectively.

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