EFFICIENT ANALOG COMMUNICATION OVER QUANTUM CHANNELS

STEWART D. PERSONICK

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Abstract

This report is concerned with the incorporation of the axioms of quantum measurements into current communication estimation theory. It is well known that classical electromagnetic theory does not adequately describe fields at optical frequencies. The advent of the laser has made the use of optical carriers for information transmission practical. Classical communication estimation theory emphasizes background noise and channel fading as primary limitations on system performance. At optical frequencies, quantum effects may totally dominate performance. Estimation theory is formulated using the quantum theory so that this type of system limitation can be understood, and optimal receivers and systems designed.

The equations determining the optimal minimum mean-square-error estimator of a parameter imbedded in a quantum system are derived. Bounds analogous to the Cramér-Rao and Barankin bounds of classical estimation theory are also derived, and then specialized to the case of an electromagnetic field in a bounded region of space. Cramér-Rao-type bounds for estimation of parameters and waveforms imbedded in known and fading channels are derived.

In examples optimal receivers for the commonly used classical modulation schemes, such as PPM, PAM, PM, DSBSC, are derived. The differences between classical and quantum systems in implementation and performance are emphasized.

It is apparent from the examples and from the structure of the bounds, that quantum effects often appear as an additive white noise arising in heterodyne and homodyne structure receivers. These receivers are not always optimal in performance or in implementation simplicity. Other receivers employing detection by photon counting are sometimes optimal or near optimal.

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I. INTRODUCTION

1.1 MOTIVATION FOR THIS RESEARCH

Before the advent of the laser, optical communication was limited to the auspicious use of what engineers commonly call noise. Now that coherent light sources are available, interest in optical communication systems has developed. It is well known that classical electromagnetic theory does not adequately describe many optical phenomena. In particular, the statistical outcomes of optical measurements can be understood only with the help of the quantum theory. Engineers are often interested in estimation of parameters imbedded in electromagnetic fields. The performance attainable at classical frequencies is often limited by thermal noise and channel fading. At optical frequencies, quantum effects may completely dominate thermal noise in limiting performance. It is the purpose of this work to incorporate the axioms of quantum measurements into classical communication estimation theory in order to develop the tools necessary to analyze and design good quantum communication systems.

I shall use the concept of a conditional density operator along with the axioms of quantum mechanics to answer the following questions. What performance can be attained at optical frequencies by using familiar modulation schemes on laser carriers? Do quantum receivers look any different from their classical analogs? How much can be gained by optimal processing rather than heterodyning as a first stage to demodulation? Do systems that perform equally well at classical frequencies perform equally well at optical frequencies?

1.2 CLASSICAL vs QUANTUM FORMULATION

I shall briefly discuss the two formulations here. A typical classical communication system is shown in Fig. 1. A quantum system is shown in Fig. 2. The classical system comprises an analog data source that is modulated onto a carrier. The output of the transmitter travels over a channel. The channel distorts, attenuates, and adds random noise to the field. The receiver converts the field impinging upon its antenna into a waveform. We can represent a time-limited portion of this waveform by a stochastic Fourier series. The coefficients of this expansion are random variables that are dependent upon the message back at the analog source. Using the probabilistic relationship (usually a conditional density) between the message and the received field data coefficients, we make our estimate of the message. The performance of our estimation scheme is specified in terms of a mapping of all possible messages and estimates into the real line.

In the quantum case, the system up to the channel output is the same as the classical system. The field at the receiver is specified quantum-mechanically, however, in terms of the density operator conditioned upon the message. The correspondence

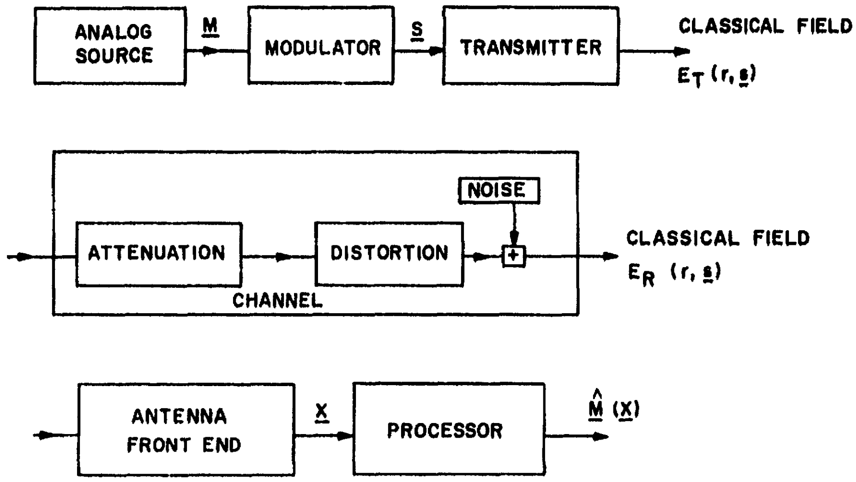


Fig. 1. Classical communication system.

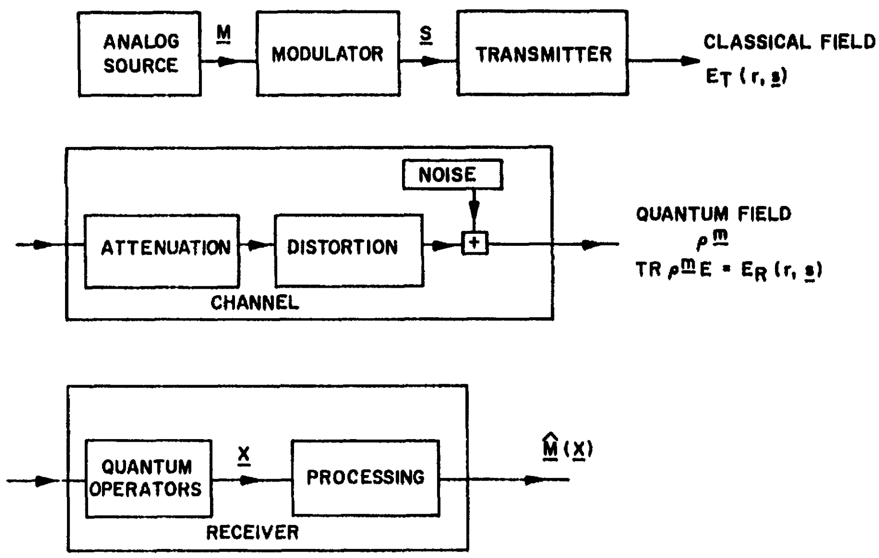


Fig. 2. Quantum communication system.

between the classical and quantum systems is that the expectation of a measurement of the quantum E-field operator is the classical received field. We must specify the quantum measurements that the optimal receiver will make upon the received field. What constitutes a measurement, and the relationship between measurement outcome and message are matters to be treated.

1.3 PREVIOUS WORK

Much of the work of Carl Helstrom[1] on quantum estimation has been used as a foundation for this work. The results of Glauber[2] on the representation of quantum fields have also been used extensively. Of course, the classical communication theory that I am trying to extend is a basic tool. Much of my work follows the results of Van Trees.[3] The work of Messiah[4] and Louisell[5] will be called upon in the discussions on quantum measurements and fields. For a treatment of quantum detection theory, I refer the reader to the work of Jane W. S. Liu.[6]

1.4 SUMMARY

The optimal quantum estimator of a random variable coupled to a quantum field will be derived. Bounds similar to the classical Cramér-Rao bounds for the estimation of parameters imbedded in quantum fields will also be derived. Applications to fading and nonfading channels at optical frequencies will be made. Optimal receivers will be derived for modulation schemes, such as PAM, PM, PPM, DSB.

From the results of this work, we can make some statements about optimal receivers and performance. From the Cramér-Rao bounds it is clear that in any classical communication system in which an efficient or asymptotically efficient receiver, incorporating homodyning as a first step in demodulation, for a known-phase nonfading channel, exists, a quantum receiver incorporating homodyning as a first stage will also be efficient or asymptotically efficient. Examples include DSBSC and PPM. This does not mean however, that a simpler receiver, or one that performs better when efficiency does not occur, cannot exist.

From the section on applications, which includes a large number of examples of commonly used classical modulation schemes, it is apparent that efficient receivers do exist for strong signals. For the examples studied, photon counting, perhaps with a local oscillator, was always employed in the first stage of demodulation.

II. FORMULATION

2.1 TUTORIAL MATERIAL

Some material that is necessary for understanding the results of this research will now be presented. This is meant as an aid to the reader who does not wish to consult the references and wants to get the gist of the results. The author feels that the concepts of communication estimation and quantum theory are not usually well understood simultaneously by every given person. Experience with reading the preliminary drafts indicates that a tutorial section is justified.

2.1.1 Estimation Theory

Classically, we are often presented with the following estimation problem. A source produces a message $\underline{M}$ which is a set of numbers called the message. As a result of this message, a receiver obtains a sequence of numbers $\underline{X}$ called the data. There is a probabilistic relationship between the data and the message, usually expressed as a conditional density $p(\underline{X}/\underline{M})$. The receiver must generate, based upon the data, an estimate of the message. There is a specified cost functional relating the message and data ensembles to the real line. The receiver must pick his estimate to minimize the average cost. We assume here that the a priori message density $p(\underline{M})$ is known. If we denote our estimate $\hat{\underline{M}}(\underline{X})$, the average cost that we shall use here is $E[(\hat{M}(X)-M)^2]$, which is known as the mean-squared error. The symbol $E[\ ]$ denotes expectation or ensemble average. We shall now develop some classical results that will soon be extended to the quantum case.

Suppose $\hat{\underline{M}}_{opt}(X)$ is the optimal estimator for the mean-square-error cost functional. We then have

$$C(\hat{\underline{M}}_{opt}) = E\left[(\hat{\underline{M}}_{opt}(\underline{X})-\underline{M})^T\ (\hat{\underline{M}}_{opt}(\underline{X})-\underline{M})\right]$$

$$= \int p(\underline{X},\underline{M})[(\ )^T(\ )]\ d\underline{X}d\underline{M}$$

$$C(\hat{\underline{M}}_{opt}) \leq C(\hat{\underline{M}}) \qquad \text{for all } \hat{M}. \tag{1}$$

It is a simple application of the calculus of variations to show that

$$\hat{\underline{M}}_{opt}(\underline{X}) = \frac{\int \underline{M}p(\underline{X},\underline{M})\ d\underline{M}}{\int p(\underline{X},\underline{M})\ d\underline{M}} = \int \underline{M}p(\underline{M}/\underline{X})\ d\underline{M}. \tag{2}$$

That is, the optimal estimate, given $\underline{X}$, is the conditional mean of the message, given $\underline{X}$. The conditional mean of the message is in general difficult to calculate. There are several bounds to the cost associated with any estimator. Classically, we can often find estimators that come close enough to these bounds to call these estimators quasi-optimal.

We shall now derive the Cramér-Rao bound.

Define the bias vector as

$$\underline{B}(\underline{M}) = \int p(\underline{X}/\underline{M})(\hat{\underline{M}}(\underline{X})-\underline{M})\, d\underline{X}. \tag{3}$$

$\underline{B}(\underline{M})$ is the average difference between the estimate and the message, given that the message has realized value $\underline{M}$.

Require

$$\underline{B}(\underline{M})\, p(\underline{M})\big|_{m_j=\pm\infty} = 0 \qquad \text{for all } j. \tag{4}$$

We then have

$$d/dm_i[p(\underline{M})b_j(\underline{M})] = \int [\hat{m}_j(\underline{X})-m_j]\, d/dm_i[p(\underline{M},\underline{X})]\, d\underline{X} - \delta(i,j)\, p(\underline{M}), \tag{5}$$

where b_j is the j^{th} component of $\underline{B}$, and m_j is the j^{th} component of $\underline{M}$. Integrate both sides of (5) on $\underline{M}$ using (4).

$$0 = \int [\hat{m}_j(X)-m_j]\, p(\underline{X},\underline{M})\, d/dm_i[\ln p(\underline{X},\underline{M})]\, d\underline{X}d\underline{M} - \delta(i,j). \tag{6}$$

Define

$$L_i = d/dm_i[\ln P(\underline{X},\underline{M})]. \tag{7}$$

Form the vector $\underline{Z}$

$$\underline{Z} = \begin{bmatrix} \hat{m}_1(\underline{X}) - m_1 \\ L_1 \\ L_2 \\ \vdots \\ L_k \end{bmatrix}. \tag{8}$$

Form the matrix G

$$\underline{G} = \int p(\underline{X},\underline{M})\, \underline{Z}\underline{Z}^T\, d\underline{X}d\underline{M} = \begin{bmatrix} E(\hat{m}_1(\underline{X})-m_1)^2 & 1 & 0 & 0 & 0 & 0 \\ 1 & H_{11} & & & & \\ 0 & & & H_{ij} & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}. \tag{9}$$

The reader can convince himself that $\underline{G}$ is semipositive definite, since the expectation of the square of a quantity is non-negative. Thus Det $\underline{G}$ is greater than or equal to zero. Expanding the determinant along the left column, we obtain

$$E\left[(\hat{m}_1(\underline{X})-m_1)^2\right] \geq H^{11}, \tag{10}$$

where H^{11} is the (1, 1) element of the inverse of matrix H. Similarly, we obtain the general bound

$$E\left[(\hat{m}_i(\underline{X})-m_i)^2\right] \geq H^{ii} \tag{11}$$

$$H_{ij} = E[L_i L_j]. \tag{12}$$

The usefulness of this bound will become apparent as we progress.

2.1.2 Axioms of Quantum Mechanics

Quantum mechanics, in its formalism, concerns itself with the state $x\rangle$ of an abstract system. It is concerned, too, with operations performed upon the system which yield real-number outcomes. If we make a correspondence between a physical system and a quantum system, then we can use the axioms of quantum mechanics to predict the outcomes of measurements performed upon the physical system in terms of operations performed upon the quantum system which correspond to those measurements. The quantum state $x\rangle$ lies in a Hilbert space. A Hilbert space is a linear space with the following properties.

1. Properties of Linear Space

If $x\rangle$, $y\rangle$ and $z\rangle$ are elements of the space and a, b, c are real or complex numbers, then

A. There is an operation called addition (+)
   a. $x\rangle + y\rangle = y\rangle + x\rangle$
   b. $x\rangle + (y\rangle + z\rangle) = (x\rangle + y\rangle) + z\rangle$

B. There is a unique element 0 of the space
   $x\rangle + 0 = x\rangle$

C. For every $x\rangle$ there is a unique $-x\rangle$ such that
   $x\rangle + -x\rangle = 0$

D. $c(x\rangle + y\rangle) = cx\rangle + cy\rangle$ $\quad$ $(c+d)x\rangle = cx\rangle + dx\rangle$
   $cdx\rangle = c(dx\rangle)$ $\quad$ $1x\rangle = x\rangle$

2. Properties of an Inner Product Space

A. X is a linear space

B. There exists an operation on pairs of elements denoted $\langle x, y\rangle$ called the inner product

a. $\langle x, x\rangle^{1/2} \geq 0$; equality implies $x\rangle = 0$

b. $\langle x+y, x+y\rangle^{1/2} \leq \langle x, x\rangle^{1/2} + \langle y, y\rangle^{1/2}$;
where $x+y\rangle = x\rangle + y\rangle$

c. $\langle x, y\rangle = \langle y, x\rangle^*$

d. $\langle cx+y, z\rangle = c\langle x, z\rangle + \langle y, z\rangle$

3. Properties of a Complete Inner Product Space

A. X is an inner product space

B. If a sequence in X converges in the Cauchy sense in norm $\langle x, x\rangle^{1/2}$ then a limit in X for that sequence exists.

A complete inner product space is a Hilbert space.

As we have mentioned, our quantum state $x(t)\rangle$ will be an element of a Hilbert space. It is possible to formulate the quantum system such that the states are time-invariant. Since most elementary texts formulate quantum mechanics first in the Schrödinger picture, in which the state varies in time, we shall start this way. For any given system there is an operator (which maps elements of the space into elements of the space) called the Hamiltonian. In the Schrödinger picture, the time evolution of the system state is

$$i\hbar d/dt\ x(t)\rangle = Hx(t)\rangle \quad \text{(Schrödinger wave equation)}, \tag{13}$$

where $\hbar$ is Planck's constant $/2\pi$.

Suppose that we define the transition operator by

$$\begin{aligned} i\hbar d/dt\ \theta(t, t_o) &= H\theta(t, t_o) \\ \theta(t, t) &= I \\ x(t)\rangle &= \theta(t, t_o)\ x(t_o)\rangle. \end{aligned} \tag{14}$$

Define the transformation

$$X_H\rangle = \theta(t_o, t)\ x(t)\rangle = x(t_o)\rangle, \tag{15}$$

where t_o is an arbitrary initial time. In this representation, called the Heisenberg picture, the state is time-invariant. This is the representation that we shall use throughout this report. We shall therefore omit the subscript H hereafter.

We are now ready to discuss measurements. A Hermitian operator is defined by the condition

$$\langle xMy\rangle = \langle yMx\rangle^* \qquad \text{for all } x\rangle \text{ and } y\rangle. \tag{16a}$$

A Hermitian operator may be expanded in one of the following forms

$$M = \Sigma\ e_i\ e_i\rangle\langle e_i \quad \text{or} \quad M = \int m(e)\ e\rangle\langle e\ de, \tag{16b}$$

where

$$\langle e_i e_j \rangle = \delta(i, j) \qquad \langle ef \rangle = \delta(e, f)$$

(Kronecker delta) (Dirac delta)

(the e_i or $m(e)$ are real)

It is possible to interpret the right-hand case as a limiting case of the left-hand case.[4] We shall prove some of our theorems in the discrete case, but will apply the results to the continuous case.

One way of expressing the axioms of measurement is as follows. To every measurement there corresponds a Hermitian operator called an observable. If we make a measurement corresponding to an observable M, the outcome of the measurement will be one of the eigenvalues of the operator. The probability of outcome e_i [see (16b)], given the state is $x\rangle$, under the assumption that the e_i are distinct, is

$$\text{prob}\ (e_i / x\rangle) = \frac{\langle e_i, x\rangle \langle x, e_i\rangle}{\langle x, x\rangle}. \tag{17}$$

Furthermore, after the measurement, if the outcome is e_i, the state will be $e_i\rangle$. Therefore, assuming that the e_i are distinct, we know the state after the measurement. No more information about the state before measurement is available through further measurements.

It is certainly possible that the state of the system before measurement is not known exactly. We may only have a probabilistic knowledge of the a priori state. This is expressed by a Hermitian operator called the density operator. It has the following eigenstate expansion

$$\rho = \Sigma\ p_i\ p_i\rangle \langle p_i, \tag{18}$$

where

$$p_i \geqslant 0; \qquad \Sigma\ p_i = 1.$$

The probability of outcome e_i of a measurement M, given the a priori knowledge summarized in ρ is

$$\text{Pr}\ (e_i/\text{a priori knowledge}) = \sum_j p_j \langle e_i p_j\rangle \langle p_j e_i\rangle. \tag{19}$$

The average value of a measurement of M is

$$E(M) = \sum_j p_j \langle p_j M p_j\rangle = \text{TR}\ \rho M. \tag{20a}$$

(The notation $E(f(X))$ when X is an operator means the expectation of f (outcome of measurement X).) The transform of the conditional density of the outcome of a measurement of M is

$$E(e^{isM}) = TR\ \rho\ e^{isM}. \tag{20b}$$

2.1.3 Concept of a Quantum Estimator

We now know that measurements performed upon a quantum system are probabilistically determined in terms of the conditional density of the outcome, given the measurement. Our measurement outcomes will be used to estimate a message. Our problem is to determine what measurements to make, and how to transform these measurements into an estimate. Let us introduce the concept of a conditional density operator. If the message takes on a particular value $\underline{M}$, then the density operator of the quantum space is given by $\rho^{\underline{M}}$. If we specify a measurement L, the conditional density of the measurement outcome, given that the message has assumed value $\underline{M}$ (in transform) is

$$E(e^{isL}/\underline{M}) = TR\ \rho^{\underline{M}}\ e^{isL}. \tag{21}$$

Our estimation performance, given a cost functional, depends upon the choice of L, as well as the processing of the result. If we write L, its expansion is

$$L = \Sigma\ l_i\ l_i\rangle\langle l_i. \tag{22}$$

Our estimate consists in measuring L and transforming the outcome according to $\hat{m}(l_i)$. We can write the measurement plus transformation as

$$\hat{M} = \Sigma\ \hat{m}(l_i)\ l_i\rangle\langle l_i. \tag{23}$$

Thus our estimator itself is a Hermitian operator. The significance of the intermediate operator L may be that it is what we physically measure and then transform.

The description above is of single-parameter estimation. For multiparameter estimation, we transform the outcome l_i into a vector of estimates for the vector message.

2.2 PARAMETER ESTIMATION

2.2.1 Optimal Single-Parameter Estimator

Suppose we are given the following problem. A single random parameter, A, is to be estimated. We are given its a priori density p(a). We are given the conditional density operator ρ^a described above. We wish to find the optimal estimator $\hat{A}_{opt}$, which is a Hermitian operator defined upon the space of ρ^a. We wish to show under what circumstances such an estimator exists, and is unique. We shall also show that an operator measured on a product space given by the space of ρ^a and another space independent of A, at the choice of the measurer, cannot do better than the best estimator on the space of ρ^a. Our cost functional is the average squared error between message and estimate.

From the tutorial material, we know that the expectation of the square error, given the message for any operator, is

$$E(\hat{A}-a)^2 = \mathrm{TR}\ \rho^a(\hat{A}-aI)^2. \tag{24}$$

Thus our cost functional is

$$C(\hat{A}) = \int p(a)\ \mathrm{TR}\ \rho^a(\hat{A}-aI)^2\ da. \tag{25}$$

Define the operators

$$\begin{aligned} \Gamma &= \int p(a)\ \rho^a\ da \\ \eta &= \int ap(a)\ \rho^a\ da. \end{aligned} \tag{26}$$

It follows that

$$C(\hat{A}) = E(A^2) + \mathrm{TR}\ (\Gamma\hat{A}^2 - 2\hat{A}\eta). \tag{27}$$

Let D be any Hermitian operator and a any real number.

$$C(\hat{A}_{opt}+aD) \geq C(\hat{A}_{opt}). \tag{28}$$

This implies

$$\mathrm{TR}\ (\Gamma\hat{A}_{opt}D+\hat{A}_{opt}\Gamma D-2\eta D) = 0 \qquad \text{for all Hermitian D.} \tag{29a}$$

<u>Lemma</u>

If Γ is positive definite, the optimal estimator must satisfy

$$\Gamma\hat{A}_{opt} + \hat{A}_{opt}\Gamma = 2\eta. \tag{29b}$$

Furthermore, the optimal operator is uniquely given by

$$\hat{A}_{opt} = 2\int_0^\infty e^{-\Gamma a}\ \eta\ e^{-\Gamma a}\ da. \tag{30}$$

<u>Proof</u>: Suppose we call

$$\Gamma\hat{A}_{opt} + \hat{A}_{opt}\Gamma - 2\eta = K. \tag{31}$$

K is clearly Hermitian. From (29a) we have the necessary condition

$$\mathrm{TR}\ KD = 0 \qquad \text{for all Hermitian D.} \tag{32}$$

Expand K in its diagonal form

$$K = \Sigma\ k_i\ k_i\rangle\langle k_i. \tag{33}$$

We can set

$$D = k_j\rangle\langle k_j. \tag{34}$$

We obtain $k_j = 0$ for all j. Thus K is the zero operator and (29b) holds. Suppose there are two solutions to (29). Call the difference $\hat{A}_1 - \hat{A}_2 = G$. We must have

$$\Gamma G + G\Gamma = 0. \tag{35}$$

Expand G in diagonal form

$$G = \Sigma\ g_i\ g_i\rangle\langle g_i. \tag{36}$$

By (35) we have

$$\langle g_i(\Gamma G + G\Gamma)g_i\rangle = 2g_i\langle g_i\Gamma g_i\rangle = 0. \tag{37}$$

Since Γ is positive definite, $g_i = 0$ for all i. Thus G is zero, and the two solutions are really the same.

Let R be the solution that we postulate.

$$R = 2\int_0^\infty e^{-a\Gamma}\ \eta\ e^{-\Gamma a}\,da. \tag{38}$$

Multiply both sides of (38) by Γ and integrate by parts

$$\Gamma R = -2e^{-a\Gamma}\ \eta\ e^{-\Gamma a}\Big|_0^\infty - 2\int_0^\infty e^{-\Gamma a}\ \eta\ e^{-\Gamma a}\,da\,\Gamma$$

$$= 2\eta - R\Gamma. \tag{39}$$

Thus R is indeed the unique solution of (29).

The mean-squared error associated with the estimate is

$$C(\hat{A}_{opt}) = E(A^2) - TR\ \hat{A}_{opt}\eta. \tag{40}$$

We are restricted to making Hermitian measurements. We can, however, make a Hermitian measurement on a product space $\Omega_1 \times \Omega_2$, where the density operator of Ω_1 is ρ^a and the density operator of Ω_2 is ρ_2, independent of A and specified by the measurer.

Proceeding as before, we obtain

$$\begin{aligned} \Gamma &= \rho_2\Gamma_o \\ \eta &= \rho_2\eta_o, \end{aligned} \tag{41}$$

where Γ_o and η_o are the quantities defined in (26) on the space Ω_1.

$$\rho_2\Gamma_o\hat{A}_{opt} + \hat{A}_{opt}\rho_2\Gamma_o = 2\rho_2\eta_o. \tag{42}$$

The solution is clearly the original solution to (29b) which commutes with the density operator ρ_2. Thus the optimal operator and performance are unchanged by going to a product space.

The preceding derivation was of the optimal single-parameter estimator. For the multiparameter case, there is a certain difficulty involved. If we solve, as we would do classically, for the individual optimal estimators, we may find that they do not commute. That is,

$$\hat{A}_{opt}\hat{B}_{opt} \neq \hat{B}_{opt}\hat{A}_{opt}, \tag{43}$$

where $\hat{A}_{opt}$ and $\hat{B}_{opt}$ are the individual optimal estimators of two parameters A and B.

Thus we cannot simultaneously make the individual optimal estimates. We must make some compromise set of commuting estimates. We can set the problem up by using a Lagrange multiplier constraint technique with the original individual cost functionals.

$$C(\hat{A}, \hat{B}) = C(\hat{A}) + C(\hat{B}) + i[TR\,\lambda(\hat{A}\hat{B}-\hat{B}\hat{A})], \tag{44}$$

where λ is Hermitian and summarizes the constraints of commutation.

The optimal estimators that commute must satisfy

$$\begin{aligned} &\Gamma\hat{A}_{opt} + \hat{A}_{opt}\Gamma - 2\eta_A + i(\hat{B}_{opt}\lambda-\lambda\hat{B}_{opt}) = 0 \\ &\Gamma\hat{B}_{opt} + \hat{B}_{opt}\Gamma - 2\eta_B - i(\hat{A}_{opt}\lambda-\lambda\hat{A}_{opt}) = 0 \\ &\hat{A}_{opt}\hat{B}_{opt} = \hat{B}_{opt}\hat{A}_{opt}, \end{aligned} \tag{45}$$

where

$$\begin{aligned} \eta_A &= \int ap(a, b)\ \rho^{a, b}\ dadb \\ \eta_B &= \int bp(a, b)\ \rho^{a, b}\ dadb. \end{aligned} \tag{46}$$

Unfortunately, we cannot say what space to measure on. That is, for multiparameter estimation, examples in which going to a product space helps exist. Even if we knew what space to measure on, Eqs. 45 are difficult to solve.

Special Cases and Examples

Example 1. Suppose Γ and η commute. Let L be the solution of $\Gamma L = \eta$. Clearly, L is Hermitian. Furthermore, by taking the adjoint of the previous equation, we obtain

$L\Gamma = \eta$. Thus L is a solution of (29) and is the optimal estimator for this case. Writing out operators in their eigenvector expansions, we obtain

$$\Gamma = \Sigma\ \gamma_j j\rangle\langle j \tag{47}$$

$$\eta = \Sigma\ \eta_j j\rangle\langle j \tag{48}$$

$$\hat{A}_{opt} = \Sigma\ \hat{a}_{opt\ j} j\rangle\langle j, \tag{49}$$

where

$$a_{opt\ j} = \eta_j/\gamma_j. \tag{50}$$

Such a situation could arise if the density operator were diagonal in the same representation for all values of the parameter A. That is,

$$\rho^a = \Sigma\ p_j(a)\ j\rangle\langle j. \tag{51}$$

(Note that the eigenvectors do not depend upon a.)

From (26) we obtain

$$\hat{a}_{opt\ i} = \int p(a)\ p_i(a)\ a\ da / \int p(a)\ p_i(a)\ da. \tag{52}$$

This is analogous to the conditional mean of classical estimation theory. That is, given that the eigenvalue associated with eigenvector k occurs, we estimate A to be the conditional mean of A, given k.

For instance, suppose we have a cavity that is lossless and has one mode of electromagnetic oscillation. Suppose the density operator of the mode is coupled to the parameter to be estimated in the following way. Using the notation of Glauber[2] (see Appendix B), we have

$$\rho^a = \Sigma\ e^{-w(a)n}\ [1 - e^{-w(a)}]\ n\rangle\langle n, \tag{53}$$

where

$n\rangle$ is an eigenstate of the number operator b^+b

$w = \hbar\Omega/kT \quad T = T(a)$

$\hbar$ = Planck's constant/2π

k = Boltzmann's constant

Ω = resonant frequency in rad/s

T = temperature in degrees Kelvin.

Since the density operator is diagonal in the number representation for all a, we can simply count photons and then process the result for the best estimate. Suppose

$$a = \mathrm{TR}\ \rho^{a} b^{+} b = [e^{w(a)} - 1]^{-1}. \tag{54}$$

We are essentially modulating the temperature in a nonlinear fashion. $p_j(a)$ is given by

$$p_j(a) = [1 - e^{-w(a)}]\, e^{-w(a)j} = (1+1/a)^{-j}\ (1+a)^{-1}. \tag{55}$$

If we receive k photons, our estimate is given by plugging (55) into (52) with i = j.

Example 2. A simple Hilbert space is Euclidian space R^2 with the usual inner product. Suppose our random variable A can take on two values s and t, each with probability 1/2. Suppose also that, given that the random variable realizes value s or t, the state of the system is known. That is, the density operator has only one positive eigenvalue equal to unity. Since we are talking about a space that we can visualize, let us

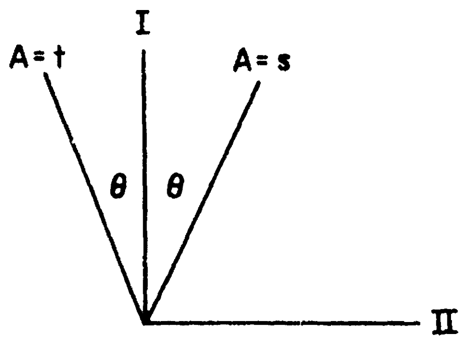


Fig. 3. Example – the space R^2.

set up a coordinate system in the space so that we can graphically show what is happening. Figure 3 shows the reference vectors I and II which are orthogonal. The states of the system, given A = s or A = t, are shown. We can represent an operator by a matrix whose elements are the inner products with the reference vectors I and II of the form $\langle i0j \rangle$, where 0 is an arbitrary operator. The density operator, given A = s, is $s\rangle\langle s$. Its matrix representation is

$$P^s = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}. \tag{56}$$

The reader should verify this. We can also interpret this in the following way. If we expand the density in terms of the eigenvectors I and II, we obtain

$$\rho^s = \Sigma\,\Sigma\, P^s_{ij} i\rangle\langle j \qquad i, j = \text{I and II}. \tag{57}$$

Similarly, we obtain

$$P^t = \begin{bmatrix} \cos^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \sin^2\theta \end{bmatrix}. \tag{58}$$

Using (26), we obtain

$$\Gamma = 1/2(P^s+P^t) = \begin{bmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{bmatrix} \tag{59}$$

$$\eta = 1/2(sP^s+tP^t) = \begin{bmatrix} 1/2(s+t)\cos^2\theta & 1/2(s-t)\cos\theta\sin\theta \\ 1/2(s-t)\cos\theta\sin\theta & 1/2(s+t)\sin^2\theta \end{bmatrix}. \tag{60}$$

The optimal estimator of A is

$$\hat{A}_{opt} = \begin{bmatrix} 1/2(s+t) & (s-t)\cos\theta\sin\theta \\ (s-t)\cos\theta\sin\theta & 1/2(s+t) \end{bmatrix}. \tag{61}$$

The eigenvectors and eigenvalues of the optimal operator are shown in Fig. 4. Note that when s = t, the optimal operator is sI, as expected.

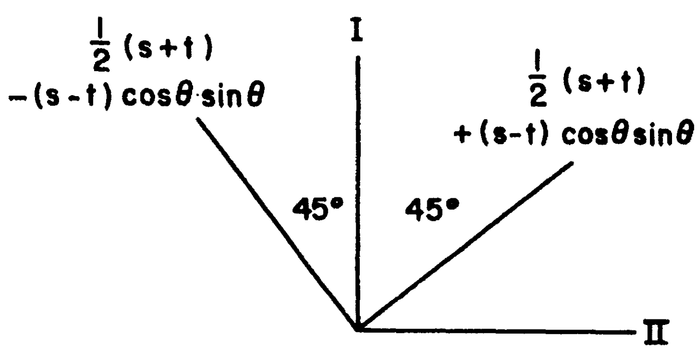


Fig. 4. Example – the space R^2.

A simple interpretation of the optimal estimator is as follows. A transmitter produces one of two possible E-fields. The received field is in an eigenstate of the number operator, with eigenvalue 1, for some mode of a bounded region, and has polarization given by one of two directions: perpendicular to the field propagation direction, and separated by angle 2θ. The optimal receiver, from the results above, passes the received field through a polarizer whose center line bisects the angle between possible received field polarizations. A photon counter after the polarizer determines which of the eigenvalues is the estimate (depending upon whether or not a count is received).

2.2.2 Cramér-Rao Bounds

The idea of a quantum equivalent for the Cramér-Rao bound is due to Helstrom.[1] I shall rederive his result, as well as a number of other bounds. The classical equivalents and applications can be found in Van Trees.[3]

a. Random Variables

Suppose the a priori density of a set of L random variables $p(\underline{a})$ is specified. Suppose we also know the multidependent conditional density operator $\rho^{\underline{a}}$. Let $\hat{\underline{A}}$ be any estimator of $\underline{A}$. That is, the eigenvalues of measurement $\hat{\underline{A}}$ are mapped into the space R^L. We can think of $\hat{\underline{A}}$ as a set of L commuting operators. Define the bias vector $\underline{B}(\underline{a})$ as

$$\underline{B}(\underline{a}) = \mathrm{TR}\ \rho^{\underline{a}}(\hat{\underline{A}}-\underline{a}I) \tag{62}$$

and call the components $b_j(\underline{a})$.

Require that

$$p(\underline{a})\ b_j(\underline{a})\Big|_{a_i=\pm\infty} = 0 \qquad \text{for all } i \text{ and } j. \tag{63}$$

It follows that

$$d/da_j[p(\underline{a})b_i(\underline{a})] = (d/da_j[p(a)])\ \mathrm{TR}\ \rho^{\underline{a}}(\hat{a}_i-a_iI) + p(a)\ \mathrm{TR}\left[\left(d/da_j\rho^{\underline{a}}\right)(\hat{a}_i-a_iI)\right] - p(a)\ \delta(i,j), \tag{64}$$

where $\hat{a}_j$ is the j^{th} component of the operator $\hat{A}$, and a_j is the j^{th} component of the random variable $\underline{A}$.

Integrate both sides of (64) over R^L. Using (63), we see that the left side vanishes. We obtain

$$\delta(i,j) = \mathrm{RL} \int p(\underline{a})\ \mathrm{TR}\ \rho^{\underline{a}}(\hat{a}_i-a_iI)\ (d/da_j \ln p(a)+L_j)\ d\underline{a}, \tag{65}$$

where

$$d/da_j\rho^{\underline{a}} = 1/2\left[L_j(\underline{a})\rho^{\underline{a}}+\rho^{\underline{a}}L_j(\underline{a})\right]. \tag{66}$$

The reader should verify (65) by substitution.

Form the vector $\underline{Z}$

$$\underline{Z} = \begin{bmatrix} \hat{a}_1 - a_1I \\ d/da_1 \ln p(\underline{a})\ I + L_1 \\ \vdots \\ d/da_L \ln p(\underline{a})\ I + L_L \end{bmatrix} \tag{67}$$

Form the matrix $\underline{G}$

$$G = RL \int p(a)\ TR\ \rho^{\underline{a}} \underline{Z}\underline{Z}^T\ d\underline{a}$$

$$= \begin{bmatrix} \int p(\underline{a})\ TR\ \rho^{\underline{a}}(\hat{a}_1 - a_1 I)^2\ d\underline{a} & 1 & 0 & 0 & 0 \\ 1 & & H_{11} & & \\ 0 & & & & \\ 0 & & & H_{ij} & \end{bmatrix} \tag{68}$$

$$H_{ij} = RL \int p(\underline{a})\ TR\ \rho^{\underline{a}}[(d/da_i \ln p(\underline{a})) + L_i][(d/da_j \ln p(\underline{a})) + L_j]\ d\underline{a}. \tag{69}$$

Since the expected value of the square of a Hermitian operator is non-negative, we have G semipositive definite. Expanding the determinant, we obtain

$$\mathrm{Det}\ G = G_{11}\ \mathrm{Det}\ H - \mathrm{Cof}\ H_{11} \geq 0, \tag{70}$$

where Cof H_{11} is the cofactor of H_{11} in matrix H. (The matrix whose elements are H_{ij}.) We get

$$G_{11} \geq \mathrm{Cof}\ H_{11}/\mathrm{Det}\ H = H^{11}, \tag{71}$$

where H^{11} is the (1, 1) element of H^{-1}. Using this same technique, we obtain the bound

$$\int p(a)\ TR\ \rho^{a}(\hat{a}_i - a_i I)^2\ d\underline{a} \geq H^{ii} \tag{72}$$

with equality iff

$$\hat{a}_i - a_i I = \Sigma\ k_j (d/da_j \ln p(\underline{a}) + L_j(\underline{a}))$$

for some set of constants k_j.

The reader may wonder about the choice of the symmetrized derivative of (66). We could have defined the more general derivative $d/da_j \rho^{\underline{a}} = 1/2\left[\rho^{\underline{a}} S_j + S_j^{+} \rho^{\underline{a}}\right]$ which has infinitely many solutions. If we use this form, we find that the unique Hermitian solution gives the tightest bound.

b. Nonrandom Variables

Suppose we wish to estimate a vector $\underline{a}$ whose a priori density we do not know. We stipulate that our estimator must be unbiased at a point $\underline{a}_o$ in a ball about $\underline{a}_o$. That is,

$$TR\ \rho^{\underline{a}}(\hat{A} - aI) = 0 \qquad \text{in some ball about } \underline{a}_o. \tag{73}$$

We then must have

$$d/da_j\left[\mathrm{TR}\ \rho^{\underline{a}}(\hat{a}_i - a_i I)\right] = 0 \qquad \text{at } \underline{a} = \underline{a}_o \tag{74}$$

$$\mathrm{RL}\ \mathrm{TR}\ \rho^{\underline{a}}[L_j(\hat{a}_i - a_i I)] = \delta(i,j) \qquad \text{at } \underline{a} = \underline{a}_o, \tag{75}$$

where L_j is defined in (66).

Proceeding exactly as in the random-variable case, except for the absence of the integration, we obtain

$$\mathrm{TR}\ \rho^a(\hat{a}_j - a_j I)^2 \geq J^{jj} \qquad \text{at } a = a_o \tag{76}$$

(J^{jj} is the (j,j) element of J^{-1})

with equality iff

$$\hat{a}_j - a_j I = \Sigma\ k_i(\underline{a})\ L_i(a),$$

where

$$J_{ij} = \mathrm{RL}\ \mathrm{TR}\ \rho^{\underline{a}} L_i L_j. \tag{77}$$

Note that

$$H_{ij} = EJ_{ij} + K_{ij}, \tag{78}$$

where

$$K_{ij} = E(d/da_i \ln p(\underline{a}))(d/da_j \ln p(\underline{a})).$$

Result (76) was originally derived by Helstrom.[1]

<u>Example</u> 3. Suppose we consider again the harmonic oscillator of Example 1. Writing the density operator in a slightly different form, we have

$$\begin{aligned} \rho^a &= (1-e^{-w})\ e^{-wb^+b} \\ a &= (e^w - 1)^{-1}. \end{aligned} \tag{79}$$

Taking the derivative, we get

$$d/da\ \rho^{\underline{a}} = (a(a+1))^{-1}\ (b^+b - a)\rho^{\underline{a}}. \tag{80}$$

In this simple case, L commutes with ρ^a and is given by

$$L(a) = (a(a+1))^{-1}\ (b^+b - aI). \tag{81}$$

We obtain the bound

$$\mathrm{TR}\ \rho^a(\hat{a}-aI)^2 \geq (\mathrm{TR}\ \rho^a L^2)^{-1} = a(a+1). \tag{82}$$

From (81) we see that we have the condition for equality and that the optimal operator is b^+b, that is, photon counting.

2.2.3 Bhattacharyya Bound

There is a tighter bound than the Cramér-Rao bound, the Bhattacharyya bound. I shall next derive its quantum equivalent.

Assume that we wish to estimate a nonrandom parameter. That is, we know ρ^a but not $p(a)$. Require that our estimate be unbiased.

$$\mathrm{TR}\ \rho^a(\hat{A}-aI) = 0 \qquad \text{in some interval around } a_o. \tag{83}$$

We have

$$\begin{aligned} &\mathrm{TR}\ (d/da\rho^a)(\hat{A}-aI) = 1 \\ &\mathrm{TR}\ (d^n/da^n\rho^a)(\hat{A}-aI) = 0 \qquad n = 2,3,4,\ldots \quad \text{at} \quad a = a_o. \end{aligned} \tag{84}$$

Define the derivative operators $L_j(a)$:

$$d^j/da^j\rho^a = 1/2\left[L_j(a)\rho^a+\rho^a L_j(a)\right]. \tag{85}$$

Form the vector

$$M = \begin{bmatrix} \hat{A} - aI \\ L_1(a) \\ L_2(a) \\ \vdots \\ L_n(a) \end{bmatrix}. \tag{86a}$$

Form the matrix

$$G = \mathrm{RL}\ \mathrm{TR}\ \rho^a MM^T = \begin{bmatrix} \mathrm{TR}\ \rho^a(\hat{A}-aI)^2 & 1 & 0 & 0 & 0 \\ 1 & F_{11} & & & \\ 0 & & F_{ij} & & \\ 0 & & & & \end{bmatrix} \tag{86b}$$

where

$$F_{ij} = \mathrm{RL}\ \mathrm{TR}\ \rho^a L_i(a)\ L_j(a). \tag{87}$$

Similar to the Cramér-Rao results we have

$$\mathrm{TR}\ \rho^a(\hat{A}-aI)^2 \geq F^{11} \qquad \text{at } a = a_o. \tag{88}$$

This is the Bhattacharyya bound, which includes the Cramér-Rao bound for the case wherein we only go to the first derivative.

2.2.4 Barankin Bound

There is a bound, called the Barankin bound, which includes the Cramér-Rao and Bhattacharyya bounds as limiting cases. When maximized over a testing function, soon to be defined, it takes on a value that is the performance of the optimal unbiased estimator, provided certain conditions are met.[7] That is, it is maximally tight. It should be emphasized that the best estimator at a fixed point a, which is unbiased in an interval about a, may not be optimal elsewhere in that interval. Once again, we state the condition for no bias:

$$\mathrm{TR}\ \rho^{a+h}(\hat{A}-aI) = h; \quad \text{in some interval } S \text{ containing the point } a. \tag{89}$$

Define the symmetrized translation operator

$$\rho^{a+h} = 1/2[L(a,h)\rho^a + \rho^a L(a,h)]. \tag{90}$$

Define the real-valued testing function $g(h_i)$ for discrete points h_i. We have

$$\mathrm{RL} \sum_1^N \mathrm{TR}\ \rho^a L(a,h_i)\ g(h_i)(\hat{A}-aI) = \sum_1^N h_i g(h_i) \tag{91}$$

for all finite N, h_i in S and testing functions $g(h_i)$. From the Schwarz inequality $|\mathrm{TR}\ AB|^2 \leq \mathrm{TR}\ A^2\ \mathrm{TR}\ B^2$, we obtain

$$\mathrm{TR}\ \rho^a(\hat{A}-aI)^2 \geq \frac{(\Sigma\, h_i g(h_i))^2}{\mathrm{TR}\ \rho^a(\Sigma\, g(h_i)L(a,h_i))^2} \qquad \text{for } h_i \text{ in } S. \tag{92}$$

We shall now show that the sup of the right side of (92) over all testing functions is achieved by the optimal unbiased estimator at the point a_o. First, we must refer to a theorem of Banach.[8]

Theorem

Let Ω be a space of Hermitian operators with inner product $\langle X, Y\rangle = \mathrm{TR}\ \rho^a XY$. (We assume that ρ^a is positive definite.) Assume Ω is complete and contains $L(a, h_i)$ for all h_i in the set S.

If there exists a constant C such that

$$\left| \sum_{1}^{N} h_i g(h_i) \right| \leq C \left[\mathrm{TR}\ \rho^a \left[\sum_{1}^{N} g(h_i)\ L(a, h_i) \right]^2 \right]^{1/2} \tag{93}$$

for all h_i in S and finite N, and real-valued $g(h_i)$. Then: There exists a linear functional on the space such that

$$F(L(a, h_i)) = h_i \qquad \text{for all } h_i \text{ in } S \tag{94}$$

and

$$\sup_{X \in \Omega} \frac{|F(X)|}{(\mathrm{TR}\ \rho^a X^2)^{1/2}} \leq C.$$

Furthermore, by the Riesz representation theorem, we have a member of Ω such that

$$\mathrm{TR}\ \rho^a F L(a, h_i) = h_i \qquad \text{for all } h_i \text{ in } S \tag{95}$$

$$\mathrm{TR}\ \rho^a F^2 \leq C^2.$$

Let C_o be the inf of all C satisfying (93). Then: We have an element of the space F_o with

$$\mathrm{TR}\ \rho^a F_o L(a, h_i) = h_i \qquad \text{for all } h_i \text{ in } S \tag{96}$$

$$\mathrm{TR}\ \rho^a F_\rho^2 \leq C_o^2.$$

But, by definition, $F_o + aI$ is an unbiased estimator. We have

$$\mathrm{TR}\ \rho^a F_o^2 \geq \sup_{\substack{h_i \in S \\ \text{and } N}} \frac{\left[\sum_{1}^{N} g(h_i)\ h_i \right]^2}{\mathrm{TR}\ \rho^a (\Sigma\ g(h_i) L(a, h_i))^2} = C_o^2. \tag{97}$$

Thus $R = F_o + aI$ is an element of Ω satisfying

$$\mathrm{TR}\ \rho^{a+h}(R - aI) = h_i \qquad \text{for all } h \text{ in } S \tag{98}$$

$$\mathrm{TR}\ \rho^a (R - aI)^2 = C_o^2.$$

We can choose Ω to be the completion of the space spanned by the $L(a, h_i)$. The optimal operator lies in this space.

III. QUANTUM FIELD

3.1 QUANTUM FIELD IN A BOUNDED REGION

I shall be concerned throughout this work with measurements that can be made at a fixed time in an imaginary box in space which I shall call the measurement region. The field in the measurement region will arise from two sources. Usually there will be a thermal-noise field, not always white in space, but always stationary. There will be a message field arising from a distant transmitter.

Quantum mechanics tells us how to describe the outcomes of commuting Hermitian operators measured on the field in such a box. We shall apply the tools of Section II to the estimation of parameters imbedded in an electromagnetic field. Such measurements may be difficult to carry out physically. We shall show that physical measurements made over a time interval at a fixed plane in space can achieve the performance of the optimal fixed-time quantum operators. Furthermore, no examples that I have found indicated that a multi-time measurement performs better than a fixed-time measurement, provided the measurement region is large enough. We may think of the multi-time estimate as a physical implementation of the quantum estimator, since performance is the only criteria of interest. Justification for using the fixed-time measurement restraint is that the concepts of multi-time measurement are not well formulated, at present. Furthermore, it seems likely that multi-time measurements described in terms of interaction Hamiltonians between system and apparatus can be put into correspondence with fixed-time Hermitian operators on the proper space. (This last statement is not verified in general.)

To discuss the quantum field, authors[2,5] usually write down the classical field in a bounded region in terms of the modal solutions to Maxwell's equations in that region. The volume of the region may in the end go to infinity. We shall expand the field in a source-free cube in terms of the plane-wave solutions to Maxwell's equations. At a fixed time t, the expansion is

$$\vec{E}(r,t) = i \sum \sqrt{(\hbar\omega_k/2V} \, e_j\left[C^j_{\underline{k}} \exp(i(\underline{k}\cdot\underline{r}-\omega_k t)) - C^{*j}_{\underline{k}} \exp(-i(\underline{k}\cdot\underline{r}-\omega_k t))\right], \tag{99}$$

where

$V = L^3$

$\underline{\vec{k}} = 2\pi/L(k_x\vec{e}_x+k_y\vec{e}_y+k_z\vec{e}_z)$

k_x, k_y, k_z are integers between $-\infty$ and ∞

$k = \left(k_x^2+k_y^2+k_z^2\right)^{1/2} = \omega_k/c$

c = speed of light

$\vec{e}_i$ = a unit vector in the i direction.

Quantum mechanics tells us to treat each mode as an independent harmonic oscillator (see Appendix B). We form the E-field operator by replacing $C_{\underline{k}}$ with the annihilation operator b_k, and replacing $C_{\underline{k}}^*$ with the creation operator b_k^+. We write down the E-field in shorthand (single polarization) as

$$E(r,t) = i \sum \sqrt{\hbar\omega_k/2V} \left[b_k \psi_k(r)\, e^{-i\omega_k t} - b_k^+ \psi_k^*(r)\, e^{i\omega_k t} \right], \tag{100}$$

where $\psi_k(r) = e^{i\underline{k}\cdot\underline{r}}$.

The commutation rules are

$$[b_k, b_j] = 0 \qquad \left[b_k, b_j^+ \right] = \delta(k,j). \tag{101}$$

We see that the E-field operator, evaluated at different points in space does not in general lead to commuting operators.

For our purposes, we shall be concerned with those modes that are contained in a band of frequencies with $\omega_k \approx \Omega$, the carrier frequency. We shall replace $\hbar\omega_k/2V$ in our equations by $\hbar\Omega/2V$.

In our first applications we shall treat plane-wave messages. That is, the components k_x and k_y of the propagation vector shall be zero for all modes excited by the message field. Eventually, we shall treat fields that have their propagation vectors $\underline{k}$ lying in a narrow cone about some mean propagation direction.

3.2 NOISE FIELD

For situations to be studied here, we shall allow the presence of a complex stationary Gaussian random process called noise. Such a field is sometimes referred to as a completely incoherent field.[1,2] The density operator for this field, as well as the other fields to be discussed, will be expanded in terms of the right eigenkets (eigenvectors) of the non-Hermitian operator b_k. Glauber has shown that physical fields can be expanded in the following form:

$$\rho = \int P\{\beta\} \prod_k \beta_k\rangle \langle\beta_k \, d^2\beta_k, \tag{102}$$

where

$$b_k\beta_k\rangle = \beta_k\beta_k\rangle.$$

This is referred to as the P-representation. The function P( ) represents a statistical mixture of states. For our purposes (but not necessarily in general) P( ) is a probability density.

For most of our discussions, we shall deal with the quantum analog of stationary complex Gaussian white noise. The density operator of such a noise field is

$$\rho_n = \prod_k \int (1/\pi\langle n\rangle)\, e^{-|\beta|^2/\langle n\rangle}\, \beta_k\rangle\langle\beta_k\, d^2\beta_k, \tag{103}$$

where $\langle n\rangle$ is the expected value of the outcome of a measurement of $b_k^+ b_k$.

For the quantum analog of colored noise, the density operator of L modes is

$$\rho_n = \int (1/\pi^L|\Lambda|)\, e^{-\beta^+\Lambda^{-1}\beta}\, \beta\rangle\langle\beta\, d^{2L}\beta, \tag{104}$$

Λ a positive-definite matrix.

We shall later see that (104) can be put in the form of (103) with $\langle n\rangle$ replaced by $\langle n_k\rangle$ if we make a transformation of modes for field representation.

3.3 CORRESPONDENCE

Glauber[9] has shown that a classical, nonstatistical current source radiating into a vacuum creates a quantum state that is an eigenstate of the operators b_k. In particular, it is an eigenstate of the operator E^+.

$$E^+(r,t) = i\,\Sigma\sqrt{\hbar\Omega/2V}\; b_k\psi_k(r,t)\; e^{-i\Omega t}$$
$$[\psi_k(r,t) = \exp(i(\underline{\vec{k}}\cdot\underline{\vec{r}}-(\omega_k-\Omega)t))] \tag{105}$$

for narrow-band fields.

The density operator for a nonstatistical classical current source is

$$\rho = \beta_1,\beta_2,\ldots,\beta_k,\ldots,\rangle\langle,\ldots,\beta_k,\ldots,\beta_2,\beta_1, \tag{106}$$

where

$$b_k\beta_k\rangle = \beta_k\beta_k\rangle \tag{107}$$

and

$$\mathrm{TR}\rho E(r,t) = 2RLi\;\Sigma\sqrt{\hbar\Omega/2V}\;\beta_k\psi_k(r,t)\,e^{-i\Omega t}. \tag{108}$$

That is, the average value of a measurement of the E-field operator is given by (108).

We make the correspondence that the classical field given by

$$E(r,t) = 2RLS(r,t)\,e^{-i\Omega t} \tag{109}$$

be equal to the average quantum field of (108).

Thus, we have the Classical-Quantum Correspondence

$$S(r,t) = i\,\Sigma\,\sqrt{\hbar\Omega/2V}\;\beta_k\psi_k(r,t). \tag{110}$$

If the message field is known statistically, we express its density operator in the form of (102), where we average over the a priori message distribution.

The density operator for message plus noise is in the form of (102), where we convolve the P( ) densities of the message field and the noise field as we would for the addition of random variables.

For the case of white noise plus nonrandom message we have the density operator

$$\rho = \prod_k \int (1/\pi\langle n\rangle)\,\exp\left(-|\alpha_k-\beta_k|^2/\langle n\rangle\right)\,\alpha_k\rangle\langle\alpha_k\,d^2\alpha_k, \tag{111}$$

where the β_k are determined by (110).

IV. PLANE-WAVE CHANNELS

We shall begin our application of the results of Section II by studying plane-wave channels. That is, we shall expand the E field at a fixed time in a measurement region that is infinite in extent in two directions and of extent L in z, the third direction. We shall consider only modes with propagation vector in the z direction. All calculations will be done on a per unit area basis in the equiphase plane. We shall expand the E-field as

$$E(z,t) = i \sum \sqrt{\hbar\Omega/2L} \left[b_k e^{i\omega_k(z/c-t)} - b_k^+ e^{-i\omega_k(z/c-t)} \right]$$

$$z \in [0, L]; \; t \text{ fixed.} \tag{112}$$

We have assumed that the field is narrow-band. That is,

$$\hbar\omega_k/2L \approx \hbar\Omega/2L. \tag{113}$$

We assume that the field arises from thermal Gaussian noise, and a message that is time-limited to an interval of length L/c. In the absence of noise, then, the classical E-field is

$$E_{\text{class}}(z,t) = 2RLS(z,t,\underline{m})\, e^{i\Omega(z/c-t)}$$

$$z \in [0, L]; \; t \text{ fixed}; \; \underline{m} = \text{message}. \tag{114}$$

We assume that $S(z,t,\underline{m})$ is of the form $S(z/c-t,\underline{m})$. This means that the field may be expanded in the modes of (112). We assume that the measurement region encloses the field at time t. Thus quantum measurements made in this region at time t have all of the field available. The message may be a single parameter, a group of parameters, or a time-limited waveform. We can let the interval T and the distance L = cT go to infinity after we solve the finite L problem.

We have the correspondence between the quantum density operator and the classical field as follows (for the white-noise case):

$$\rho \text{ signal + noise} = \prod_k \int (1/\pi\langle n\rangle)\, e^{-|a_k-\beta_k|^2/\langle n\rangle}\, a_k\rangle\langle a_k\, d^2 a_k \tag{115}$$

$$S(t,\underline{m}) = i \sum \beta_k\, e^{i(\omega_k-\Omega)t} \sqrt{\hbar\Omega/2L}$$

$$t \in [0, T]; \; T = L/c. \tag{116}$$

4.1 CRAMÉR-RAO BOUNDS

4.1.1 Single-Parameter Estimation

I shall now apply the Cramér-Rao bound of Section II to the estimation of a single parameter imbedded in a plane wave of finite duration. Consider Eqs. 112 and 114. Suppose that we have a parameter M whose probability density p(m) is known. We transmit a plane wave that is classically of the form (see Fig. 1)

$$E(z/c-t) = 2RLS(z/c-t, m)\, e^{i\Omega(z/c-t)}. \tag{117}$$

The density operator is given in (115) and (116). To apply the Cramér-Rao bound, we must first find the operator L(m) defined in Eq. 66:

$$d/dm\rho^{m} = 1/2\,[L(m)\rho^{m}+\rho^{m}L(m)].$$

Taking the derivative of (116), where the complex number β is $\beta(m)$, we obtain

$$d/dm\rho^{m} = \sum_{k} \prod_{j\neq k} \int (1/\pi\langle n\rangle) \exp\left(-|\alpha_j-\beta_j|^2/\langle n\rangle\right) \alpha_j\rangle\langle\alpha_j\, d^2\alpha_j$$

$$\cdot \int (1/\pi\langle n\rangle)\left[2RL\left((\alpha_k-\beta_k)\beta_k'^{*}/\langle n\rangle\right)\right] \exp\left(-|\alpha_k-\beta_k|^2/\langle n\rangle\right) \alpha_k\rangle\langle\alpha_k\, d^2\alpha_k, \tag{118}$$

where $\beta_k'^{*} = d/dm(\beta^{*})$. We can use the following[5]:

$$b_k\beta_k\rangle = \beta_k\beta_k\rangle, \tag{119}$$

where b_k is the anihilation operator for the k^{th} mode.

$$f\left(b_k, b_k^{+}\right)\rho^{m} = \rho^{m} f\left[(b_k+\beta_k/\langle n\rangle)e^{-w},\ \left(b_k^{+}-\beta_k/(\langle n\rangle+1)\right)e^{w}\right],$$

where $e^{w} = (\langle n\rangle+1)/\langle n\rangle$; and f(,) is any power series in b_k and b_k^{+}.

We obtain through algebraic manipulation

$$L(m) = \sum\,(\langle n\rangle+1/2)^{-1}\left[(b_k-\beta_k)\beta_k'^{*}+\left(b_k^{+}-\beta_k^{*}\right)\beta_k'\right]. \tag{120}$$

We are now ready to apply the bounds

$$\begin{aligned} &\mathrm{Var}\,(\hat{M}-m) \geq J_{11}^{-1} \quad \text{(unbiased estimator)} \\ &E(\hat{M}-M)^2 \geq (EJ_{11}+K_{11})^{-1} \quad \text{(random-variable estimator)}, \end{aligned} \tag{121}$$

where $\hat{M}$ is the estimator of M.

$$J_{11} = TR\rho^m L(m)\, L(m)$$
$$K_{11} = E(d/dm \ln p(m))^2. \tag{122}$$

It is straightforward to calculate J_{11}, provided the operators in L^2 are kept in normal order by using the commutation rules.

$$J_{11} = 2\, \Sigma\, (\langle n\rangle + 1/2)^{-1}\, \beta_k' \beta_k'^{*}. \tag{123}$$

Using (116), we see that

$$J_{11} = 2(\langle n\rangle + 1/2)^{-1}\, 2L/(\hbar\Omega T) \int_0^T |S'(t, \underline{m})|^2\, dt, \tag{124}$$

where $S'(t, \underline{m}) = d/dm S(t, \underline{m})$. Recalling that $L = cT$, we obtain

$$J_{11} = (4c/\hbar\Omega)(\langle n\rangle + 1/2)^{-1} \int_0^T |S'(t, \underline{m})|^2\, dt. \tag{125}$$

Equations (121) and (125) constitute the Cramér-Rao single-parameter bounds. Note that, except for the factor 1/2 added to the noise, these bounds are identical to the classical white-noise bounds.[3] We shall defer applications for a while.

4.1.2 Waveform Estimation

To estimate a waveform, we can use the classical approach of expanding a time-limited portion of the process in a stochastic Fourier series whose coefficients are uncorrelated random variables. We shall consider only Gaussian random processes which have been described by others.[3] To guarantee that our coefficients are uncorrelated and therefore independent, we shall use for our expansion the solutions of the following eigenvalue equation.

$$\int_0^T K_m(t, u)\, \phi_j(u)\, du = \lambda_j \phi_j(t), \tag{126}$$

where $K_m(t, u) = E[m(t)m(u)]$.

These solutions are orthogonal for different λ_j, and can be orthogonalized in cases for which more than one solution for a given λ_j exists. We shall normalize the eigenfunctions to unit square integral. Writing m(t) in its expansion, we have

$$m(t) = \Sigma\, m_j \phi_j(t); \qquad t \in [0, T]. \tag{127}$$

Assuming that m(t) is a zero-mean Gaussian random process, the m_j are zero-mean Gaussian random variables with variance λ_j.

The cost functional that we use for time-limited waveforms is

$$C(\hat{M}, M) = E \int_0^T (\hat{m}(t) - m(t))^2\, dt. \tag{128}$$

If we expand our estimate in the functions $\phi_j(t)$ (we complete the set if it is not already complete),

$$\hat{m}(t) = \sum \hat{m}_j \phi_j(t)$$
$$C(\hat{M}, M) = E \sum (\hat{m}_j - m_j)^2. \tag{129}$$

If the process m(t) has finite power, and if we estimate only a finite number of coefficients m_j to form our estimate $\hat{m}(t)$, then the cost is given by

$$\hat{m}(t) = \sum_1^N \hat{m}_j \phi_j(t)$$
$$C(\hat{M}, M) = E \sum_1^N (\hat{m}_j - m_j)^2 + \sum_{N+1}^{\infty} \lambda_j. \tag{130}$$

For a given problem, the second sum in (130) is always negligible for sufficiently large N. We shall formulate our estimation problem as an estimation of the first N coefficients. When this is done, we shall let N go to infinity.

a. Memoryless Channels

We assume that an analog message source produces a sample function of a Gaussian random process for an interval T sec long. A modulator produces the complex envelope of a plane-wave field in a no-memory manner, based on the message. That is, the classical field is given by

$$E(z, t) = 2RLS(z/c-t, m(z/c-t))\, e^{i\Omega(z/c-t)}$$
$$z \in [0, L]; \; L = cT. \tag{131}$$

The fact that S(t, m(t)) is only a function of m(t) and not m(u) for u in [0, T] is what we mean by memoryless. Let us expand m(t) in its Karhunen-Loève expansion as given by (126-127).

$$m(t) = \sum_1^N m_j \phi_j(t) \quad \text{(truncated series)}. \tag{132}$$

Call the set of N coefficients the random vector $\underline{M}$. We are now ready to apply the Cramér-Rao bound of Section II.

We have the correspondence

$$S(t, m(t)) = S(t, \underline{m}) = i\ \Sigma\ (\hbar\Omega/2L)^{1/2}\ \beta_k\ \exp(i(\omega_k-\Omega)t)$$

$$\beta_k = \beta_k(\underline{m}) \tag{133}$$

$$\rho^{\underline{m}} = \prod_k \int (1/\pi\langle n\rangle)\ \exp\left(-|\alpha_k-\beta_k|^2/\langle n\rangle\right)\ \alpha_k\rangle\langle\alpha_k\ d^2\alpha_k.$$

Define for notation

$$d/dm_j\beta_k = \beta_k^j. \tag{134}$$

To apply the bound, we need to know the operators L_j defined as in Eq. 66. Proceeding as in (118-120), we obtain

$$L_j = \sum_k (\langle n\rangle+1/2)^{-1}\left[(b_k-\beta_k)\beta_k^{j*}+\left(b_k^+-\beta_k^*\right)\beta_k^j\right]. \tag{135}$$

The quantities J_{ij} are given by

$$J_{ij} = RL\,TR\rho^{\underline{m}}L_i(m)\ L_j(m) = 2\ RL\ \Sigma\ (\langle n\rangle+1/2)^{-1}\ \beta_k^i\beta_k^{j*}. \tag{136}$$

We must now call upon Eqs. 132-133.

$$\beta_k = -(2L/\hbar\Omega)^{1/2}\ (1/T)i\int_0^T S(t, \underline{m})\ \exp(-i(\omega_k-\Omega)t)\ dt \tag{137}$$

$$\beta_k^j = -i(2L/\hbar\Omega)^{1/2}\ (1/T)\int_0^T [d/dm(t)S(t, \underline{m})]\ \phi_j(t)\ \exp(-i(\omega_k-\Omega)t)\ dt. \tag{138}$$

We need to know K_{ij}.

$$K_{ij} = E(d/dm_i \ln p(\underline{m}))(d/dm_j \ln p(\underline{m})). \tag{139}$$

Since the m_j are independent and Gaussian with variance λ_j, it follows that

$$p(\underline{m}) = \prod_1^N (2\pi\lambda_j)^{-1/2}\ \exp\left[-(m_j)^2/(2\lambda_j)\right] \tag{140}$$

$$K_{ij} = (\lambda_j)^{-1}\ \delta(i, j). \tag{141}$$

We know from Mercer's theorem that we can write the message correlation function as

$$E(m(t)m(u)) = \sum_1^\infty \lambda_j\phi_j(t)\ \phi_j(u). \tag{142}$$

Define the truncated kernel

$$ {}_N K_m(t, u) = \sum_1^N \lambda_j \phi_j(t)\, \phi_j(u). \tag{143}$$

Clearly, the inverse of this kernel over the set of functions spanned by the first N eigenfunctions is

$$ {}_N Q_m(t, u) = \sum_1^N (\lambda_j)^{-1}\, \phi_j(t)\, \phi_j(u) \tag{144}$$

$$ \int_0^T {}_N K_m(t, z)\, {}_N Q_m(z, u)\, dz = \sum_1^N \phi_j(t)\, \phi_j(u) = {}_N \delta(t, u). \tag{145}$$

Define the kernel

$$ J(t, u) = RL\, E\left[\sum_{-\infty}^{\infty} (4L/\hbar\Omega)(1/T^2)\, S'(t, \underline{m})\, S'^*(u, \underline{m}) \right.$$

$$ \left. \cdot\, (\langle n \rangle + 1/2)^{-1} \exp[-i(\omega_k - \Omega)(t-u)] \right], \tag{146}$$

where $S'(t, \underline{m}) = d/dm(t)\, S(t, \underline{m})$.

Define

$$ {}_N H(t, u) = J(t, u) + {}_N Q_m(t, u). \tag{147}$$

It follows that

$$ H_{ij} = EJ_{ij} + K_{ij} = \int_0^T \phi_i(t)\, \phi_j(u)\, {}_N H(t, u)\, dtdu \qquad \text{for } i, j = 1, 2, \ldots, N. \tag{148}$$

From previous results,

$$ \begin{aligned} G_{jj} &\geq H^{jj} \\ G_{jj} &= E(\hat{m}_j - m_j)^2. \end{aligned} \tag{149}$$

But we also have

$$ H^{jj} = \int \phi_j(t)\, \phi_j(u)\, {}_N H^{-1}(t, u)\, dtdu; \qquad i, j = 1, 2, \ldots, N, \tag{150}$$

where

$$ \int_0^T {}_N H(t, z)\, {}_N H^{-1}(z, u)\, dz = {}_N \delta(t, u). \tag{151}$$

If we multiply (147) by ${}_N H^{-1}(u, z)\ {}_N K_m(x, t)$ and integrate on t and u, we get

$$ {}_N K_m(x, z) = {}_N H^{-1}(x, z) + \int_0^T {}_N H^{-1}(u, z)\ J(t, u)\ {}_N K_m(x, t)\ dudt. \tag{152}$$

Now define

$$ D(t, u) = E(S'(t, \underline{m}) S'^*(u, \underline{m})). \tag{153}$$

Recognizing the impulse function in Mercer's form, we obtain

$$ J(t, u) = RL[D(t, u)(4L/T\hbar\Omega)\delta(t, u)(\langle n\rangle + 1/2)^{-1}]. \tag{154}$$

Plugging in and letting N go to infinity, we obtain

$$ K_m(x, z) = H^{-1}(x, z)\ \frac{+4c \int H^{-1}(u, z)\ D(u, u)\ K_m(x, u)\ du}{\hbar\Omega(\langle n\rangle + 1/2)}, \tag{155}$$

which implicitly determines the bound. Except for an additive factor of 1/2 in the noise term, (155) is identical to the classical bound of white-noise memoryless channels.[3] We have (see Eq. 128)

$$ C(\hat{M}, M) \geq \int_0^T H^{-1}(t, t)\ dt. \tag{156}$$

b. Channels with Memory

Suppose that the modulator for our plane-wave channel is preceded by a time-variant filter that operates upon the message. The input to the modulator is

$$ a(t) = \int_0^T h(t, v)\ m(v)\ dv. \tag{157}$$

The modulation now creates the envelope

$$ S(t, \underline{m}) = S(t, a(t)). \tag{158}$$

We must replace (138) with

$$ d/dm_j \beta_k = -i(2L/\hbar\Omega)^{1/2}\ (1/T) \int\int_0^T d/da(t)\ S(t, \underline{m}) $$

$$ \cdot\ h(t, v)\ \phi_j(v)\ \exp(-i(\omega_k - \Omega)t)\ \cdot\ dtdv. \tag{159}$$

We now redefine J(t, u) as

$$ J(t, u) = RL \int D_a(z, z)(4c/\hbar\Omega)(\langle n\rangle + 1/2)^{-1}\ \cdot\ h(z, t)\ h(z, u)\ dz, \tag{160}$$

where $D_a(z,z) = E|d/da(t)S(t,a(t))|^2$. Equations 147-152 still hold with (160) substituted

$$K_m(x,z) = H^{-1}(x,z) + \frac{4c \int K_m(x,t)\, h(v,t)\, D_a(v,v)\, h(v,u)\, H^{-1}(u,z)\, dudtdv}{\hbar\Omega(\langle n\rangle + 1/2)}. \qquad (161)$$

4.2 DIRECT APPROACH – DOUBLE SIDEBAND

4.2.1 Analysis

The preceding results are applications of the Cramér-Rao bound to plane-wave fields. We have not yet used the results of Section II for optimal estimators. I shall now discuss a case for which we can solve for these estimators.

Suppose we have double-sideband modulation. That is, the plane-wave complex envelope is given by

$$S(t, m(t)) = m(t); \qquad t \in [0, T]. \qquad (162)$$

Expand m(t) in its Karhunen-Loève expansion

$$m(t) = \Sigma\, m_i \phi_i(t). \qquad (163)$$

We assume that m(t) is very narrow-band compared with the carrier frequency Ω.

Thus far, we have expanded the E-field as

$$E(z,t) = i\, \Sigma \sqrt{\hbar\Omega/2L}\left[b_k \exp[i\omega_k(z/c-t)] - b_k^+ \exp[-i\omega_k(z/c-t)]\right]. \qquad (164)$$

Suppose that we expand the E-field in the functions $\phi_i(t)$ instead of the sinusoids. That is, we expand the narrow-band field operator as

$$E(z,t) = \sum_k \sqrt{\hbar\Omega/2c}\left[c_k \phi_k(z/c-t)\, e^{i\Omega(z/c-t)} + c_k^+ \phi_k^*(z/c-t)\, e^{-i\Omega(z/c-t)}\right]. \qquad (165)$$

We have performed the following transformations

$$e^{i(\omega_k-\Omega)t} = i \sum_j r_{jk}\phi_j(t)\sqrt{T}$$

$$c_j = -\sum_k b_k r_{jk}. \qquad (166)$$

The first equation defines the r_{jk}, the second defines the c_j.

Since the $\phi_j(t)$ form a complete orthonormal set and the $\exp(i(\omega_k-\Omega)t)/\sqrt{T}$ also form a complete orthonormal set, with respect to the narrow-band functions that we are considering, the c_j are a unitary transformation upon the b_k. The c_k therefore obey the commutation rules

$$[c_j, c_k] = 0 \qquad \left[c_j, c_k^+\right] = \delta(j, k). \tag{167}$$

Therefore we have the same operator algebra as before. In particular, the E-field for a nonrandom source is in an eigenstate of the c_k operators. We assume no noise. For white noise, we can expand the density operator in the right eigenkets of the c_k.

$$\rho^{\underline{m}} = \prod_k \int (1/\pi\langle n\rangle) \exp\left(-|a_k-\beta_k|^2/\langle n\rangle\right) a_k\rangle\langle a_k \, d^2 a_k, \tag{168}$$

where $c_k a_k\rangle = a_k a_k\rangle$.

We have the correspondence

$$\mathrm{TR}\ \rho^{\underline{m}} E(z,t) = 2\ \mathrm{RL}\ m(z/c-t)\ e^{i\Omega(z/c-t)}.$$

Therefore we must have

$$\beta_k(\underline{m}) = (2c/\hbar\Omega)^{1/2}\ m_k \qquad \text{for all } k.$$

It is apparent then that each message coefficient affects one mode of a product space of many modes. The optimal estimator for coefficient k is an operator on mode k. Since the optimal estimators are on different modes, they all commute. We shall now solve for these individual optimal estimators. Our message estimate is the set of individual estimators whose outcomes are used with the message process eigenfunctions.

Our optimal estimator for mode k must be a solution of

$$\hat{M}_k \Gamma_k + \Gamma_k \hat{M}_k = 2\eta_k$$

$$\Gamma_k = \int p(m_k)\ \rho^{m_k}\ dm_k$$

$$\eta_k = \int m_k p(m_k)\ \rho^{m_k}\ dm_k$$

$$\rho^{m_k} = \int (1/\pi\langle n\rangle) \exp\left(\frac{-|a_k-\beta_k|^2}{\langle n\rangle}\right) a_k\rangle\langle a_k \ d^2 a_k$$

$$p(m_k) = (2\pi\lambda_k)^{-1/2} \exp\left(-(m_k)^2/2\lambda_k\right). \tag{169}$$

Fortunately, it is not too difficult to convolve Gaussian functions. We obtain

$$\Gamma_k = \int (1/\pi\langle n\rangle)^{1/2} (1/\pi(\langle n\rangle + 2\lambda_k/x))^{1/2}$$

$$\cdot \exp\left[-(\mathrm{RL}\, a_k)^2/(\langle n\rangle + 2\lambda_k/x)\right]$$

$$\cdot \exp\left[-(\mathrm{IM}\, a_k)^2/\langle n\rangle\right] a_k\rangle\langle a_k \, d^2 a_k, \tag{170a}$$

where

$$x = (\hbar\Omega/2c)$$

$$\eta_k = \int \frac{\mathrm{RL}\, a_k \lambda_k/(x)^{1/2}}{\langle n\rangle/2 + \lambda_k/x} \text{ (integrand of Eq. 170a) } d^2 a_k. \tag{170b}$$

Using the algebra of boson operators,[5] we find

$$\hat{M}_k = \frac{\left[\left(c_k + c_k^+\right)/2\right]\left(\lambda_k/(x)^{1/2}\right)}{\langle n\rangle/2 + \lambda_k/x + 1/4}. \tag{171}$$

The error associated with the optimal estimator is given by

$$E(\hat{M}_k - M_k)^2 = \frac{\lambda_k(\langle n\rangle/2 + 1/4)}{\lambda_k/x + \langle n\rangle/2 + 1/4}. \tag{172}$$

Although we have not yet interpreted what the optimal estimator is physically, we see that the estimation error is the same as in the white-noise classical case, with the exception of the added term 1/4.

The obvious question is, What does (171) mean physically? To answer this, we turn to a section on measurements involving photon counters and local oscillators.

4.2.2 Implementation

a. Heterodyning and Homodyning

We wish to find a physical measurement corresponding to the quantum operator $c_k + c_k^+$. We shall call a physical measurement an implementation of a quantum measurement if the moment-generating function of the physical measurement outcome is the same as the moment-generating function of the quantum measurement outcome. In Appendix A, it is shown that the output of a photon counter which has a plane wave comprising a message field, Gaussian noise, and a strong local oscillator impinging upon it is one of the following classical signals (after normalization):

$$g(t) = RL(S(t, \underline{m})) + n(t) \qquad \text{(homodyne case)}$$

$$E(n(t)n(u)) = (\hbar\Omega/2c)(\langle n\rangle/2+1/4)\ \delta(t, u)$$

or (173)

$$g(t) = 2\,RL(S(t)\,e^{ivt}) + n(t) \qquad \text{(heterodyne case)}$$

$$E(n(t)n(u)) = (\hbar\Omega/2c)(\langle n\rangle+1)\ \delta(t, u),$$

where v is a classical frequency highpass compared with $S(t, \underline{m})$.

The choice depends upon whether the local oscillator is adjusted for heterodyning or homodyning.

Suppose we homodyne, and then correlate the classical output g(t) against the function $\phi_j(t)$. The number thus obtained will be

$$g_i = \int_0^T g(t)\ \phi_i(t)\ dt$$

$$g_i = s_i + n_i, \tag{174}$$

where

$$s_i = \int_0^T S(t, \underline{m})\ \phi_i(t)\ dt \ \ \text{(for real } S(t, \underline{m})\text{)},$$

and n_i is a zero-mean Gaussian random variable with variance

$$E(n_i)^2 = (\hbar\Omega/2c)\,(\langle n\rangle/2+1/4). \tag{175}$$

Let us now compare this random variable g_i with the outcome of a measurement of $\left(b_j+b_j^+\right)/2$, the sum of the annihilation and creation operators previously called c_j and c_j^+. We have the quantum relationship

$$E(e^{-s(\text{outcome})}) = TR\,\rho_j^m\ e^{-s\left(b_j^++b_j\right)/2}$$

$$\rho_j^m = \int (1/\pi\langle n\rangle)\ e^{-|\alpha_j-\beta_j|^2/\langle n\rangle}\ \alpha_j\rangle\langle\alpha_j\ d^2\alpha_j \tag{176}$$

$$\beta_j = s_j(2c/\hbar\Omega)^{1/2}.$$

Performing the calculation, we obtain

$$E(e^{-s(\text{outcome})}) = e^{-s\beta_j}\ e^{s^2\{[\langle n\rangle/2+1/4]/2\}}. \tag{177}$$

Thus the outcome is a Gaussian random variable with mean β_j and variance $\langle n\rangle/2 + 1/4$, for real $S(t, \underline{m})$.

Comparing this with (174), we see that the random variable is the same random variable as g_i, except for the multiplicative constant $(2c/\hbar\Omega)^{1/2}$. Thus homodyning and correlating corresponds to a measurement of the operator $b_k + b_k^+$, where the index k is determined by the function against which we correlate.

We can obtain the imaginary part of the complex envelope for complex or imaginary $S(t, \underline{m})$ by homodyning with a local oscillator 90° out of phase with the carrier. This, combined with correlation against one of the mode functions, corresponds to a measurement of $\left(b_j - b_j^+\right)/2i$. We can make one or the other of these two measurements, but not both. This is not surprising, since the operators $b_j + b_j^+$ and $\left(b_j - b_j^+\right)/i$ do not commute. If we wish to measure the real and imaginary parts of the complex envelope $S(t, \underline{m})$, we can try heterodyning. We then multiply the classical waveform $g(t)$ by $\sin(vt)$ to obtain

$$g_1(t) = -(\mathrm{IM}\, S(t, \underline{m}))(1 - \cos 2vt) + (\mathrm{RL}\, S(t, \underline{m}))\sin 2vt + n_1(t). \tag{178}$$

Similarly, multiplying by $\cos(vt)$, we obtain

$$g_2(t) = \mathrm{RL}\, S(t, \underline{m})(1 + \cos 2vt) - (\mathrm{IM}\, S(t, \underline{m}))\sin 2vt + n_2(t), \tag{179}$$

where

$$E(n_1(t)n_1(u)) = E(n_2(t)n_2(u)) = (\hbar\Omega/2c)(\langle n\rangle/2 + 1/2)\ \delta(t, u), \tag{180}$$

and $n_1(t)$ and $n_2(t)$ are independent Gaussian random processes.

Since the envelope $S(t, \underline{m})$ is lowpass compared with frequency v, we can correlate against $\phi_j(t)$ to obtain

$$\begin{aligned} g_{1j} &= s_{1j} + n_{1j} \\ s_{1j} &= \mathrm{IM} \int s(t, \underline{m})\, \phi_j(t)\, dt, \end{aligned} \tag{181}$$

where

$$E(n_{1j})^2 = (\hbar\Omega/2c)(\langle n\rangle/2 + 1/2).$$

We obtain the real part of the coefficient in a similar fashion by using the waveform $g_2(t)$.

Comparing (181) with (175), we see that the penalty for heterodyning rather than homodyning in order to obtain the real and imaginary parts of the envelope coefficients is to have twice as much quantum noise on each coefficient. That is, we have the factor 1/2 rather than 1/4 added to the thermal noise on each part of the complex coefficient.

We might ask what operator heterodyning corresponds to. Authors sometimes talk of a noisy measurement of the non-Hermitian operator b_j. In a paper by Gordon and Louisell[10] measurement of an operator that has a complete set of right eigenkets (but not necessarily orthonormal) is discussed. The moment-generating function for the two parts of the outcome of measurement b_j is

$$E\, e^{uRL(\text{outcome})}\, e^{vIM(\text{outcome})}$$

$$= TR\, \rho^m\, e^{sb_j}\, e^{s^* b_j^+}$$

$$= \int (1/\pi\langle n\rangle)\, e^{-|\alpha_j-\beta_j|^2/\langle n\rangle}\, e^{uRL\alpha_j}\, e^{vIM\alpha_j}\, d^2\alpha_j\, e^{(u^2+v^2)/4}$$

$$= e^{uRL\beta_j}\, e^{vIM\beta_j}\, e^{(u^2+v^2)(\langle n\rangle/2+1/2)/2}, \tag{182}$$

where $u = 2RL(s)$ and $v = -2IM(s)$.

Thus we see that except for a multiplicative factor, the two parts of the measurement of b_j are the same random variables as the two parts of the physical heterodyne-correlation measurement. We can always multiply by the proper constant to make the two measurements exactly the same. There is another interpretation of the heterodyne measurement in terms of operators. We wish to, but cannot, measure the two noncommuting operators $b_j + b_j^+$ and $i\left(b_j-b_j^+\right)$. Specify a mode that has no signal on it. Call the creation and annihilation operators of that mode b_f and b_f^+. Preshield the mode so that its density operator is

$$\rho_f = \int (1/\pi\langle x\rangle)\, \exp\left(-|\alpha_f|^2/x\right)\, \alpha_f\rangle\langle\alpha_f\, d^2\alpha_f, \tag{183}$$

where $\langle x\rangle$ is much less than one and can be made arbitrarily small by adjusting the black-body temperature associated with the signal-free mode f.

On the product space of modes j and f, measure the two commuting operators

$$\begin{aligned} G_1 &= 1/2\left(b_k+b_k^+\right) + 1/2\left(b_f+b_f^+\right) \\ G_2 &= 1/2\, i\left(b_k-b_k^+\right) - 1/2\, i\left(b_f-b_f^+\right). \end{aligned} \tag{184}$$

Since the operators commute on the product space, they can be measured simultaneously.

$$E\left(e^{uG_1(\text{outcome})}\ e^{vG_2(\text{outcome})}\right)$$

$$= TR\rho^m \rho_f \left(e^{uG_1+vG_2}\right)$$

$$= TR\rho^m \exp\left\{\left(u\left(b_k+b_k^+\right)/2\right) + iv\left(b_k-b_k^+\right)/2\right\}$$

$$\cdot\ TR\rho_f \exp\left\{\left(u\left(b_f+b_f^+\right)/2\right) - iv\left(b_f-b_f^+\right)/2\right\}. \tag{185}$$

We can call upon a fact of operator algebra

$$e^A\, e^B = e^{A+B}\, e^{1/2[A,B]} \quad \text{iff} \quad [A,[A,B]] = 0 \quad \text{and} \quad [B,[A,B]] = 0. \tag{186}$$

It follows that

$$E\left(e^{uG_1(\text{outcome})}\ e^{vG_2(\text{outcome})}\right)$$

$$= \exp(uRL\beta_j - vIM\beta_j)\ \exp\{[(u^2+v^2)/2(\langle n\rangle/2+\langle x\rangle/2+1/2)]\}. \tag{187}$$

Since $\langle x\rangle$ is negligible compared with unity by choice of the measurer, we see that G_1 and G_2 correspond to the heterodyne-correlation measurement of the envelope coefficients, except for a multiplicative factor that we can provide. We can think of mode f as the image band of a heterodyne measurement that can be shielded against thermal noise and extraneous signals, but not against zero-point fluctuations contributing quantum noise.

b. Implementation

With a physical interpretation of the operator $b_j + b_j^+$ now at hand, we can interpret the optimal receiver given by the operators specified by (171). What we must do is homodyne the received field in the interval $[0, T]$. The resulting classical signal will then be

$$g(t) = m(t) + n(t)$$
$$E(n(t)n(u)) = (\hbar\Omega/2c)(\langle n\rangle + 1/2)/2\,\delta(t,u). \tag{188}$$

We could then obtain the coefficients $m_j + n_j$ by correlation; multiply each coefficient by the constant of (171) and then reconstruct the estimate by forming the series with the message eigenfunctions $\phi_j(t)$.

$$g_i = \int_0^T g(t)\, \phi_i(t)\, dt$$

$$\hat{m}(t) = \Sigma\, \hat{m}_i \phi_i(t) \tag{189}$$

$$\hat{m}_i = g_i \lambda_i / [(\hbar\Omega/2c)(\langle n\rangle/2+1/4)+\lambda_i].$$

The net effect of these operations corresponds to putting g(t) through a filter which is the optimal white-noise filter for the noise given in (188). From our knowledge of classical systems, the performance is clearly given by (172). Thus if we use double-sideband modulation at optical frequencies, the optimal receiver is the classical receiver

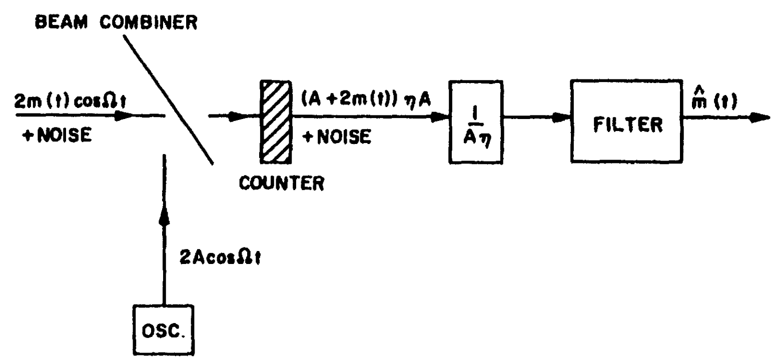


Fig. 5. Double-sideband receiver.

with a quantum homodyne operation at the front end. (See Fig. 5.) The optimal performance is the classical performance with the added term 1/4 added to the noise covariance.

Examination of the Cramér-Rao bound of (155) yields the same result, as expected. The term D(u, u) is unity of DSBSC modulation. If we hypothesize a solution of the form $H^{-1}(t, u)$ given by

$$H^{-1}(t, u) = \Sigma\, h_i \phi_i(t)\, \phi_i(u), \tag{190}$$

we find that such a solution does in fact exist, and that the h_i are given by

$$\lambda_i = h_i + h_i \lambda_i [4c/(\hbar\Omega(\langle n\rangle+1/2))]$$

$$h_i = \lambda_i / [1+\lambda_i 4c/(\hbar\Omega(\langle n\rangle+1/2))]. \tag{191}$$

The lower bound to the estimation error is the same as the estimation error for the optimal operator of (172), as in the classical case.

4.3 OTHER APPLICATIONS

4.3.1 Pulse-Position Modulation

Suppose we wish to use the classical analog of a pulse-position modulation system for parameter communication. The received complex envelope is

$$S(m,t) = S(t-m); \qquad t - m \in [0,T]. \tag{192}$$

That is, the envelope consists in a displaced pulse that is contained in the measurement region for all possible displacements. We shall assume that S(t, m) is a real function. (This assumption will be explained in Section VII.) We have the bound

$$\begin{aligned} &\mathrm{Var}\,(\hat{M}-m) \geq J_{11}^{-1} \\ &J_{11}^{-1} = (\hbar\Omega/4c)(\langle n\rangle+1/2)\left(\int (S'(m,t))^2\, dt\right)^{-1}, \end{aligned} \tag{193}$$

where

$$S'(m,t) = d/dm\, S(t-m) = -d/dt\, S(t-m).$$

The question remains about the implementation of a measurement that has performance close to the bound, if such a measurement exists. Suppose we homodyne as in Appendix A to obtain

$$\begin{aligned} &g(t) = S(t-m) + n(t) \\ &E(n(t)n(u)) = (\hbar\Omega/4c)(\langle n\rangle+1/2)\ \delta(t,u). \end{aligned} \tag{194}$$

It is clear that the classical Cramér-Rao bound for this baseband problem is the same as the bound of (193). Furthermore, under high signal-to-noise (thermal plus quantum contributions to baseband noise) conditions, the classical maximum-likelihood estimate is efficient. That is, we correlate g(t) against S(t−x) and pick the value of x that gives the highest correlation, where x lies in a region where m is expected to be a priori. There may be a better estimate that performs better at low signal-to-noise ratios, or which is easier to implement. (This will be discussed in Section VII.)

4.3.2 Phase Modulation

The optimal classical phase-modulation receiver at high signal-to-noise ratios has been shown[11] to have the form of a phase-locked loop. We shall next show that a similar structure is optimal for the quantum case at high signal-to-noise ratios. First, we must evaluate the bound of (155) for pulse modulation. We have the classical envelope

$$S(m(t),t) = \sqrt{P}\ e^{i\beta m(t)}. \tag{195}$$

From (153) we obtain

$$D(u, u) = P\beta^2. \tag{196}$$

Thus the Cramér-Rao bound is the same that would result from double-sideband modulation at baseband with a classical received signal of the form

$$\begin{aligned} r_{baseband}(t) &= (P)^{1/2}\,\beta m(t) + n(t) \\ E(n(t)n(u)) &= (\hbar\Omega/4c)(\langle n \rangle + 1/2)\,\delta(t, u). \end{aligned} \tag{197}$$

Suppose that we use the receiver structure shown in Fig. 6. The output of the photon counter, by an analysis quite similar to that in Appendix A is

$$1/2\, g(t) = (P)^{1/2} \sin[\beta(m(t)-\hat{m}(t))] + n(t) + A/2, \tag{198}$$

where the noise is given in (197).

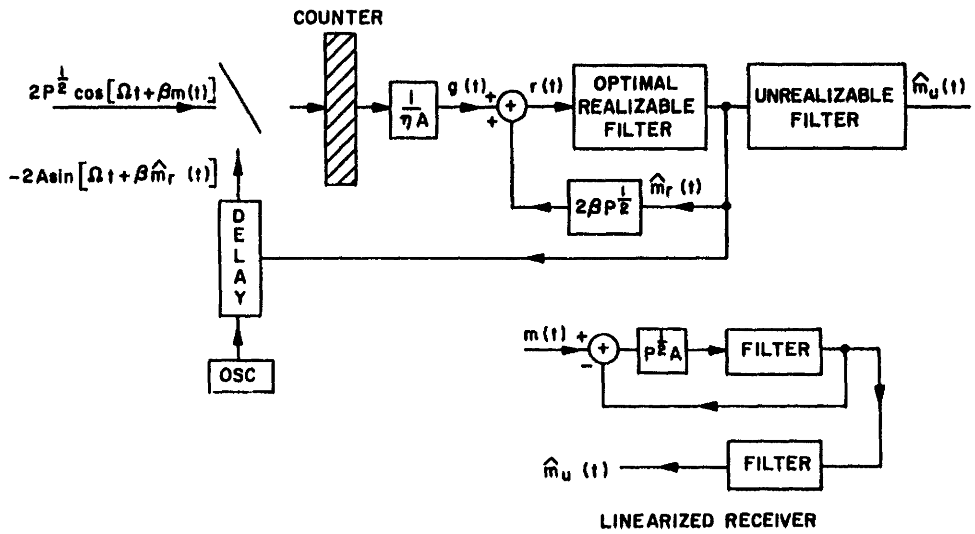


Fig. 6. Pulse-modulation receiver.

As in the classical case, we make the assumption that the realizable estimate is good enough, so that with high probability

$$\sin[\beta(m(t)-\hat{m}(t))] \approx (m(t)-\hat{m}(t))\beta. \tag{199}$$

Thus

$$1/2\, r(t) = (P)^{1/2}\,\beta m(t) + n(t) + A/2, \tag{200}$$

where n(t) is given in (197).

From our discussion of double-sideband modulation, as well as from our knowledge of classical estimation theory, we know that the unrealizable filter (realizable with delay) achieves the Cramér-Rao bound, provided (199) holds. Quantitative analysis of the loop for the classical case is available.[11] These analyses hold here if we replace classical noise by thermal plus quantum noise given in (197).

V. SPATIAL-TEMPORAL CHANNELS

5.1 CRAMÉR-RAO BOUNDS

We shall now extend the results of Section IV to channels in which the received field is of the form

$$E_{\text{Class}}(r,t) = 2\,\text{RL}\; S(r,t,\underline{m})\; e^{-i\Omega t} + 2\,\text{RL}\; n(r,t)\; e^{-i\Omega t}. \tag{201}$$

We shall make measurements of that portion of the field which is contained in a bounded region of space called the "measurement region." The field modes in which we are interested will be those that are narrow-band around the carrier frequency Ω. Therefore we shall expand the field operator as follows:

$$E(r,t) = \Sigma\, \sqrt{\hbar\Omega/2V}\left[b_k\psi_k(r,t)\; e^{-i\Omega t} + b_k^+\psi_k^*(r,t)\; e^{i\Omega t}\right], \tag{202}$$

where b_k and b_k^+ satisfy the commutation rules

$$[b_k, b_j] = 0 \qquad \left[b_k, b_j^+\right] = \delta(j,k),$$

and the $\psi_k(r,t)$ are orthogonal functions, where

$$\int_{\text{M.R.}} \psi_k(r,t)\psi_j^*(r,t)\; d^3r = V\delta(k,j),$$

where subscript M.R. indicates the measurement region. Our problem is to estimate the parameter $\underline{m}$ by quantum measurements in the measurement region at fixed time t.

5.1.1 Single-Parameter Estimation

a. White-Noise Case

The problem of estimation of a single parameter in white noise is very similar to the plane-wave case. Our measurement corresponds to a Hermitian operator measured at fixed time t. We can write the density operator of the field in terms of the right eigenkets of the b_k.

$$\rho^m = \prod_k \int (1/\pi\langle n\rangle)\exp\left(-|a_k-\beta_k|^2/\langle n\rangle\right) a_k\rangle\langle a_k\; d^2a_k, \tag{203}$$

where

$$S(r,t,\underline{m}) = \Sigma\, \sqrt{\hbar\Omega/2V}\; \beta_k\psi_k(r,t).$$

Using (118), (120), and (123), we obtain

$$J_{11} = 2\Sigma\ (\langle n\rangle + 1/2)^{-1}\ \beta'_k\beta'^*_k. \tag{204}$$

From (203) and the orthogonality of the mode functions we obtain

$$J_{11} = (4/\hbar\Omega)(\langle n\rangle + 1/2)^{-1} \int_{M.R.} S'(r,t,m)\, S'^*(r,t,m)\, d^3r, \tag{205}$$

where

$$S'(r,t,m) = d/dm\ S(r,t,m).$$

The bound is

$$\begin{aligned} &\operatorname{var}(\hat{M}-m) \geq (J_{11})^{-1} \\ &E(\hat{M}-M)^2 \geq (J_{11}+K_{11})^{-1}, \end{aligned} \tag{206}$$

where

$$K_{11} = E[d/dm \ln p(m)]^2.$$

It remains to specify the relationship between m and the complex envelope $S(t, r, m)$. We shall defer this for the present.

b. Colored Noise

Suppose that we do not have spatially white noise. That is, suppose the classical noise has the following correlation function:

$$\begin{aligned} &E[n(r_1)n(r_2)] = 0 \qquad (n(r) \text{ is noise envelope}) \\ &E\left[n(r_1)n^*(r_2)\right] = R_n(r_1, r_2). \end{aligned} \tag{207}$$

The covariance R_n is a complex function in general. Let us pick the mode functions upon which we shall expand the field to be solutions of

$$\frac{1}{V}\int_{M.R.} R_n(u,v)\,\psi_k(v)\,dv = N_k\psi_k(u). \tag{208}$$

We shall make measurements at a fixed time, so we have suppressed time dependence.

The density operator of the field consisting in signal plus noise is

$$\rho^m = \prod_k \int (1/\pi\langle n_k\rangle) \exp\left(-|\alpha_k-\beta_k|^2/\langle n_k\rangle\right) |\alpha_k\rangle\langle\alpha_k|\, d^2\alpha_k, \tag{209}$$

where

$$\langle n_k \rangle = (2V/\hbar\Omega) N_k.$$

The classical field is related to the β_k by (203). Similarly to the white-noise case we have

$$J_{11} = 2\Sigma\,(\langle n_k \rangle + 1/2)^{-1}\,\beta_k'\beta_k'^{*}. \tag{210}$$

Define the kernel

$$Q_n(u, v) = (1/V)\,\Sigma\,(2V/\hbar\Omega)(\langle n_k \rangle + 1/2)^{-1}\,\psi_k(t)\,\psi_k^{*}(u) \tag{211}$$

$$\int_{M.R.} Q_n(u,v)\,[1/VR_n(v,w) + (\hbar\Omega/4V)\delta(v, w)]\,d^3v = \delta(u, w). \tag{212}$$

Using (203), we get

$$J_{11} = (2/V)\int_{M.R.} S'^{*}(u, t, m)\,Q_n(u, v)\,S'(v, t, m)\,d^3u\,d^3v. \tag{213}$$

This clearly reduces to the white-noise case when all of the N_k are equal.

5.1.2 Waveform Channels

a. Memoryless

We shall solve for the waveform Cramér-Rao bound. We assume that an analog source produces a sample from a zero-mean Gaussian random process in the interval (0, T). We expand the message m(t) in its Karhunen-Loève expansion

$$m(t) = \Sigma\, m_i \phi_i(t), \tag{214}$$

where the m_i are independent Gaussian random variables of variance λ_i.

From the results on plane-wave memoryless channels and from the results on spatial channel single-parameter estimation, we obtain (for colored-noise expansion)

$$J_{ij} = 2\,RL\,\Sigma\,(\langle n_k \rangle + 1/2)^{-1}\,\beta_k^{j*}\beta_k^{i}, \tag{215}$$

where J_{ij} has been defined in (136), and β_k^j has been defined in (134).

We assume that the message undergoes a no-memory modulation given by

$$F(t, m(t)) = \text{modulator output}. \tag{216}$$

We now assume that the complex envelope of the field in the measurement region is a linear functional upon the modulator output. That is,

$$S(r, \underline{m}) = \int_0^T h_c(r, t)\ F(t, m(t))\ dt. \tag{217}$$

We have suppressed the time of the measurement.

Using the correspondence

$$S(r, \underline{m}) = \Sigma\ \sqrt{\hbar\Omega/2V}\ \beta_k \psi_k(r), \tag{218}$$

we obtain

$$\beta_k^j = (2/\hbar\Omega V)^{1/2} \int [d/dm(u) F(u, m(u))]\ h_c(r, u)\ \psi_k^*(r)\ \phi_j(u)\ d^3r du. \tag{219}$$

Define

$$D(u, t) = E[d/dm(u) F(m(u), u) d/dm(t) F^*(m(t), t)]. \tag{220}$$

Define $Q_n(t, u)$ as in (211). Define $J(t, u)$

$$J(t, u) = \frac{1}{V} \int_{MR} h_c^*(r_1, t)\ h_c(r_2, u)\ Q_n(r_1, r_2)\ D(u, t)\ d^3r_1 d^3r_2. \tag{221}$$

It will be found that $J(t, u)$ satisfies

$$E\, J_{ij} = 2\ RL \int_0^T \phi_i(t)\ \phi_j(u)\ J(t, u)\ dt du. \tag{222}$$

Proceeding as in Eqs. 147-155, we obtain

$$K_m(t, u) = H^{-1}(t, u) + 2\ RL \int_0^T K_m(t, v)\ J(v, w)\ H^{-1}(w, u)\ dv dw \tag{223}$$

$$E \int_0^T (\hat{m}(t) - m(t))^2\ dt \geq \int_0^T H^{-1}(u, u)\ dt.$$

For the special case of white noise

$$K_m(t, u) = H^{-1}(t, u) + 4\ RL \int_0^T K_m(t, v) (\hbar\Omega(\langle n \rangle + 1/2))^{-1}\ \Lambda(w, v)\ D(v, w)$$
$$\cdot\ H^{-1}(w, u)\ dv du, \tag{224}$$

where

$$\Lambda(w, v) = \int_{M.R.} h_c(r, w)\ h_c^*(r, v)\ d^3r.$$

b. Channels with Memory

The function $h_c(r, t)$ constitutes a memory operation of the channel after modulation. There may be a memory operation before modulation, too. Suppose we have

$$a(t) = \int_0^T m(u)\, h(t,u)\, du \qquad F(t, a(t)) = \text{modulation}$$

$$S(r, \underline{m}) = \int_0^T h_c(r,t) \quad F(t, a(t))\, dt. \tag{225}$$

If we define

$$J(t,u) = \int h(v,t)\, h_c^*(r_1, v)\, h(w,u)\, h_c(r_2, w)$$

$$\cdot\, (1/V)\, Q_n(r_1, r_2)\, D_a(v,w)\, d^3r_1 d^3r_2 dvdw, \tag{226}$$

where

$$D_a(v,w) = E[d/da(v) F(v, a(v)) d/da(w) F^*(w, a(w))], \tag{227}$$

Eq. 220 and therefore (221) still hold.

Recall that although these equations look abominable, they can sometimes be formulated in state variables,[12,13] or Fourier-transformed for infinite-time stationary process problems.

5.2 APPLICATIONS

We are interested in the following situation. A transmitter emits a classical plane wave of the form (see Fig. 7)

$$E_T(z,t) = 2\, RL\, S_T(t-z/c)\, e^{i\Omega(z/c-t)}, \tag{228}$$

where the envelope depends upon the message. The classical field received over an aperture is

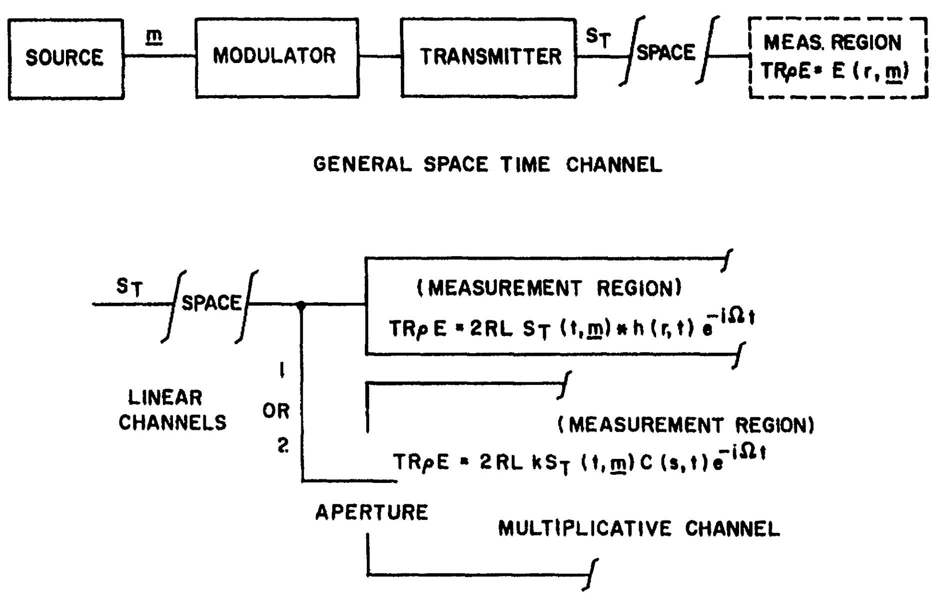


Fig. 7. Spatial temporal channels.

$$E_R(s,t) = 2\,RL\,S_R(t,m)\,C(s,t)\,e^{-i\Omega t}. \tag{229}$$

We assume that the coherence bandwidth[14] of the channel is large compared with the envelope bandwidth so that, neglecting propagation delay time, we have

$$S_R(u) = k\,S_T(u), \tag{230}$$

where k represents attenuation. Occasionally, we shall have the aperture field spatially dependent upon the message, too.

$C(s,t)$, which represents multiplicative fading, is assumed to be known here. Eventually, we shall treat $C(s,t)$ as a random process.

We assume that the received field at the aperture is composed of plane waves, all of whose propagation vectors lie in a cone such that none of the vectors deviates much from the perpendicular to the aperture. If we call the angle of deviation from perpendicular θ, we require θ small enough that $\cos\theta$ is approximately unity for all vectors.

The field propagates through the aperture into the free space behind the aperture. At some fixed time, we shall make a measurement in the space behind the aperture which comprises our measurement region. If we assume that the aperture field is time-limited, then if we wait long enough, all of this field will propagate into the space behind the aperture. We assume no reflection from the space behind the aperture. The field in the measurement region at some time u is given by the impulse response

$$S(r,u) = \int_{\text{aper}} \int S_R(t,\underline{m})\,C(s,t)\,h(r,u,s,t)\,d^2s\,dt, \tag{231}$$

where the time integration ranges over the duration of the aperture field.

Now consider the white-noise case for the single-parameter and waveform Cramér-Rao bounds. We are interested in the relationship between the message and the measurement region field. For parameter estimation we need

$$\begin{aligned}\int_{\text{M.R.}} S'(r,\underline{m})\,S'^*(r,\underline{m})\,d^3r = \iiiint\!\!\int\, & h(r,u,s_1,t_1)\,h^*(r,u,s_2,t_2) \\ & \cdot S_R'(t_1,s_1,\underline{m})\,C(s_1,t_1)\,S_R'^*(t_2,s_2,\underline{m}) \\ & \cdot C^*(s_2,t_2)\,d^3r\,d^2s_1\,d^2s_2\,dt_1\,dt_2.\end{aligned} \tag{232}$$

We have allowed for the possibility that the message affects both the spatial and temporal character of the aperture field. For waveform estimation we need

$$\begin{aligned}\int h(r,u,t)\,h^*(r,u,v)\,d^3r &= \Lambda(t,v) \\ &= \iint h(r,u,s_1,t)\,h^*(r,u,s_2,v)\,C(s_1,t)\,C^*(s_2,v)\,d^3r\,d^2s_1\,d^2s_2,\end{aligned} \tag{233}$$

where we have associated $F(m(t), t)$ with $S_R(t, \underline{m})$, and Λ is defined in (224).

Theorem

If the field impinging upon the aperture is composed of plane waves whose propagation vectors are all nearly normal ($\cos\theta \approx 1$), then with respect to such a field

$$\int h(r, u, s_1, t_1)\, h^*(r, u, s_2, t_2) = c\delta^2(s_1, s_2)\, \delta(t_1, t_2). \tag{234}$$

Proof: Let $S(s, t)$ be the envelope of an incident field satisfying the conditions of the theorem. Expand the field in an orthonormal series with real mode functions.

$$S(s, t) = \Sigma\, s_k \phi_k(s, t) \qquad s \in \text{(aperture)}, \quad t \in (0, T)$$
$$\int \phi_k(s, t)\, \phi_j(s, t)\, d^2s dt = \delta(k, j). \tag{235}$$

We know that the energy that enters the measurement region behind the aperture is given by

$$W = c \int_0^T \int_{aper} |S(s, t)|^2\, d^2s dt = c\, \Sigma\, s_k s_k^*. \tag{236}$$

This follows from the restriction upon the propagation vectors. Since the measurement region is assumed to be empty space with no absorption or reflection, the energy in the measurement region after all of the field has propagated through the aperture must be

$$\begin{aligned} W &= \int |S(r, u)|^2\, d^3r \\ &= \iiint h(r, u, s_1, t_1)\, h^*(r, u, s_2, t_2) \\ &\quad \cdot \sum_j \sum_k s_j s_k^* \phi_k(s_1, t_1)\, \phi_j(s_2, t_2) \\ &\quad \cdot d^3r d^2s_1 d^2s_2 dt_1 dt_2. \end{aligned} \tag{237}$$

Since this must hold true for all complex numbers s_j and the mode functions are complete over the type of field that the theorem allows, the statement of the Theorem (234) must be true. Remember that the impulse is interpreted with respect to the type of field that the theorem allows. This important result follows:

$$\Lambda(t, u) = c \int_{aper} C(s, t)\, C^*(s, t)\, d^2s\, \delta(t, u). \tag{239}$$

If $C(s, t)$ is a constant, then Eq. 224 reduces to the plane-wave equation, where

$D(v, v)$ is multiplied by the constant squared times the area of the aperture. The single-parameter bound becomes

$$J_{11} = (4c/\hbar\Omega)(\langle n\rangle + 1/2)^{-1} \iint |S_R'(t, s, \underline{m})\, C(s, t)|^2\, d^2s\,dt. \tag{240}$$

The waveform bound becomes

$$K_m(t, u) = H^{-1}(t, u)\ \frac{+4c \int K_m(t, v)\, B(v)\, D(v, v)\, H^{-1}(v, u)\, dv}{\hbar\Omega(\langle n\rangle + 1/2)}, \tag{241}$$

where

$$B(t) = \int_{\text{aper}} C(s, t)\, C^*(s, t)\, d^2s.$$

Remember that both of these bounds are for the white-noise case. In the event that the waveform is coupled to the spatial character of the envelope according to

$$S(r, u, \underline{m}) = \int h(r, u, s, t)\, F(s, t, m(t))\, C(s, t)\, d^2s\,dt, \tag{242}$$

then the Cramér-Rao bound is given by

$$K_m(t, u) = H^{-1}(t, u)\ \frac{+4c \int K_m(t, v)\, D_s(v, v)\, H^{-1}(v, u)\, dv}{\hbar\Omega(\langle n\rangle + 1/2)}, \tag{243}$$

where

$$D_s(v, v) = E \int |d/dm(t)\, F(s, t, m(t)) C(s, t)|^2\, d^2s.$$

5.2.1 Pulse-Position Modulation

Suppose that the modulation of (229) is given by

$$E_R(s, t, m) = 2\, RL\, f(t-m)\, C(s)\, e^{-i\Omega t}. \tag{244}$$

The Cramér-Rao bound is given by (240).

$$J_{11} = (4c/\hbar\Omega)(\langle n\rangle + 1/2)^{-1} \int C(s)\, C^*(s)\, d^2o \int_0^T |f'(t)|^2\, dt. \tag{245}$$

If we assume that $f(t)$ is real and that the signal-to-noise ratio is sufficiently large, then we can achieve the bound as follows. Homodyne with a local oscillator of the form

$$L.O.(s, t) = 2\, RL\, A\, C(s)\, e^{-i\Omega t}. \tag{246}$$

The output of the photon counter is obtained in a manner similar to the derivation of Appendix A. After some normalization it is given by

$$g(t) = f(t-m) \int |C(s)|^2 \, d^2s + n(t)$$

$$E(n(t)n(u)) = (\hbar\Omega/2c)(\langle n\rangle/2+1/4) \int |C(s)|^2 \, d^2s \; \delta(t,u). \tag{247}$$

We now process classically by correlating g(t) against f(t-b) and picking the value of b that gives highest correlation as the estimate. At high signal-to-noise ratios, this estimate achieves (245), which is also the classical bound for (247).

5.2.2 Double-Sideband and Phase Modulation

For DSBSC or PM, use the local oscillator of (246). The performance is the same as for the plane-wave case, except that we must multiply D(v, v) of the plane-wave channel by $\int |C(s)|^2 \, d^2s$ for the spatial aperture channel. As before, for PM we require high signal-to-noise ratio for the phase-locked loop to be operating efficiently. The optimal receivers are those of the plane-wave case, except for the different local oscillator (see Figs. 5 and 6).

5.2.3 Estimation of the Angle of Arrival of a Plane Wave

Suppose that the classical aperture field envelope is

$$S(s,t,\theta) = (p)^{1/2} f(t) \, e^{-i(\Omega/c)x \sin\theta}. \tag{248}$$

The aperture has length L and M in the x and y directions, respectively. The system is shown in Fig. 8.

We wish to estimate the angle of arrival θ. We shall assume that the angle of arrival is known approximately a priori so that $\sin\theta \approx \theta$.

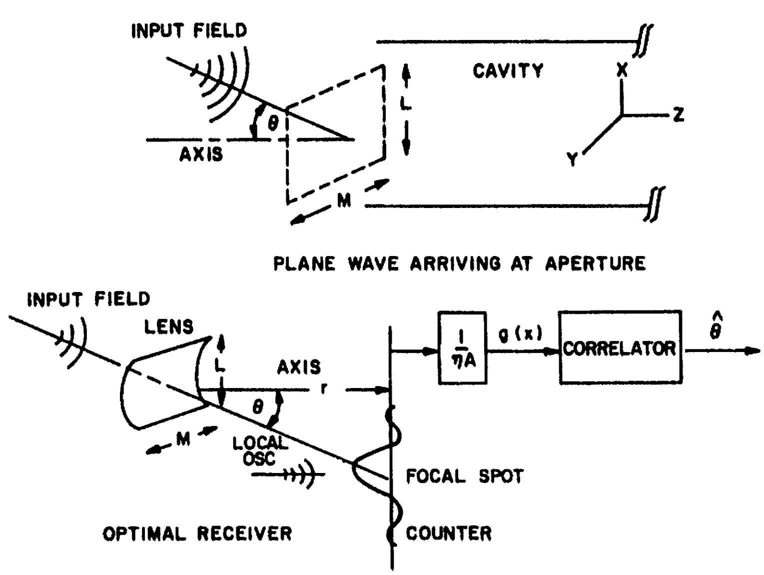


Fig. 8. Angle-of-arrival modulation.

We have the Cramér-Rao bound given by (240).

$$J_{11} = (4c/\hbar\Omega)(\langle n\rangle + 1/2)^{-1}\, p \int_0^T |f(t)|^2\, dt$$

$$\cdot \int_{-1/2L}^{1/2L} \int_{-1/2M}^{1/2M} (\Omega/c)^2\, x^2\, dxdy \tag{249}$$

$$= (4c/\hbar\Omega)(\langle n\rangle + 1/2)^{-1}\, p \int_0^T |f(t)|^2\, dt\, (L/\lambda)^2\, ML\pi^2/3,$$

where $\Omega/c = 2\pi/\lambda$.

The bound can be achieved by using a lens and a photon counter as follows. Place an ideal rectangular lens in the aperture. The focal plane field classically is given by

$$S(x', y', t) = (1/\lambda r) \int_{\text{aper}} S(s, t)\, e^{i(2\pi/\lambda r)(xx'+yy')}\, dxdy$$

$$= \frac{f(t)}{\lambda r} ML \frac{\sin(c_1 x'')\sin(x_2 y')}{(c_1 x'')(c_2 y')}, \tag{250}$$

where $c_1 = \pi L/(\lambda r)$; $c_2 = \pi M/(\lambda r)$; $x'' = x' - r\theta$, with r = focal length. Homodyne with the local oscillator

$$\text{L.O.}(y, t) = 2\, RL\, A\, f(t) \frac{\sin(c_2 y')\, e^{-i\Omega t}}{(c_2 y')}, \tag{251}$$

where we integrate the photon counter output over time and the y coordinate, and look at this count as a function of x. We find in a manner similar to the derivation of Appendix A that the counter output as a function of x is

$$g(x') = (p)^{1/2} \int_0^T |f(t)|^2\, dt \int_{-\infty}^{\infty} \frac{\sin^2(c_2 y')}{(c_2 y')^2}\, dy'\, \frac{ML}{\lambda r} \frac{\sin(c_1 x'')}{(c_1 x'')} + n(x'), \tag{252}$$

where

$$E(n(x_1')n(x_2')) = \int_0^T |f(t)|^2\, dt \int_{-\infty}^{\infty} \frac{\sin^2(c_2 y')}{(c_2 y')^2}\, dy'\, \hbar\Omega/(8c)\, \delta(x_1', x_2').$$

Here we have neglected thermal noise. Thus we now have a spatial position-modulation process in spatial white noise. The optimal receiver correlates the waveform $g(x')$ against the function

$$\frac{\sin(c_1 x' - z)}{c_1(x' - z)},$$

and estimates $r\theta$ as the value of z with the largest correlation. We know that in the case of high signal-to-noise, the performance of this estimator is given by

$$\mathrm{var}\,(\hat{\theta}-\theta) = (\hbar\Omega/8c)\left[p\int |f(t)|^2\,dt \int_{-\infty}^{\infty} \frac{\sin^2(c_2 y')}{(c_2 y')^2}\,dy'\right]^{-1}$$

$$\cdot\left[\int_{-\infty}^{\infty} [d/dx\,(\sin(c_1 x)/(c_1 x))]^2\,dx\,(ML/\lambda)^2\right]^{-1} \tag{253}$$

(that is, the classical Cramér-Rao bound). Evaluating the integrals, we obtain

$$\mathrm{var}\,(\hat{\theta}-\theta) = (\hbar\Omega/8c)\left[p\int_0^T |f(t)|^2\,dt\,\frac{LM\pi^2}{3}\,(L/\lambda)^2\right]^{-1}. \tag{254}$$

Comparing (254) with (249), we see that when thermal noise is negligible, and when the baseband signal-to-noise ratio is high, the receiver of Fig. 8 achieves the Quantum Cramér-Rao bound. As we shall see, it is reasonable to neglect thermal noise at optical frequencies.

VI. FADING CHANNELS

In the previous sections, the classical field, given the message, was assumed known, except for an additive Gaussian noise process. It is often the case that the classical field, given the message, is a random process, whose parameters depend upon the message. In the multiplicative fading channel, when we do not assume that we know the channel (that is, we do not assume knowledge of C(s, t) of Eq. 229), we have such a circumstance. Here bounds on the estimation of the parameters of a field with the parameters imbedded in the covariance of the envelope of the field (which will be assumed a zero-mean Gaussian random process) will be derived. We shall assume that at a fixed time, the E-field in a region of space to be called the measurement region is

$$E(r,u) = 2\ \mathrm{RL}\ S(r,u)\, e^{-i\Omega u}. \tag{255}$$

We shall assume that S(r, u) is a complex Gaussian random process satisfying

$$E[S(r,u)] = 0; \qquad E[S(r,u)S(r',u)] = 0$$

$$E[S(r,u)S^*(r',u)] = K_s(r,r',u). \tag{256}$$

We can expand the classical field at fixed time u in the measurement region in terms of orthogonal spatial functions

$$S(r,u) = \Sigma\, s_i\psi_i(r), \tag{257}$$

where

$$(1/V)\int_{\mathrm{M.R.}} K_s(r,r',u)\,\psi_i(r')\,dr' = k_i\psi_i(r),$$

where M.R. indicates measurement region. By using (256) and (257), it is straightforward to show that

$$E(s_i) = 0$$

$$E(s_is_j) = 0$$

$$E\left[s_is_j^*\right] = k_i\delta(i,j) \tag{258}$$

and the real and imaginary parts of s_i are uncorrelated.

Furthermore, since the process is assumed Gaussian, all of these uncorrelated coefficients are also independent. The reader should note that not as in previous sections, any additive noise has been included in $S(r, u)$.

6.1 CRAMÉR-RAO BOUNDS

6.1.1 Single-Parameter Estimation

Given Eqs. 255-258 and assuming that we expand the quantum field operator as

$$E(r,t) = \Sigma \sqrt{\hbar\Omega/2V}\left[b_k\psi_k(r,t)\,e^{-i\Omega t} + b_k^{+}\psi_k^{*}(r,t)\,e^{i\Omega t}\right], \tag{259}$$

we have the density operator given by

$$\rho^{\underline{m}} = \prod_k \int (1/\pi\langle n_k\rangle)\, e^{-|\alpha_k|^2/\langle n_k\rangle}\, \alpha_k\rangle\langle\alpha_k\, d^2\alpha_k, \tag{260}$$

where

$$E[S(r,u,\underline{m})S^{*}(r',u,\underline{m})] = K_s(r,r',u,\underline{m})$$

$$(1/V)\int K_s(r,r',u,\underline{m})\,\psi_i(r',u,\underline{m})\,dr' = k_i(\underline{m})\,\psi_i(r,u,\underline{m})$$

$$\langle n_j\rangle = (2V/\hbar\Omega)\,k_j(\underline{m}). \tag{261}$$

The reader is cautioned that the eigenmodes, eigenvalues and thus the operators b_k and b_k^{+} depend upon the message $\underline{m}$. We shall drop the time dependence u in the eigenmodes, since we are concerned with a fixed time measurement. We shall set $\underline{m} = m$ signifying the single-parameter case.

For the density operator of (260), Helstrom[1,15] has shown that the quantity J_{11} is given by

$$J_{11} = \Sigma\,\Sigma\,[1/2(n_k+n_j)+n_k n_j]^{-1}\, n'_{kj} n'_{jk}. \tag{262}$$

(Note that we drop the brackets around the n_k for convenience), where

$$(1/V)^2\int \psi_k^{*}(r,m)[d/dm\,K_s(r,r',m)]\,\psi_j(r',m)\,dr\,dr' = k'_{kj} = (\hbar\Omega/2V)n'_{kj}. \tag{263}$$

Define the kernel

$$T(r_1,r_2,r_3,r_4) = (1/V)^2\,\Sigma\,\Sigma\,\psi_k(r_1)\psi_k^{*}(r_2)\psi_m(r_3)\psi_m^{*}(r_4)[1/2(n_k+n_m)+n_k n_m]^{-1}. \tag{264}$$

Using the fact that

$$K_s(r, r', m) = \Sigma\, k_j \psi_j(r, m)\, \psi_j^*(r, m)$$

$$(1/V) \int \psi_k(r, m)\, \psi_j^*(r, m)\, dr = \delta(k, j), \tag{265}$$

we obtain

$$J_{11} = (2/\hbar\Omega)^2 \iint T(r_1, r_2, r_3, r_4)[d/dm\, K_s(r_2, r_3, m)][d/dm\, K_s(r_4, r_1, m)]\, dr_1 dr_2 dr_3 dr_4. \tag{266}$$

If we define

$$\iint T(r_1, r_2, r_3, r_4)\, T^{-1}(r_2, r_5, r_4, r_6)\, dr_2 dr_4 = \delta(r_1, r_5)\, \delta(r_3, r_6), \tag{267}$$

we must then have

$$\begin{aligned} T^{-1}(r_1, r_2, r_3, r_4) = (2/\hbar\Omega)^2 &[K_s(r_1, r_2) K_s(r_3, r_4)] \\ &+ (2/\hbar\Omega)[K_s(r_1, r_2)\delta(r_3, r_4) \\ &+ \delta(r_1, r_2) K_s(r_3, r_4)]. \end{aligned} \tag{268}$$

When the field in the measurement region is due to a field propagating through turbulence and impinging upon an aperture we have

$$S(r, m) = \int h(r, s, t)\, S_R(s, t)\, C(s, t)\, dsdt + \int h(r, s, t)\, n(s, t)\, dsdt \;\; \text{(noise term)}. \tag{269}$$

This situation is described in section 5.2. We shall assume that the aperture field satisfies the conditions for (234) to hold. (We have suppressed the measurement time u.) We shall assume that $S_R(s, t, m)$ is known, given m, and that $C(s, t)$ is a Gaussian random process with

$$E[C(s, t)C(s', t')] = 0$$

$$E[C(s, t)C^*(s', t')] = K_c(s, s', t, t'). \tag{270}$$

It follows that

$$K_s(r, r') = K_n(r, r') + \iint h(r, s, t)\, h^*(r', s', t')\, S_R(s, t)\, S_R^*(s', t')\, K_c(s, s', t, t')\, dsds'dtdt', \tag{271}$$

where

$$K_n(r, r') = \int h(r, s, t)\, h^*(r', s', t') \cdot K_n(s, s', t, t')\, ds ds' dt dt'$$

and

$$E[n(s, t) n^*(s', t')] = K_n(\;)$$

$$\int_{aper} \left[S_R(s, t) K_c(s, s', t, t') S_R^*(s', t') + K_n(s, s', t, t') \right] \gamma_k(s', t')\, ds' dt' = l_k \gamma_k(s, t)$$

$$k_j = l_j\, c/V$$

$$\psi_k(r, \underline{m}) = \int h(r, s, t)\, \gamma_k(s, t)\, ds dt\, (V/c)^{1/2}. \tag{272}$$

Define the kernel

$$T(s_1, t_1, \ldots, s_4, t_4) = \Sigma\, \Sigma\; \gamma_k(s_1, t_1)\, \gamma_k^*(s_2, t_2)\, \gamma_j(s_3, t_3)\, \gamma_j^*(s_4, t_4) [1/2(n_k + n_j) + n_k n_j]^{-1}. \tag{273}$$

It can be shown that

$$J_{11} = (2c/\hbar\Omega)^2 \int T(s_1, t_1, \ldots, s_4, t_4)$$

$$\cdot\; d/dm\, K_g(s_2, t_2, s_3, t_3)$$

$$\cdot\; d/dm\, K_g(s_4, t_4, s_1, t_1)\, ds_1, \ldots, dt_4, \tag{274}$$

where

$$K_g(s_1, t_1, s_2, t_2) = K_n(s_1, t_1, s_2, t_2) + S_R(s_1, t_1, m)\, K_c(s_1, s_2, t_1, t_2)\, S_R^*(s_2, t_2, m)$$

and

$$T^{-1}(s_1, t_1, \ldots, s_4, t_4) = (2c/\hbar\Omega)^2 [K_g(s_1, t_1, s_2, t_2) K_g(s_3, t_3, s_4, t_4)]$$

$$+ (2c/\hbar\Omega)[K_g(s_1, t_1, s_2, t_2)\delta(s_3, s_4)\delta(t_3, t_4)$$

$$+ \delta(s_1, s_2)\delta(t_1, t_2) K_g(s_3, t_3, s_4, t_4)].$$

The reader should not be discouraged by the notation. After we discuss waveform estimation, we shall apply this bound.

6.1.2 Waveform Estimation

As usual, we shall expand a time-limited sample function of a Gaussian random process, which is the message, as

$$m(t) = \Sigma\, m_i \phi_i(t); \qquad t \in (0, T), \tag{275}$$

where the m_i are independent Gaussian random variables.

For the density operator given in (260), by analogy to (262), we have

$$J_{lm} = \Sigma\, \Sigma\, [1/2(n_j+n_k)+n_j n_k]^{-1}\, n_{jk}^{l} n_{jk}^{*m}, \tag{276}$$

where

$$n_{jk}^{l} = (2V/\hbar\Omega)(1/V)^2 \int \psi_j^*(r_1)\, d/dm_l\, K_s(r_1, r_2, \underline{m})\, \psi_k(r_2)\, dr_1 dr_2. \tag{277}$$

Now assume that the measurement field arises from an aperture field as follows. A modulator produces the waveform $F[t, m(t)]$. The received aperture field envelope is

$$R(s, t, \underline{m}) = F[t, m(t)]\, C(s, t) + n(s, t)$$

$$E[R(s, t, \underline{m}) R^*(s', t', \underline{m})] = K_g(s, t, s', t', \underline{m}). \tag{278}$$

Exactly as in the single-parameter case, we obtain

$$J_{ij} = (2c/\hbar\Omega)^2 \int T(s_1, t_1, \ldots, s_4, t_4)\, K_g^i(s_2, t_2, s_3, t_3)\, K_g^j(s_4, t_4, s_1, t_1)\, ds_1, \ldots, dt_4, \tag{279}$$

where $T(\ )$ is defined by (272-274), and K_g^i denotes differentiation with respect to m_i.

Define

$$Q_{km}(t, s, z, s') = \gamma_k^*(s, t)[d/dm(t)\, K_g(s, t, s', z)]\, \gamma_m(s', z)$$

$$+ \gamma_m(s, t)[d/dm(t)\, K_g(s', z, s, t)]\, \gamma_k^*(s', z). \tag{280}$$

Define

$$J(t, u) = E\, \Sigma\, \Sigma \int [1/2(n_k+n_j)+n_k n_j]^{-1}\, Q_{kj}(t, s, z, s')\, Q_{jk}(u, r, v, r')\, ds ds' dr dr' dz dv. \tag{281}$$

It follows from (279) that

$$E\, J_{ij} = (2c/\hbar\Omega)^2 \int \phi_i(t)\, \phi_j(u)\, J(t,u)\, dtdu. \tag{282}$$

Following the derivation of section 4. 1. 2, we obtain

$$K_m(t,u) = H^{-1}(t,u) + (2c/\hbar\Omega)^2 \int K_m(t,z)\, J(z,v)\, H^{-1}(v,u)\, dzdv. \tag{283}$$

Recall that

$$K_c(s_1,t_1,s_2,t_2) = E[C(s_1,t_1)C^*(s_2,t_2)]. \tag{284}$$

Define

$$D_1(t,u) = d/dm(t)\, F[t, m(t)]\; d/dm(u)\, F[u, m(u)]$$

$$D_2(t,u) = d/dm(t)\, F[t, m(t)]\; d/m(u)\, F^*[u, m(u)]. \tag{285}$$

We obtain

$$\begin{aligned} J(t_2,t_4) = 2\, \mathrm{RL}\, E \int & T(s_1,t_1,s_2,t_2,s_3,t_3,s_4,t_4) \\ & \cdot D_1(t_2,t_4)\, F^*[t_3, m(t_3)]\, F^*[t_1, m(t_1)]\, K_c(s_2,t_2,s_3,t_3) \\ & \cdot K_c(s_4,t_4,s_1,t_1) + T(s_4,t_4,s_2,t_2,s_3,t_3,s_1,t_1) \\ & \cdot D_2(t_2,t_4)\, F[t_1, m(t_1)]\, F^*[t_3, m(t_3)]\, K_c(s_2,t_2,s_3,t_3) \\ & \cdot K_c(s_1,t_1,s_4,t_4)\, ds_1 ds_2 ds_3 ds_4 dt_1 dt_3. \end{aligned} \tag{286}$$

In spite of the way (286) looks, we shall use it to study intensity modulation, phase modulation in Section VII, and at the end of this section.

6. 2 APPLICATIONS

6. 2. 1 Estimation of the Level of a Gaussian Random Process

Suppose the aperture field containing a parameter to be estimated is given by

$$R(s,t) = m^{1/2} C(s,t) + n(s,t), \tag{287}$$

where m is to be estimated.

$$K_g(s_1,s_2,t_1,t_2) = m\, K_c(s_1,s_2,t_1,t_2) + K_n(s_1,s_2,t_1,t_2).$$

Expand $K_c(\ )$ in its Karhunen-Loève expansion

$$K_c(s_1, s_2, t_1, t_2) = \Sigma\, c_k \gamma_k(s_1, t_1)\, \gamma_k^*(s_2, t_2). \tag{288}$$

Assume that the noise is spatial-temporally white so that we can expand $K_n(\ )$ as

$$K_n(\ ) = \Sigma\, N\gamma_k(s_1, t_1)\, \gamma_k^*(s_2, t_2). \tag{289}$$

We then have

$$K_g(\ ) = \Sigma\, (mc_k+N)\, \gamma_k(s_1, t_1)\, \gamma_k^*(s_2, t_2).$$

Using (274), we obtain

$$\begin{aligned} J_{11} &= (2c/\hbar\Omega)^2\, \Sigma\, (c_k)^2/[(2c/\hbar\Omega)(N+mc_k)[1 + (2c/\hbar\Omega)(N+mc_k)]] \\ &= \Sigma\, (c_k)^2/[(N+mc_k)(\hbar\Omega/2c + N + mc_k)]. \end{aligned} \tag{290}$$

Consider the case wherein all of the eigenvalues c_k equal b for $k = 1, 2, \ldots, K$ and are zero otherwise.

$$J_{11} = Kb^2/[(N+mb)(\hbar\Omega/2c + N + mb)]. \tag{291}$$

We can achieve the performance of the bound by counting photons in the aperture plane, in the modes that correspond to the $\gamma_k(s_1, t_1)$. That is, we measure

$$X = \sum_1^K b_k^+ b_k \tag{292}$$

in the spatial modes corresponding to the $\gamma_k(s_1, t_1)$ by letting the field propagate into a counter-filter system. (The problem of mode separation can be simple or complicated, depending upon what the modes look like.) After processing, the optimal estimator is

$$\hat{M} = \left[\sum_1^K b_k^+ b_k (\hbar\Omega/2cK) - N \right] \Big/ b. \tag{293}$$

To show that this operator achieves the bound, we must recall that the density operator of the signal modes is

$$\rho^m = \prod_{k=1}^K \int (1/\pi x)\, e^{-|\alpha_k|^2/x}\, \alpha_k\rangle\langle\alpha_k\, d^2\alpha_k, \tag{294}$$

where

$$x = (2c/\hbar\Omega)(N+mb).$$

Therefore

$$\mathrm{TR}\ \rho^{m}\hat{M} = m$$

$$\mathrm{TR}\ \rho^{m}(\hat{M}-mI)^{2} = J_{11}^{-1}. \tag{295}$$

Examples of situations in which the photon counting could be implemented are as follows. We could have a case in which thermal noise is negligible. Then we could focus the entire received field on the aperture onto a photon counter, and count in the region of the focal plane where we expect message signal, and during times when we expect message. (These restrictions are required, since the noise is not identically zero.) As we shall see in Section VII, thermal noise is in fact often negligible. In another case, we may have only temporal fading. We transmit K pulses that are narrow compared with the channel coherence time and widely spaced compared with the coherence time. We focus the field onto a photon counter, count in the central focal spot, and only during times when pulses are present. Each pulse corresponds to a mode.

6.2.2 Optimal Diversity

Suppose the function $C(s,t)$ of (287) is at the control of the communicator. Such a situation could occur if, for example, $C(s,t) = f(t)\, C_1(s,t)$. That is, it is the product of a message function at the control of the communicator, and an uncontrollable channel process. For different $f(t)$, the magnitude of b and the number of diversity paths K will vary. Suppose we constrain the product bK to be fixed. That is, we receive a fixed amount of energy. For a given message m, we can minimize the error variance by maximizing J_{11} on K, keeping $Kb = P$ fixed. The optimal diversity is given by

$$\hat{K}_{\mathrm{opt}} = mP/[N(N+\hbar\Omega/2c)]^{1/2}. \tag{296}$$

The performance at optimal diversity is

$$\mathrm{Var}\,(\hat{M}-m) = \frac{[N(N+\hbar\Omega/2c)]^{1/2}}{mP}\left[m^{2}\left(2+\frac{2N+\hbar\Omega/2c}{[N(N+\hbar\Omega/2c)]^{1/2}}\right)\right]$$

$$\begin{aligned}\mathrm{Var}\,(\hat{M}-m) &= 4Nm/P \qquad \text{for} \quad N \gg \hbar\Omega/2c \\ &= \hbar\Omega m/2cP \qquad \text{for} \quad N \ll \hbar\Omega/2c.\end{aligned} \tag{297}$$

In Section VII we shall discuss under what circumstances we can achieve near optimal diversity over a wide range of m.

6.2.3 Radar Ranging of a Point Target

Suppose the aperture envelope is given by

$$R(s,t) = f(t-a)\, C(s) + n(s,t)$$

$$\int f(t-a)\, f^*(t-a)\, dt = 1. \tag{298}$$

That is, the coherence time of the channel is much larger than the transmitted pulse f(t-a). We wish to estimate a. Such a situation might occur in a radar problem when we wish to range a slowly fluctuating target,[11] or in a PPM problem on a slowly fluctuating channel. We can expand the covariance of the envelope as follows:

$$K_g(s_1,t_1,s_2,t_2) = f(t_1-a)\, \Sigma\, b_k \gamma_k(s_1)\, \gamma_k^*(s_2)\, f^*(t_2-a)$$

$$+ N\, \Sigma\, \psi_j(s_1,t_1)\, \psi_j^*(s_2,t_2), \tag{299}$$

where

$$K_c(s_1,s_2) = \Sigma\, c_k \gamma_k(s_1)\, \gamma_k^*(s_2)$$

$$K_n(s_1,s_2,t_1,t_2) = f(t_1-a)\, \Sigma\, N\gamma_k(s_1)\, \gamma_k^*(s_2)\, f^*(t_2-a)$$

$$+ N\, \Sigma\, \psi_j(s_1,t_1)\, \psi_j^*(s_2,t_2)$$

and $b_k = c_k + N$. That is, the functions $f(t-a)\,\gamma_k(s)$ and $\psi_j(t,s)$ form a complete set. It is a matter of algebraic manipulation to plug (299) into (274), keeping T( ) in its eigenfunction expansion of (264). We obtain for <u>real</u> f(t)

$$J_{11} = 2 \sum_k \frac{c_k^2 \int [f'(t)]^2\, dt}{N(N+c_k) + (\hbar\Omega/2c)(N+c_k/2)}. \tag{300}$$

This is the same as the classical result[11] for the case $\hbar\Omega/2c \ll N$. We shall discuss receiver structures that perform close to the bound in Section VII.

6.2.4 Coherently Unestimable Parameter Case

There is a situation of significance classically when we have a white noise contribution to each mode which is much stronger than the signal contribution. Suppose we have

$$T(r_1, r_2, r_3, r_4) = (1/V)^2 \Sigma \Sigma \psi_k(r_1) \psi_k^*(r_2) \psi_j(r_3) \psi_j^*(r_4)[1/2(n_j+n_k)+ n_j n_k]^{-1}, \quad (301)$$

where

$$\left.\begin{array}{l} n_k = n + c_k \\ n \gg c_k. \end{array}\right. \quad \text{for all } k$$

Then we can clearly write

$$T(r_1, \ldots, r_4) = (n^2+n)^{-1} \delta(r_1, r_2) \delta(r_3, r_4). \quad (302)$$

It follows that

$$J_{11} = (2/\hbar\Omega)^2 (n^2+n)^{-1} \int |d/dm\, K_s(r_1, r_2)|^2 dr_1 dr_2. \quad (303)$$

Similarly for the aperture case

$$J_{11} = (2c/\hbar\Omega)^2 (n^2+n)^{-1} \int |d/dm\, K_g(s_1, t_1, s_2, t_2)|^2 ds_1 ds_2 dt_1 dt_2. \quad (304)$$

Classically, for $n \gg 1$, this is called the <u>coherently</u> <u>unestimable</u> <u>parameter</u> or C. U. P. case. The application to optical fields is probably not great. This is because at optical frequencies, the noise photon number per mode is much less than 1 for most cases. If the signal c_k is to be much less than the noise photon number, then we need a very large number of modes for reasonable performance.

6.2.5 Angle-of-Arrival Estimation with No Thermal Noise

Consider the case when the aperture envelope is given by

$$R(s, t) = e^{-(2\pi/\lambda)i\theta x} C(t) e^{i\phi}$$

$$x \in (-1/2L, 1/2L); \quad y \in (-1/2M, 1/2M) \quad t \in (0, T). \quad (305)$$

That is, we are trying to estimate the angle of arrival of a plane wave with no spatial fading, and no thermal noise. The situation is similar to that shown in Fig. 8. Examining (274), it is easy to show that

$$J_{11} = (2c/\hbar\Omega)^2 \int T(s_1, t_1, \ldots, s_4, t_4)$$
$$\cdot\, e^{-(2\pi/\lambda)i\theta(x_2-x_3)} K_c(t_2, t_3)\, e^{-(2\pi/\lambda)i\theta(x_4-x_1)}$$
$$\cdot\, K_c(t_4, t_1)(2\pi/\lambda)^2 (x_2-x_3)(x_1-x_4)\, ds_1, \ldots, dt_4. \quad (306)$$

where

$$K_g(s_1, s_2, t_1, t_2) = \frac{e^{-(2\pi/\lambda)i\theta(x_1-x_2)} \Sigma c_k \gamma_k(t_1)\, \gamma_k^*(t_2)}{ML} + 0 \sum \sum \psi_j(s_1, t_1)\, \psi_j^*(s_2, t_2)$$

and

$$K_c(t_1, t_2) = \Sigma\, c_k \gamma_k(t_1)\, \gamma_k^*(t_2)/ML.$$

Expanding T( ) in the modes $(1/ML)^{1/2}\, \gamma_k(t)$ and $\psi_j(s, t)$, we get

$$J_{11} = \Sigma\, c_k (2c/\hbar\Omega)(2\pi L/\lambda)^2/3$$

$$= (8c/\hbar\Omega) ML \int_0^T K_c(t, t)\, dt\, (\pi L/\lambda)^2/3. \tag{307}$$

Comparison of (307) and (249) for the case of negligible thermal noise, shows that except for the replacement of $p \int f(t)\, f^*(t)\, dt$ with $\int K_c(t, t)\, dt$ the two bounds are the same. The optimal receiver converts to a spatial PPM problem by sending the received field through a rectangular lens. In the Section VII, we shall discuss PPM receivers for the temporal case. Extension to this spatial problem is straightforward.

6.2.6 Angle Modulation

As a final example let us evaluate the waveform bound for pulse modulation. Assume that the aperture field is given by

$$R(s, t) = e^{i\beta m(t)}\, C(s) + n(s, t); \qquad t \in (0, T). \tag{308}$$

The noise is spatial-temporal white, and we are assuming a slowly fading channel. We have

$$K_g(s_1, t_1, s_2, t_2) = \Sigma\, \Sigma\, g_{kj} \gamma_k(s_1)\, \sigma_j(t_1)\, \gamma_k^*(s_2)\, \sigma_j^*(t_2), \tag{309}$$

where

$$g_{k1} = N + c_k T; \qquad g_{kj} = N \qquad j \neq 1$$

$$\sigma_1(t) = (1/T)^{1/2}\, e^{i\beta[m(t)]}$$

$$K_c(s_1, s_2) = E[C(s_1) C^*(s_2)] = \Sigma\, c_k \gamma_k(s_1)\, \gamma_k^*(s_2).$$

As defined in (286), we have

$$D_1(t_2, t_4) = -\beta^2 e^{i\beta[m(t_2)+m(t_4)]}$$

$$D_2(t_2, t_4) = \beta^2 e^{i\beta[m(t_2)-m(t_4)]}$$

$$F[t, m(t)] = e^{i\beta[m(t)]}. \tag{310}$$

$$T(s_1, t_1, \ldots, s_4, t_4) = \Sigma\,\Sigma\,\Sigma\,\Sigma\, \gamma_k(s_1)\, \sigma_j(t_1)\, \gamma_k^*(s_2)\, \sigma_j^*(t_2)$$

$$\cdot\, \gamma_l(s_3)\, \sigma_m(t_3)\, \gamma_l^*(s_4)\, \sigma_m^*(t_4)$$

$$\cdot\, [1/2(n_{kj}+n_{lm}) + n_{kj}n_{lm}]^{-1}. \tag{311}$$

From (286), we obtain

$$J(t_2, t_4) = 2\text{ RL E}\Big\{-\beta^2 \sum_k (c_{kl})^2/[n_{kl}(n_{kl}+1)]$$

$$+ e^{i\beta[m(t_2)-m(t_4)]}\, \beta^2 T \sum_k \sum_m (c_{kl})^2$$

$$[1/2(n_{kl}+n_{km}) + n_{kl}n_{km}]^{-1} \cdot \sigma_m^*(t_2)\, \sigma_m(t_4)\Big\}, \tag{312}$$

where $n_{km} = (2c/\hbar\Omega)g_{km}$.

If we assume

$$c_k = \begin{cases} P/K; & k = 1, 2, \ldots, K \\ 0; & \text{otherwise} \end{cases}$$

we get

$$J(t_2, t_4) = \frac{2\text{ RL}(\hbar\Omega/2c)^2\, \beta^2 P^2 T/K\ \delta(t_2, t_4)}{N(N+PT/K) + (\hbar\Omega/2c)(N+PT/2K)} + \text{constant}. \tag{313}$$

Assuming no message energy at DC, we obtain

$$K_m(t, u) = H^{-1}(t, u) + \frac{(2\beta^2 P^2/K)T \int K_m(t, v)\, H^{-1}(v, u)\, dv}{N(N+PT/K) + (\hbar\Omega/4c)(2N+PT/K)}. \tag{314}$$

Notice that (314) approaches the known channel bound when P/K is much larger than N/T, provided we interpret the average power P as the known fixed power of the non-fading case. Implementation of the bound with a physical receiver will be discussed in Section VII.

VII. PRACTICAL EXAMPLES

I shall now apply previous results to a number of popular modulation schemes. Sometimes, I shall assume that thermal noise is negligible. I shall discuss discrete-time and continuous-time systems. I shall compare the performance of those systems with their classical counterparts, and compare the various quantum systems to each other. I shall comment upon the reasonableness of assumptions regarding signal strength and numbers of diversity paths.

Occasionally, formulas and conclusions of previous sections will be repeated. This redundancy makes reading easier than it would be with constant references to previous equations and comments. I shall not attempt to deceive the reader into believing that the examples presented here include all cases of practical interest. In fact, these examples are of systems that are performing near the lower bounds derived previously. This often means that they are performing fairly well. Although these examples are of considerable interest, many times one might be interested in systems that do not perform well because of low signal strength, bad fading, and so forth.

I hope that the results given here will provide the reader with insight into what types of modulation systems and what types of processing would be reasonable in such situations. Of course, given a demodulation system that is good at high signal levels, we could calculate its performance at low signal levels. There is no guarantee that some scheme that could never achieve Cramér-Rao performance at high levels could not be better than the efficient system (at high levels) when the signal strength is low. For example, I shall present a phase-locked loop demodulation scheme that is efficient at high signal-to-noise levels for PM and FM. At other signal-to-noise ratios, other demodulation schemes such as the prism-lens discriminator (also to be discussed) might be better. Thus these sections, although of interest, are far from the last word on analog demodulation.

The material will be presented as follows. First a discussion of noise and turbulence will be given. Then we shall discuss the various modulation schemes under known-channel and fading-channel conditions. After that, systems will be compared as to performance. The results of Jane W. S. Liu on PCM systems will be included in this comparison.

7.1 QUANTITATIVE RESULTS

7.1.1 Data on Noise

Our first consideration will be the specification of the background noise N_o for the white-noise case. From our formulation, N_o is proportional to the mean number of noise photons per mode that are due to background radiation. If we assume that the background radiation is black-body radiation at absolute temperature T, then N_o is given by

$$N_o = \hbar\Omega/2c(e^{\hbar\Omega/kT} - 1)^{-1}, \tag{315}$$

where

$k = 1.38 \times 10^{-23}$ J/deg

$\hbar = 1.05 \times 10^{-34}$ J-s

$c = 3 \times 10^{8}$ m/s

$\Omega = 1 - 10 \times 10^{15}$ rad/s for typical lasers.

Then for T varying between 1-300 and for Ω between 10^{15}-10^{16}, $\hbar\Omega/kT$ varies approximately between 10 and 10^5. Thus for the lowest frequency a black-body temperature of 10^4 °K is required for kT to equal $\hbar\Omega$.

At this point, I would like to make some comments upon units. Assuming that we expand the classical and quantum fields in the same eigenmodes, we have

$$E(r,t)_{\text{class.}} = 2RL \sum B_k \psi_k(r,t)\, e^{-i\Omega t} \tag{316}$$

$$r \in V \qquad t \text{ fixed} \qquad \langle \psi_k, \psi_j \rangle_V = V\delta(k,j)$$

$$E(r,t)_{\text{quant.}} = \sum \sqrt{\hbar\Omega/2V}\, b_k\psi_k(r,t)\, e^{-i\Omega t} + b_k^+\psi_k^*(r,t)\, e^{i\Omega t}.$$

Assume for the moment that the density operator is that of a pure state. That is, the mean number of photons and the density operator are given by

$$\text{TR}\, \rho b_k^+ b_k = \beta_k \beta_k^*$$

$$\rho = \beta_1, \beta_2, \ldots \beta_k \ldots \rangle \langle \ldots \beta_k \ldots \beta_2, \beta_1. \tag{317}$$

For a classical plane wave, the energy contained in a mode is

$$W_k = 2B_k B_k^* V\epsilon_o, \tag{318}$$

where ϵ_o is the permittivity of free space, 8.854×10^{-12} Fd/m.

Therefore, the classical number of photons in the field is given by

$$n_k = 2B_k B_k^* V\epsilon_o/\hbar\Omega. \tag{319}$$

We have the correspondence

$$\text{TR}E(r,t)\,\rho = E(r,t)_{\text{class.}} \tag{320}$$

That is,

$$(\hbar\Omega/2V)^{1/2}\, 2RL\beta_k\psi_k(r,t)\, e^{-i\Omega t} = 2RLB_k\psi_k(r,t)\, e^{-i\Omega t}.$$

Therefore

$$(\hbar\Omega/2V)^{1/2}\, \beta_k = B_k.$$

Thus (318)-(320) imply that we are working in a system of units where $\epsilon_o = 1$. Therefore the numerical value of the E-field in Volts/meter is 3.35×10^5 times the numerical value of the field amplitude in the units used here. (Remember that we are measuring energy in Joules.)

Another comment that I wish to make concerns the appearance of the factor c (speed of light) in equations like (315). If we performed all of our integrations in space, this factor would not appear. Often, however, we transform spatial problems into spatial-temporal problems. We know that

$$\int_a^b f(z)\, dz = c \int_{a/c}^{b/c} f(ct)\, dt. \tag{321}$$

Thus for plane-wave and aperture problems, the factor c keeps appearing in equations. For example, for a Gaussian noise field with mean photon number per mode $\bar{n}$ the spatial correlation function is $R_n(r, r') = \Sigma\, \hbar\Omega\bar{n}/2V\psi_k(r)\psi_k^*(r')$. When we go to the plane-wave case, the spatial correlation is $\bar{n}\hbar\Omega/2\delta(z-z')$. If we then transfer to time integrations, we get

$$R_n(t, t') = \frac{\hbar\Omega\bar{n}}{2}\, \delta(ct-ct') = \bar{n}\hbar\Omega/2c\delta(t-t') = N_o\delta(t-t'). \tag{322}$$

This is the logic leading to (315). When we solve a spatial problem, the factor c in (315) is replaced by unity.

7.1.2 Data on Turbulence

There have been articles describing fading optical signals in turbulence — both from the theoretical[16] and experimental[17] points of view.

Theoretical studies indicate that the fading process should be log-normal. That is, the multiplicative fading process is of the form $e^{s(t)}$, where s(t) is a complex Gaussian random process. The results that follow will be limited, for the most part, to Gaussian fading except when extensions to other types of fading are discussed in section 7.7. We shall now discuss some experimental data of interest for practical examples.

If we send an unmodulated carrier through the turbulent atmosphere, the spatial correlation function is given by

$$|\langle E(r+s, t), E^*(r, t)\rangle| = \exp 1.45\, C_n^2 L s^{5/3} (2\pi/\lambda)^2, \tag{323a}$$

where s is a displacement in the plane perpendicular to the direction of propagation, and L is the path length between transmitter and observation plane. C_n is called the structure constant. A representative value is[17]

$$C_n = 2.3 \times 10^{-8}/\mathrm{cm}^{1/3}. \tag{323b}$$

Another measure of turbulence is the effective diameter. If a plane wave is transmitted through free space and the received signal is heterodyned with a plane-wave local oscillator, the signal-to-noise ratio is proportional to the area of the receiving aperture. If, on the other hand, there is turbulence, the signal-to-noise ratio will increase asymptotically to some maximum value as the area is increased. The diameter at which the signal-to-noise ratio is within 3 dB of its maximum is called the effective diameter. It is given by

$$D_{eff} = \left(.058\, \lambda^2/C_n^2 L\right)^{3/5}. \tag{324}$$

D_{eff} is typically between 0.5 cm and 13 cm for path lengths between 4-24 km, depending on atmospheric conditions (see Goldstein, Miles, and Chabot[17]).

7.2 PULSE AMPLITUDE MODULATION SYSTEMS

7.2.1 PAM with No Fading and Known Phase

Assume that by some means (perhaps an auxiliary locked-loop system) we know the arrival phase of the received signal which is given classically by

$$E(r, \rho) = 2RL\, A(E/s)^{1/2}\, f(t)\, e^{i\Omega t} + n(t, \rho)\, e^{i\Omega t}$$

$$\int_0^T f(t)\, f^*(t)\, dt = 1; \qquad \int_{aper} 1\, d^2\rho = s$$

$$t \in (0, T) \qquad \rho \in (\text{aperture})$$

$$E\left[n(t, \rho) n^*(t', \rho')\right] = N_o \delta(t, t')\, \delta(\rho, \rho'). \tag{325}$$

From the results on amplitude modulation, we know that an unbiased estimate of A is obtained by homodyning the received field to baseband and correlating the result against f(t), provided f(t) is real. If f(t) is not real, our homodyning must be done with an oscillator matched to f(t) in time. The variance of the error is given by

$$\mathrm{Var}\,(\hat{A}-A) = \frac{N_o/2 + 1/4\hbar\Omega/2c}{E}. \tag{326}$$

Example 1. Suppose we assume a black-body radiation of 300°K. From (315) we see that we can neglect thermal noise at typical laser frequencies in (326). Suppose we wish our error variance to be 10^{-2}. (This might be reasonable if our a priori variance of A is 1.) Then we require an "energy" of $E = 10^{-26}$ J s/m (for pink ruby laser). The quotation marks mean that the quantity E is not really energy. If the pulse duration is 10^{-6} s, we have the actual power across the aperture of 3×10^{-12} W, for A = 1. For an aperture of area of 0.01 m^2 we require 3×10^{-10} W/m^2. (The E-field is 3.35×10^{-4} V/m.)

7.2.2 PAM with Random Phase

Laser oscillators can be made stable to drifts of 1 Hz/s in the laboratory. It is not clear whether one will always know the carrier phase or wish to track it. Suppose the incoming carrier phase for the signal of (325) is randomly distributed. We shall neglect thermal noise in light of the discussion above. We shall sometimes keep the noise in the equations, letting it go to zero when convenient.

The density operator for the field, when it is expanded so that f(t) is one of the temporal modes, is

$$\rho^A = \int_{-\pi}^{\pi} \int (1/\pi\langle n\rangle) \exp[-|\beta e^{i\phi} - A\sqrt{E2c/\hbar\Omega}|^2/\langle n\rangle] \cdot \beta\rangle\langle\beta \, d^2\beta d\phi. \tag{328}$$

We can expand (328) in the number representation

$$\rho^A = \sum \frac{\Lambda^j e^{-\Lambda}}{j!} j\rangle\langle j \qquad \Lambda = A^2 E2c/\hbar\Omega, \tag{329}$$

where we have set the noise to zero.

The Cramér-Rao bound is given by

$$d/dA\rho = \left(-\rho + \frac{b^+ b}{\Lambda}\rho\right) d/dA\Lambda$$

$$L = 4cE/A\hbar\Omega \left(\frac{b^+ b\hbar\Omega}{2cE} - A^2\right)$$

$$\mathrm{TR}\,\rho L^2 = (4cEA/\hbar\Omega)^2 \left(1 - 2\Lambda/\Lambda + \frac{\Lambda^2 + \Lambda}{\Lambda}\right) = 8cE/\hbar\Omega. \tag{330}$$

We see therefore that the bound of (326) for $N_o = 0$ is the same as (330). The unknown phase causes estimation ambiguities unless A is restricted to positive values. The maximum-likelihood receiver measures the number operator by focusing the received field onto a photon counter that is sensitive in the region of the focal spot. The count is processed by multiplying it by $\hbar\Omega/2cE$ and taking the square root of that number. The maximum-likelihood estimate is not unbiased. If A is a random variable, the optimal estimator, given that we receive j counts, is

$$\hat{A}_j = \int A\Lambda^j e^{-\Lambda} p(A)\, dA / \int \Lambda^j e^{-\Lambda} p(A)\, dA. \tag{331}$$

It should be emphasized that because the density operator is diagonal in the number operator representation for all values of A, the optimal unbiased estimator and the minimum mean-square-error estimator must commute with the number operator. That is, counting photons is optimal, provided we process the count optimally.

Note that the estimation error is independent of the parameter. Suppose the message parameter is A^2 rather than A. That is, the message linearly modulates the intensity of the signal field. Call the message B. The Cramér-Rao bound to the estimation of B is

$$d/dB\rho = \left(-\rho + \frac{b^+b}{\Lambda}\rho\right) d/dB\Lambda$$

$$L = 2cE/b\hbar\Omega\left(\frac{b^+b\hbar\Omega}{2cE} - B\right)$$

$$\mathrm{TR}\,\rho L^2 = 2cE/\hbar\Omega B. \tag{332}$$

Here, from the form of L, we see that the maximum-likelihood estimate is efficient and is given by counting photons and multiplying by $\hbar\Omega/2cE$. Note that the estimation error is proportional to the unknown B. This may not be desirable, but it is not always true that an estimation error independent of the parameter is desirable. For instance, practical systems often employ companding to make the estimation error proportional to the signal squared. This results in weak noise for weak signals and makes the performance the same for weaker and stronger messages, when the criterion is signal-to-noise ratio at the output.

It is not clear whether the modulation will commonly be linear in intensity or E-field. Internal cavity modulators usually employ intensity modulation around a bias. Polarization modulation combined with an analyzer can be used to generate linear E-field modulation.

For any single-valued relationship between the message parameter and the intensity B, the Cramér-Rao bound is given by

$$\mathrm{TR}\,\rho^m L_m^2 = 2cE/\hbar\Omega B(dB/dm)^2. \tag{333}$$

How close the maximum-likelihood estimate is to being efficient depends upon the relationship between B and m. Essentially, for an efficient estimate to exist the average error magnitude must be small enough so that with high probability we have $\hat{m} \approx m + (\hat{B}-B)dm/dB$. This condition, if met, over the range of a priori m will guarantee that the estimate is nearly unbiased and nearly efficient as can be checked by the reader.

Example 2. Suppose that we are using intensity modulation with B varying between 0.1 and 1. We wish the error variance not to exceed 10^{-2} for all B. Using the same frequency as in Example 1, we require 4×10^{-26} J s/m (calculated for the worst case B = 1). For B = 1, an aperture of 0.01 m^2 and a pulse length of 10^{-6} s, we need 12×10^{-10} W/m^2. Comparing this with Example 1, it is clear that for intensity modulation we need $4(B_{max}/B)$ times the power for each value of B, to achieve the same error variance at that value of B. If B varies in a small increment, the power disadvantage is 6 dB.

Example 3. Suppose that we wish to use linear E-field modulation on a random-phase channel. We shall estimate using the maximum-likelihood scheme. We restrict A to the interval (1, 2); therefore, B varies between 1 and 4. From (330) we see that provided the maximum-likelihood estimate is nearly efficient, and assuming that we desire an error variance of 10^{-4}, we need 100 times the energy quoted in Example 1. We must check to see whether the conditions for the maximum-likelihood estimate to be nearly efficient are satisfied. For the energy above, the error variance of the estimate of B is 4×10^{-4} B. Thus with high probability, the error in the estimate of B is less than $0.1\ B^{1/2}$ (5 standard deviations). Since B is restricted to the internal (1, 4), we have

$$A = (B)^{1/2} = (B+\Delta B)^{1/2} = B^{1/2} + 1/2\ \Delta B/B^{1/2}$$

$$+ (-1/8)(\Delta B)^2/B^{3/2} \ldots \approx A + \Delta B/(2A). \tag{334}$$

That is, we may neglect terms in powers of $\Delta B/B$ because with high probability this number is less than $0.1\ B^{-1/2}$ which never exceeds 0.1 for the a priori range of B.

We observe therefore, that for this type of modulation, the condition for efficient estimation is coupled to the lower limit of a priori B (or A). We require that the standard deviation of the B estimate error evaluated at the minimum a priori B (or A) be less than 0.02 times that value of B. That is, $2B_{min}^{1/2}$ (standard deviation of error in estimating A) divided by B_{min} must be less than 0.02. Therefore A_{min}^{-1} (standard deviation of the A estimate < 0.01.

7.2.3 PAM with Gaussian Fading

In Section VI, PAM communication over Gaussian fading channels with intensity modulation was discussed. In particular, for reasonable performance diversity was required (see Eqs. 296-297). For equal-strength diversity systems, with total average received "energy" E J sec/m and M diversity paths, we have

$$\mathrm{Var}\ (\hat{B}-B) = 1/M(B+N_oM/E)(B+N_oM/E+M\hbar\Omega/2cE). \tag{335}$$

The optimal estimator is

$$\hat{B} = 1/E \left\{ \sum_1^M b_j^+ b_j \hbar\Omega/2c \right\} - MN_o/E. \tag{336}$$

If we assume $N_o \ll \hbar\Omega/2c$, we obtain

$$\mathrm{Var}\ (\hat{B}-B) = B^2/M + \hbar\Omega B/2cE + (M/E^2)N_o\hbar\Omega/2c. \tag{337}$$

If we further assume that for all a priori expected B we have $B/M \ll \hbar\Omega/2cE$ and $MN_o/E \ll B$, we get

$$\mathrm{Var}\,(\hat{B}-B) = \hbar\Omega B/2cE, \tag{338}$$

provided

$$M \gg \max_{B} [2cEB/\hbar\Omega]$$

$$N_o \ll \min_{B} [EB/M].$$

This performance is the same as the intensity-modulated random-phase channel with fixed "energy" E. As in the random phase case we can let $B = A^2$, in which case the bound becomes

$$\mathrm{Var}\,(\hat{A}-A) \geqslant \hbar\Omega/8cE. \tag{339}$$

Example 4. Suppose B varies in the range 0.1-1.0. We wish the error variance to be 10^{-2}. To insure this, we require $\hbar\Omega/2cE$ to be 10^{-2}. Thus we require at least $M = 10^3$ diversity paths, and we must have N_o less than $10^{-3}\ \hbar\Omega/2c$. (If M is larger than 10^3, the noise must be proportionately smaller.) The condition on the noise is easily met at room temperatures for laser frequencies between 10^{15}-10^{16} rad/s. Remember that the total diversity is the product of the spatial and temporal diversity. The conditions of (338) are clearly not always satisfied. In other cases we should use (335). In order for an M to exist such that (338) is satisfied, it is necessary that N_o be less than $10^{-2}\ \hbar\Omega/2c(B_{max}/B_{min})$ (if we define much less than by 10^{-1} for this case).

7.3 PULSE POSITION MODULATION SYSTEMS

I shall now discuss systems in which the received waveform is displaced in time (or position for spatial problems) according to the value of a parameter to be estimated.

7.3.1 PPM with No Fading and Known Phase

If we know the phase of the carrier, possibly because of some channel-estimation scheme, and if the envelope f(t-a) is real, the optimal PPM processor, at high signal-to-noise ratios, homodynes the received signal to baseband and processes the result in a matched filter-correlator device. The estimate is the peak time of the matched filter, or the displacement that gives the best correlation for the correlator.

$$E(t,\rho) = 2RL(E/s)^{1/2}\, f(t-a)\, e^{i\Omega t} + n(t,\rho)\, e^{i\Omega t}$$

$$E(n(t,\rho)n^*(t',\rho')) = N_o\delta(t,t')\,\delta(\rho,\rho')$$

$$s = \text{aperture area}$$

$$\int f(t)\, f^*(t)\, dt = 1. \tag{340}$$

The homodyne output is

$$g(t) = (E)^{1/2}\, f(t-a) + w(t)$$

$$E(w(t)w(t')) = (N_o/2+\hbar\Omega/8c)\ \delta(t, t'). \tag{341}$$

The estimate is given by

$$d/d\gamma \int g(t)\ f(t-\gamma) \Big|_{\gamma=\hat{a}} = 0$$

$$\mathrm{Var}\ (\hat{a}-a) \geq (N_o/2+\hbar\Omega/8c)/\{E(\int f'(t)\, f'(t)\, dt)\}. \tag{342}$$

For the maximum-likelihood estimate to be efficient we require the following approximations to hold.

$$\begin{aligned} d/d\gamma \int g(t)\ f(t-\gamma)\ dt \Big|_{\gamma=\hat{a}} &= -\int g(t)\ f'(t-\gamma) \Big|_{\gamma=\hat{a}} \\ &= -\int (\sqrt{E}\ f(t-a) + w(t))\ f'(t-\gamma) \Big|_{\gamma=\hat{a}} \\ &= -\int \sqrt{E}\ \ (f(t-\gamma) + f'(t-\gamma)(a-\gamma) + f''(t-\gamma)(a-\gamma)^2/2 \ldots) \\ &\quad \cdot f'(t-\gamma)\ dt - \int w(t)\ f'(t-\gamma)\ dt \Big|_{\gamma=\hat{a}} \\ &\approx -(\hat{a}-a)\ \sqrt{E}\ \int f'(t)\ f'(t)\ dt - \int f'(t-\gamma)\ w(t)\ dt \end{aligned} \tag{343}$$

which implies

$$\mathrm{EXPECT}\ \{(\hat{a}-a)^2\, E(\int f'(t)\, f'(t)\, dt)^2\} = (N_o/2 + \hbar\Omega/8c) \cdot \int f'(t)\ f'(t)\ dt.$$

That is,

$$\mathrm{Var}\ (\hat{a}-a) \approx \frac{N_o/2 + \hbar\Omega/8c}{E \int f'(t)\, f'(t)\, dt}.$$

Whether or not the approximation holds depends upon the signal-to-noise ratio and the function f(t). A check on consistency is that the variance of the error obtained from the Cramér-Rao bound should be much less than the ratio of the integral of the square of the first derivative of f(t) to 1/6 the integral of the square of the second derivative of f(t). To see this, notice that the integral of the product of the first and second derivatives is zero. Thus the first term of approximation is the one checked above.

Example 5. Suppose f(t) is the unit energy Gaussian pulse

$$f(t-a) = (2\beta^2/\pi)^{1/4}\ e^{-\beta^2(t-a)^2}. \tag{344}$$

Suppose that we neglect thermal noise when compared with quantum noise. The integral of f'(t) squared is β^2. The integral of f''(t) squared is $\beta^4/3$. Therefore for consistency, we must specify the Cramér-Rao error variance as much less than $18\ \beta^{-2}$.

Examining (342), we see that the bound predicts an error of $0.18\,\beta^{-2}$ for a signal-to-noise ratio $8cE/\hbar\Omega$ of 5.55. Again, we must emphasize that we have not proved that the maximum-likelihood estimate is efficient but only made efficiency plausible. We must comment that if we have no a priori knowledge of A, we would surely have anomalous errors caused by correlation peaks that are due to noise alone which exceed the signal peak. The bound and approximations implicitly assume that we look at the right peak. That is, the bound specifies the performance in the absence of anomaly. As the signal-to-noise ratio goes up and the a priori range of A goes down, the probability of anomaly becomes small.

In the discussion above, we required that f(t) be real. If f(t) were complex, we would have to use a heterodyne receiver to obtain the real and imaginary parts. When thermal noise is much greater than quantum noise, there is no performance degradation, provided the signal is strong enough, because of using a complex envelope. The estimator obtains the real and imaginary parts and correlates the signals, which have added noise, against their displaced counterparts. The correlator outputs are added in the square. The displacement which maximizes this sum is the estimate. For the high signal-to-noise case, such a scheme is efficient classically. The quantum noise added to each phase because of heterodyning is twice as large as the quantum noise added to the desired phase when homodyning. Thus, when quantum noise dominates, the heterodyne system cannot be efficient. One might ask whether the use of a complex f(t) has any compensating advantages. The performance predicted by the Cramér-Rao bound is governed by the integral of the magnitude squared of the derivative of the envelope. Assume that we use a complex envelope

$$f(t) = f_1(t) + if_2(t), \tag{345}$$

where

$$\int (f_1(t)^2\,dt = \int (f_2(t))^2\,dt = 1/2 \int f(t)f^*(t)\,dt = 1/2$$

$$\int (f_1'(t))^2\,dt = \int (f_2'(t))^2\,dt = 1/2 \int f'(t)f'^*(t)\,dt, \tag{346}$$

that is, equal-energy and equal-bandwidth pulses. We could achieve the same Cramér-Rao bound by using the pulse $\sqrt{2}\,f_1(t)$ which has the same energy and bandwidth as f(t). Therefore when energy, bandwidth, and performance are the only criteria, and it is at the convenience of the designer, we should use a real envelope for f(t).

7.3.2 PPM Random Phase

If we use a real envelope f(t) and if the carrier phase is random, we could heterodyne with a local oscillator and put the output through a bandpass matched filter or correlate each phase against a displaced copy of f(t) and add the two correlations in the square. The peak of the filter output power or the displacement with the best correlation is the estimate. Classically, for high signal-to-noise ratios, this scheme performs as well as the known-phase case. When quantum noise dominates, the disadvantages of heterodyning make this impossible. It is possible that the estimator will use a small portion of the received energy to estimate the phase and then homodyne. There is another technique that is simpler when thermal noise is negligible compared with quantum noise.

We assume that f(t) is real. Focus the incoming field onto a photon counter and record the number of counts and their arrival times. The incoming classical field is given in (340). (We are not going to use the carrier phase, so we need not worry about it.) The counts form a Poisson process with rate given by

$$\gamma(t) = (2c/\hbar\Omega)\, E(f(t-a))^2. \tag{347}$$

Let me state a priori that the anomalous behavior of the system is governed by the possibility of receiving no counts at all. The reader will see this eventually. The probability of receiving no counts is $e^{-2cE/\hbar\Omega}$. We can write down the probability of obtaining N counts at times t_i, given that we have at least one count. (This last conditioning will soon become reasonable.)

$$\Pr\left[N, \{t_i\} \middle| a, N\neq 0\right] = \prod_{i=1}^{N} \frac{\gamma(t_i)\, e^{-\lambda}}{N!} \Big/ (1-e^{-\lambda}), \qquad \lambda = 2cE/\hbar\Omega. \tag{348}$$

We can then obtain the classical Cramér-Rao bound to the performance of any estimate of a based upon the counts.

$$d/da \ln \Pr[\;] = L(a) = -\sum_{1}^{N} \left.\frac{d/dt\, \gamma(t)}{\gamma(t)}\right|_{t=t_i} \tag{349}$$

$$J_{11} = \sum_{N=1}^{\infty} \int \Pr[\;](L(a))^2 = \frac{8cE/\hbar\Omega \int (f'(t))^2\, dt}{1-e^{-\lambda}}$$

$$\mathrm{Var}((\hat{a}-a)/N\neq 0) \geq (J_{11})^{-1}.$$

Provided we have at least one count, the maximum-likelihood estimate is clearly

$$0 = \sum_{1}^{N} \left.\frac{d/dt\, \gamma(t_i, a)}{\gamma(t_i, a)}\right|_{\hat{a}=a} = \sum_{1}^{N} d/dt \ln f(t_i - a)\Big|_{\hat{a}=a}. \tag{350}$$

Note that when the probability of zero counts is small, the bound of (349) is the same as the quantum bound of (342) for $N_o = 0$. Of course, if we receive no counts, the estimate cannot be unbiased. If we are estimating a Gaussian random variable with a priori variance T^2, then the Cramér-Rao bound and the MAP estimate are given by

$$\mathrm{Var}\,(\hat{a}-a) \geq \frac{T^2(1-e^{-\lambda})^2}{(1-e^{-\lambda}) + T^2 8cE/\hbar\Omega \int (f'(t))^2\, dt} + T^2 e^{-\lambda} \tag{351}$$

$$0 = \left\{ 2 \sum_{1}^{N} d/dt \ln f(t_i - a) \right\} + a/T^2 \Bigg|_{a=\hat{a}} \qquad N \neq 0 \tag{352}$$

$$\hat{a} = 0 \quad \text{if} \quad N = 0.$$

If f(t) is symmetric and unimodal, the maximum-likelihood and MAP estimates are unbiased (in the nonrandom parameter case, provided we get at least one count). For high signal-to-noise ratios (351) will become an equality. This equation displays explicitly the errors caused by anomaly (no counts).

Example 6. Suppose we use a Gaussian pulse as in (344). The MAP estimate then becomes

$$-4\beta^2 \sum_{1}^{N} (t_i - \hat{a}) + \hat{a}/T^2 = 0$$

$$\hat{a}_{map} = \bar{t}\{4N\beta^2/(1/T^2 + 4N\beta^2)\}. \tag{353}$$

If we use the MAP estimate, we obtain the following results

$$E(\hat{a}|\, a) = \sum_{N=1}^{\infty} \frac{e^{-\lambda}}{N!} \prod_{i=1}^{N} \int (2c/\hbar\Omega)\, \sqrt{2/\pi}\; \beta\, e^{-2\beta^2 (t_i - a)^2}$$

$$\cdot\, (1/N) \sum_{j=1}^{N} t_j [4N\beta^2/(1/T^2 + 4N\beta^2)]\, dt_i\, .$$

$$= \sum_{N=1}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!}\, 4aN\beta^2/[1/T^2 + 4N\beta^2]. \tag{354}$$

$$E(\hat{a})^2 = \int p(a) \sum_{N=1}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!} \left\{ \frac{(N^2+N)a^2}{N^2} + \frac{N}{N^2}\,(a^2 + 1/(4\beta^2)) \right\}$$

$$\cdot\, [4N\beta^2/(1/T^2 + 4N\beta^2)]^2\, da$$

$$= \sum_{N=1}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!}\, [T^2 + 1/(4\beta^2 N)]\, [4N\beta^2/(1/T^2 + 4N\beta^2)]^2. \tag{355}$$

Suppose that instead of the MAP estimator we simply use the estimate $\bar{t}$ (the average photon arrival time). With slight modification of (354) and (355) we obtain for this estimator

$$E(\hat{a}|\, a) = (1 - e^{-\lambda})a$$

$$E(\hat{a})^2 = (1 - e^{-\lambda})T^2 + (1/4\beta^2) \sum_{1}^{\infty} e^{-\lambda}\, \lambda^N/N!\, N. \tag{356}$$

It then follows that by using this estimator the error variance for the estimator $\bar{t}$ is

$$E(\hat{a}-a)^2 = T^2\, e^{-\lambda} + (1/4\beta^2) \sum_1^{\infty} e^{-\lambda}\, \lambda^N/N!\, N. \tag{357}$$

Suppose that we can make the following approximation

$$(1/4\beta^2) \sum_1^{\infty} e^{-\lambda}\, \lambda^N/N!\, N \approx 1/4\beta^2\lambda = \hbar\Omega/8cE\beta^2. \tag{358}$$

Table 1 shows when the approximation is valid. The error variance is then

$$E(\hat{a}-a)^2 = \hbar\Omega/8cE\beta^2 + T^2\, e^{-\lambda}. \tag{359}$$

Table 1. $\lambda\, e^{-\lambda} \sum_1^{\infty} \lambda^N/N!\, N$ vs λ.

$\lambda\, e^{-\lambda} \sum_1^{\infty} \lambda^N/N!\, N$	λ
1.32	4
1.24	6
1.13	10
1.06	20
1.01	100

Now observe that if the probability of no photons is small and if $8cE\beta^2/\hbar\Omega \gg 1/T^2$, the bound of (351) becomes

$$E(\hat{a}-a)^2 \geq \hbar\Omega/8cE\beta^2 + T^2\, e^{-\lambda}. \tag{360}$$

Thus under these circumstances $\bar{t}$ is an efficient estimator. (Remember that this is still Example 6 with a Gaussian pulse shape.)

As a final comment, it should be emphasized that if the interval we are looking at becomes too large, the possibility of a thermal noise count may effect performance, since thermal noise is not zero.

7.3.3 PPM Gaussian Fading Channels

We have the Cramér-Rao bound for a Gaussian fading channel given by (Eq. 266)

$$TR\, \rho L^2 = (2/\hbar\Omega)^2 \int_{M.R.} T(t,u,z,w)\, d/da\, K_s(u,z)\, d/da\, K_s(w,t)\, dtdudwdz.$$

Assume that the field impinging upon an aperture is

$$R(t, \rho) = \gamma(t, \rho)\, f(t-a) + \text{noise}. \tag{361}$$

Assume

$$f(t-a) = \sum_1^L f_k(t-a), \tag{362}$$

where the $f_k(t)$ are real and

$$\int f_k(t)\, f_j(t)\, dt = \delta(k, j).$$

Assuming that the field is composed of plane waves lying in a cone concentrated about normal incidence, we use the results of section 5.1 to obtain

$$\begin{aligned} TR\, \rho L^2 = (2c/\hbar\Omega)^2 \int & T(t_1, \rho_1, t_2, \rho_2, t_3, \rho_3, t_4, \rho_4) \\ & \cdot d/da\, K_g(t_2, \rho_2, t_3, \rho_3)\, d/da\, K_g(t_4, \rho_4, t_1, \rho_1) \\ & \cdot dt_1 d^2\rho_1 \ldots dt_4 d^2\rho_4, \end{aligned} \tag{363}$$

where

$$\begin{aligned} K_g(t_1, \rho_1, t_2, \rho_2) &= E\, R(t_1, \rho_1)\, R^*(t_2, \rho_2) \\ &= \hbar\Omega/2c \sum \psi_k(t_1, \rho_1)\, \psi_k^*(t_2, \rho_2)\, \lambda_k \end{aligned} \tag{364}$$

and

$$\begin{aligned} T(t_1, \ldots \rho_4) = \sum_k \sum_m & \psi_k(t_1, \rho_1)\, \psi_k^*(t_2, \rho_2)\, \psi_m(t_3, \rho_3)\, \psi_m^*(t_4, \rho_4) \\ & \cdot [\lambda_k \lambda_m + 1/2(\lambda_k + \lambda_m)]^{-1}. \end{aligned}$$

Assume that the turbulence is slowly varying enough so that, under the assumption that the $f_k(t)$ are short compared with the correlation time of the channel and disjoint at wide intervals compared with the correlation time, we may write

$$\begin{aligned} K_g(t_1, \rho_1, t_2, \rho_2) = & \sum_1^L \sum f_k(t_1 - a)\, \gamma_j(\rho_1)\, \gamma_j^*(\rho_2)\, f_k(t_2 - a)\, b_{kj} \\ & + N_o \delta(t_1 - t_2)\, \delta(\rho_1 - \rho_2), \end{aligned} \tag{365}$$

where

$$\int \gamma_j(\rho)\, \gamma_k^*(\rho)\, d^2\rho = \delta(k, j).$$

From (364 and (365) we obtain

$$T(x_1, x_2, x_3, x_4) = \Gamma_o \delta(x_1-x_2)\, \delta(x_3-x_4)$$

$$+ \sum_j \sum_k \Gamma_{jk} \delta(x_3-x_4)\, f_k(t_1-a)\, f_k(t_2-a)\, \gamma_j(\rho_1)\, \gamma_j^*(\rho_2)$$

$$+ \delta(x_1-x_2)\, f_k(t_3-a)\, f_k(t_4-a)\, \gamma_j(\rho_3)\, \gamma_j^*(\rho_4)$$

$$+ \sum_j \sum_k \sum_l \sum_m \Gamma_{jklm} f_k(t_1-a)\, f_k(t_2-a)\, f_m(t_3-a)\, f_m(t_4-a)$$

$$\cdot\, \gamma_j(\rho_1)\, \gamma_j^*(\rho_2)\, \gamma_l(\rho_3)\, \gamma_l^*(\rho_4), \tag{366}$$

where

$$x_i = (t_i, \rho_i)$$

$$\Gamma_o = \left[(2c/\hbar\Omega)N_o + ((2c/\hbar\Omega)N_o)^2\right]^{-1}$$

$$\Gamma_{jk} = 1 \Big/ \left[(2c/\hbar\Omega)(N_o+b_{jk}/2) + (2cN_o/\hbar\Omega)\left(\frac{2c}{\hbar\Omega}(N_o+b_{jk})\right)\right] - \Gamma_o$$

$$\Gamma_{jklm} = 1 \Big/ \left[\frac{2c}{\hbar\Omega}(N_o+b_{kj}/2+b_{ml}/2) + \frac{4c^2}{(\hbar\Omega)^2}(N_o+b_{kj})(N_o+b_{ml})\right]$$

$$- \Gamma_{jk} - \Gamma_{lm} - \Gamma_o.$$

Plugging into (363) and taking advantage of the fact that the $f_k(t)$ are disjoint in time, we obtain

$$\mathrm{TR}\, \rho L^2 = \sum_j \sum_k \frac{2b_{kj}^2 \int (f_k'(t))^2\, dt}{N_o(N_o+b_{kj}) + (\hbar\Omega/2c)(N_o+b_{kj}/2)}. \tag{367}$$

If we assume no thermal noise, we obtain

$$\mathrm{TR}\, \rho L^2 = \sum\sum (8cb_{kj}/\hbar\Omega) \int (f_k'(t))^2\, dt. \tag{368}$$

Having the bound of (368), we can try the estimation scheme used previously for the case of random phase and no thermal noise.

Suppose we focus the incident field onto a photon counter and record the counts and their arrival times. (Since thermal noise is not identically zero, we should count only in the region of the focal plane where we expect signal energy.) We can write down the probability of N counts at arrival times t_i, given that we have at least one count, and given the message value a.

$$\Pr\left[N,\{t_i\} \mid N\neq 0, a\right] = \int p(\underline{r}) \prod_{i=1}^{N} \gamma(t_i, r) \frac{e^{-\lambda(\underline{r})}\, d(\underline{r})}{N!\,(1-e^{-\lambda(\underline{r})})}, \tag{369}$$

where

$$\underline{r} = \{r_{kj}\}$$

$$\lambda(\underline{r}) = \Sigma\Sigma\, r_{kj} r_{kj}^{*} 2c/\hbar\Omega$$

$$\gamma(t, \underline{r}) = \Sigma\Sigma\, r_{kj} r_{kj}^{*} (f_k(t-a))^2\, 2c/\hbar\Omega$$

$$p(\underline{r}) = \prod_k \prod_j (\pi b_{kj})^{-1}\, e^{-\left(r_{kj} r_{kj}^{*}/b_{kj}\right)}.$$

That is, we write down the density, given the mode amplitudes, and average over these amplitudes. We have used the fact that the $f_k(t)$ are time disjoint.

$$d/da \Pr[\ ] = \Pr[\ ] \sum_{1}^{N} \sum_{1}^{L} - 2d/dt \ln f_k(t-a)\Big|_{t=t_i}. \tag{370}$$

From this we obtain the Cramér-Rao bound

$$\int \Pr[\ ]\,(d/da \ln \Pr[\ ])^2 = \int p(\underline{r})(8c/\hbar\Omega) \frac{\Sigma\Sigma\, r_{kj} r_{kj}^{*} \int (f_k'(t))^2\, dt}{(1-e^{-\lambda})}\, d(\underline{r}). \tag{371}$$

If we assume that we have enough diversity and that the average energy is high enough so that the probability of no counts is small, we can set the denominator to unity to obtain

$$\mathrm{Var}\,(\hat{a}-a) \geq \frac{\hbar\Omega}{8c\,\Sigma\Sigma\, b_{kj} \int (f_k'(t))^2\, dt}, \tag{373}$$

provided we get at least one count. If the probability of no counts is small enough (373) gives the same bound as (368).

If our estimate is of a Gaussian random variable with a priori variance T^2, we obtain the bound

$$E(\hat{a}-a)^2 \geq \frac{T^2(1-e^{-\lambda})^2}{(1-e^{-\lambda}) + \frac{T^2 8c}{\hbar\Omega} \Sigma\Sigma\, b_{kj}\left[\int (f_k'(t))^2\, dt\right]} + T^2 e^{-\lambda}, \tag{374}$$

where $\lambda = \Sigma\Sigma\, b_{kj} 2c/\hbar\Omega$, which displays the anomalous behavior explicitly. The maximum-likelihood estimate is

$$\sum_{1}^{N} \sum_{1}^{L} d/dt \ln f_k(t_i - a) \bigg|_{\hat{a}=a} = 0. \tag{375}$$

The MAP estimate is given by

$$2 \sum_{1}^{N} \sum_{1}^{L} d/dt \ln f_k(t_i - a) + a/T^2 \bigg|_{\hat{a}=a} = 0. \tag{376}$$

We observe that for enough diversity, the performance bound of (373) is the same as the random-phase bound (349), provided the bandwidth of $f_k(t)$ is the same as the bandwidth of $f(t)$ and we set the average energy for the fading case equal to the fixed energy of the nonfading case.

We could have derived the Cramér-Rao bound without the condition that there be at least one count. In that case, the term in the denominator of (371) would be unity, regardless of diversity. The resulting bound would be identical to (368). This is a more just comparison anyway, since (368) is not exclusive of anomalous behavior.

One should also note that although (368) was derived for the Gaussian case, (371) is the Cramér-Rao bound for this type of measurement, regardless of the nature of the fading. That is, (371) holds even if the mode amplitudes are correlated, nonzero mean, and so forth.

Example 7. Suppose that we use only spatial diversity and we use $f(t)$ given in (344), that is, a Gaussian pulse with bandwidth β^2. Suppose that we have 150 diversity paths in space. Suppose that the total average energy $\Sigma\, b_j$ is 10 $\hbar\Omega/2c$. Assume that the diversity paths are nearly equal-strength. The actual received energy is the sum of 300 independent Gaussian random variables of approximately equal variance (sum in the square). The standard deviation of the sum of the squares of 2N independent GRV's is $(3/2N)^{1/2}$ times the expected value of the sum. Thus we see that for N = 150 the standard deviation is 1/10 the mean. Thus we see that with high probability, we shall not have a deep fade. For an exact analysis we need the distribution of a chi-square random variable with 2N degrees of freedom. The MAP estimate, under the assumption of a Gaussian parameter, is given by (376). If we use the estimate $\bar{t}$ instead, we obtain

$$E(\hat{a}|\,a) = a\left(1 - \int p(\underline{r})\, e^{-\lambda(\underline{r})}\, d\underline{r}\right) = a\,\overline{\left(1 - e^{-\lambda(\underline{r})}\right)}$$

$$E(\hat{a}-a)^2 = T^2\,\overline{\left(e^{-\lambda(\underline{r})}\right)} + 1/(4\beta^2)\,\Sigma\,\overline{\frac{e^{-\lambda(\underline{r})}\,\lambda(\underline{r})^N}{N!\,N}}, \tag{377}$$

similar to (354) and (355). If the fading is Gaussian, we have

$$\overline{e^{-\lambda(\underline{r})}} = \prod_1^J \int (\pi b_j)^{-1} \exp\left[-r_j r_j^*(2c/\hbar\Omega + 1/b_j)\right] d^2 r_j$$

$$= \prod_1^J (2c/\hbar\Omega + 1/b_j)^{-1} (b_j)^{-1} = \prod_1^J (1+2cb_j/\hbar\Omega)^{-1}$$

$$= \prod_1^J (1+2cE/\hbar\Omega J)^{-1} \geq e^{-2cE/\hbar\Omega} = e^{-\bar{\lambda}}, \tag{378}$$

where E is the total average energy, and J is the number of diversity paths. The inequality becomes an equality for J sufficiently large. For J = 150 and $2cE/\Omega = 10$ we can make (378) an equality.

The inequality follows from the familiar compound interest formula. The larger J is, the larger is the reciprocal of the left side of the inequality. Its limit is the reciprocal of the right side of the inequality.

To evaluate the second half of (377), note that

$$\sum_1^\infty \lambda^N/N!\,N = \int_0^1 \frac{e^{\lambda u} - 1}{u}\,du; \tag{379}$$

therefore

$$\overline{\sum_1^\infty \frac{e^{-\lambda}\lambda^N}{N!\,N}} = \overline{\int_0^1 \frac{e^{\lambda(u-1)} - e^{-\lambda}}{u}\,du}$$

$$= \int_0^1 \frac{\prod_1^J (1+2cE(1-u)/\hbar\Omega J)^{-1} - \prod_1^J (1+2cE/\hbar\Omega J)^{-1}}{u}\,du$$

$$\approx \int_0^1 \frac{e^{-2cE/\hbar\Omega(1-u)} - e^{-2cE/\hbar\Omega}}{u}\,du = \sum_1^\infty \bar{\lambda}^N e^{-\bar{\lambda}}/N!\,N \tag{380}$$

for $2cE/\hbar\Omega J$ small.

Thus, given a large number of diversity paths making the signal-to-noise ratio per diversity path small, the fading channel performance is the same as that of the non-fading channel for the estimator $\bar{t}$ and a Gaussian pulse (provided we substitute average energy where we had fixed energy).

7.4 PULSE FREQUENCY MODULATION – THE PRISM-LENS DISCRIMINATOR

I shall now discuss the estimation of the frequency of a burst of sinusoid with a combination of a prism and a lens.

Suppose that we receive a plane wave over an aperture for the period $(-1/2T, 1/2T)$ given by

$$E(t, \rho) = 2RL(P/s)^{1/2}\, e^{i(\Omega+\beta a)t}, \tag{381}$$

where the aperture area is s, and there is no noise.

We wish to estimate the parameter A. The Cramér-Rao bound is given by (125):

$$TR\, \rho L^2 = \frac{8cP\beta^2}{\hbar\Omega} \int_{-1/2T}^{1/2T} t^2\, dt = \frac{2cPT^3\beta^2}{3\hbar\Omega}. \tag{382}$$

Examine the following estimation scheme. Send the plane wave through a prism that disperses the spectrum. The dispersion relationship is

$$\Delta\theta = \gamma\Delta\omega. \tag{383}$$

Send the dispersed field through a lens that is small enough that for all prism outputs its aperture is completely covered with field. The arrangement is shown in Fig. 9. We

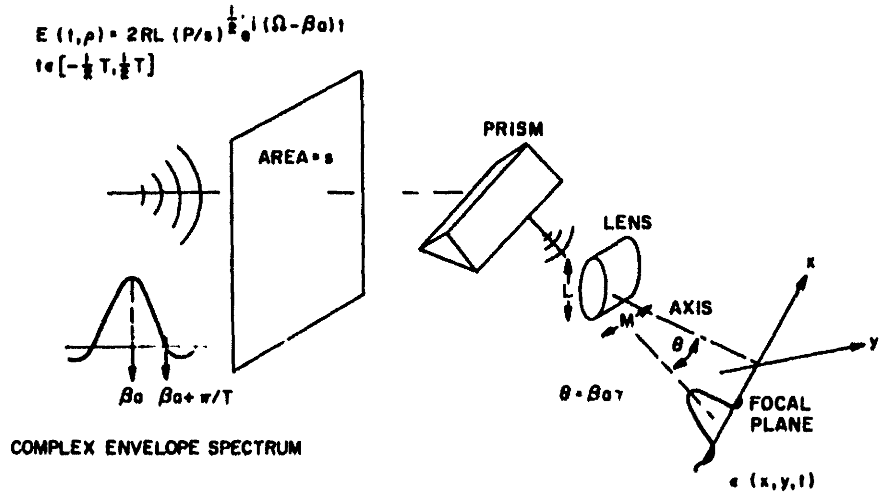


Fig. 9. Prism-lens discriminator for pulse frequency modulation.

can write the classical field in the focal plane of the lens, assuming a rectangular aperture and that for all values of A the dispersed field is contained in a cone not too far from normal incidence.

$$\epsilon(x, y, t) = \frac{T(P/s)^{1/2}\, ML}{2\pi\lambda r^2\gamma} \frac{\sin(\pi My/\lambda r)}{\pi My/\lambda r}\, e^{i(\Omega-\beta a)t}$$

$$\cdot \frac{\sin T/2\gamma r(x-\beta\gamma ra)}{T/2\gamma r(x-\beta\gamma ra)}\, e^{i\phi(x,t)} * \frac{\sin \pi Lx/\lambda r}{\pi Lx/\lambda r}, \tag{384}$$

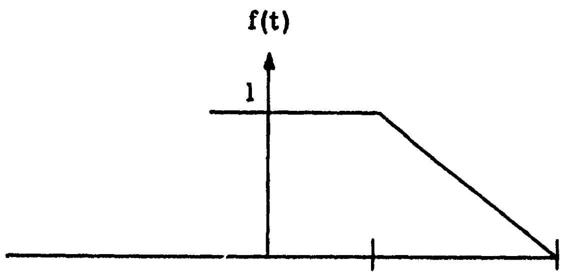
where $\phi(t, x) = t/r\gamma(x-\beta\gamma ra)$, and the star denotes convolution. Convolutions are easier to perform in the transform domain where the factor ϕ is a positional shift. We obtain the following results.

If:

$$T/2\gamma < \pi L/\lambda \tag{385}$$

Then:

$$\epsilon(x, y, t) = \frac{TM(P/s)^{1/2}}{2\pi\gamma r} \frac{\sin \pi My/\lambda r}{\pi My/\lambda r} \frac{\sin (x-\beta\gamma ra)(T/2\gamma r)\, f(t)}{(T/2\gamma r)\, f(t)(x-\beta\gamma ra)}\, f(t)\, e^{i(\Omega+\beta a)t}$$



$$\int f(t)\, dt = 2\pi\gamma L/\lambda$$

$$\int f^3(t)\, dt = \frac{2\pi L\gamma}{\lambda} - T/2$$

$$\frac{r\gamma}{2}\left(\frac{2\pi L}{\lambda r} - \frac{T}{\gamma r}\right) \qquad \frac{r\gamma}{2}\left(\frac{2\pi L}{\lambda r} + \frac{T}{r\gamma}\right)$$

(symmetric)

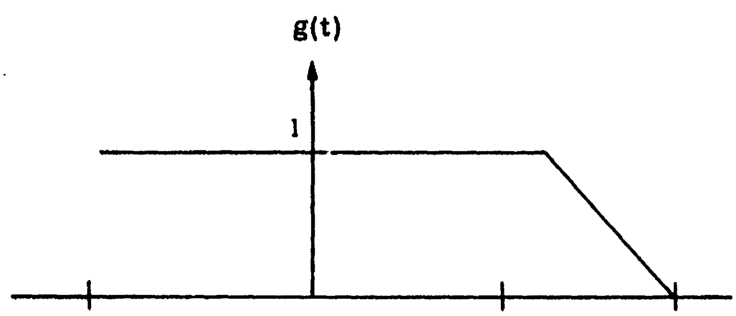
(386)

If:

$$T/2\gamma > \pi L/\lambda$$

Then:

$$\epsilon(x, y, t) = \frac{LM(P/s)^{1/2}}{\lambda r} \frac{\sin \pi My/\lambda r}{\pi My/\lambda r} \frac{\sin (\pi L/\lambda r)(x-\beta\gamma ra)\, g(t)}{(\pi L/\lambda r)(x-\beta\gamma ra)\, g(t)}\, g(t)\, e^{i(\Omega-\beta a)t}$$



$$\int g(t)\, dt = T$$

$$\int g^3(t)\, dt = T - \pi L\gamma/\lambda$$

$$\frac{r\gamma}{2}(T/\gamma r - 2\pi L/\lambda r) \qquad \frac{r\gamma}{2}(T/\gamma r + 2\pi L/\lambda r)$$

(symmetric)

Suppose that we now process the signal by counting photons in the focal plane and noting their x coordinate. We use the processing described for PPM systems except in position rather than time. The Cramér-Rao bound for an estimate based upon the counts is

If: $T/2\gamma < \pi L/\lambda$

$$\mathrm{Var}\,(\hat{a}-a) \geq \frac{3\hbar\Omega}{2c}\left[(\beta^2 T^3 PM/s)(\lambda/2\pi\gamma)\left(\frac{2\pi L\gamma}{\lambda} - \frac{T}{2}\right)\right]^{-1}. \tag{387a}$$

If: $T/2\gamma > \pi L/\lambda$

$$\mathrm{Var}\,(\hat{a}-a) \geq \frac{3\hbar\Omega}{2c}\left[\frac{PL^2}{\lambda^2}\, 4\beta^2\gamma^2\pi^2\, \frac{LM}{s}\left(T - \frac{\pi L\gamma}{\lambda}\right)\right]^{-1}. \tag{387b}$$

From (387) we see that the first bound is optimized when γ is as large as necessary for the term $T/2$ to be negligible. In the second equation the denominator is maximized for $T/2\gamma = 3/4\ \pi L/\lambda$ which is outside the allowed range. Thus for γ sufficiently large we obtain

$$\mathrm{Var}\,(\hat{a}-a) \geq \frac{3\hbar\Omega}{2c\beta^2 T^3 P\ ML/s}. \tag{388}$$

Thus if ML/s is close to unity, we can achieve quantum efficiency at high signal-to-noise ratios.

7.5 CONTINUOUS TIME SYSTEMS

7.5.1 Angle Modulation

Before discussing angle modulation, I shall first restate the results previously derived for estimation of the message of a PM system. Suppose that we have a slowly varying known spatial envelope multiplying the received signal. That is,

$$E(t,\rho) = 2RL\gamma(\rho)(P/s)^{1/2}\, e^{i\beta m(t)} + n(t,\rho)\, e^{i\Omega t}$$

$$E(n(t,\rho)n^*(t',\rho')) = N_o\,\delta(t,t')\,\delta(\rho,\rho')$$

$$\int \gamma(\rho)\,\gamma^*(\rho)\, d^2\rho = s. \tag{389}$$

The ratio P/s is fixed at A^2. Therefore $P = A^2 s$. The Cramér-Rao bound is given by

$$K_m(x,z) = H^{-1}(x,z) + \frac{P\beta^2 \int H^{-1}(t,z)\, K_m(x,t)\, dt}{\hbar\Omega/8c + N_o/2}. \tag{390}$$

From the results of Section IV, we know that for sufficient "power," the optimal receiver can take the form of a phase-locked loop with feedback signal given by

$$2RL\gamma(\rho)\,C\, e^{+i\beta\hat{m}_r(t)+i[\Omega t+\pi/2]}, \tag{391}$$

where C is large.

The arrangement is shown in Fig. 6. Provided the loop spends most of its time in the linear region, the estimate is efficient.

Suppose there is a phase term added to the message-induced phase because of carrier drift. Provided that the spectrum of this process is disjoint from the message spectrum and the drift is not so rapid as to drive the loop out of its linear region, we can estimate the message and drift terms, and separate the two in the post loop filter.

A more difficult problem than phase drift is channel fading. From (308)-(314) we obtain the Cramér-Rao bound of the estimation performance for a slowly varying channel with diversity in space. It is

$$K_m(x, z) = H^{-1}(x, z) + \frac{(2\beta^2 TP^2/K) \int K_m(x, t)\, H^{-1}(t, z)\, dt}{N_o(N_o+PT/K) + \frac{\hbar\Omega}{2c}(N_o+PT/2K)}, \tag{392}$$

where K is the number of diversity paths, and P is average receiver "power." We can obtain another lower bound by assuming that we know the envelope $\gamma(\rho)$. We solve (390) in terms of the random parameter $P = A^2 s$ and average the bound over the probability density of s. We make the observation that for N_o much less than the ratio TP/K, (392) is identical to (390) with the replacement of fixed power by average power.

Consider the special case when the signal-to-noise ratio on each diversity path is large enough so that we can estimate the spatial envelope. If we have K diversity paths, then s is the sum of the squares of 2K Gaussian random variables. For K sufficiently large, the probability of a deep fade is small. That is, the standard deviation of the sum is $(3/2K)^{1/2}$ times the mean of the sum.

The performance of the phased-locked loop is obtained for s fixed by solving (390). We then average this performance over s to obtain the fading-channel performance. This performance is conditional on there being enough diversity paths and enough energy so that the loop stays in its linear region most of the time. Furthermore, it is conditioned on obtaining a perfect channel estimate without disturbing much of the received energy. Remember that channel estimation involves heterodyning, which is not compatible with message estimation in the quantum case. Therefore we must estimate the channel by using a small fraction of the signal on each mode.

We know that the channel estimate will not be perfect. Let us consider how sensitive this scheme is to errors in this estimate. We shall express the estimate as an expansion in terms of the uncorrelated Karhunen-Loève spatial functions for a slowly varying spatial envelope.

$$\hat{\gamma}(t, \rho) = \Sigma\, \gamma_i(t)\, \psi_i(\rho)$$

$$\int \psi_i(\rho)\, \psi_j^*(\rho)\, d^2\rho = \delta(i, j)$$

$$\hat{\gamma}_i(t) = |\hat{\gamma}_i(t)|\, e^{i\hat{\theta}_i(t)}. \tag{393}$$

The output of the phase-locked loop, after normalization so that the perfect channel estimate would yield unity gain, is

$$\text{Error signal} = \frac{\Sigma\, |\hat{\gamma}_i| \left[|\gamma_i| \left[\hat{m}(t) - m(t) + (\hat{\theta}_i - \theta_i)/\beta \right] + n_i(t)/\beta \right]}{\Sigma\, |\hat{\gamma}_i|^2} . \tag{394}$$

Equation (394) assumes that the loop is operating in its linear region. We see from this equation that the phase errors add up randomly to form a noise process that is assumed lowpass compared with the message. Thus this signal will not pass through the loop filter or the unrealizable (realizable with delay) post loop filter. If these errors do not affect the linearization approximation, they will cause no problems. Because of amplitude errors, the signal coming out of the counter is multiplied by a gain different from unity. If we examine the linearized loop in Fig. 6, we see that this affects the forward gain of the linearized filter transfer function. If we assume that the signal-to-noise ratio is high, then the transfer function, under the assumption of a perfect estimate, must be near unity at message frequencies. Thus, a small change in the forward gain will not affect the over-all transfer function which is (forward gain/1 + forward gain). If on the other hand, the transfer function is not close to unity at some frequencies, under the assumption of a perfect channel estimate, then the over-all gain will be accordingly modified. Thus, at frequencies for which the signal-to-noise ratio is high, amplitude errors do not affect the gain of the estimate to a great extent. There is another effect that is due to amplitude estimation errors. The over-all loop gain and the loop plus post loop unrealizable filter will no longer be matched to the message plus noise. Furthermore, since the combining is not optimal for the different paths, the over-all signal-to-noise ratio will be reduced. It is these effects that degrade performance.

If the channel estimation is not good enough for the approximations leading to a linear loop to hold, perhaps we should use a different demodulation scheme.

7.5.2 FM Discrimination with a Prism-Lens System

We have seen that the estimation of the frequency of a burst of a sinusoid could be accomplished by passing the received signal through a prism or diffraction grating, and

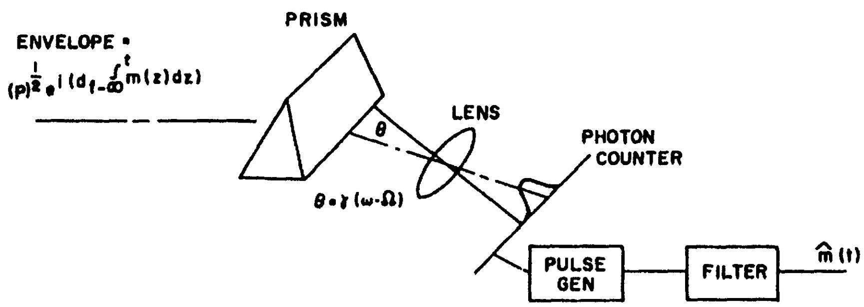


Fig. 10. Prism-lens discriminator for frequency modulation.

then through a lens, to convert to a spatial position modulation problem. I shall now investigate the prism-lens system as an FM discriminator.

The system is shown in Fig. 10. Note that the lens aperture has a Gaussian rather than abrupt changing transmittance. This is more than a mathematical convenience. It is imperative for this system that the focal spot caused by a plane wave impinging upon the aperture have a finite second moment when squared. The abrupt aperture has a focal spot intensity that falls off as $1/x^2$. This is not fast enough for good estimation.

The field impinging upon the prism or diffraction grating is given by

$$E(t, \rho) = 2RL\ P^{1/2} \exp\left(i\Omega t + id_f \int_{-\infty}^{t} m(z)\, dz\right), \tag{395}$$

where no thermal noise has been assumed. The aperture transmittance is given by

$$T(x, y) = e^{-(x^2+y^2)}/(LM/2\pi). \tag{396}$$

The corresponding focal spot is

$$h(x, y) = (ML)^{1/2}\ (1/\sigma)\left(e^{-(x^2+y^2)/(2\sigma^2/\pi)}\right), \tag{397}$$

where $\sigma^2 = (\lambda r)^2/LM$, $L = M$, and r is the focal length.

The field at the focal plane is the prism output angular spectrum convolved with the focal spot. It is given by

$$\begin{aligned} E(x, y, \tau) = {} & 1/2\pi \iint \sqrt{P}\ e^{id_f \int_{-\infty}^{t} m(z)\, dz}\ e^{i\omega(t-\tau)} \\ & \cdot \left(\frac{ML}{\sigma^2}\right)^{1/2} e^{-\left[(x-\gamma r\omega)^2+y^2\right]/(2\sigma^2/\pi)} \cdot dt\, d\omega \\ = {} & \left(\frac{PML}{2\pi r}\right)^{1/2} (\sigma)^{-1/2} \left(\frac{\sigma}{\pi\gamma r}\right)^{1/2} \\ & \cdot \int e^{id_f \int_{-\infty}^{t} m(z)\, dz}\ e^{-ix(t-\tau)/\gamma r}\ e^{-\sigma^2(t-\tau)^2/2\pi\gamma^2 r^2} \\ & \cdot e^{-y^2/(2\sigma^2/\pi)}\ dt. \end{aligned} \tag{398}$$

Note that

$$\int\int E(x, y, \tau)\ E^*(x, y, \tau)\ dx dy = PML \tag{399}$$

as expected.

With this field impinging upon the focal plane, the output of the photon counter will be a Poisson process in time. That is, every so often, a photoelectron will be emitted.

The information that it contains about the message is in its x coordinate relative to the lens axis. Suppose that we use each count to drive a pulse generator. The amplitude of the pulse will be proportional to the x position of the count. The width of the pulse will be small compared with the mean interarrival time of the counts. The random process thus generated will be used as the input to a linear filter which will generate the estimate. We must determine the optimal (Wiener) filter.

First, let us establish two facts about the statistics of the arrival position of a count, given the message.

If a count occurs, the expected value of its x coordinate is

$$\bar{x}\Big|_m = \frac{\int x|E(x,y,\tau)|^2\,dxdy}{\int |E(x,y,\tau)|^2\,dxdy}$$

$$= r\gamma d_f \int_{-\infty}^{\infty} m(t)(1/2\pi u^2)^{1/2}\, e^{-(t-\tau)^2/2u^2}\,dt. \tag{400}$$

The variance of its x coordinate is

$$\overline{x^2}\Big|_m = (r\gamma d_f)^2 \int m^2(t)(1/2\pi u^2)^{1/2}\, e^{-(t-\tau)^2/2u^2}\,dt + \lambda^2 r^2/2\pi L^2$$

$$u^2 = L^2\gamma^2\pi/2\lambda^2. \tag{401}$$

Call the random process that we generate v(t). To specify the optimal filter, we need two correlations:

$$\text{Expect}\,[v(t_1)v(t_2)]$$

$$\text{Expect}\,[v(t_1)m(t_2)]. \tag{402}$$

Arrival positions at different times are independent. We assume that the pulses generated are infinitely narrow. Assuming that the message is stationary, we obtain

$$E[v(t_1)m(t_2)] = \frac{2cPML}{\hbar\Omega}\, r\gamma d_f \int_{-\infty}^{\infty} K_m(t_1-t_2-t_3)(1/2\pi u^2)^{1/2} \exp\left(-t_3^2/2u^2\right) dt_3 \tag{403}$$

$$E[v(t_1)v(t_2)] = \frac{2cPML}{\hbar\Omega}\left[(r\gamma d_f)^2 K_m(0) + \lambda^2 r^2/2\pi L^2\right]\delta(t_1,t_2)$$
$$+ (r\gamma d_f 2cPML/\hbar\Omega)^2 \int K_m(t_1-t_2-t_3)(1/4\pi u^2)^{1/2} \exp\left(-t_3^2/4u^2\right) dt_3. \tag{404}$$

The optimal filter is then

$$H_{opt}(\omega) = S_{vm}(\omega)/S_{vv}(\omega)$$

$$= \frac{\hbar\Omega/[2cP\gamma r d_f ML]\, S_m(\omega)\, e^{u^2\omega^2/2}}{S_m(\omega) + \hbar\Omega/[2cPML]\, e^{u^2\omega^2}\left[K_m(0) + 1/\left(4d_f^2u^2\right)\right]}. \tag{405}$$

The error variance is

$$\epsilon_u = \int \frac{S_m(\omega)\left[\omega^2\hbar\Omega/\left(8cPMLd_f^2\right)\right]\left[\left(1+4K_m(0)\,d_f^2u^2\right)\left(e^{u^2\omega^2}/(\omega^2u^2)\right)\right]}{S_m(\omega) + \omega^2\hbar\Omega\Big/\left(8cPMLd_f^2\right)\left[\left(1+4K_m(0)\,d_f^2u^2\right)\left(e^{\omega^2u^2}/(\omega^2u^2)\right)\right]}\,\frac{d\omega}{2\pi}. \tag{406}$$

We have written (406) in the form that would occur if the message m(t) were imbedded at baseband in noise of spectral density

$$S_n(\omega) = S_{C.\,R.}(\omega)\left[\left(1+4K_m(0)\,d_f^2u^2\right)\left(e^{u^2\omega^2}/(u^2\omega^2)\right)\right], \tag{407}$$

where $S_{C.\,R.}(\omega)$ is the noise predicted by the Cramér-Rao bound for FM. Suppose the message spectrum is narrow-band around some center frequency. We could optimize

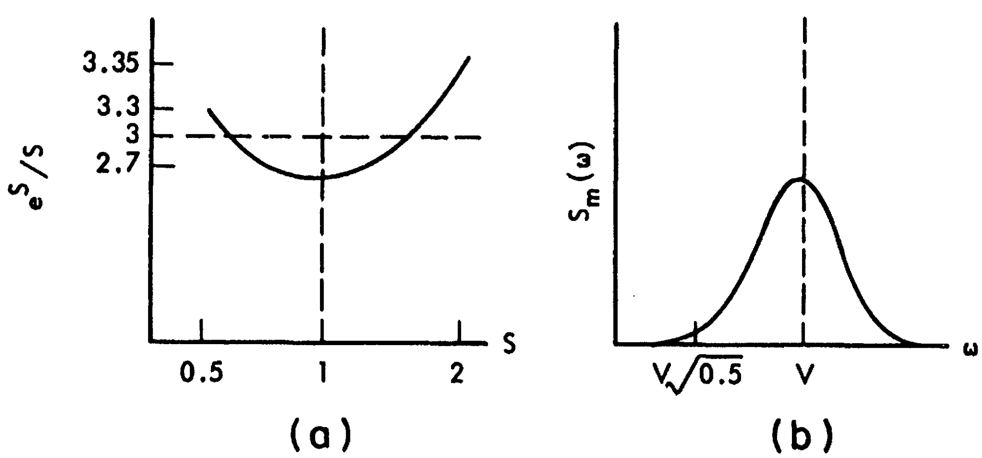


Fig. 11. Illustrations for the prism-lens discriminator.

(407) by adjusting u^2. Consider the function e^s/s. It has a minimum at s = 1 (see Fig. 11a). Suppose that $S_m(\omega)$ is as shown in Fig. 11b. If we set uv = 1, we obtain

$$S_n(\omega) \approx 3\left[1 + 4K_m(0)\,d_f^2/v^2\right] S_{C.\,R.}(\omega). \tag{408}$$

If in addition $4K_m(0)(d_f/v)^2 \ll 1$, our performance is within 5 dB of the Cramér-Rao bound. If, on the other hand, u^2 is very small, then we can replace $\left(e^{u^2\omega^2}/u^2\omega^2\right)$ by $(1+1/u^2\omega^2)$ in (406).

Thus we see that under certain conditions, it is possible for this system to operate close to efficiency in spite of its simplicity.

7.5.3 Amplitude Modulation

From our discussion in Section IV, we know that the optimal DSBSC demodulator for the known-phase case homodynes the received field to baseband and then processes optimally the resulting classical signal plus noise.

If we have a random-phase angle, possibly because of an unstable carrier, we can track the phase in a phase-locked loop, using part of the incoming energy (amplitude and phase measurements are not compatible). If we do not have a perfect phase estimate, the output of the photon counter will include the same noise component, but a reduced signal component. The amplitude of the signal component will be proportional to the cosine of the phase error. This means that our performance is degraded.

Classically, we can use a single-sideband or quadrature modulation to put two signals in the same band. Either of these schemes would require heterodyning as a first step in demodulation. When quantum noise dominates, the noise added to each of the two estimates would be twice as large as the noise added to each if we used disjoint frequency bands and homodyned. In the classical case there is no difference. Thus, if the bandwidth is available, it is to our advantage to avoid SSB or quadrature modulation in the quantum case, perhaps by frequency-multiplexing.

If we have a fading channel, but can still estimate the spatial envelope, then the performance is given by the nonfading Cramér-Rao bound averaged over the fading. The optimal receiver homodynes with an oscillator matched to the estimated fading envelope. This holds true whether the fading is Gaussian, log-normal or whatever.

If we do not have a perfect channel estimate there will be two effects. Estimate errors in the phase of the spatial envelope will cause a reduction of the signal component of the output as previously discussed. Estimation errors in the amplitude of the envelope will cause changes in the noise and signal components at the output. Both effects lead to nonoptimal combining and thus higher noise-to-signal ratio. Furthermore, the filter will not be properly matched to this new signal-to-noise ratio.

It is not likely that we can get a good channel estimate, because of severe fading. In that case, we should avoid this type of modulation.

7.5.4 Intensity Modulation

There is a type of modulation that does not require channel estimation in cases in which there is a large number of diversity paths. This is intensity modulation. It is certainly possible that the signal-to-noise ratio overall is high, while the signal-to-noise ratio per diversity path is small. In such a case, a channel estimate may not be possible. We shall now investigate this scheme. Using Eq. 286, we modulate the intensity of the field around an operating point as follows:

$$E(t, \rho) = 2RL(m(t)+B)^{1/2} C(\rho) A e^{i\Omega t} + n(t, \rho) e^{i\Omega t}$$

$$R_c(\rho, \rho') = E\, C(\rho)\, C^*(\rho') = \Sigma\, b_k \gamma_k(\rho)\, \gamma_k^*(\rho')$$

$$t \in (0, T), \qquad \rho \in (\text{aperture})$$

$$\text{Prob}\,(m(t) \ll B) \approx 1 \tag{409}$$

$$K_m(t, u) = H^{-1}(t, u) + (2c/\hbar\Omega)^2 \int K_m(t, v)\, J(v, x)\, H^{-1}(x, u)\, dx du \tag{410}$$

$$J(t, u) = E\left\{\frac{\delta(t, u)(\hbar\Omega/2c)^2}{B + m(t)} \sum \frac{E_m b_k^2 A^4}{2\left[\left(N_o + 1/2 b_k A^2 E_m\right)(\hbar\Omega/2c) + N_o\left(N_o + A^2 b_k E_m\right)\right]}\right\}, \tag{411}$$

where

$$E_m = \int_0^T (B+m(t))\, dt$$

if we neglect thermal noise $J(t, u) \approx A^2 \delta(t, u)(\hbar\Omega/2cB)\, \Sigma\, b_k$.

Now consider the output of a photon counter in the focal plane of a lens that focuses the incoming field. The process is Poisson if we condition it upon knowledge of the mode amplitudes r_k. The mean of the process is

$$\mu(t)\Big|_{r_k} = (2c/\hbar\Omega) A^2\, \Sigma\, r_k r_k^* (B+m(t)). \tag{412a}$$

Its covariance is

$$R_{\text{counts}}(t_1, t_2)\Big|_{r_k} = \mu(t_1)\, \mu(t_2) + \mu(t_1)\, \delta(t_1, t_2). \tag{412b}$$

We consider each photoelectron as an impulse.

If we call the photon counter output v(t), we have

$$\text{Expect}\,(v(t_1)v(t_2)) = (K_m(t_1-t_2) + B^2)[A^2(2c/\hbar\Omega)]^2 \left[\Sigma\, b_k^2 + (\Sigma b_k)^2\right]$$
$$+ B\, \Sigma\, b_k (2cA/\hbar\Omega)^2\, \delta(t_1, t_2) \tag{413}$$

$$\text{Expect}\,(v(t_1)m(t_2)) = A^2(2c/\hbar\Omega)\, K_m(t_1-t_2)\, \Sigma\, b_k.$$

We can use these results to determine the optimal (Wiener) linear filter. That is, we form the process v(t) by letting the counts drive a narrow pulse generator. The performance will be that which is associated with an additive noise channel at baseband with spectral height $(\hbar\Omega/2c)(B/A^2 \Sigma b_k) + S_m(\omega) \Sigma b_k^2/(\Sigma b_k)^2$. Examining (411) and (410) we see that the first term above is the noise of the Cramér-Rao bound. If we assume that we have fixed total energy and a large number of diversity paths, the second term above will be negligible (that is, $b_k = P/K$ for $k = 1, \ldots, K$; K sufficiently large). Thus our estimate is efficient.

7.6 SYSTEM COMPARISON

We can compare the sampled data systems discussed previously with each other and with a simple PCM system. We shall call upon a fact of rate distortion theory.[18] If we wish to transmit a Gaussian random variable with a priori variance T over a channel such that the average error variance of the estimate is U, then the minimum channel capacity required to perform this task is

$$C_{min} = 1/2 \ln (1+T/U) \text{ bits.} \tag{414}$$

Suppose that we use one of our schemes to transmit N independent Gaussian random variables each of a priori variance T over our quantum channel so that the error variance of each estimate is U. We shall assign a utility to the scheme equal to the minimum channel capacity necessary to perform this task for any system

$$F = (\text{utility}) = (N/2) \ln (1+T/U). \tag{415}$$

Let us compare four systems that could be used over a nonfading channel. We require that each scheme use a fixed or average energy E and operate in an interval of 1 second. We shall write the utility of each system in terms of the bandwidth B in Hz that each uses and the energy E. The aperture field is

$$E(t, \rho) = 2\mathrm{Rl}(E/s)^{1/2} f(t, m(t)) e^{i\Omega t} \tag{416}$$

$t \in (0, 1) \qquad \rho \in (\text{aperture})$

aperture area = s

Average energy in $f(t, m(t)) = 1$.

Systems

1. We use PAM to communicate N pulses over the channel. Each pulse is modulated in amplitude by a GRV of variance T. The energy of each pulse is $(E/NT)A_j^2$, where A_j is the parameter of pulse j. Thus the average total energy is E.

2. We use PPM to communicate N GRV's. Each pulse is modulated in an interval 1/N s long. Each pulse has energy E/N. The modulation is adjusted so that as the random variable takes on values between plus and minus two standard deviations, the pulse moves from the bottom to the top of its interval. We assume that we use a Gaussian-shaped pulse whose bandwidth is $2\beta/\pi$ Hz as discussed in (344). We assume that β is sufficiently large so that with the modulation constraint, the probability of a pulse

extending beyond its interval is small. We assume that the energy per pulse is large enough so that our system achieves the Cramér-Rao performance.

3. We use PFM with a sequence of pulses of length $1/N$. We define the bandwidth as 8π times the modulation index times the standard deviation of the parameter $T^{1/2}$, under the assumption that N is not so large that the bandwidth is governed by the pulse duration.

4. We use PCM to transmit M orthogonal waveforms N times. The information transmitted, under the assumption of correct decisions, is $N \ln M$. We shall detect the pulses by counting photons in each orthogonal mode. The probability of error on each transmission is the probability of receiving no counts in the excited mode $-e^{-2cE/\hbar\Omega N}$. The bandwidth is MN.

Applying the results derived previously, we obtain the comparisons listed in Table 2.

Table 2. System comparison.

System	Utility (F)
PAM	$B/2 \ln (1+8cE/\hbar\Omega B)$ where $B = N$
PPM	$N/2 \ln (1+\pi^2 cB^2E/8\hbar\Omega N^3)$
PFM	$N/2 \ln (1+\pi^2 cB^2E/6\hbar\Omega N^3)$
PCM	$(1-e^{-2cE/\hbar\Omega N})N \ln (B/N)$
PAM (intensity)	$B/2 \ln (cE/8\hbar\Omega B+1)$

We see that the utility increases as ln B for the PPM, PFM, and PCM systems, provided we have sufficient energy. The utility function of the PAM system looks like the Shannon capacity with N_o replaced by $\hbar\Omega/8c$, and with a factor 1/2 in front. If a derivation of the capacity of a known-phase quantum channel becomes available (some work has been done along these lines), we can normalize F to obtain an efficiency-of-operation rating.

If we transmit PPM, PFM or PCM over a fading channel with adequate diversity, the utility ratings remain the same. We must realize, however, that we are using bandwidth B on L paths.

If we transmit PAM intensity modulation over a fading channel with diversity, and if we operate around a bias such that the intensity transmitted when the message parameter is zero is twice the intensity transmitted when the message is minus two standard deviations, then we obtain the result indicated in Table 2. The difference between intensity and E-field PAM is a factor of 64 in average energy (with the bias given above).

7.7 EXTENSIONS TO NON-GAUSSIAN FADING

The assumption that the fading channel can be modeled as a Gaussian random process multiplying the signal envelope is a mathematical convenience. We know that the fading of light signals attributable to turbulence is more often described as a multiplication by a log-normal process. That is, multiplication by a process whose complex logarithm is Gaussian. Does this destroy the applicability of the results obtained herein for fading channels?

The Cramér-Rao bounds derived for Gaussian fading are certainly not directly applicable. For many of the examples discussed, however, we used these bounds only in passing. Often we looked at the known-channel bound, which is a function of the "energy" or "power" that we receive. We assume that we could measure the channel, and then estimate the message optimally based on those results. Our performance would be the performance of the known channel as a function of the received "energy" averaged over the fading process. Even if we could not measure the channel, the result above is a lower bound to what any other technique could do. Sometimes, measuring the channel was not necessary at all, for example, PPM, provided we had enough diversity.

In light of these statements, let us recall the relationship between a received "energy" and the fading process on a channel that is spatially fading and slowly varying in time. The received "energy" is the sum of the squares of the mode amplitudes times the energy we would receive if the spatial envelope were unity. Thus the "energy" is proportional to the sum of the squares of N uncorrelated random variables (which are complex Gaussian for the Gaussian fading case), where N is the number of significant eigenvalues of the spatial envelope correlation function. We can make the statement that if these random variables have finite mean-square values, and if there are enough that are independent, then the sum will converge to its average by the same arguments used in the Gaussian fading case. Thus the probability of a deep fade will be small, and the performance will approach the known-channel bound with fixed energy replaced by average energy. All of the discussion above is predicated on the fact that either we can estimate the channel, or a channel estimate is not required for demodulation (for example, intensity modulation).

In principle, of course, we can directly obtain the Cramér-Rao or Barankin bounds for any type of fading. Unfortunately, analytical problems make this difficult.

For the special case of PAM intensity modulation, where the density operators commute for all parameter values, under the assumption of completely random phase, the optimal estimator can be determined directly by straightforward computation.

VIII. CONCLUSION

It is apparent that the bounds obtained in this report do have considerable use in designing optimal receivers, at least in the case of strong signals. It is also apparent that very often photon counting, perhaps with a local oscillator, will be employed in the receiver structure.

The Barankin bound, although a tighter bound, is difficult to evaluate for the quantum case. Although it would be a useful tool, its form is not as simple as the classical equivalent.[19] I have been able to evaluate it for the single-mode case. Extension to the multimode case poses no great obstacles, but seems to lead to cumbersome expressions.

A problem that needs more investigation is multiparameter estimation, when individual optimal estimators do not commute.[20] The restraint of commutation is not difficult to apply for a mean-square-error cost functional. The difficulty is in the determination of the space upon which the optimal commuting operators should measure. What is needed is more study into so-called noisy measurements of noncommuting operators, and measurement of non-Hermitian operators characterized by a complete set of eigenkets.

There is a classical tool that has no present meaning for the quantum problem. This is the maximum-likelihood estimator. It would be of great value to interpret some maximum-likelihood operator. This will probably evolve with more experience and quantitative results.

Extension of the fading channel results to log-normal fading will probably be possible, once classical tools for handling log-normal processes become available, and more insight into such processes in optical channels makes reasonable engineering approximations possible.

APPENDIX A

Homodyning and Heterodyning with a Local Oscillator and a Photon Counter

R. J. Glauber has shown[9] that the moment-generating function of the sum of the counts of a photon counter subject to an incident plane-wave field that is narrow-field and arrives in the interval $(0, t)$ is given by

$$M_c(s, t) = E[\exp -s(\text{counts in } (0, t))]$$

$$= \int P(\underline{\beta}) \exp\left[(e^{-s}-1)\eta \int_0^t E(\underline{\beta}, u)\, E^*(\underline{\beta}, u)\, du\right] \prod_k d^2\beta_k, \tag{A.1}$$

where the field density operator is

$$\rho = \int P(\underline{\beta}) \prod_k \beta_k\rangle\langle\beta_k\, d^2\beta_k,$$

and the complex field is

$$E(\beta, u) = i \sum \sqrt{\hbar\Omega/2L}\ \beta_k\, e^{-i\omega_k u},$$

where Ω is the center frequency. Suppose the incident plane wave, which is time-limited to the interval $(0, T)$, where $L = cT$, consists in the sum of two fields. The signal field has density operator

$$\rho^m = \prod_k \int (1/\pi\langle n\rangle) \exp\left(-|a_k-\beta_k|^2/\langle n\rangle\right) \cdot a_k\rangle\langle a_k\, d^2 a_k, \tag{A.2}$$

where

$$i \sum \sqrt{\hbar\Omega/2L}\ \beta_k\, e^{-i\omega_k u} = S(m(u), u)\, e^{-i\Omega u}.$$

The added local-oscillator field is

$$\rho^1 = \prod_k \int (1/\pi\langle x\rangle) \exp\left(-|a_k-1_k|^2/\langle x\rangle\right) \cdot a_k\rangle\langle a_k\, d^2 a_k, \tag{A.3}$$

where

$$i \sum \sqrt{\hbar\Omega/2L}\ 1_k\, e^{-i\omega_k u} = A\, e^{-i(\Omega - v)u}.$$

Here, v is a classical frequency, either zero, or large compared with the bandwidth of S(m(u), u).

The total field is given by

$$\rho_{total} = \prod_k \int (1/\pi\langle z\rangle) \exp\left(-|\alpha_k - r_k|^2/\langle z\rangle\right) \cdot \alpha_k\rangle\langle\alpha_k \, d^2\alpha_k, \tag{A.4}$$

where

$$\langle z\rangle = \langle n\rangle + \langle \gamma\rangle$$

and

$$C(u) = i\sum \sqrt{\hbar\Omega/2L}\; r_k e^{-i\omega_k u} = (S(m(u), u) + A\, e^{ivu})\, e^{-i\Omega u}.$$

Plugging (A.4) into (A.1), we get

$$E\left[\exp\left(-s\sum \text{counts in } (0,t)/\eta A\right)\right]$$

$$= M_c(s, t, A)$$

$$= \int \exp\left[(\exp(-s/\eta A)-1)\eta \int_0^t \left|\left(i\sum \sqrt{\hbar\Omega/2L}\;\beta_k\, e^{-i\omega_k u} + C(u)\right)\right|^2 du\right]$$

$$\cdot \prod_k (1/\pi\langle z\rangle) \exp\left(-|\beta_k|^2/\langle z\rangle\right) d^2\beta_k, \tag{A.5}$$

where

$$\beta_k = \alpha_k - r_k.$$

Suppose now that we let A get very large. Equation A.5 becomes (A.6).

$$M_c = \exp\left[-s\int_0^t 2RL(S(m(u), u)\, e^{-ivu})\, du\right] \exp(s^2 t/2\eta)$$

$$\cdot \int \exp(-sAt) \exp\left[-s2RL\sqrt{\hbar\Omega/2L}\; i\sum \beta_k \int_0^t e^{-i(\omega_k - \Omega + v)u}\, du\right]$$

$$\cdot \prod_k (1/\pi\langle z\rangle) \exp\left(-|\beta_k|^2/\langle z\rangle\right) d^2\beta_k. \tag{A.6}$$

We can perform the integration over β_k to obtain

$$M_c(s,t) = \exp\left[-s\int_0^t 2RL(S(m(u),u)\,e^{-ivu})\,du\right]\exp(-sAt)$$

$$\cdot\ \exp[s^2t/2\eta]\,\exp\left[s^2/(2\hbar\Omega/2L)\int_0^t\int_0^t R_n(u,w) + R_n^*(u,w)\,du dw\right], \quad (A.7)$$

where

$$R_n(u,w) = \sum_k \langle z\rangle\, e^{-i(\omega_k - \Omega + v)(u-w)} . \quad (A.8)$$

The index k runs over the modes when we allow our photon counter to count. We shall now investigate that index.

A. 1 HOMODYNE CASE

Suppose we are performing a homodyne measurement. This is, the frequency v is zero. Since the message field envelope $S(m(u),u)$ contains both positive and negative frequencies, the index k must extend over frequencies on both sides of Ω. Thus R_n is an impulse with respect to narrow-band signals about Ω.

$$R_n(t,u) = T\langle n\rangle\ \delta(t,u). \quad (A.9)$$

A. 2 HETERODYNE CASE

If we are performing a heterodyne measurement, the frequency v is higher than the largest frequency in $S(m(u,u))$. The index k must extend over all frequencies such that $\omega_k - \Omega + v \geq 0$. This insures that the entire message plus local-oscillator field is within the sensitive region of the counter. We shield other frequencies, to prevent their noise contributions, from entering the photon counter. We get

$$R_n(t,u) + R_n^*(t,u) = T\langle z\rangle\ \delta(t,u). \quad (A.10)$$

Plugging (A.9) and (A.10) into (A.7), we see that we have two cases. If we think of the output of the counter as a continuous g(t) signal (the counting rate before normalization was high because of the local oscillator. Any smoothing by the apparatus will make g(t) look continuous) and if we subtract off the bias A, then this waveform which is the counts per second, for an ideal photon counter with $\eta = 2c/\hbar\Omega$, is

$$g(t) = RL(S(m(t),t)+n(t))$$

$$E(n(t)n(u)) = (\hbar\Omega/2c)\left(\frac{\langle n\rangle}{2} + \frac{1}{4}\right)\delta(t,u), \quad (A.11)$$

the homodyne case. We have divided the output by two. And

$$g(t) = 2RL(S(m(t), t)\, e^{i\nu t}) + n(t)$$

$$E(n(t)n(u)) = (\hbar\Omega/2c)(\langle n\rangle+1)\ \delta(t, u), \tag{A. 12}$$

the heterodyne case, where we assume $\langle z\rangle \approx \langle n\rangle$. That is, the local oscillator contributes negligible noise.

APPENDIX B

Field Operators

Quantum field theory tells us to treat each mode of the electromagnetic field in a bounded region of space as a quantum harmonic oscillator. We shall discuss some of the salient features here. For a more complete treatment, several authors may be consulted.[2,4,5] We must first define the Hilbert space in which our operators operate and which represents our quantum harmonic oscillator. Strictly speaking, since our operators may have continuous spectra, our space is larger than a Hilbert space; and is sometimes called a "ket" space. One way to define the space is to specify a set of orthogonal vectors that span the space, and the effects of the space operators on this set of vectors; that is, specify a coordinate system. Let N be a Hermitian operator on our space. Our reference vectors will be the eigenstates of N that we denote $n\rangle$.

$$N\, n\rangle = n\, n\rangle. \tag{B. 1}$$

Thus far we are fairly general. We make no restriction on the real numbers n. We assume that the $n\rangle$ form a complete set. Now consider two non-Hermitian operators b and its adjoint b^+. We require

$$[b, b^+] = 1. \tag{B. 2}$$

Require next that

$$b\, 0\rangle = 0, \tag{B. 3}$$

where $0\rangle$ is one of the number states $n\rangle$ that are eigenstates of N. Furthermore, require the rest of the eigenstates to be generated as follows:

$$n\rangle = \frac{(b^+)^n\, 0\rangle}{(n!)^{1/2}}. \tag{B. 4}$$

From (B. 2) and (B. 4) we have

$$b^+\, n\rangle = (n+1)^{1/2}\, (n+1\rangle)$$

$$b\, n\rangle = (n)^{1/2}\, (n-1\rangle),$$

since $[b, b^{+k}] = kb^{+k-1}$, and

$$b^+b\, n\rangle = n\, n\rangle. \tag{B. 5}$$

Thus b^+b which is clearly Hermitian is the number operator N. Furthermore, this guarantees that the $n\rangle$ defined in (B. 4) are in fact orthogonal.

The operators b and b^+ are called the annihilation and creation operators, respectively. They are also called the boson operators. The eigenstates $n\rangle$ of b^+b are called the number states, since they correspond to the integer number of photons associated with an oscillator if it is in one of these states. Measuring the operator b^+b will result in an integer outcome. (That the eigenvalues are integers is guaranteed by (B.7).) In general the state of the oscillator will be a linear combination of the number states. Furthermore, the state may only be known statistically. For the case of a harmonic oscillator in thermal equilibrium at temperature T, the density operator is

$$\rho = \sum_j (\langle n\rangle + 1)^{-1} \left(\frac{\langle n\rangle}{\langle n\rangle + 1}\right)^j j\rangle\langle j, \tag{B.6}$$

where

$$b^+b\, j\rangle = j\, j\rangle$$

$$\langle n\rangle = \mathrm{TR}\ \rho b^+ b = (e^{\hbar\Omega/kT} - 1)^{-1}$$

Ω = oscillator frequency in rad/s.

When the field mode arises from the radiation of a classical (strong, negligibly reacting with the radiation process) source, it has been shown by Glauber[9] to be in an eigenstate of the non-Hermitian operator b. The eigenstates of b are complete, but not orthogonal. They are linear combinations of the number states given by

$$\beta\rangle = e^{-1/2|\beta|^2} \sum_n \frac{\beta^n}{(n!)^{1/2}}\, n\rangle, \tag{B.7}$$

where

$$b\,\beta\rangle = \beta\,\beta\rangle$$

β is a complex number

$$\langle \alpha\beta\rangle = \exp(\alpha^*\beta - 1/2|\alpha|^2 - 1/2|\beta|^2).$$

The density operator of a nonstatistical classical current source field, for the mode that we are considering, is

$$\rho = \alpha\rangle\langle\alpha|, \tag{B.8}$$

where the value of α is determined by the correspondence discussed in Section III.

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13. ABSTRACT

This report is concerned with the incorporation of the axioms of quantum measurements into current communication estimation theory. It is well known that classical electromagnetic theory does not adequately describe fields at optical frequencies. The advent of the laser has made the use of optical carriers for information transmission practical. Classical communication estimation theory emphasizes background noise and channel fading as primary limitations on system performance. At optical frequencies, quantum effects may totally dominate performance. Estimation theory is formulated using the quantum theory so that this type of system limitation can be understood, and optimal receivers and systems designed.

The equations determining the optimal minimum mean-square-error estimator of a parameter imbedded in a quantum system are derived. Bounds analogous to the Cramér-Rao and Barankin bounds of classical estimation theory are also derived, and then specialized to the case of an electromagnetic field in a bounded region of space. Cramér-Rao-type bounds for estimation of parameters and waveforms imbedded in known and fading channels are derived.

In examples optimal receivers for the commonly used classical modulation schemes, such as PPM, PAM, PM, DSBSC, are derived. The differences between classical and quantum systems in implementation and performance are emphasized.

It is apparent from the examples and from the structure of the bounds, that quantum effects often appear as an additive white noise arising in heterodyne and homodyne structure receivers. These receivers are not always optimal in performance or in implementation simplicity. Other receivers employing detection by photon counting are sometimes optimal or near optimal.

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