Transform/Subband Analysis and Synthesis of Signals

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Abstract

A general relationship between transform representation and subband representation of a signal is described in a unifying and tutorial manner. First, the transform and subband representations of a signal are described. It is then shown how one representation can be obtained from a simple rearrangement of the other representation. Specific examples illustrate how a block transform can be implemented as an FIR subband filter bank, and vice-versa. The extension of the results is made for the two-dimensional case. Finally, applications to signal coding and adaptive amplitude modulation are discussed.

Acknowledgements

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Transform/Subband Analysis and Synthesis of Signals

1 Introduction

In many signal processing applications, it is convenient to decompose a signal into a more suitable form for processing. For instance, in transform image coding, an image is commonly decomposed using a discrete cosine transform operation. This representation of an image may be more convenient for data rate reduction. The decomposition of a signal is referred to as the analysis operation. The reconstruction of the signal is referred to as the synthesis operation.

Two methods of decomposing a signal are transform analysis [1,2] and subband analysis [2,3,4,5]. Even though these two types of signal analysis are often considered different, they result in the same time/frequency representation of a signal. For a given transform analysis, an equivalent subband analysis can be found, and vice-versa. One representation may be more useful than the other in a particular application, but both offer a useful perspective.

This paper describes a general relationship between transform analysis/synthesis and subband analysis/synthesis of a signal. Let $x[n]$ denote a one-dimensional signal and $X_r[k]$ denote a two-dimensional time-frequency representation of $x[n]$. The function $X_r[k]$ is a function of a frequency variable k and a time variable r and represents some form of local spectral contents of $x[n]$. When $X_r[k]$ is viewed as a function of

k, it is a transform analysis of $x[n]$. When $X_r[k]$ is viewed as a function of r, it is a subband analysis of $x[n]$. An example of $x[n]$ and $|X_r[k]|$ is shown in Figure 1. The function $X_r[k]$ used in Figure 1 is the short-time Fourier transform of $x[n]$, which has been used [6,7] extensively in speech processing applications.

The relationships between transform analysis/synthesis and subband analysis/synthesis have been discussed in the recent literature [5,8,9,10,11]. In this paper, we present a general relationship between the two in a unifying and tutorial manner. The outline of this paper is as follows. The transform and subband representations for one-dimensional signals are described in Sections 2 and 3, respectively. Section 4 then describes the general relationship between the transform and subband representations. Section 5 generalizes the results to the two-dimensional case. Section 6 discusses some applications of these ideas. Section 7 concludes this paper.

2 Transform Representation

One method of representing a signal is using a transform operation. In this method, a signal is decomposed by segmenting the signal and performing block transformations (mappings) on each segment. Let the input signal $x[n]$ be written as a column vector $\mathbf{x}$. Then the transform coefficient representation of $\mathbf{x}$ is given by $\mathbf{X}$:

$$\mathbf{X} = \mathbf{T}\mathbf{x} \tag{1}$$

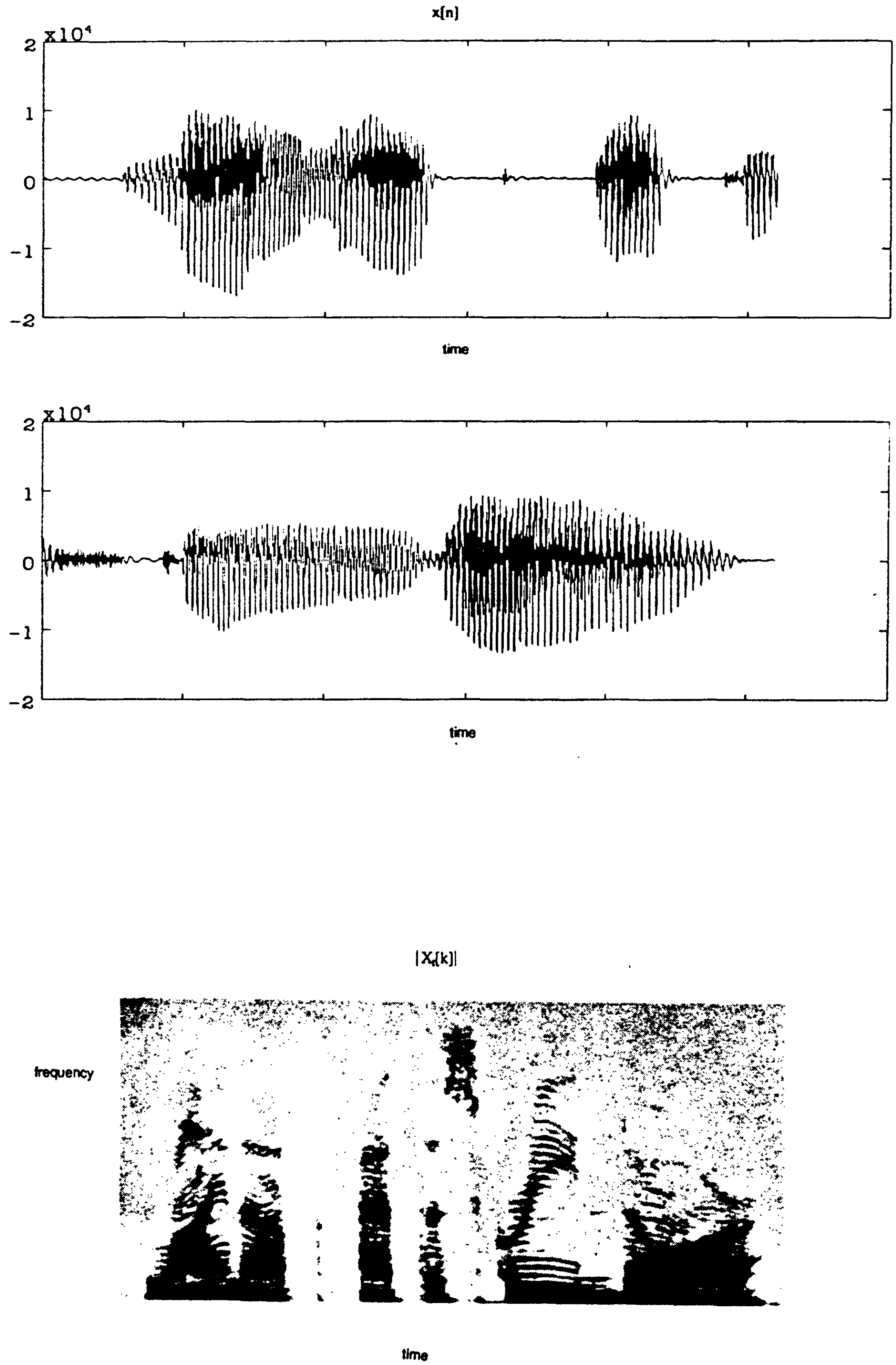

Figure 1: Example of short-time Fourier transform analysis.

where $\mathbf{T}$ is the transform analysis matrix. The transformation, or decomposition, from $\mathbf{x}$ to $\mathbf{X}$ is the analysis operation. In general, if the input $\mathbf{x}$ is segmented into N blocks, $\mathbf{T}$ will be of the form

$$\mathbf{T} = \mathbf{diag}(\mathbf{T_0}, \mathbf{T_1}, ...\mathbf{T_{N-1}}) = \begin{bmatrix} \mathbf{T_0} & & & \\ & \mathbf{T_1} & \mathbf{0} & \\ & & \cdot & \\ & \mathbf{0} & \cdot & \\ & & & \mathbf{T_{N-1}} \end{bmatrix} \tag{2}$$

where each $\mathbf{T_r}$, $0 \leq r \leq N-1$, is a matrix and $\mathbf{T}$ is block diagonal. If $\mathbf{T} = \mathbf{T_0}$, then the transformation is over the entire signal. The dimensions of the matrices $\mathbf{T_r}$ will determine whether the input segments are disjoint or overlapped.

The synthesis operation is the transformation from $\mathbf{X}$ to $\mathbf{y}$, where:

$$\mathbf{y} = \mathbf{UX} \tag{3}$$

and $\mathbf{U}$ is the transform synthesis matrix. In general, $\mathbf{U}$ will be of the form:

$$\mathbf{U} = \mathbf{diag}(\mathbf{U_0}, \mathbf{U_1}, ...\mathbf{U_{N-1}}) \tag{4}$$

where each $\mathbf{U_r}$ is a matrix and $\mathbf{U}$ is block diagonal. If $\mathbf{U} = \mathbf{U_0}$, then the transformation is over the entire signal. If $\mathbf{U} = \mathbf{T^{-1}}$, then perfect reconstruction of $\mathbf{x}$ is achieved; that is, $\mathbf{y} = \mathbf{x}$, and $y[n] = x[n]$.

The transform operations can be classified into two basic types: those in which the input segments are disjoint, and those in which the input segments overlap. We first consider the case of disjoint input segments. Disjoint input segments can be obtained by partitioning the one-dimensional signal $x[n]$ of length NM into N smaller, disjoint signals $x_r[n]$, $0 \leq r \leq N-1$, of length M, as follows:

$$\hat{x}_r[n] = x[n + Mr] \quad , \quad 0 \leq n \leq M-1 \tag{5}$$

If each of the N signals $\hat{x}_r[n]$ is written as an M-dimensional column vector $\hat{\mathbf{x}}_\mathbf{r}$, then the composite NM-dimensional column vector $\mathbf{x}$ is formed by $\mathbf{x} = [\hat{\mathbf{x}}_\mathbf{0} \; \hat{\mathbf{x}}_\mathbf{1} \; \ldots \; \hat{\mathbf{x}}_\mathbf{N-1}]^\mathbf{T}$. When the input segments are disjoint, the corresponding transform analysis/synthesis matrices $\mathbf{T_r}$ and $\mathbf{U_r}$ in Equations (2) and (4) are square MxM matrices, where the transform block size is M.

The transform coefficient representation of $\mathbf{x}$ is given by $\mathbf{X}$ in Equation (1), where $\mathbf{X} = [\mathbf{X_0} \; \mathbf{X_1} \; \ldots \; \mathbf{X_{N-1}}]^\mathbf{T}$. Viewed as a function of k, the signal $X_r[k]$ associated with the vector $\mathbf{X_r}$ contains the transform coefficients for the r^{th} subblock $\hat{x}_r[n]$. The collection of the signals $X_r[k]$ provides a transform representation of $x[n]$. This is illustrated in Figure 2. Note that a short segment of the input $\hat{\mathbf{x}}_\mathbf{r}$ is transformed into a short segment of the output $\mathbf{X_r}$ using the transformation matrix $\mathbf{T_r}$. The fact that a small portion of the output is obtained from transformation of a small portion of the input is the idea behind "short-time transforms."

Typical examples of non-overlapping block transformations include the discrete Fourier transform (DFT) and the discrete cosine transform (DCT). For the DFT, the analysis and synthesis operations in Equations (1)-(4) are defined by the $\mathbf{T_r}$ and $\mathbf{U_r}$

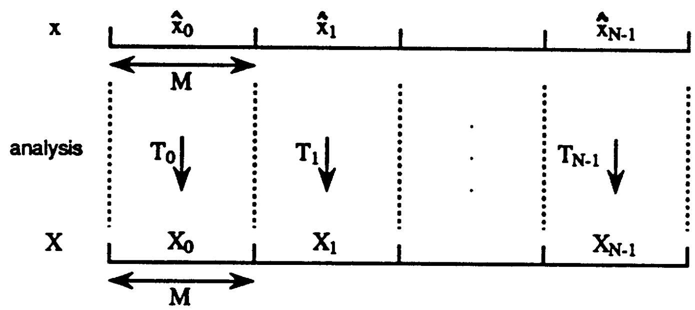

Figure 2: Transform representation using disjoint input segments.

matrices with elements (row = a, column = b):

$$(\mathbf{T_r})_{ab} = e^{-j\frac{2\pi ab}{M}} \tag{6}$$

$$(\mathbf{U_r})_{ab} = \frac{1}{M}e^{+j\frac{2\pi ba}{M}} \tag{7}$$

where $0 \leq a, b \leq M-1$ and $0 \leq r \leq N-1$. With the DFT transform analysis/synthesis, perfect reconstruction is achieved. This transform is also referred to as the short-time Fourier transform (STFT).

The analysis/synthesis operations for the DCT are defined by the following $\mathbf{T_r}$ and $\mathbf{U_r}$ matrices:

$$(\mathbf{T_r})_{ab} = 2cos(\frac{\pi a}{2M}(2b+1)) \tag{8}$$

$$(\mathbf{U_r})_{ab} = \frac{1}{M}(\frac{1}{2}\delta[b] + u[b-1])cos(\frac{\pi b}{2M}(2a+1)) \tag{9}$$

where $\delta[\,]$ is the unit-sample function, $u[\,]$ is the unit-step function, $0 \leq a, b \leq M-1$, M is an even integer (even length, even symmetric DCT/IDCT), and $0 \leq r \leq N-1$.

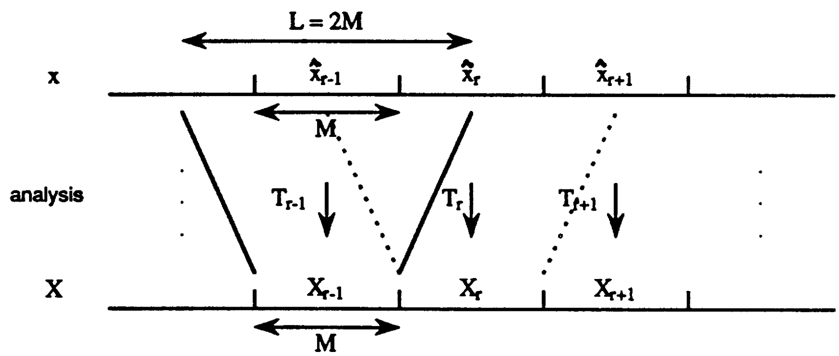

Figure 3: Transform representation using lapped input segments.

The DCT transform analysis/synthesis also achieves perfect reconstruction, and has been used extensively in image coding applications [1,2].

If the input segments overlap, then the transform analysis/synthesis matrices $\mathbf{T_r}$ and $\mathbf{U_r}$ are rectangular. These types of transforms are referred to [12,13] as "lapped orthogonal transforms" (LOT). In one particular LOT, the dimension of the transformation analysis matrix $\mathbf{T_r}$, $1 \leq r \leq N-2$ (away from the signal edge boundaries), is MxL, where $L = 2M$. In this case, the overlap into adjacent segments is by M samples. This is illustrated in Figure 3. Note that overlapping L-sample input segments are represented by M-sample output segments. When the signal edge boundaries are also taken into account, the number of output values is the same as the number of input values. Perfect reconstruction of the entire signal (vector) $\mathbf{x}$ requires invertibility of the matrix $\mathbf{T}$.

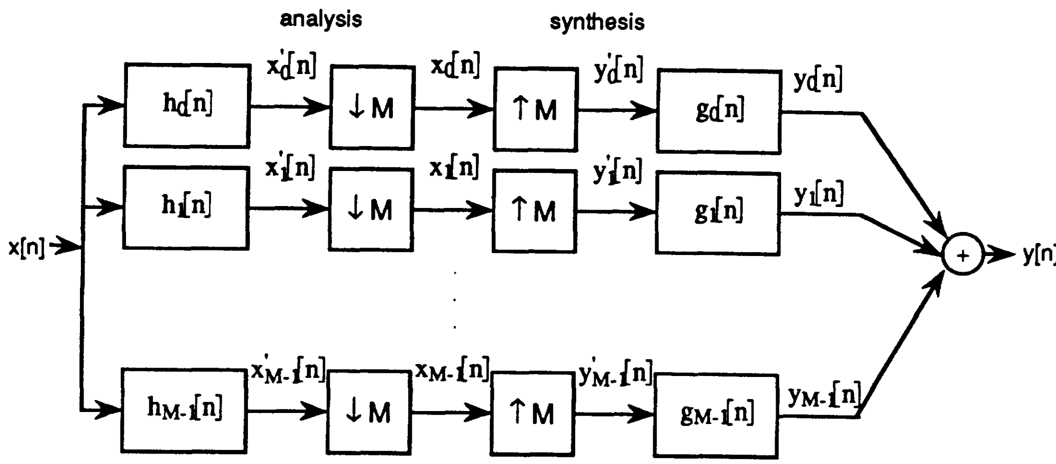

Figure 4: Subband filter analysis/synthesis bank.

3 Subband Representation

Another method of representing a signal is to use subband analysis. The basic idea behind subband analysis of a signal is to decompose the signal into a set of different frequency components, or subbands. Figure 4 illustrates this with a typical M-band subband filter bank, also referred to as a "critically sampled filter bank."

In the subband analysis operation, the input signal $x[n]$ is passed through a bank of M filters $h_i[n]$, $0 \leq i \leq M-1$, typically bandpass and disjoint in nature, and downsampled by M. The decimated output signal $x_i[n]$, viewed as a function of n, is referred to as the i^{th} subband. Because these outputs are effectively bandpass versions of the input, they typically have features similar to the input. The collection of the signals $x_i[n]$ provides a subband representation of $x[n]$.

The analysis filters $h_i[n]$ and the synthesis filters $g_i[n]$ in Figure 4 can be chosen such that the output $y[n]$ is a perfect reconstruction of the input $x[n]$, i.e. $y[n] = x[n]$. One class of these filters is referred to as the quadrature mirror filter (QMF) bank [3,14,15,16,17].

Consider the case when the analysis filters $h_i[n]$ are L-tap FIR filters. When $L = M$ (the decimation factor), the set of downsampled values in the i^{th} subband $\{..., x_i[n-1], x_i[n], x_i[n+1], ...\}$ is obtained from separate (disjoint) M-length portions of the input $x[n]$. However, when $L > M$, the set of downsampled values is obtained from *overlapping* L-length portions of the input $x[n]$.

One example of a subband decomposition of a signal when $L = M$ (no overlap) is the STFT (or DFT). When implemented as a subband filter bank, the appropriate analysis/synthesis filters (perfect reconstruction) for an M-band decomposition as in Figure 4 are given by:

$$h_i[n] = \sum_{m=0}^{M-1} e^{-j\frac{2\pi im}{M}} \delta[n+m] \tag{10}$$

$$g_i[n] = \sum_{m=0}^{M-1} \frac{1}{M} e^{+j\frac{2\pi im}{M}} \delta[n-m] \tag{11}$$

where $0 \leq i \leq M-1$. The M subbands $x_i[n]$ provide a subband signal representation of $x[n]$.

Another example of a subband representation of a signal is using a modulated filter bank [10,18]. A general modulated filter bank is obtained by modulation of a base-band filter. Modulated filter banks can be designed in which $L > M$.

4 Relationship between Transform and Subband Representations

In this section, we discuss a simple relationship between transform representation and subband representation. Let $X_r[k]$ represent the k^{th} transform coefficient in the r^{th} transform block, and let $x_i[n]$ represent the n^{th} subband signal value in the i^{th} subband. Then the transform representation $X_r[k]$ of a signal $x[n]$ can be related to the subband representation of $x_i[n]$ of the signal by

$$x_i[n] = X_r[k]|_{r=n,\ k=i} \tag{12}$$

and

$$X_r[k] = x_i[n]|_{n=r,\ i=k} \tag{13}$$

This means that the subband representation is a simple rearrangement of the transform representation, and vice-versa.

To discuss the relations given by Equations (12) and (13), we first discuss the short-time Fourier transform (STFT). We then illustrate the relation given by Equation (12) using the STFT, DCT, and LOT as examples. The relation given by (13) is illustrated using the STFT and the modulated filter bank as examples.

4.1 Short-Time Fourier Transform

The STFT [6,19] discussed in Sections 2 and 3 is a specific example of a time-frequency representation of a signal. It can be viewed as a transform operation or a filtering (subband) operation. Specifically, as discussed in Section 2, a STFT representation can be obtained by using the transform matrix of Equation (6) in Equations (1) and (2). The resulting signals $X_r[k]$, viewed as a function of k (r fixed), provide a transform representation of $x[n]$. For a fixed r, $X_r[k]$ gives a frequency decomposition for the r^{th} temporal segment.

If viewed as a function of r (with k fixed), the signals $X_r[k]$ provide a subband representation of $x[n]$. For a fixed k, $X_r[k]$ gives a temporal decomposition of $x[n]$ into the k^{th} frequency band. This suggests that $X_r[k]$, for a fixed k, can be obtained as the output of some bandpass filter. This is, in fact, what is performed by the subband filter bank, as shown in Figure 4. As will be shown later, the analysis filters for the STFT given by Equation (10) will yield an equivalent representation for $x[n]$ as does the transform analysis matrix given by Equation (6).

4.2 Subband Representation from Transform Representation

By viewing $X_r[k]$ as a function of k, a transform representation of $x[n]$ is obtained. By viewing $X_r[k]$ as a function of r, a subband representation of $x[n]$ is obtained. Denoting the n^{th} subband signal value in the i^{th} subband as $x_i[n]$, $x_i[n]$ can be obtained from $X_r[k]$ by Equation (12). The subband representation, therefore, is a simple rearrangement of the transform representation. An example is illustrated in Figure 5.

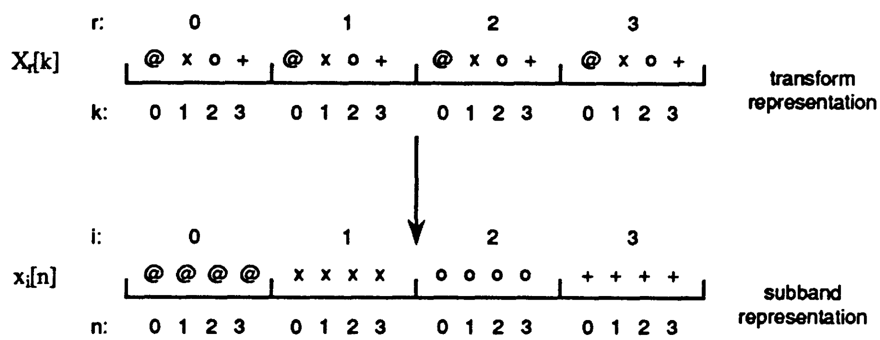

Figure 5: Subband representation from transform representation.

For a given transform analysis matrix, an equivalent subband analysis filter bank can be found. The analysis filters $h_i[n]$ in Figure 4 will be referred to as the "transform analysis filters."

The validity of Equation (12) is now illustrated for the STFT, or the nonoverlapping DFT. The transform analysis matrix for an M-point DFT is defined in Equation (6). The matrix analysis operation defined by Equations (1), (2), (5), and (6) can be written as:

$$X_r[k] = \sum_{u=0}^{M-1} x[u + Mr] e^{-j\frac{2\pi ku}{M}} \tag{14}$$

When viewed as a function of k, this results in a transform representation of $x[n]$. The following shows that by using the FIR transform analysis filters of Equation (10), the resulting subband filtered signals $x_i[n]$ (refer to Figure 4) are simply rearrangements of the transform coefficients $X_r[k]$:

$$x_i'[n] = x[n] * h_i[n] \tag{15}$$

$$= \sum_{m=-\infty}^{\infty} x[m]h_i[n-m] \tag{16}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \sum_{u=0}^{M-1} e^{-j\frac{2\pi iu}{M}} \delta[n-m+u] \tag{17}$$

$$= \sum_{u=0}^{M-1} \sum_{m=-\infty}^{\infty} x[m]\delta[n-m+u]e^{-j\frac{2\pi iu}{M}} \tag{18}$$

$$= \sum_{u=0}^{M-1} x[n+u]e^{-j\frac{2\pi iu}{M}} \tag{19}$$

$$\Rightarrow x_i[n] = x_i'[nM] = \sum_{u=0}^{M-1} x[nM+u]e^{-j\frac{2\pi iu}{M}} = X_r[k]|_{r=n,\ k=i} \tag{20}$$

Q.E.D.

Note that the ensemble of $x_i'[n]$ for a fixed n is the short-time M-point DFT of $x[n]$ (i^{th} frequency component) starting at time n.

In Figure 6, the magnitude frequency responses $|H_i(\omega)|$ for the DFT transform analysis filters are shown when the number of subbands (or transform block size) is $M = 8$. Note that these magnitude responses are not symmetric about $\omega = 0$. This is because the DFT analysis filters $h_i[n]$ in Equation (10) are not purely real.

It can be shown that the DFT FIR synthesis filters $g_i[n]$ of Equation (11) perform the IDFT operation (perfect reconstruction) defined by the IDFT matrix of Equation (7). The filters $g_i[n]$ in Figure 4 will be referred to as the "transform synthesis filters."

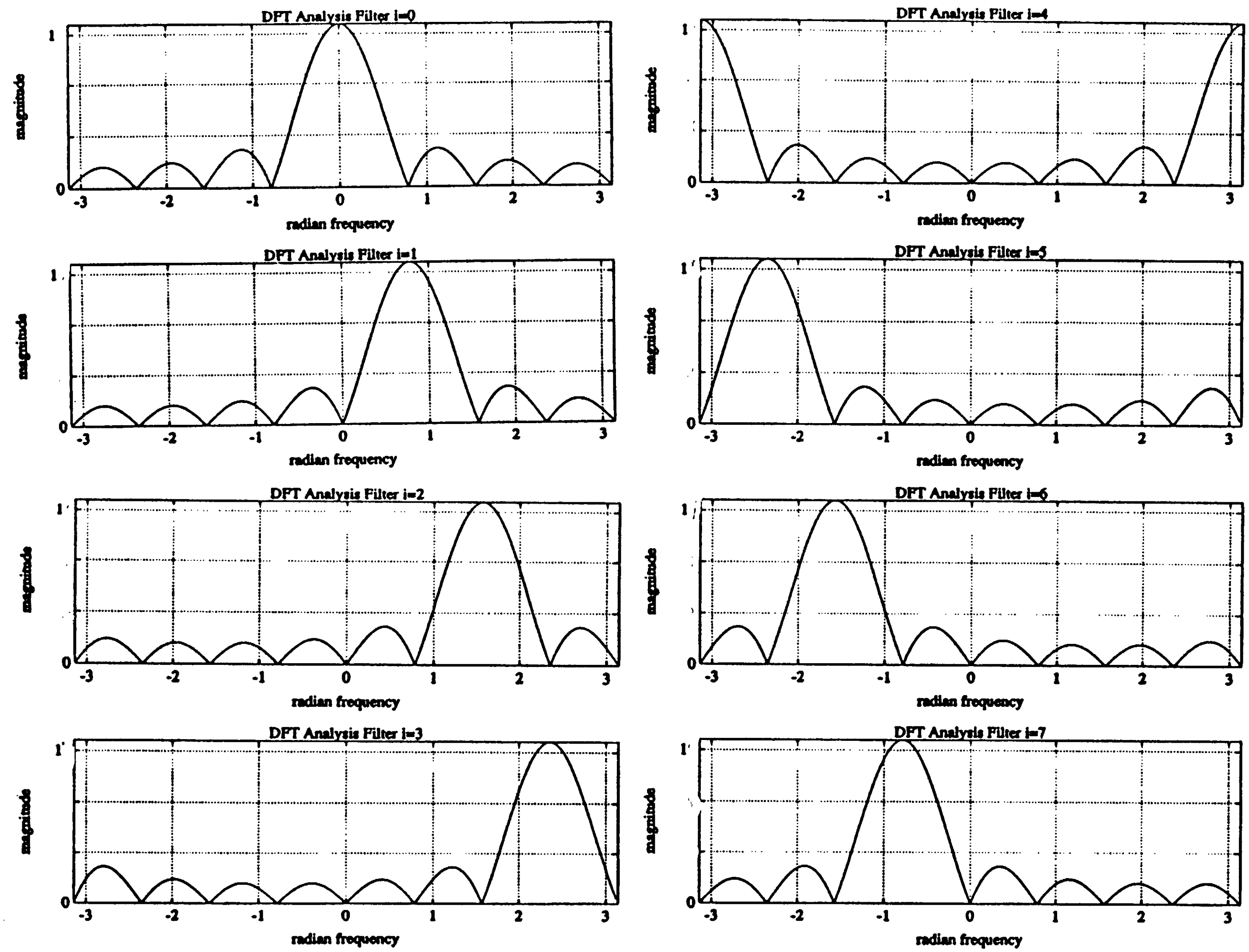

Figure 6: Magnitude responses for the 8-band DFT transform analysis filters.

An equivalent transform analysis filter can be found for the nonoverlapping DCT transform analysis matrix of Equation (8). The matrix analysis operation defined by Equations (1), (2), (5), and (8) can be written as:

$$X_r[k] = \sum_{u=0}^{M-1} 2x[u + Mr] cos(\frac{\pi k}{2M}(2u + 1)) \tag{21}$$

where M is an even integer (even length, even symmetric DCT). When viewed as a function of k, this results in a transform representation of $x[n]$. An equivalent subband representation of $x[n]$ can be obtained by using the following FIR DCT transform analysis filters:

$$h_i[n] = \sum_{m=0}^{M-1} 2cos(\frac{\pi i}{2M}(2m + 1))\delta[n + m] \tag{22}$$

The following shows that this bank of analysis filters satisfies Equation (12) (refer to Figure 4):

$$\begin{aligned}
x_i'[n] &= x[n] * h_i[n] && (23)\\
&= \sum_{m=-\infty}^{\infty} x[m]h_i[n-m] && (24)\\
&= \sum_{m=-\infty}^{\infty} x[m] \sum_{u=0}^{M-1} 2cos(\frac{\pi i}{2M}(2u+1))\delta[n-m+u] && (25)\\
&= \sum_{u=0}^{M-1} \sum_{m=-\infty}^{\infty} 2x[m]cos(\frac{\pi i}{2M}(2u+1))\delta[n-m+u] && (26)\\
&= \sum_{u=0}^{M-1} 2x[n+u]cos(\frac{\pi i}{2M}(2u+1)) && (27)
\end{aligned}$$

$$x_i[n] = x_i'[nM] = \sum_{u=0}^{M-1} 2x[nM + u]cos(\frac{\pi i}{2M}(2u + 1)) = X_r[k]|_{r=n,\ k=i} \tag{28}$$

Q.E.D.

Note that the ensemble of $x_i'[n]$ for a fixed n is the short-time M-point DCT of $x[n]$ (i^{th} frequency component) starting at time n.

The magnitude frequency responses $|H_i(\omega)|$ for the DCT transform analysis filters $h_i[n]$ are shown in Figure 7 when the number of subbands (or transform block size) is $M = 8$. Notice how the higher frequency band filters (for example $i = 6, 7$) are good at stopping the frequencies outside their "passbands"; that is, they do not have much interband leakage. Therefore, since typical images tend to be low frequency in content, these higher frequency subbands, or transform coefficients, will tend not to contain as much energy relative to the lower frequency ones. This is one explanation for the good energy compaction property of the DCT that makes it attractive in image coding applications.

It can be shown that the DCT FIR synthesis filters that perform the IDCT operation (perfect reconstruction) defined by the IDCT matrix of Equation (9) are given by:

$$g_i[n] = \sum_{m=0}^{M-1} \frac{1}{M} w[i] cos(\frac{\pi i}{2M}(2m+1))\delta[n-m] \tag{29}$$

where $w[i] = \frac{1}{2}\delta[i] + u[i-1]$, and $0 \leq i \leq M-1$.

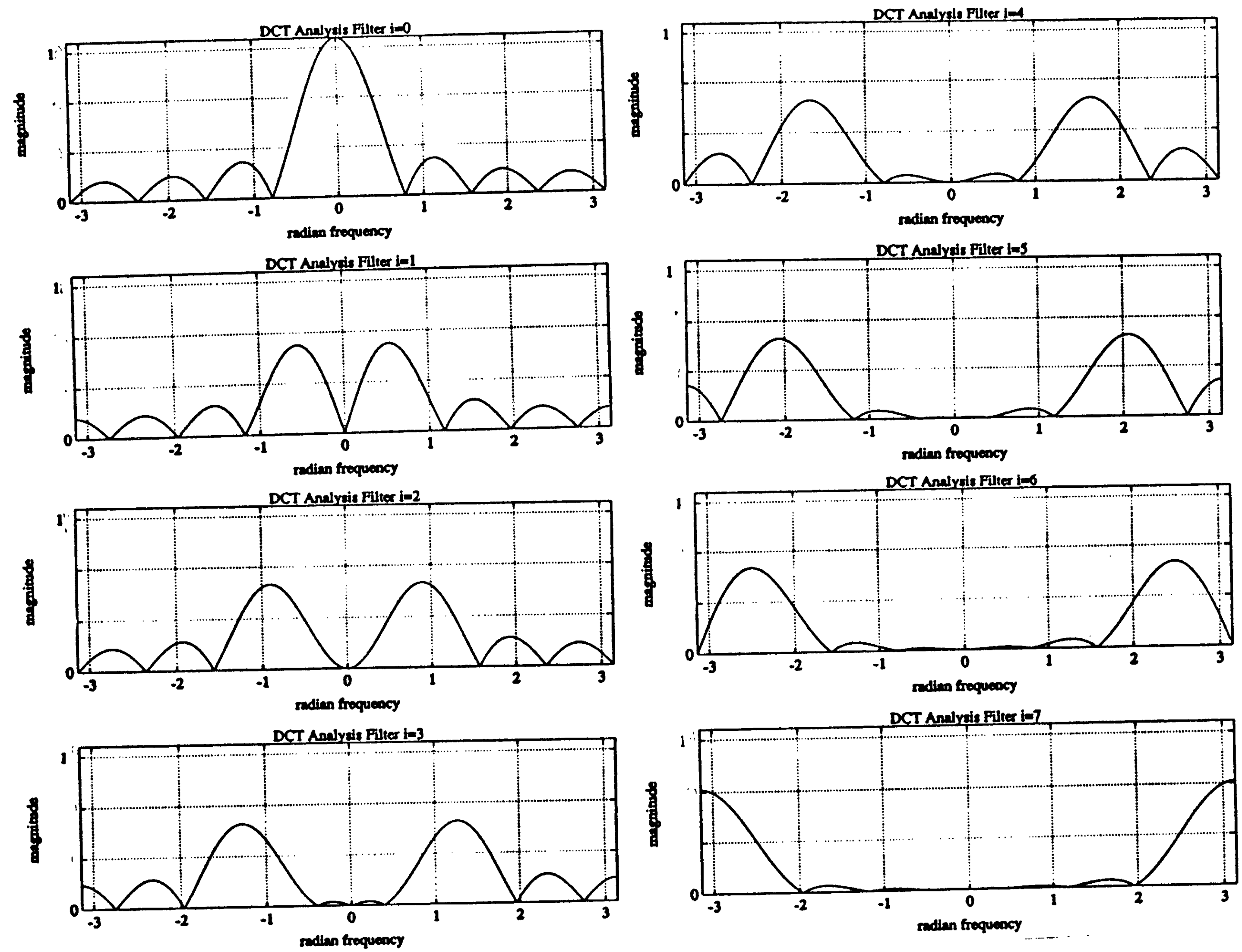

Figure 7: Magnitude responses for the 8-band DCT transform analysis filters.

For the case of a lapped transform such as the LOT, the number of taps in the FIR transform analysis filter, L, will be greater than M, the decimation factor (refer to Figure 4). Consider a particular LOT with transformation analysis matrix elements (row $= a$, column $= b$) $(\mathbf{T_r})_{ab}$, away from signal edge boundaries. Also let the dimensions of $\mathbf{T_r}$ be MxL, where $L = 2M$, and M is an even integer. If the overlap for a given input segment is by $\frac{M}{2}$ samples on each adjacent segment, then the appropriate transform analysis filter is:

$$h_i[n] = \sum_{m=0}^{L-1} (\mathbf{T_r})_{im} \delta[n + m - \frac{M}{2}] \tag{30}$$

where $0 \leq i \leq M - 1$ and $0 \leq m \leq L - 1$. Figure 8 shows an example [18] of the magnitude frequency responses $|H_i(\omega)|$ for a lapped transform analysis with $M = 8$ and $L = 2M = 16$.

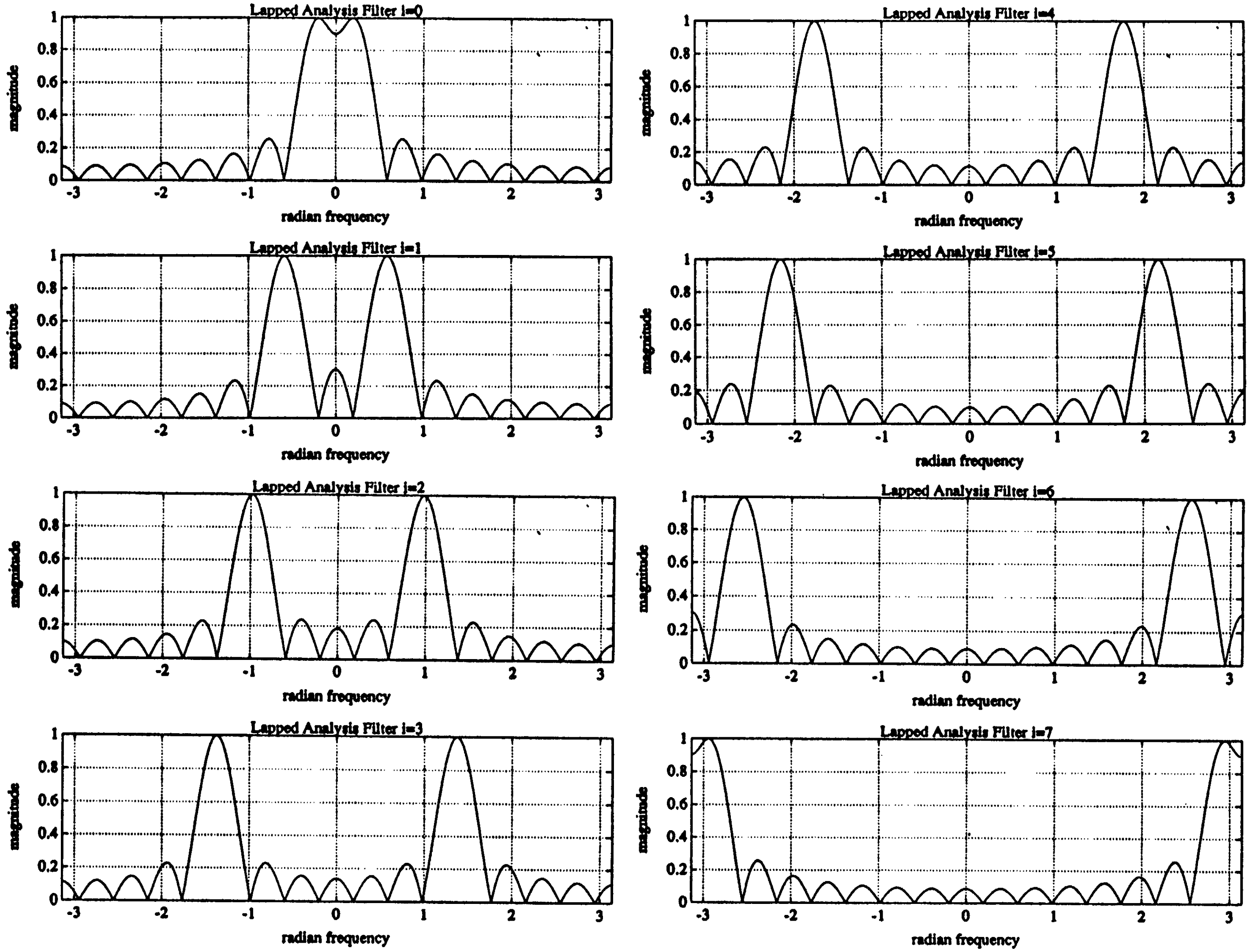

Figure 8: Magnitude responses for the 8-band lapped transform analysis filters.

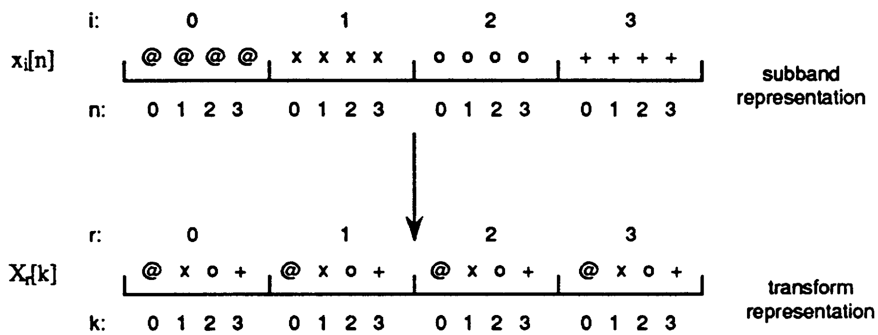

Figure 9: Transform representation from subband representation.

4.3 Transform Representation from Subband Representation

Let $x_i[n]$ represent the n^{th} subband signal value in the i^{th} subband, and let $X_r[k]$ represent the k^{th} transform coefficient in the r^{th} transform block. By viewing $x_i[n]$ as a function of i, a transform coefficient representation of $x[n]$ is obtained. The transform representation $X_r[k]$ of $x[n]$ can be obtained from the subband representation by Equation (13). The transform representation is simply a rearrangement of the subband representation. An example is illustrated in Figure 9.

For a given subband analysis filter bank, an equivalent transform analysis matrix can be found. An example of a subband analysis filter bank (no overlap) is given by the STFT (DFT) transform analysis filters in Equation (10). Equations (15)-(20) show that these filters result in the subband filtered signals of Equation (20).

Equation (14) shows the transform coefficients that result from the transform analysis matrix of Equation (6). Comparison of Equations (20) and (14) shows that in fact, $X_r[k] = x_i[n]|_{i=k,\ n=r}$. Therefore, the STFT transform analysis matrix is given by Equation (6).

The validity of the relationship in Equation (13) can also be illustrated for the case of a modulated filter bank. Consider the case for a modulated filter bank where $L > M$, M is an even integer, and an overlap between adjacent input segments of $\frac{M}{2}$ samples. This can then be considered as a type of "lapped" transformation. Equation (30) shows how the transform analysis filter can be written in terms of a given transform analysis matrix. Alternatively, for a given modulated filter bank, the corresponding transform analysis matrix can be found using Equation (30).

4.4 Discussions

We have discussed that subband and transform representations are rearrangements of each other. We have also discussed how a subband representation can be obtained from a transform representation. Equivalent transform analysis filters for the DFT, DCT, and an LOT were determined. We have also discussed how a transform representation can be obtained from a subband representation. Equivalent transform analysis matrices for the STFT and a modulated filter bank were determined.

Intuitively, it makes sense to think of a linear FIR filtering operation as a block transform operation, and vice-versa. Since a linear FIR filter performs linear convolution on the input, this operation can be formulated as a block transformation– or equivalently, a matrix transformation– of the input. Conversely, a block transformation performed on segments of the input (short-time analysis) can be formulated as a linear FIR filtering operation on the input.

For practical considerations, such as computational efficiency, an implementation based on block transformations may be preferred over an implementation based on subband filtering, and vice-versa. In comparing computational efficiency, two interesting observations are worth noting about the subband filtering approach. First, the analysis/synthesis filtering operations may be implemented efficiently using fast Fourier transform (FFT) algorithms. However, this does not give any substantial computational savings unless L (the number of taps in the FIR filter) is large. Second and more significantly, since the filtered input signals (analysis) are effectively sampled, not all the values are actually needed, and therefore these "extraneous" values need not even be *computed.* This can result in substantial computational savings. By comparison, the block transformation implementation does not compute any "extraneous" values (coefficients). Further reduction in computations using the subband approach can also be achieved by exploiting the fact that many input values to the synthesis filter are zero.

Because each analysis filter in the subband filtering implementation processes the entire input signal prior to sampling, the subband analysis/synthesis system can logically be thought of as a generalization of a block transformation. This is because in block transformations, the input is typically segmented into spatially localized disjoint portions, and the block transformation is applied to each separately. That is, the transformation is commonly a disjoint "short-time" process, and no "extraneous" values are computed. On the other hand, the subband filter implementation performs processing on the entire input, and these values are subsequently sampled. Consequently, if the number of taps, L, in the FIR analysis filters is larger than the decimation factor M, then the downsampled values will be obtained from *overlapping* L portions of the input.

5 Extension to the Two-Dimensional Case

For separable two-dimensional transforms or separable two-dimensional subband filters, which is common in many applications, the extension of the results of the previous section is straightforward. The results are simply applied to each dimension. Let $x_{ij}[n_1, n_2]$ represent the $(n_1, n_2)^{th}$ subband signal value in the $(i, j)^{th}$ subband, and let $X_{rs}[k_1, k_2]$ represent the $(k_1, k_2)^{th}$ transform coefficient in the $(r, s)^{th}$ transform block. The transform/subband relationship is given by:

$$x_{ij}[n_1, n_2] = X_{rs}[k_1, k_2]|_{r=n_1,\ s=n_2,\ k_1=i,\ k_2=j} \tag{31}$$

$$X_{rs}[k_1, k_2] = x_{ij}[n_1, n_2]|_{i=k_1,\ j=k_2,\ n_1=r,\ n_2=s} \tag{32}$$

Again, the subband representation is simply a rearrangement of the coefficients in the transform representation, and vice-versa; this is illustrated in Figure 10. To

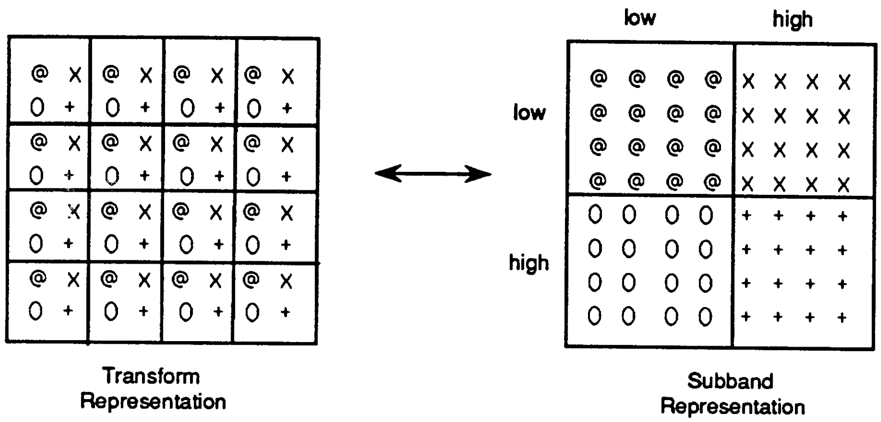

Figure 10: Relationship between transform/subband coefficient representations, two-dimensional case.

illustrate this relationship with an image, consider an original 512x512 image shown in Figure 11. Figure 12(a) illustrates a transform representation of the image using a separable two-dimensional DCT, where the block size is $M = 8$ (no overlap). Figure 12(b) shows the equivalent subband representation of the image by rearranging the transform coefficients, according to Equation (31). As expected, each subband tends to have features similar to the original image. In Figure 12, coefficient magnitude is displayed, all scaled by the same factor to illustrate detail in the higher frequency subbands. As a consequence, the DC band (upper left) has been "clipped" to the maximum display value.

Figure 11: Original 512x512 GIRL.

(a)

(b)

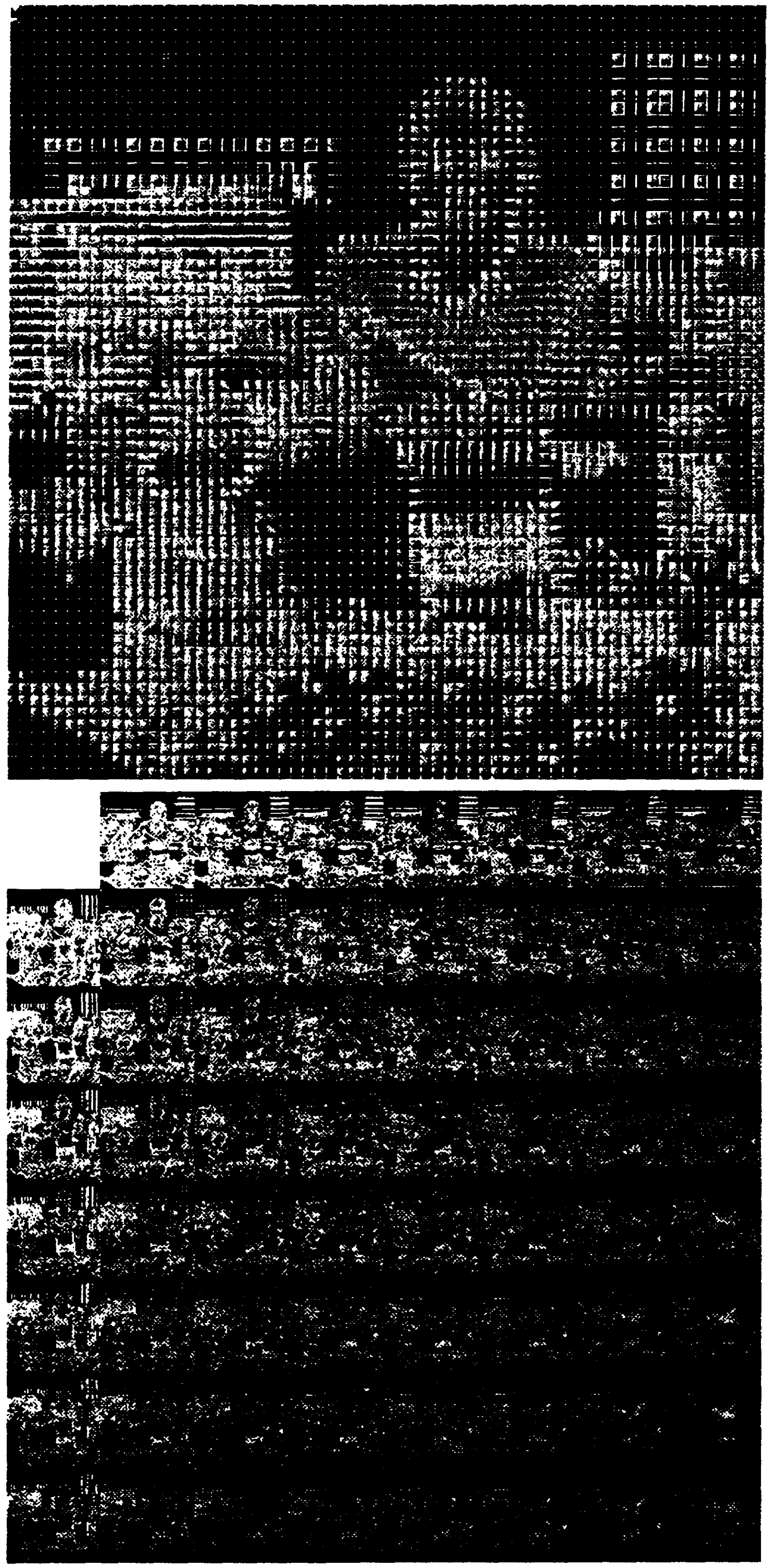

Figure 12: Transform/subband representations of GIRL. (a) transform representation (b) subband representation

6 Applications

The notion that transform coefficient representation and subband signal representation can be viewed as rearrangements of each other can be very useful in applications. Operations in one representation can be translated into operations in the other representation, and this may offer useful insights.

One application of a transform or subband analysis of a signal is in signal coding. For example, typical images tend to contain spatial redundancy; that is, adjacent pixels tend to be highly correlated. Because typical transform or subband decompositions of an image tend to decorrelate the signal values and compact most of the energy in a small fraction of coefficients, a transform or subband representation of the image is typically more appropriate for performing data rate reduction. Only the coefficients that contain significant energy need to be coded, while the corresponding reconstructed image can still be perceptually indistinguishable from the original image. Image coding systems are discussed in [1,2,5,20].

Figure 13 shows a typical transform/subband coding system. The input signal x is analyzed using an MxM decomposition, and the resulting transform/subband coefficients are selected, quantized and then resynthesized. In a transform coding system, two methods of selecting the transform coefficients are threshold coding and zonal coding. In threshold coding, only the transform coefficients that exceed some threshold are selected. This then requires the selected coefficient locations be known at reconstruction. In zonal coding, only the coefficients within a specified region

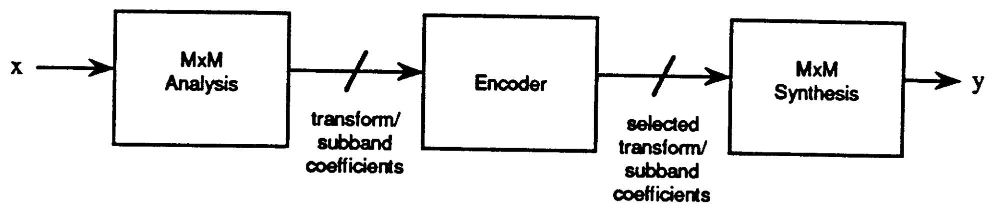

Figure 13: Transform/subband coding system.

are coded. Even though some transform coefficients with small magnitudes may be selected while those with large magnitudes may be discarded, zonal coding does not require location information of the selected coefficients. Some combination of threshold coding and zonal coding have also been considered. Both threshold coding and zonal coding originated from transform coding literature. Because of the simple relationship between transform and subband representations, they can be applied very simply to subband coding.

When disjoint input segments are used in a transform signal coding system, "blocking effects" tend to occur due to independent processing of adjacent segments, or blocks. Because a subband analysis can be easily extended to account for the processing of overlapping input segments, a subband signal coding system offers a useful approach to reducing the blocking effects. In this case, the FIR analysis filters can be designed to have a length larger than the decimation factor. The idea of overlapping input segments in a subband analysis is in fact equivalent to the idea of "lapped transforms."

Both transform and subband representations are obtained by decomposing a signal into smaller "sub-signals". These sub-signals are called "transform blocks" in the previous case and "subbands" in the latter. Each fixed transform block represents a fixed spatial location, but contains all the different frequency components. On the other hand, each fixed subband represents a fixed frequency, over the entire spatial region. For these reasons, transform coders are commonly classified as frequency-domain coders, whereas subband coders are commonly classified as waveform coders. However, as far as signal representation is concerned, both coding schemes are identical. Both representations offer useful perspectives in designing a transform/subband coding system.

As another application of the simple relationship between transform and subband representations, we consider the design of an adaptive amplitude modulation/demodulation (AM/DM) system for noise reduction. Adaptive amplitude modulation is an adaptive gain control technique that increases the signal-to-noise ratio. This is achieved by multiplying the signal at the transmitter by a factor $\alpha > 1$. The factor α generally varies with the local properties of the signal, and is known as the adaptive modulation factor. By determining the AM factors so that the modulated signal does not exceed some maximum level, the SNR can be increased without increasing the peak-to-peak signal range. The received signal is divided by α. As a result, any additive noise introduced in the channel is effectively reduced by the factor α, while the signal portion is unaffected.

Adaptive modulation has been effectively applied to image processing systems [21,22], including the design of a high definition television (HDTV) system for terrestrial broadcasting. When a typical image is decomposed using a transform or subband analysis, many of the values are small and are suitable for adaptive modulation. In blank regions, the high frequency transform coefficients or subband values are small, and AM can be succesfully applied to reduce noise. This is important because in blank regions, noise is more visible.

Adaptive modulation requires the AM factors to be transmitted. It is important to reduce this side information in systems that are bandwidth limited. By exploiting the properties of the transform or subband representations of an image, the AM side information can be reduced. In an earlier approach [21], the AM side information is reduced by estimating the AM factors for high frequency subbands from low frequency subbands. This exploits the property that higher frequency subbands for typical images tend to decrease in amplitude. In a later approach [22], properties of the transform representation are exploited in reducing the AM side information. In this approach, the transform coefficients are modeled with a few parameters. Since the AM factors can be determined from the model, only the model *parameters* need to be transmitted. This enables the amount of AM side information to be significantly reduced. Even though the AM method has been originally applied to subband-filtered signals, performance improvement of the method was possible through the transform representation viewpoint.

7 Conclusion

A transform and subband decomposition of a signal offer a time-frequency representation of the signal. Because both representations are rearrangements of each other, they are equivalent in terms of their *information content* or representation of the signal. For a given transform analysis matrix, an equivalent transform analysis filter can be found, and vice-versa. One representation may be more useful than the other in a particular signal processing application, but both offer a perspective in which useful insights can be gained.

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