Fault-Tolerant Round Robin A/D Converter System

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Paul E. Beckmann and Bruce R. Musicus

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Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, MA 02139 USA

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Paul E. Beckmann
Bruce R. Musicus*
Massachusetts Institute of Technology
Research Laboratory of Electronics
Room 36-797
Cambridge, MA 02139

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Abstract

We describe a robust A/D converter system which requires much less hardware overhead than traditional modular redundancy approaches. A modest amount of oversampling is used to generate information which can be exploited to achieve fault tolerance. A generalized likelihood ratio test is used to detect the most likely failure and also to estimate the optimum signal reconstruction. The error detection and correction algorithm reduces to a simple form and requires only a slight amount of hardware overhead. We present a derivation of the algorithm followed by simulation results for both ideal and optimized FIR processing.

Keywords: A/D converter, fault-tolerance, oversampling A/D converters.

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1 Introduction

In this paper we describe a round robin A/D converter system that provides a high sampling rate and that can tolerate converter failures. Such a system could be used in high stress environments where continuous operation is needed or in remote sensing applications where servicing faulty units is impractical or even impossible.

Traditional approaches to fault tolerance have focused upon using modular redundancy [8]. Several identical copies of the hardware operate in parallel using the same input. Their outputs are compared with one another and agree if no errors have occurred. Using one extra copy (100% overhead), a single fault can be detected; with two complete extra copies (200% overhead), the faulty system can be identified and disabled. This amount of overhead required for fault tolerance is much greater than that required for other applications such as data transmission, where error coding techniques can be used.

In a communications system where N bit symbols are transmitted through a noisy channel, it would be very wasteful to retransmit each symbol several times in order to achieve robustness. Instead, each N bit symbol is mapped into a slightly larger $N + C$ bit symbol, and this is transmitted. Uncorrupted data occupies a fraction of the $N + C$ dimensional space and the receiver tests each received symbol to see if it lies within this subset. If a received symbol is not within the allowable subspace, an error has occurred and the most likely transmitted symbol can be determined.

Musicus and Song [7, 6, 5] have applied this technique to multiprocessor architectures computing linear functions. Their approach combines weighted linear checksums similar to [1, 2, 3, 4], with an optimal statistical fault test. They begin with N processors which compute the same linear function of different inputs. Next, C extra processors are added which also compute this function. These extra processors each use a different weighted sum of the inputs to the N processors as their input.

When all processors are operating correctly, the subspace spanned by the output of the $N + C$ processors will have dimension N. If the output does not lie within this subspace, a processor has failed and the failure can be identified and corrected using a generalized likelihood ratio test.

In this paper, we apply their basic idea to A/D conversion. We start with a number of slower A/D converters operated in round robin fashion, and introduce linear redundancy through oversampling. A generalized likelihood ratio test is used to detect and correct errors. The algorithm reduces to a simple form with a complexity comparable to an FIR filter. A high pass filter is used to detect converter failure, and the output of working converters is used to interpolate samples from the faulty converter. If N converters are needed to achieve the Nyquist sampling rate, then adding one extra converter will allow single faults to be detected. Using two extra converters allows single fault correction with ideal filters; using three extra converters permits single fault correction for practical systems based on finite order filters.

This paper also extends work done by Wolf [9]. He shows that under certain conditions, discrete-time sequences carry redundant information which allows detection and correction of errors. Specifically, sequences whose discrete Fourier transforms contain zeros can be protected from impulse noise. Wolf's error detection and correction scheme is based on coding theory, while ours utilizes a generalized likelihood ratio test. Both methods use out-of-band energy to detect errors.

We begin by describing a model of a round robin A/D converter system. Then we develop the fault correction/detection algorithm using a generalized likelihood ratio test, and present computer simulation results of the ideal system. Next, we consider the restrictions imposed by real systems, and discuss how a practical system can best be implemented.

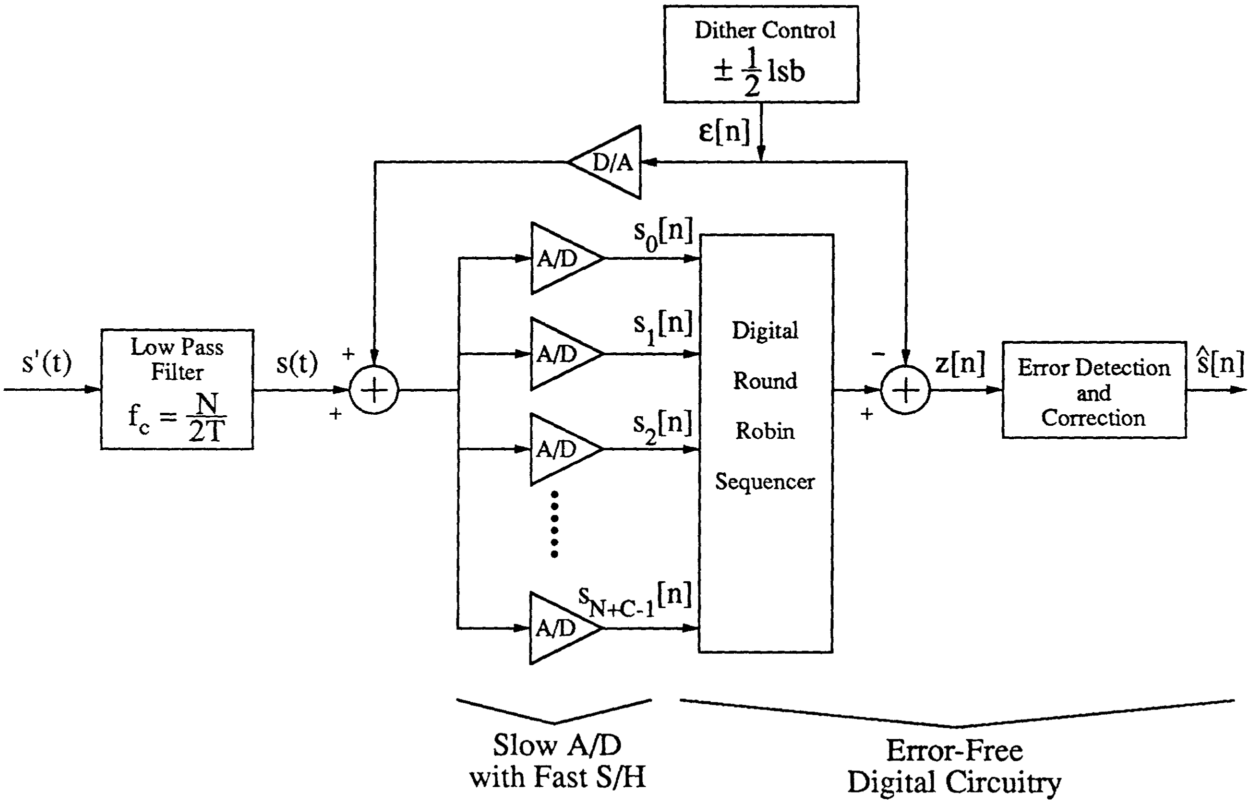

Figure 1: Round robin A/D converter system.

2 Round Robin A/D Converter

A round robin A/D converter system is shown in Figure 1. It contains N slow A/D converters each with fast sample and hold circuitry. The first converter samples and holds the analog input signal and then begins a conversion. After a fixed delay, the second converter samples the signal and begins a conversion. This repeats for all N converters and by the time the $N^{\underline{th}}$ converter starts, the first converter has finished and is ready to accept another sample. Operation continues in this circular fashion. If a conversion requires T seconds for a single converter then the overall sampling rate for the round robin system would be N/T samples/sec., and the input can contain frequencies up to $f_{max} = N/2T$ Hz.

To decorrelate the quantization noise from the signal, we use a small amount of dither. Dither circuitry adds a random analog voltage uniformly distributed between $\pm 1/2$ lsb (least significant bit) to the sample. After conversion, it subtracts this same quantity from the digital signal. As a result of dither, each output sample contains white quantization noise uniformly distributed between $\pm 1/2$ lsb which is uncorrelated with the signal.

We assume that the converters must operate in a stressed environment and that they are the only components subject to failure. We model converter failures as being independent and assume that only one converter fails at a time. (Multiple failures could also be handled properly but with much more difficulty.) We assume that the dither circuitry, digital processing, and output busses always function properly. If necessary, these components could be protected against failure by triple modular redundancy or the digital processing may be performed remotely in a less stressful environment. Also, failures in the dither circuitry can be restricted to cause no more than a $\pm\frac{1}{2}$ lsb error in the samples.

This system is made robust by introducing redundant information. Keep the analog input signal bandlimited to $\pm f_{max}$ Hz, but add C extra converters to increase the sampling rate to $\frac{N+C}{T}$ samples/sec. The input signal is now somewhat oversampled.

When a fault occurs in the $k^{\underline{th}}$ converter, the output signal will contain a noise spike every $N + C$ samples. In the frequency domain, this fault noise has a periodic spectrum with $N + C$ complete copies of the fault spectrum contained in an interval of width 2π. C copies of the fault spectrum are contained in the high frequency region where there is no energy from the low pass input signal. The phase shift between these copies depends on which converter failed. This is illustrated in Figure 2.

Our optimal fault detection/correction algorithm essentially measures high frequency energy to determine if a fault has occurred. It identifies the broken converter from the phase difference between high frequency copies of the fault spectrum. Finally

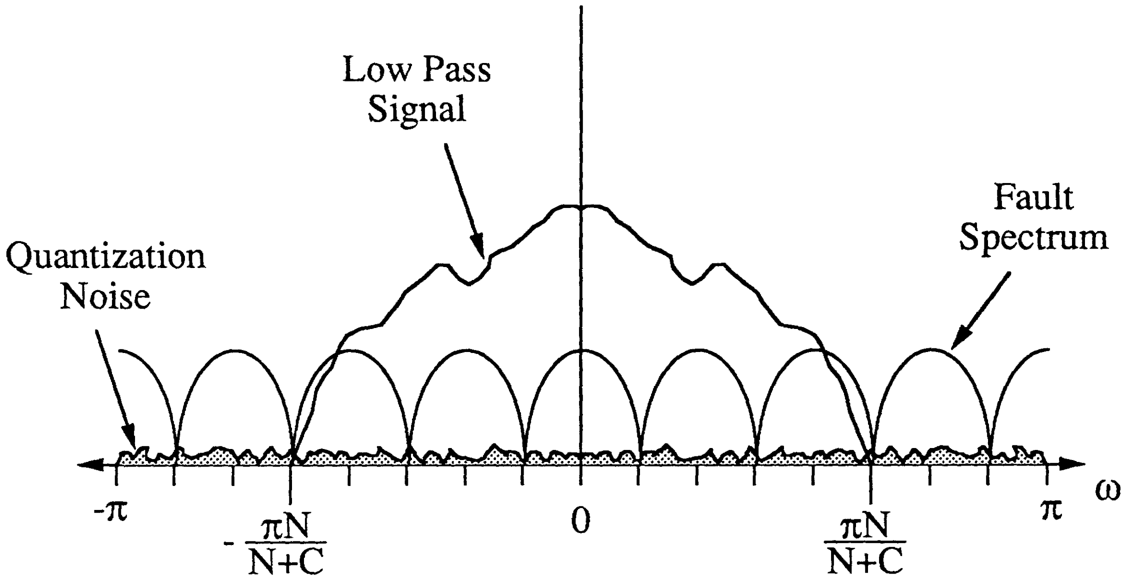

Figure 2: Output frequency spectrum of round robin A/D converter system.

the fault is reconstructed by averaging the C copies in the high frequency region, and it is subtracted from the observations to estimate the fault-free signal.

3 Algorithm Development

This section describes a model of the A/D converter system and then develops the fault detection/correction algorithm using a generalized likelihood ratio test. The output of the round robin system shown in Figure 1 can be written as

$$z[n] = s[n] + \epsilon[n] + \phi_k[n] \tag{1}$$

where $s[n]$ is the original low pass input signal, $\epsilon[n]$ is white quantization noise which is uncorrelated with $s[n]$, and $\phi_k[n]$ is noise due to a failure of the $k^{\underline{th}}$ converter. $\phi_k[n]$ is zero except for samples which came from the $k^{\underline{th}}$ converter. Let $\underline{s}$ be a vector of all samples $s[n]$, and define $\underline{z}$, $\underline{\phi}_k$, and $\underline{\epsilon}$ similarly.

3.1 Generalized Likelihood Ratio Test

We will use a generalized likelihood ratio test to determine the most likely fault hypothesis and to correct the fault if necessary. Let H_* represent the hypothesis that no converter has failed, and let H_k represent a failure in the $k^{\underline{th}}$ converter where $0 \leq k \leq N + C - 1$. Define $p(H_*)$ and $p(H_k)$ as the a priori probabilities of these events, which we assume are independent of the signal or fault,

$$p(H_k) = p\left(H_k | \underline{s}, \underline{\phi}_k\right). \tag{2}$$

We must compute the likelihood L_k of each hypothesis H_k given the observed data $\underline{z}$. For hypothesis H_*, the likelihood unfortunately depends on the unknown signal $\underline{s}$. Therefore, we will maximize the likelihood over $\underline{s}$ to determine the likelihood L_* of hypothesis H_*,

$$L_* = \max_{\underline{s}} \log\left[p\left(\underline{z} \mid H_*, \underline{s}\right) p\left(H_*\right)\right]. \tag{3}$$

For hypotheses H_k, $k = 0, \ldots, N+C-1$, the likelihoods depend on both the unknown signal $\underline{s}$ and the unknown fault $\underline{\phi}_k$. We therefore maximize over both $\underline{s}$ and $\underline{\phi}_k$ to determine the likelihood L_k of H_k,

$$L_k = \max_{\underline{s}, \underline{\phi}_k} \log \left[p\left(\underline{z} \mid H_k, \underline{s}, \underline{\phi}_k \right) p\left(H_k\right) \right]. \tag{4}$$

The most likely failure hypothesis is chosen by finding the largest likelihood. The failure estimate $\hat{\underline{\phi}}_k$ (if any) and a clean signal estimate $\hat{\underline{s}}$ are the values at which the likelihood is maximized.

The bulk of our derivation will be performed in the frequency domain where the distribution of samples of the noise transform is approximately white Gaussian noise. We will exploit this by approximating the noise as being zero mean Gaussian in both time and frequency:

$$p\left(\epsilon[n]\right) = N\left(0, \sigma_\epsilon^2\right). \tag{5}$$

where the variance $\sigma_\epsilon^2 = lsb^2/12$.

We begin by solving for L_*. Using the distribution of the quantization noise, (3) can be written as

$$L_* = \max_{\underline{s}} \left[\log p\left(H_*\right) - \frac{1}{2} \sum_{n=0}^{M-1} \log\left(2\pi\sigma_\epsilon^2\right) - \frac{1}{2\sigma_\epsilon^2} \sum_{n=0}^{M-1} \left(z[n] - s[n]\right)^2 \right] \tag{6}$$

where M is the number of samples available. The first two terms are constants and will be denoted as η_*. Define $Z(\omega_r)$, $S(\omega_r)$, and $\Phi_k(\omega_r)$ to be the M point DFTs of $z[n]$, $s[n]$, and $\phi_k[n]$. For long time intervals, Parseval's theorem can be applied to the third term in (6), giving

$$L_* = \max_{\underline{s}} \left[\eta_* - \frac{1}{2\sigma_\epsilon^2} \frac{1}{M} \sum_{r=0}^{M-1} \left| Z(\omega_r) - S(\omega_r) \right|^2 \right] \tag{7}$$

where ω_r is a frequency index with value $\omega_r = 2\pi r/M$.

We will work with frequencies in the range of 0 to 2π, and divide the frequency samples into a low frequency region, Ω_L, which contains signal energy, and a high frequency region, Ω_H, which does not. The regions will be divided as follows:

$$\begin{aligned} \Omega_L &= \{\omega_r \,|r = 0, \ldots, M_L - 1 \text{ and } r = M_H, \ldots, M-1\} \\ \Omega_H &= \{\omega_r \,|r = M_L, \ldots, M_H - 1\} \end{aligned} \tag{8}$$

where

$$M_L = \left(\tfrac{M}{N+C}\right)\tfrac{N}{2} \quad \text{and} \quad M_H = M - \left(\tfrac{M}{N+C}\right)\tfrac{N}{2}. \tag{9}$$

Assume that $M/2$ is a multiple of $N + C$ so that the frequency samples can be easily divided into the two regions.

Since $S(\omega_r)$ is bandlimited, (7) can be rewritten as

$$L_* = \max_{\underline{s}} \left[\eta_* - \frac{1}{2\sigma_\epsilon^2}\frac{1}{M}\left\{ \sum_{\omega_r \in \Omega_L} |Z(\omega_r) - S(\omega_r)|^2 + \sum_{\omega_r \in \Omega_H} |Z(\omega_r)|^2 \right\}\right]. \tag{10}$$

Now maximize with respect to $S(\omega_r)$ to obtain,

$$\hat{S}(\omega_r) = \begin{cases} Z(\omega_r) & \text{for } \omega_r \in \Omega_L \\ 0 & \text{else.} \end{cases} \tag{11}$$

This can also be written as,

$$\hat{S}(\omega_r) = H_{LP}(\omega_r) Z(\omega_r) \tag{12}$$

where $H_{LP}(\omega_r)$ is an ideal low pass filter with frequency response

$$H_{LP}(\omega_r) = \begin{cases} 1 & \text{for } \omega_r \in \Omega_L \\ 0 & \text{else.} \end{cases} \tag{13}$$

If there are no faults, so that hypothesis H_* is true, then this optimally "clean" signal estimate $\hat{S}(\omega_r)$ is found by low pass filtering $Z(\omega_r)$ to remove high frequency quantization noise. Substituting (11) into (10), the likelihood of H_* becomes,

$$L_* = \eta_* - \frac{1}{2\sigma_\epsilon^2}\frac{1}{M}\sum_{\omega_r \in \Omega_H} |Z(\omega_r)|^2 . \tag{14}$$

It is convenient to define $Z_H(\omega_r)$ as a high pass version of $Z(\omega_r)$,

$$Z_H(\omega_r) = H_{HP}(\omega_r) Z(\omega_r) \tag{15}$$

where $H_{HP}(\omega_r)$ is an ideal high pass filter with,

$$H_{HP}(\omega_r) = \begin{cases} 1 & \text{for } \omega_r \in \Omega_H \\ 0 & \text{else.} \end{cases} \tag{16}$$

Then since (14) only depends on the high frequency samples of $Z(\omega_r)$, we can write

$$L_* = \eta_* - \frac{1}{2\sigma_\epsilon^2}\frac{1}{M}\sum_{r=0}^{M-1} |Z_H(\omega_r)|^2 . \tag{17}$$

Applying Parseval's theorem and returning to the time domain results in,

$$L_* = \eta_* - \frac{1}{2\sigma_\epsilon^2}\sum_{n=0}^{M-1} z_H^2[n]. \tag{18}$$

Now consider the case of a failure in the $k^{\underline{th}}$ converter. As before, we apply

Parseval's theorem to the time domain likelihood expression for L_k and obtain

$$L_k = \max_{\underline{s},\underline{\phi}_k} \left[\log p\,(H_k) - \frac{1}{2}\sum_{n=0}^{M-1}\log\left(2\pi\sigma_\epsilon^2\right) - \frac{1}{2\sigma_\epsilon^2}\frac{1}{M}\sum_{r=0}^{M-1}|Z(\omega_r) - S(\omega_r) - \Phi_k(\omega_r)|^2\right]. \tag{19}$$

The first two terms of this expression are constants and we will denote them as η_k. Divide the summation into low and high frequency regions and maximize with respect to $S(\omega_r)$ to obtain

$$\hat{S}(\omega_r) = H_{LP}(\omega_r)\,[Z(\omega_r) - \Phi_k(\omega_r)]\,. \tag{20}$$

Substituting this back into (19), the log likelihood function becomes

$$L_k = \max_{\underline{\phi}_k} \left[\eta_k - \frac{1}{2\sigma_\epsilon^2}\frac{1}{M}\sum_{\omega_r\in\Omega_H}|Z(\omega_r) - \Phi_k(\omega_r)|^2\right]. \tag{21}$$

Since the summation is only over high frequencies, substitute $Z_H(\omega_r)$ for $Z(\omega_r)$. Next, extend the summation over all frequencies and subtract any newly introduced terms,

$$L_k = \max_{\underline{\phi}_k} \left[\eta_k - \frac{1}{2\sigma_\epsilon^2}\frac{1}{M}\left\{\sum_{r=0}^{M-1}|Z_H(\omega_r) - \Phi_k(\omega_r)|^2 - \sum_{\omega_r\in\Omega_L}|\Phi_k(\omega_r)|^2\right\}\right]. \tag{22}$$

In order to reduce this expression further, we must exploit the structure of $\Phi_k(\omega_r)$. $\phi_k[n]$ only contains samples from the faulty converter, and it is zero except for one out of every $N+C$ samples. Its transform $\Phi_k(\omega_r)$ consists of $N+C$ copies of the fault spectrum, each with a phase shift that depends on which converter failed,

$$\Phi_k\left(\omega_r + \frac{2\pi l}{N+C}\right) = \Phi_k(\omega_r)e^{-\frac{j2\pi l}{N+C}k}. \tag{23}$$

Therefore we may write

$$\left|\Phi_k\left(\omega_r + \frac{2\pi l}{N+C}\right)\right| = |\Phi_k(\omega_r)| \tag{24}$$

for all ω_r and all integers l. We use this to extend the second summation in (22) to include all frequencies,

$$L_k = \max_{\underline{\phi}_k} \left[\eta_k - \frac{1}{2\sigma_\epsilon^2}\frac{1}{M}\left\{\sum_{r=0}^{M-1} |Z_H(\omega_r) - \Phi_k(\omega_r)|^2 - \frac{N}{N+C}\sum_{r=0}^{M-1} |\Phi_k(\omega_r)|^2\right\}\right]. \tag{25}$$

Applying Parseval's theorem and combining terms we obtain,

$$L_k = \max_{\underline{\phi}_k} \left[\eta_k - \frac{1}{2\sigma_\epsilon^2}\sum_{n=0}^{M-1}\left(z_H^2[n] - 2\phi_k[n]z_H[n] + \frac{C}{N+C}\phi_k^2[n]\right)\right]. \tag{26}$$

We can now maximize this expression and solve for the fault estimate. For a failure in the $k^{\underline{th}}$ converter, the only nonzero samples of $\phi_k[n]$ are those for which $n \equiv k$ (we will use this notation as shorthand for $n \bmod (N+C) = k$.) Therefore maximizing (26) yields,

$$\hat{\phi}_k[n] = \begin{cases} \frac{N+C}{C} z_H[n] & \text{for } n \equiv k \\ 0 & \text{else} \end{cases} \tag{27}$$

as the optimal fault estimate.

Now, to obtain the likelihood of H_k, substitute (27) back into (26). After some algebra we are left with,

$$L_k = \eta_k - \frac{1}{2\sigma_\epsilon^2}\sum_{n=0}^{M-1} z_H^2[n] + \frac{1}{2\sigma_\epsilon^2}\frac{N+C}{C}\sum_{n\equiv k} z_H^2[n] \tag{28}$$

Assume that all a priori failure probabilities $p(H_k)$ are the same for $k = 0, \ldots, N+$

$C-1$, so that the constants η_k are all equal,

$$\eta_k = \eta \text{ for } k = 0, \ldots, N+C-1. \tag{29}$$

Then it is convenient to use scaled relative likelihoods defined by,

$$L'_k = 2\sigma_\epsilon^2 \left(\frac{N+C}{C}\right) [L_k - L_* - \eta + \eta_*] \text{ for } k = * \text{ and } 0, \ldots, N+C-1. \tag{30}$$

The scaled relative likelihoods reduce to,

$$L'_* = \gamma \tag{31}$$

$$L'_k = \left(\frac{N+C}{C}\right)^2 \sum_{n \equiv k} z_H^2[n] = \sum_{n \equiv k} \hat{\phi}_k^2[n] \tag{32}$$

where

$$\gamma = 2\sigma_\epsilon^2 \left(\frac{N+C}{C}\right) \left[\log \frac{p(H_*)}{p(H_k)}\right] \tag{33}$$

is a constant. For hypothesis $k = 0, \ldots, N+C-1$, the best estimate of the original signal is then,

$$\hat{S}(\omega_r) = H_{LP}(\omega_r) \left[Z(\omega_r) - \hat{\Phi}_k(\omega_r)\right]. \tag{34}$$

The complete generalized likelihood ratio test fault correction system is shown in Figure 3. The dithered sampled signal $z[n]$ is first scaled by $\frac{N+C}{C}$ and filtered by $H_{HP}(\omega_r)$. The output of this filter, $\frac{N+C}{C} z_H[n]$, is sorted into $N+C$ interleaved streams, with the $k^{\underline{th}}$ stream receiving every $(N+C)^{\underline{th}}$ sample starting with sample k. Each of these streams is simply $\hat{\phi}_k[n]$, the best least squares estimate of the fault in the $k^{\underline{th}}$ converter, assuming that converter k is faulty. We measure the energy in each of these fault estimates, and if any energy is greater than the threshold γ, we declare that a converter has failed. The largest energy indicates the best guess $\hat{k}$ of which converter has failed. The signal is then reconstructed by fixing the incorrect

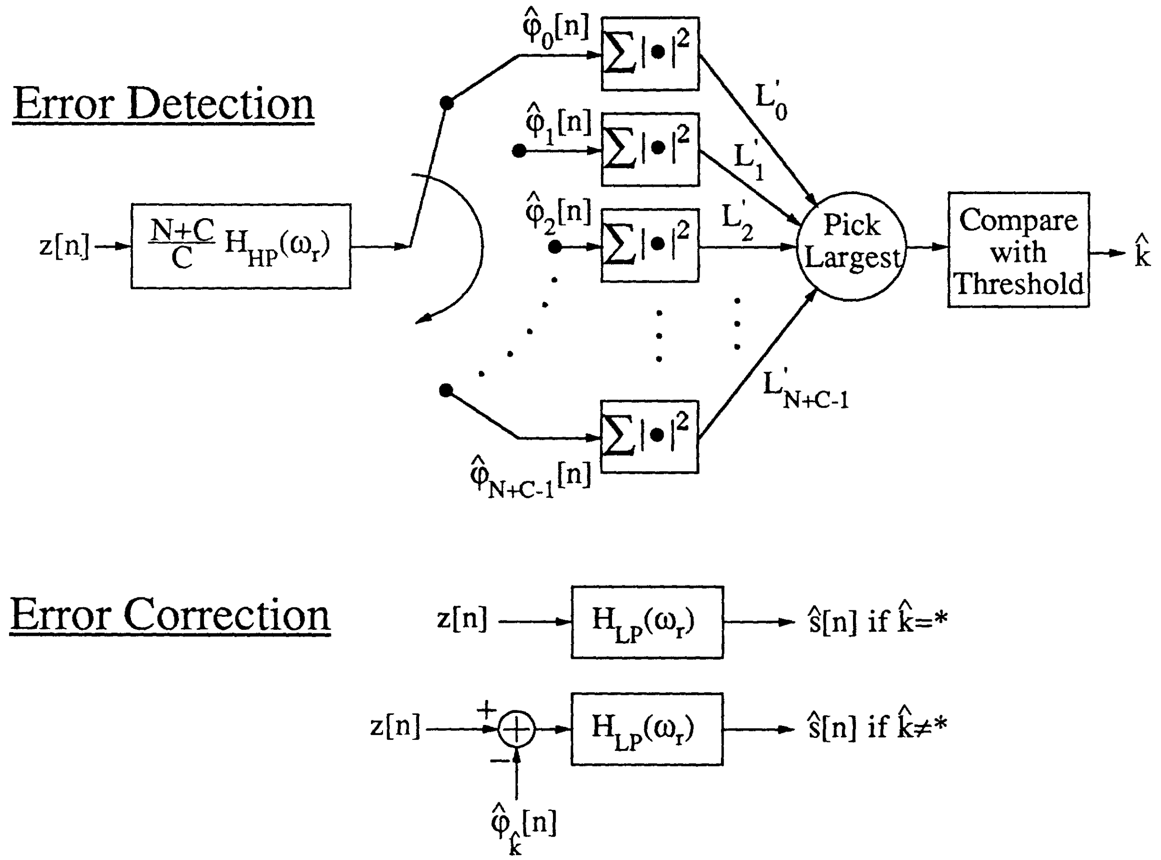

Figure 3: Ideal error detection and correction system.

samples (if any) and low pass filtering.

3.2 Required Number of Extra Converters

Insight into choosing an appropriate number of extra converters C can be gained by writing (27) in the frequency domain,

$$\hat{\Phi}_k(\omega_r) = \frac{1}{C} \sum_{l=0}^{N+C-1} Z_H\left(\omega_r + \frac{2\pi l}{N+C}\right) e^{\frac{j2\pi k}{N+C} l}. \tag{35}$$

The magnitude of this function is periodic in ω_r with period $\frac{2\pi}{N+C}$. Since $Z_H(\omega_r)$ is high pass, for a given ω_r only C terms in this sum are nonzero. Each $\frac{M}{N+C}$ point "period" of $\hat{\Phi}_k(\omega_r)$ is thus formed by averaging C sections of $Z_H(\omega_r)$ with appropriate phase shifts.

If $C = 1$, no averaging takes place. The fault estimates, $\hat{\Phi}_k(\omega_r)$ for $k = 0, \ldots, N + C - 1$ will differ only by a phase shift, and the energy in each will be the same. The system will only be able to decide if a fault has occurred, but will not be able to determine the specific faulty converter. If $C \geq 2$, then the energy in $\hat{\Phi}_k(\omega_r)$ is maximized when the proper phase shift is applied so that all C non-zero terms add coherently. Thus, with $C \geq 2$ we can achieve single fault correction.

3.3 Probability of False Alarm

The hypothesis testing procedure can make several errors. A "false alarm" occurs when H_* is true but the algorithm selects H_k as most likely, for some $k \neq *$. "Detection" occurs if converter q has failed and the method chooses any hypothesis except H_*. "Misdiagnosis" occurs when converter q has failed, hypothesis H_k is diagnosed, and $k \neq q$. Let P_F, P_D, and P_M represent the probabilities of these events.

We can develop a Neyman–Pearson test which adjusts the threshold γ to achieve a given probability of false alarm, P_F. Suppose H_* is true and that the quantization noise $\epsilon[n]$ is white zero mean Gaussian with variance σ_ϵ^2. Now

$$z_H[n] = \sum_{l=0}^{M-1} h_{HP}[n-l]\epsilon[l]. \tag{36}$$

Since the samples $\epsilon[n]$ are Gaussian, so are the samples $z_H[n]$ with mean and covariance given by,

$$\begin{aligned} \mathrm{E}\left[z_H[n] | \, H_* \right] &= 0 & (37) \\ \mathrm{Cov}\left[z_H[n], z_H[m] | \, H_* \right] &= \sum_{l=0}^{M-1} h_{HP}[n-l] h_{HP}[m-l] \sigma_\epsilon^2 & \\ &= h_{HP}[n-m]\sigma_\epsilon^2 & (38) \end{aligned}$$

where the last line follows because $h_{HP}[n] * h_{HP}[n] = h_{HP}[n]$, where '$*$' represents circular convolution with period $N+C$. Since $h_{HP}[n-m]$ has zeroes spaced $(N+C)/C$ apart, samples of $z_H[n]$ spaced by multiples of $(N+C)/C$ are statistically independent. Equation (27) thus implies that the non-zero samples of $\hat{\phi}_k[n]$ are independent Gaussian random variables with zero mean and variance $\left(\frac{N+C}{C}\right)\sigma_\epsilon^2$. Equation (32) implies that under hypothesis H_*, each L_k' is the sum of the squares of $M/(N+C)$ independent Gaussian random variables, $\hat{\phi}_k[n]$. Each L_k' is thus Chi-square distributed with $D = \frac{M}{N+C}$ degrees of freedom, and

$$P\left(L_k' \geq \gamma | \, H_* \right) = \frac{1}{\Gamma\left(D/2\right)} \int_{\frac{\gamma C}{2\sigma_\epsilon^2 (N+C)}}^{\infty} t^{D/2-1} e^{-t} dt \tag{39}$$

where $\Gamma(x) = (x-1)!$ is the normalization factor.

A false alarm occurs when H_* is true, but one or more likelihoods are greater than γ. Thus

$$P_F = 1 - P\left(L_0' \leq \gamma, L_1' \leq \gamma, \ldots, L_{N+C-1}' \leq \gamma \middle| H_*\right). \tag{40}$$

Since $z_H[n]$ is a bandpass signal with bandwidth $\pm\pi C/(N+C)$, the L_k' are highly correlated with each other. Samples of $z_H[n]$ spaced by $(N+C)/C$ will be independent of each other, and the others are determined by linear interpolation. This implies that likelihoods spaced by $(N+C)/C$ are independent of each other. Since there are C of these,

$$\begin{aligned} P_F &\approx 1 - P\left(L_0' \leq \gamma, L_{\frac{N+C}{C}}' \leq \gamma, L_{2\left(\frac{N+C}{C}\right)}' \leq \gamma, \ldots, L_{(C-1)\left(\frac{N+C}{C}\right)}' \leq \gamma \middle| H_*\right) \\ &= 1 - P\left(L_0' \leq \gamma | H_*\right)^C \\ &= 1 - \left[1 - P\left(L_0' \geq \gamma | H_*\right)\right]^C \\ &\approx C P\left(L_0' \geq \gamma | H_*\right) \end{aligned} \tag{41}$$

where the approximation in the last line is valid for small P_F. We can thus use the chi-squared formula in (39) to set γ to achieve any desired level for P_F.

If the integration interval M is much larger than the number of converters $N+C$, then the number of degrees of freedom D will be large, and the distribution of L_k' can be approximated as Gaussian. In this case, a good approximation for P_F is in terms of an error function:

$$P_F \approx \frac{C}{2}\mathrm{erfc}\left(\frac{\gamma - \mathrm{E}\left[L_0' | H_*\right]}{\sqrt{2\mathrm{Var}\left[L_0' | H_*\right]}}\right) \tag{42}$$

where:

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt \tag{43}$$

and where the mean and variance of the likelihoods are given in Appendix B:

$$\mathrm{E}\left[L_0' \middle| H_*\right] = \frac{M}{N+C}\left(\frac{N+C}{C}\right)\sigma_\epsilon^2 \tag{44}$$

$$\mathrm{Var}\left[L_0' \middle| H_*\right] = 2\frac{M}{N+C}\left(\frac{N+C}{C}\right)^2\sigma_\epsilon^4. \tag{45}$$

In other words, γ should be set to:

$$\gamma = \mathrm{E}\left[L_0' \middle| H_*\right] + \beta\sqrt{\mathrm{Var}\left[L_0' \middle| H_*\right]} \tag{46}$$

where:

$$\beta = \sqrt{2}\,\mathrm{erfc}^{-1}\left(\frac{2P_F}{C}\right) \tag{47}$$

A reasonable approximation to $\mathrm{erfc}(x)$ is $\frac{1}{x+\sqrt{x^2+2}}e^{-x^2}$. Therefore P_F falls rapidly as γ increases. It is usually better to set γ somewhat too high, rather than too low.

3.4 Probability of Detection

The probability of detection P_D is defined as the probability that a fault is declared on some converter given that a fault has occurred on converter q. In practice, this is approximately equal to the probability that likelihood $L_q' \geq \gamma$:

$$\begin{aligned} P_D &= 1 - P\left(L_0' \leq \gamma, \ldots, L_{N+C-1}' \leq \gamma \middle| H_q, \underline{\phi}_q\right) \\ &\approx P\left(L_q' \geq \gamma \middle| H_q, \underline{\phi}_q\right) \end{aligned} \tag{48}$$

If the number of terms summed in the likelihoods is large, $M/(N+C) \gg 1$, then we can approximate L'_q as Gaussian. Appendix B shows that:

$$P_D \approx 1 - \frac{1}{2}\text{erfc}\left(\frac{\text{E}\left[L'_q \middle| H_q, \underline{\phi}_q\right] - \gamma}{\sqrt{2\text{Var}\left[L'_q \middle| H_q, \underline{\phi}_q\right]}}\right) \tag{49}$$

where:

$$\text{E}\left[L'_q \middle| H_q, \underline{\phi}_q\right] = \frac{M}{N+C}\left(\frac{N+C}{C}\right)\sigma_\epsilon^2\left[1 + \text{FNR}_q\left(\frac{C}{N+C}\right)\right] \tag{50}$$

$$\text{Var}\left[L'_q \middle| H_q, \underline{\phi}_q\right] = 2\frac{M}{N+C}\left(\frac{N+C}{C}\right)^2\sigma_\epsilon^4\left[1 + 2\text{FNR}_q\left(\frac{C}{N+C}\right)\right] \tag{51}$$

and where FNR_q is the fault-to-noise ratio on converter q:

$$\text{FNR}_q = \frac{N+C}{M}\sum_{l=0}^{M-1}\frac{\phi_q^2[l]}{\sigma_\epsilon^2} \tag{52}$$

Since $\text{erfc}(x)$ decays faster than rate e^{-x^2}, we expect P_D to approach 1 exponentially as FNR_q increases or as the integration length $M/(N+C)$ increases.

3.5 Probability of Misdiagnosis

A misdiagnosis occurs when a fault occurs on converter q, a fault is declared, but the wrong converter is identified. This will cause the algorithm to "correct" the wrong converter. The probability of converter misdiagnosis is difficult to compute analytically, but it is possible to gain some insight by examining a simpler measure. We can compute instead the probability that some likelihood L'_k is greater than L'_q given that H_q is true and that the fault is $\underline{\phi}_q$. Assuming that the number of terms summed in the likelihoods is large, $M/(N+C) \gg 1$, then the likelihoods L'_k and L'_q

can be approximated as jointly Gaussian. Then Appendix B shows that:

$$P\left(L'_k \geq L'_q \middle| H_q, \underline{\phi}_q\right) \approx \frac{1}{2}\mathrm{erfc}\left(\frac{\mathrm{E}\left[L'_q - L'_k \middle| H_q, \underline{\phi}_q\right]}{\sqrt{2\mathrm{Var}\left[L'_q - L'_k \middle| H_q, \underline{\phi}_q\right]}}\right) \tag{53}$$

where

$$\begin{aligned} \mathrm{E}\left[L'_q - L'_k \middle| H_q, \underline{\phi}_q\right] &= \frac{M}{N+C}\sigma_\epsilon^2\left(1 - S^2(k-q)\right)\mathrm{FNR}_q & (54) \\ \mathrm{Var}\left[L'_q - L'_k \middle| H_q, \underline{\phi}_q\right] &= \\ & 4\frac{M}{N+C}\left(\frac{N+C}{C}\right)^2\sigma_\epsilon^4\left(1 - S^2(k-q)\right)\left(1 + \mathrm{FNR}_q\left(\frac{C}{N+C}\right)\right) & (55) \end{aligned}$$

where

$$S(l) = \frac{\sin\left(\frac{\pi C}{N+C}l\right)}{C\sin\left(\frac{\pi}{N+C}l\right)} \tag{56}$$

is a circular sinc function with $S(0) = 1$, and FNR_q is defined in (52).

For fixed N, C, and M, and for $k \neq q$, $S(k-q)$ is maximized for $k = q \pm 1$. Thus, the most common misdiagnosis declares an adjacent neighbor of the faulty converter to be faulty. Also note that increasing the number of samples contributing to the likelihood, $M/(N+C)$, increasing the fraction of extra converters $C/(N+C)$, or increasing the size of the fault, FNR_q, all decrease the misdiagnosis probability.

3.6 Variance of Low Pass Signal Estimate

In this section we consider the accuracy of our system in estimating the original low pass signal. The equations presented are derived in Appendix A. First, consider the case when all converters are functioning properly and hypothesis H_* is chosen. We can show that the expected value and variance of our estimator equals:

$$
\begin{aligned}
\mathrm{E}\left[\hat{s}[n] \mid H_*\right] &= s[n] & (57) \\
\mathrm{Var}\left[\hat{s}[n] \mid H_*\right] &= \frac{N}{N+C}\sigma_\epsilon^2. & (58)
\end{aligned}
$$

Now suppose that the $q^{\underline{th}}$ converter is broken with actual fault $\phi_q[n]$, and that hypothesis q is correctly chosen as the most likely. Under these conditions, we can show that:

$$
\begin{aligned}
\mathrm{E}\left[\hat{s}[n] \mid H_q, \underline{\phi}_q\right] &= s[n] & (59) \\
\mathrm{Var}\left[\hat{s}[n] \mid H_q, \underline{\phi}_q\right] &= \begin{cases} \frac{N}{C}\sigma_\epsilon^2 & \text{for } n \equiv q \\ \frac{\sigma_\epsilon^2}{N+C}\left[N + CS^2(k-q)\right] & \text{else} \end{cases} & (60)
\end{aligned}
$$

Thus we see that our estimator $\hat{s}[n]$ is unbiased. Signal estimate samples for the faulty converter have variance N/C times larger than the quantization noise of a working converter, σ_ϵ^2. All the other signal estimate samples, however, have variance below σ_ϵ^2. In fact, for $C \geq 2$, the average signal estimate variance,

$$
\frac{1}{M}\sum_{n=0}^{M-1} \mathrm{Var}\left[\hat{s}[n] \mid H_q, \underline{\phi}_q\right] = \left(\frac{N}{N+C}\right)\left(\frac{C+1}{C}\right)\sigma_\epsilon^2 \qquad (61)
$$

is less than σ_ϵ^2.

4 Simulation of Ideal System

In this section we present computer simulation results for the ideal generalized likelihood ratio test. The system studied had $N = 5$, $C = 3$, and $M = 1024$. The A/D converters were modeled as having a dynamic range of $\pm\frac{1}{2}$ volt and $B = 12$ bit resolution. Quantization noise was modeled as uniformly distributed between $\pm\frac{1}{2}lsb$, with variance $\sigma_\epsilon^2 = lsb^2/12$. Despite this non-Gaussian noise, the likelihoods are formed from so many independent terms, $M/(N+C) = 128$, that they can be accurately modeled as having a Gaussian distribution.

Substituting into our formula for P_F in (42) gives:

$$P_F \approx 1.5\text{erfc}\left(\frac{\gamma - 341\sigma_\epsilon^2}{60.3\sigma_\epsilon^2}\right) \tag{62}$$

For example, a value of $\beta = 3.85$ in (46) yields $\gamma = 505\sigma_\epsilon^2$ and $P_F = 10^{-4}$. To test this formula, we performed 10,000 simulations without faults and recorded the likelihoods. A graph of P_F vs. γ as predicted by (41) is compared with these computer simulation results in Figure 4. As expected, the simulation and analytic results are very close, and thus either our chi-squared formula (41) or Gaussian approximation formula (42) can be used to set γ.

Next, in order to measure P_D, we simulated the system with a faulty converter. A fault of F bits corresponds to adding random noise uniformly distributed over a range $\pm 2^{F-B-1}$ to the input before quantizing. When $F = B$ the A/D converter has completely failed; when $F = 1$ only the lsb is broken. Approximating the likelihoods as Gaussian, formula (49), and recognizing that a 1 bit fault corresponds to $\text{FNR}_q = 4$,

$$P_D \approx 1 - 0.5\text{erfc}\left(\frac{853\sigma_\epsilon^2 - \gamma}{121\sigma_\epsilon^2}\right) \tag{63}$$

We performed $10,000$ simulations with 1 bit faults and a graph of the ideal versus

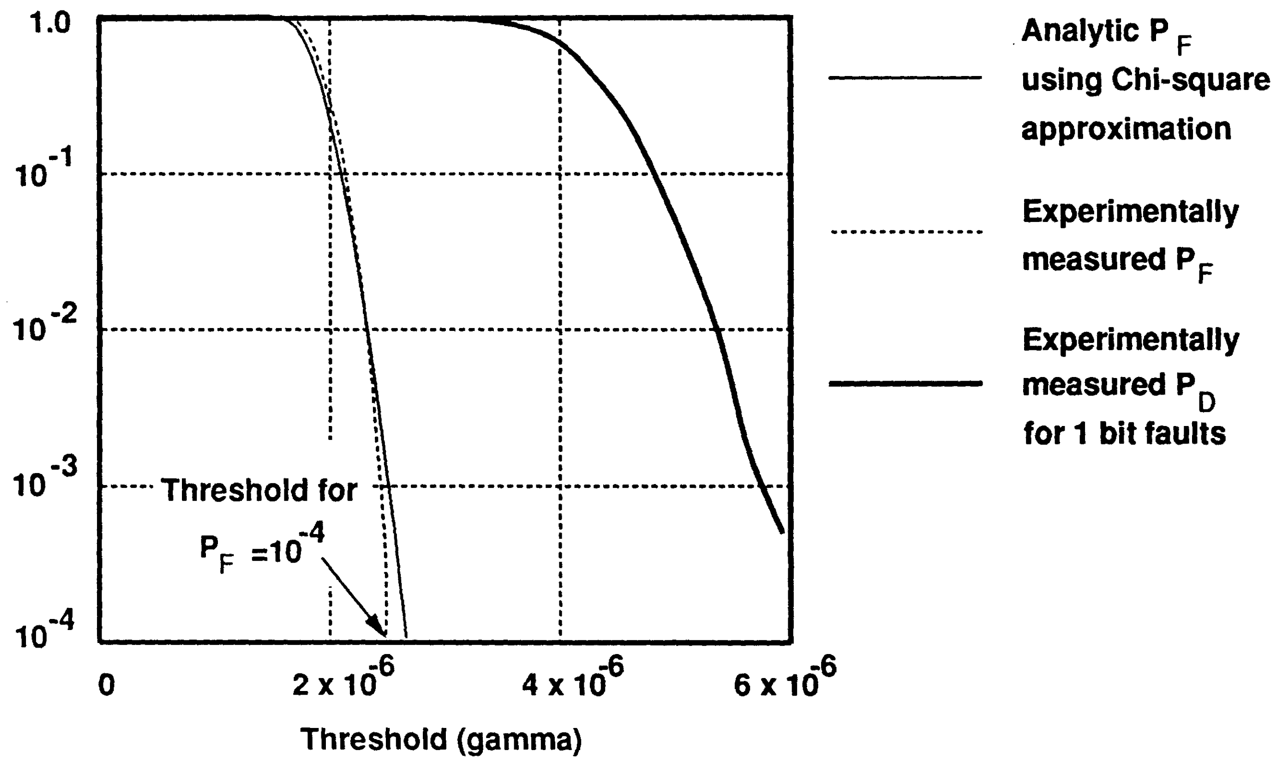

Figure 4: Comparison of analytic and measured likelihoods for the ideal system.

the experimental P_D vs. γ is shown in Figure 4. Fault detection is highly reliable. For example, the same value of γ which gives $P_F = 10^{-4}$ in our example yields $P_D = 1 - 10^{-5}$. In fact, with this γ, every fault was detected in our 10,000 tests.

The probability of misdiagnosis is also low. During 10,000 simulations with 1 bit faults, we found that converter misdiagnosis occurred only 5 times, giving $P_M \approx 5 \times 10^{-4}$. When a mistake did occur, one of the faulty converters' immediate neighbors was always identified as being broken. These results are consistent with formula (53), which gives for our example:

$$P\left(L'_{q+1} \geq L'_q \middle| H_q, \underline{\phi}_q\right) \approx 4 \times 10^{-4} \tag{64}$$

The simulation results also matched the predictions of signal variance made in section 3.6. When H_* is true, the average variance of $\hat{s}[n]$ was found to be $0.62\sigma_\epsilon^2$, and this agrees well with (58). When any size fault occurred, the variance of $\hat{s}[n]$ was found to be $0.832\sigma_\epsilon^2$, which agrees with (61).

5 Realistic Systems

In practice, the data stream to be processed is infinite in length, $M = \infty$, and so our batch oriented ideal system is unrealizable. In this section we consider several modifications to the ideal scheme in order to make it practical. First, we remove the low pass filter on the signal estimate, since it reduces the error by less than 1 bit. Second, we remove the dither system. Third, we replace the unrealizable ideal high pass filter with an FIR filter designed to minimize the variance of the signal estimate in case of failure. Fourth, we replace the infinite length integration to compute the likelihoods with finite length boxcar or IIR filters. Finally, we attempt to develop a robust real-time, causal strategy for declaring and correcting faults which works well for both continuous and transient faults.

5.1 Eliminate the Low Pass Filter

We can substantially reduce the computational complexity of estimating the signal in (12) or (34) by omitting the low pass filter on the final signal estimate, and instead using the approximate signal estimate $\hat{s}_A[n]$ defined by:

$$\hat{s}_A[n] = \begin{cases} z[n] & \text{if } H_* \text{ appears to be true} \\ z[n] - \hat{\phi}_k[n] & \text{if } H_k \text{ appears to be true.} \end{cases} \tag{65}$$

Appendix A shows that the low pass filtering operation leaves the sample estimates from the faulty converter unchanged, $\hat{s}[n] = \hat{s}_A[n]$ for $n \equiv q$, and only affects samples from working converters. It also derives the mean and variance of this unfiltered estimator, assuming the fault is correctly diagnosed:

$$\mathrm{E}\left[\hat{s}_A[n] \,\middle|\, H_q, \underline{\phi}_q\right] = s[n] \tag{66}$$

$$\mathrm{Var}\left[\hat{s}_A[n]\,\middle|\,H_q, \underline{\phi}_q\right] = \begin{cases} \frac{N}{C}\sigma_\epsilon^2 & \text{if } n \equiv q \\ \sigma_\epsilon^2 & \text{else.} \end{cases} \tag{67}$$

The estimator is unbiased, and for typical values of N and C, the average variance is only slightly greater than that of the low pass filtered estimator. Thus, eliminating the low pass filter cuts the computational load by almost half, with little loss in accuracy.

We repeated our previous simulation and estimated the variance of the unfiltered signal estimate. We found that regardless of the size of the fault, the average variance was $1.086\sigma_\epsilon^2$ without the low pass filter, compared to $0.832\sigma_\epsilon^2$ with the low pass. This matches our theoretical predictions, and is a strong argument for eliminating the low pass.

5.2 Eliminate the Dither System

The dither system at the front end of the A/D system was used to ensure that the quantization noise was white and uncorrelated with the signal. In practice, if the signal is sufficiently random, without long slow ramps or constant portions, then the noise will be nearly white even without dither. In our simulations with quantized Gaussian signals, we could detect little difference between systems using dither and those without. We can thus eliminate the dither system, simplifying the hardware and eliminating a potential failure without severely degrading performance.

5.3 Finite Length Filter

In a practical system with an infinite data stream it is convenient to replace the ideal high pass filter $H_{HP}(\omega_r)$ in (16) with a finite order FIR filter. This has an important impact on the choice of C. Unlike the ideal high pass, an FIR filter will have a finite width transition region. To avoid signal leakage, the filter's stopband will have to fill the low frequency region Ω_L. The transition band will have to be inside the high frequency region Ω_H, which leaves only a portion of the high frequency region for the passband. At least two complete copies of the fault spectrum are needed in the filter's passband in order to be able to correct any faulty converter. To achieve this with an FIR filter will require at least $C = 3$ extra converters.

There are many possible ways to design the FIR high pass filter; we could window the ideal high pass, we could use Parks-McClellan, and so forth. In the following, we consider an alternative design strategy based on minimizing the variance of the fault estimate.

Let $h[n]$ be the impulse response of a high pass filter. Let us assume it has odd length $2K+1$, and that it is centered about the origin, $h[n] = 0$ for $|n| > K$. Let us assume that a fault occurred on the $q^{\underline{th}}$ converter, H_q, with unknown value $\phi_q[n]$, and that the faulty converter has been properly identified. As before, let us assume the quantization noise $\epsilon[n]$ is zero mean and white, with sample variance σ_ϵ^2. Unlike the previous sections, let us also assume that the signal is described as a wide sense stationary stochastic process with zero mean and covariance $R[n] = \mathrm{E}\left[s[j]s[n+j]\right]$. We assume that the signal and noise are uncorrelated, and that the signal power spectrum $P(\omega)$ (the Fourier transform of $R[n]$) is low pass, $P(\omega) = 0$ for $\omega \in \Omega_H$. Now suppose we generate a fault estimate $\hat{\phi}_q[n]$ by high pass filtering the observations,

$$\begin{aligned}
\hat{\phi}_q[n] &= \begin{cases} \sum_{l=-L}^{L} h[l]z[n-l] & \text{for } n \equiv q \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \sum_{l=-K}^{K} h[l]\left(s[n-l] + \phi_q[n-l] + \epsilon[n-l]\right) & \text{for } n \equiv q \\ 0 & \text{else} \end{cases}
\end{aligned} \tag{68}$$

(Note that we have absorbed the scaling factor $\frac{N+C}{C}$ shown in (27) into $h[n]$.) Now for samples $n \equiv q$,

$$\mathrm{E}\left[\hat{\phi}_q[n] \middle| H_q, \underline{\phi}_q\right] = \sum_{l=-K}^{K} h[l]\phi_q[n-l]. \tag{69}$$

To eliminate any bias, design $h[n]$ so that

$$h\left[p(N+C)\right] = \begin{cases} 1 & \text{for } p = 0 \\ 0 & \text{for } p = \pm 1, \pm 2, \ldots \end{cases} \tag{70}$$

Note that this implies that a filter of length $l(N+C)-1$ will do just as well as a filter of length $l(N+C)+1$. A "natural" length for the FIR filter, therefore, is $2K+1 = l(N+C)-1$ for some l.

Subject to the above constraint, let us now choose the remaining coefficients of $h[n]$ in order to minimize the variance of the fault estimate. For $n \equiv q$:

$$\begin{aligned} \mathrm{Var}\left[\hat{\phi}_q[n] \middle| H_q, \underline{\phi}_q\right] &= \mathrm{Var}\left[\sum_{l=-K}^{K} h[l]s[n-l] + \sum_{l=-K}^{K} h[l]\epsilon[n-l] \middle| H_q, \underline{\phi}_q\right] \\ &= \sum_{l=-K}^{K}\sum_{p=-K}^{K} h[l]h[p]R[p-l] + \sum_{l=-K}^{K} h^2[l]\sigma_\epsilon^2 \\ &= \underline{h}^T\left(\mathbf{R} + \sigma_\epsilon^2\mathbf{I}\right)\underline{h} \end{aligned} \tag{71}$$

where $\underline{h} = (h[-K]\cdots h[K])^T$ and $\mathbf{R}$ is a $(2K+1)\times(2K+1)$ Toeplitz correlation matrix with entries $\mathbf{R}_{n,m} = R[m-n]$. For convenience, number the rows and columns of $\underline{h}$ and $\mathbf{R}$ from $-K$ through $+K$. We now optimize this variance over all possible sets of filter coefficients $\underline{h}$ subject to the constraints in (70). Since the samples of $h[n]$ for $n \equiv 0$ are already specified, we will remove them from $\underline{h}$ and include them explicitly in the equation. Form the reduced vector $\underline{\tilde{h}}$ from $\underline{h}$ by removing rows $0, \pm(N+C), \pm 2(N+C), \ldots$. Similarly, form the reduced matrix $\tilde{\mathbf{R}}$ from $\mathbf{R}$ by removing rows $0, \pm(N+C), \pm 2(N+C), \ldots$ and columns $0, \pm(N+C), \pm 2(N+C), \ldots$. The variance then becomes,

$$= R[0] + \sigma_\epsilon^2 + 2\underline{\tilde{h}}^T \underline{\tilde{r}} + \underline{\tilde{h}}^T \left(\tilde{\mathbf{R}} + \sigma_\epsilon^2 \mathbf{I}\right) \underline{\tilde{h}} \tag{72}$$

where $\underline{\tilde{r}} = (R[-K] \cdots R[K])^T$ is column 0 of matrix $\mathbf{R}$ with rows $0, \pm(N+C), \pm 2(N+C), \ldots$ removed. Set the derivative with respect to $\underline{\tilde{h}}$ to zero to find the minimum variance filter:

$$\underline{\hat{\tilde{h}}} = -\left(\tilde{\mathbf{R}} + \sigma_\epsilon^2 \mathbf{I}\right)^{-1} \underline{\tilde{r}} \tag{73}$$

To obtain the optimal FIR high pass filter $\underline{\hat{h}}$, reinsert the samples specified by (70). This method is guaranteed to have a unique minimum solution since $\tilde{\mathbf{R}} + \sigma_\epsilon^2 \mathbf{I}$ is positive definite. Also, the optimal filter $\hat{h}[n]$ will have even symmetry because of the symmetry of $\tilde{\mathbf{R}} + \sigma_\epsilon^2 \mathbf{I}$ and $\underline{\tilde{r}}$.

A different γ threshold is needed for the FIR high pass than for the ideal high pass. Including contributions from both signal leakage and quantization error, equation (71) gives the variance of $\hat{\phi}_q[n]$, with the optimal filter $\underline{\hat{h}}$ substituted for $\underline{h}$. If we approximate samples $\hat{\phi}_q[n]$ and $\hat{\phi}_q[n + l(N+C)]$ as being uncorrelated for any $l \neq 0$ (this is strictly true only for an ideal high pass), then formula (42) for P_F continues to apply, but with:

$$\begin{aligned} \mathrm{E}\left[L'_k \middle| H_*\right] &= \frac{M}{N+C}\left[\underline{\hat{h}}^T \left(\mathbf{R} + \sigma_\epsilon^2 \mathbf{I}\right) \underline{\hat{h}}\right] \\ \mathrm{Var}\left[L'_k \middle| H_*\right] &= 2\frac{M}{N+C}\left[\underline{\hat{h}}^T \left(\mathbf{R} + \sigma_\epsilon^2 \mathbf{I}\right) \underline{\hat{h}}\right]^2 \end{aligned} \tag{74}$$

We reconstruct the signal as in (65), without using a low pass filter. We can then show that:

$$\begin{aligned} \mathrm{E}\left[\hat{s}_A[n] - s[n] \middle| H_q, \underline{\phi}_q\right] &= 0 \\ \mathrm{Var}\left[\hat{s}_A[n] - s[n] \middle| H_q, \underline{\phi}_q\right] &= \begin{cases} \underline{\tilde{r}}^T \left(\tilde{\mathbf{R}} + \sigma_\epsilon^2 \mathbf{I}\right)^{-1} \underline{\tilde{r}} & \text{for } n \equiv q \\ \sigma_\epsilon^2 & \text{else.} \end{cases} \end{aligned} \tag{75}$$

A particularly useful case is where the signal power spectrum is low pass and flat in the pass band, $P(\omega) = \sigma_s^2$ for $\omega \in \Omega_L$ and $P(\omega) = 0$ for $\omega \in \Omega_H$. (This is the signal model used in all our simulations.) The covariance of $s[n]$ is then

$$R[n] = \sigma_s^2 \frac{\sin\left(\frac{\pi N}{N+C} n\right)}{\pi n}. \tag{76}$$

where $\sigma_s^2 = 2^{2B}\sigma_\epsilon^2$ is the variance of a signal filling the available dynamic range of B bits. Using the same system arrangement described earlier with $N = 5$, $C = 3$, and $B = 12$, we computed the optimal filter solution (73) for lengths 23, 31, and 63 samples respectively. Graphs of the frequency responses for the filters are shown in Figure 5.

Once again, we collected $M = 1024$ samples so that each likelihood was formed from $M/(N + C) = 128$ terms. 10^5 independent trials were performed, and we used (42) with (46) and (74) to set the threshold γ. We found that a value of $\beta = 3.85$ in (46) gave a false alarm probability of $P_F \approx 10^{-4}$. Results for the three filters tested are shown in Table 1.

All filters tested were able to accurately detect and diagnosis a faulty converter. The longer the filter, the more accurate its estimate $\hat{\phi}_k[n]$, and thus the better its performance. Although none of the FIR filters were as sensitive as the ideal filter in detecting errors in the least significant bit, larger faults could be detected and diagnosed accurately. The variances of the signal estimates are also shown in Table 1. The variances are quite low, and their values match match those predicted earlier. The false alarms which occur do not significantly raise the average signal estimate variance for these filter lengths.

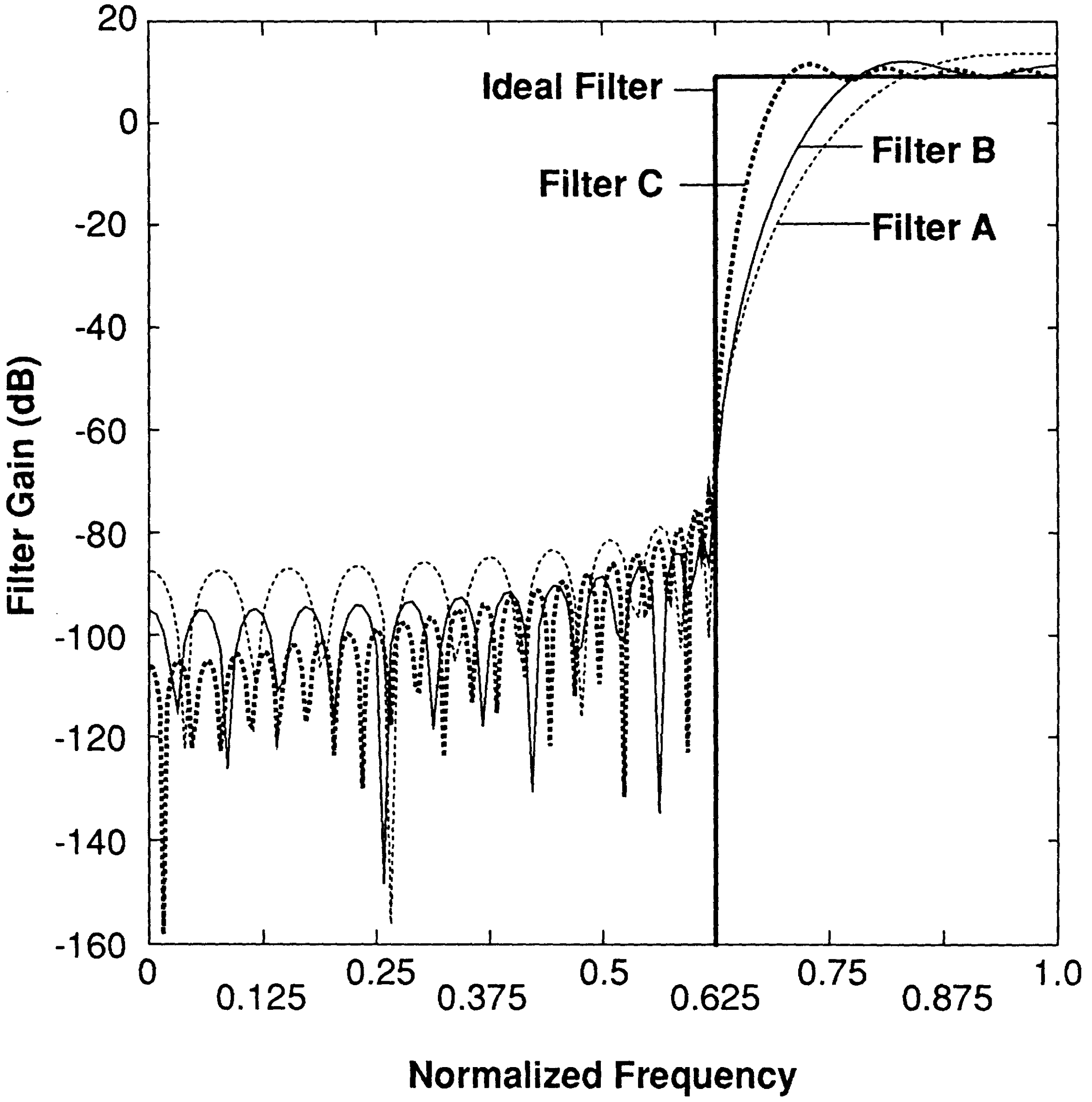

Figure 5: Graphs of the frequency responses of filters tested. A:23 pt., B:31 pt., C:63 pt., optimal filters. D:ideal filter.

		Filter			
		A 23 points	B 31 points	C 63 points	Ideal ∞ points
$\mathrm{Var}[\hat{s}_A[n]]$		$1.39\sigma_\epsilon^2$	$1.22\sigma_\epsilon^2$	$1.15\sigma_\epsilon^2$	$1.08\sigma_\epsilon^2$
Calculated γ		$968\sigma_\epsilon^2$	$716\sigma_\epsilon^2$	$612\sigma_\epsilon^2$	$505\sigma_\epsilon^2$
Measured P_F		2.5×10^{-4}	2.2×10^{-4}	9.6×10^{-5}	1×10^{-4}
1 bit	$1 - P_D$	5.89×10^{-2}	2.80×10^{-3}	3.8×10^{-4}	10^{-5}
faults	P_M	1.58×10^{-1}	5.42×10^{-2}	6.39×10^{-3}	5×10^{-4}
2 bit	$1 - P_D$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$
faults	P_M	4.3×10^{-4}	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$
3 bit	$1 - P_D$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$
faults	P_M	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$

Table 1: Results for the three optimal filters tested.

5.4 Finite Order Integrators

The exact likelihood algorithm computes the likelihoods by integrating the energy in the fault estimates over all time, and then makes a single fault diagnosis for the entire batch of data. It would be useful to develop a real-time scheme which makes fault diagnoses and corrections based only on the most recently received data. One change needed is to compute the likelihoods using finite order integrators: we consider rectangular windows and single pole IIR filters. At time n, we use the high pass filter output $\hat{\phi}_k[n]$ to update the k^{th} converter's likelihood $L'[n]$ where $n \equiv k$:

$$\begin{aligned} &\text{Rectangular:} \quad L'[n] = \sum_{l=0}^{K_{int}-1} \hat{\phi}_k^2[n - l(N+C)] \\ &\text{IIR update:} \quad L'[n] = \alpha L'[n-(N+C)] + \hat{\phi}_k^2[n] = \sum_{l=0}^{\infty} \alpha^l \hat{\phi}_k[n - l(N+C)] \end{aligned} \tag{77}$$

where K_{int} is the length of the rectangular window integrator, and $0 < \alpha < 1$ is the decay rate of the IIR integrator. The IIR approach has the advantage of requiring only $\mathbf{O}(1)$ storage, while the rectangular filter requires $\mathbf{O}(K_{int})$ storage. Appendix C shows that for long integration windows (i.e. large K_{int} or $\alpha \approx 1$), our formulas for P_F, P_D, and P_M are still valid, but the likelihood means and variances are different. The expected values of the likelihoods have the same forms as in (45), (51), (54), and (74), except that the factor $M/(N+C)$ is replaced by K_{int} for the rectangular window likelihood, and by $1/(1-\alpha)$ for the IIR likelihood. Similarly, the variances of the likelihoods have the same forms as in (45), (51), (55), and (74), except that the factor $M/(N+C)$ is replaced by K_{int} for the rectangular window likelihood, and by $1/(1-\alpha^2)$ for the IIR likelihood. Careful study of the formulas suggests that similar performance for these two integrators should result if we choose $K_{int} = (1+\alpha)/(1-\alpha)$, and pick thresholds $\gamma_{rect} = \gamma_{IIR}(1+\alpha)$.

We ran the same simulation as in the previous section to compare the performance of several rectangular integrators and IIR integrators, using the 31 point FIR high pass filter B described earlier. The thresholds γ were chosen using (42), (46), and (74) to achieve $P_F \approx 10^{-4}$. The decays α and thresholds γ_{IIR} of the IIR integrators were chosen to match the performance of the rectangular integrators. The results for rectangular windows are shown in Table 2, and those for IIR integrators are shown in Table 3. As expected, the rectangular and IIR integrators have similar performance, and the results shown in the tables behave in the expected way. Although the observed P_F is much larger than predicted, especially for short integration lengths and IIR integrators, raising γ by 10-20% would drop the observed P_F down to about 10^{-4}. Performance is much worse than expected for integration lengths shorter than those shown in the tables; the reason for this is discussed in the next section.

5.5 Real-Time Fault Detection Algorithm

The system sketched earlier is batch oriented; it collects all the outputs of the high pass filter, sums up the likelihoods for the entire batch, then makes one fault decision

		Integration Length K_{int} (samples)		
		128	64	32
Calculated γ		$716\sigma_\epsilon^2$	$406\sigma_\epsilon^2$	$237\sigma_\epsilon^2$
Measured P_F		2.2×10^{-4}	1.21×10^{-3}	1.89×10^{-3}
1 bit	$1 - P_D$	2.80×10^{-3}	1.00×10^{-1}	3.96×10^{-1}
faults	P_M	5.40×10^{-2}	1.82×10^{-1}	3.48×10^{-1}
2 bit	$1 - P_D$	$< 10^{-5}$	$< 10^{-5}$	2×10^{-5}
faults	P_M	$< 10^{-5}$	1.58×10^{-3}	2.25×10^{-2}
3 bit	$1 - P_D$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$
faults	P_M	$< 10^{-5}$	$< 10^{-5}$	2×10^{-5}
4 bit	$1 - P_D$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$
faults	P_M	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$

Table 2: Results for filter B using rectangular windows.

		Effective Integration Length (samples)		
		128	64	32
α		0.9845	0.9692	0.9394
Calculated γ		$361\sigma_\epsilon^2$	$206\sigma_\epsilon^2$	$122\sigma_\epsilon^2$
Measured P_F		2.13×10^{-3}	2.13×10^{-3}	4.07×10^{-3}
1 bit	$1 - P_D$	1.95×10^{-3}	9.88×10^{-2}	3.97×10^{-1}
faults	P_M	6.63×10^{-2}	2.06×10^{-1}	3.66×10^{-1}
2 bit	$1 - P_D$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$
faults	P_M	$< 10^{-5}$	2.18×10^{-3}	2.75×10^{-2}
3 bit	$1 - P_D$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$
faults	P_M	$< 10^{-5}$	$< 10^{-5}$	3×10^{-5}
4 bit	$1 - P_D$	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$
faults	P_M	$< 10^{-5}$	$< 10^{-5}$	$< 10^{-5}$

Table 3: Simulation results for filter B using IIR integrators.

for all the data. Several modifications are necessary to achieve a causal, real-time fault tolerant A/D system capable of responding correctly to both transient and continuous faults. The simplest approach would update one likelihood with each new high pass filter output, then if this likelihood is greater than γ and also greater than all the other likelihoods, a fault would be declared in the corresponding converter. Unfortunately, this strategy doesn't work properly for transient failures. Figure 6 illustrates the problem, showing various failures $\phi_0[n]$ on converter 0, together with the sequence of likelihoods $L'[n]$ generated by the IIR update formula (77). The dotted lines on the graph identify the samples corresponding to converter 0.

For a continuous failure, the correct likelihood $L'[n]$ for $n \equiv 0$ is always larger than any of the other likelihoods. For a single sample failure on converter 0 at time n_0, however, the output of the high pass filter starts rising at time $n_0 - K$ and continues oscillating until time $n_0 + K$. In fact, because $h[n] = 0$ for n equal to multiples of $N + C$, the likelihoods corresponding to converter 0 are unaffected by the failure until time n_0. Thus for K samples before the failure, all the likelihoods except the correct one start building in energy and may cross the threshold γ. This implies that a fault decision algorithm may have to wait up to K samples from the time the first likelihood rises above threshold before it decides which converter is at fault.

Another problem shows up with faults whose amplitude increases rapidly over time. The figure illustrates a fault whose amplitude rises at an exponential rate. Notice that the correct likelihood is consistently smaller than some of the other likelihoods. Eventually, when the fault amplitude stops growing, the correct likelihood will become largest, but this may take much more than K samples from the time the first likelihood crosses threshold.

It seems difficult to design a single decision algorithm which produces the correct fault diagnoses in all these cases. One possible procedure might use a three state approach. At time n, the incoming sample $z[n]$ is processed through the high pass

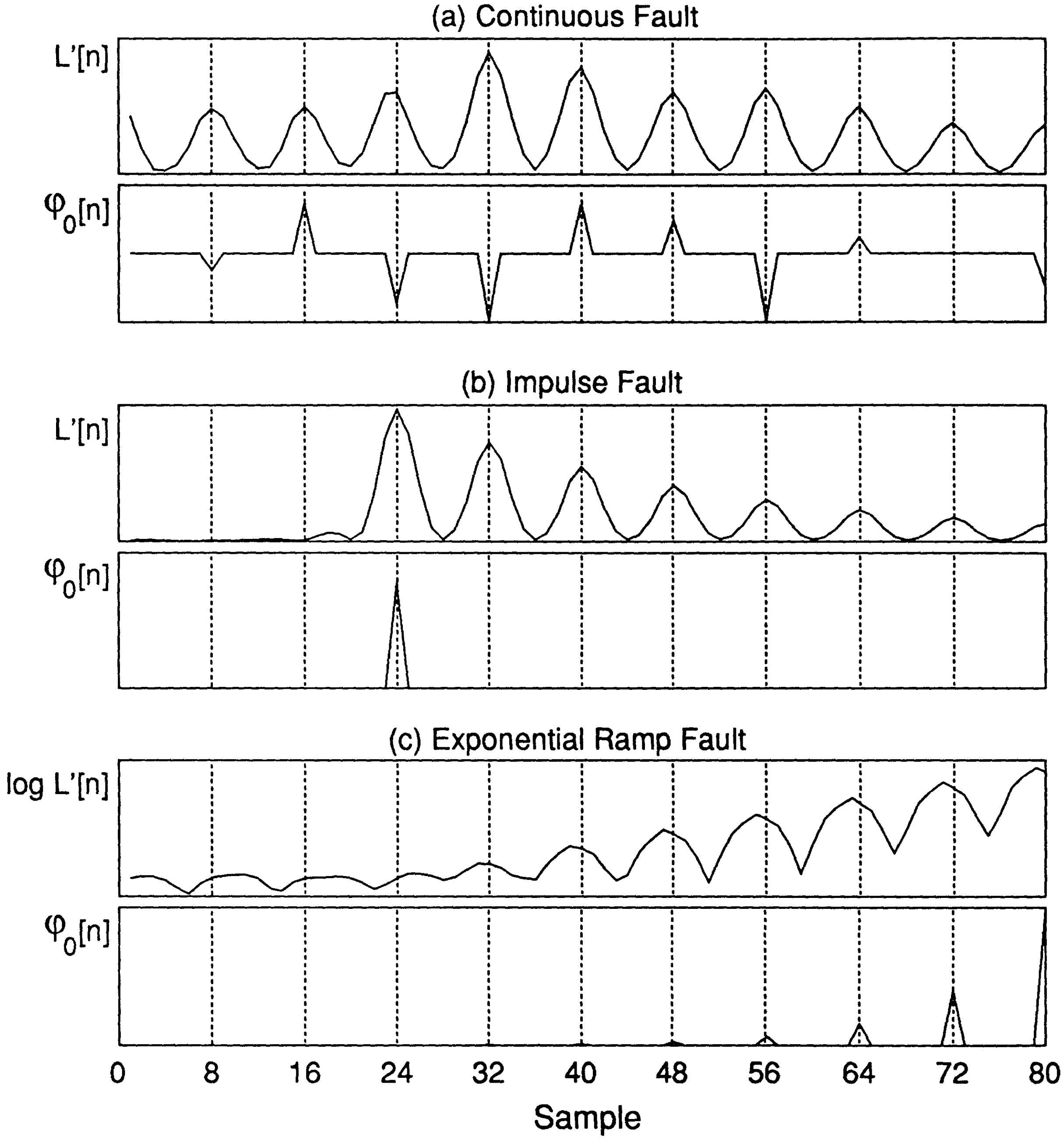

Figure 6: IIR Likelihoods $L'[n]$ versus fault $\phi_0[n]$: (a) Continuous fault (b) impulse fault (c) exponential ramp fault.

filter, and the output of the filter $\hat{\phi}_k[n-K]$ is used to update likelihood $L'[n-K]$. Initially the detector starts in state "OK", and it remains there as long as all likelihood values are below the threshold γ. If one goes above threshold, go to state "detected" and wait for at least K more samples. Then pick the largest likelihood from the last $N+C$, and if it is greater than threshold go to state "corrected" and declare that the corresponding converter has failed. Also start correcting the samples from that converter. Note that because we do not identify a failure until at least $2K$ samples after it has occurred, we will need to save at least the most recent $2K$ converter samples in a buffer so that these can be corrected in case we enter the "correction" state. This imposes a minimum latency of $2K$ in our fault correction system.

Further work is needed to refine this algorithm to incorporate realistic fault scenarios. If a small continuous fault starts, it may take awhile for the likelihood energies to build up enough to cross threshold, thus delaying the correction. If exponential ramp failures can occur, then it may be necessary to change which converter is identified as faulty while in the "correction" state. Rules on when to exit the "correction" state must be developed. It will also be necessary to compute the probabilities of false alarm, detection, and misdiagnosis for whatever algorithm is eventually chosen.

Choice of the threshold γ is complicated by the fact that P_D depends on the total energy in the fault, as reflected by the fault-to-noise ratio FNR. A continuous fault of low amplitude can give rise to a large FNR, since the likelihood integrates the fault energy over a large number of samples. A transient fault which affects only one sample, however, must be quite large to result in a comparable FNR. Setting a low threshold γ and using short integration lengths allows smaller transient errors to be detected and corrected. However, low γ leads to high probability of false alarm, and short integration lengths causes the likelihoods to have high variance, which causes high probability of misdiagnosis. Choosing the best parameters in the algorithm, therefore, involves some delicate performance tradeoffs.

6 Conclusion

In this paper we described a robust oversampling round robin A/D converter system which uses the redundancy inherent to low pass signals to provide fault tolerance. The system was able to identify converter failures reliably and to correct the output accurately. The hardware needed to add robustness is minimal: a few extra converters and an amount of computation comparable to an FIR filter. A disadvantage of our approach is that we rely on a statistical test to detect and correct faults, and therefore have certain probabilities of missing or misdiagnosing a fault, or of declaring a fault where none exists. More fundamental concerns relate to our use of round-robin scheduling of multiple slow A/D converters, a technique which requires careful attention to calibration, sample/holds, and timing issues. Despite these potential problems, our approach is considerably cheaper than traditional approaches to fault tolerance such as modular redundancy, yet can achieve comparable protection against single faults.

A Signal Estimates: Mean and Variance

Assume that fault H_q has occurred with value $\phi_q[n]$. From (27) we know that $\hat{\phi}_k[n]$ is formed from a high pass version of $z[n]$. We can expand this filtering operation as a convolution sum,

$$\hat{\phi}_k[n] = \left(\frac{N+C}{C} \sum_{l=0}^{M-1} h_{HP}[l] z[n-l] \right) u_k[n] \tag{78}$$

where we define

$$u_k[n] = \begin{cases} 1 & \text{for } n \equiv k \\ 0 & \text{else} \end{cases} \tag{79}$$

and where

$$h_{HP}[n] = \frac{(-1)^n \sin\left(\frac{\pi C}{N+C} n\right)}{M \sin\left(\frac{\pi}{M} n\right)} \tag{80}$$

is the impulse response of $H_{HP}(\omega_r)$. Substitute (1) into (78), and since $s[n]$ is a low pass signal, we are left with

$$\hat{\phi}_k[n] = \left(\overline{\phi}_q[n] + \overline{\epsilon}[n] \right) u_k[n] \tag{81}$$

where $\overline{\phi}_q[n]$ and $\overline{\epsilon}[n]$ are high pass versions of $\phi_q[n]$ and $\epsilon[n]$:

$$\begin{aligned} \overline{\phi}_q[n] &= \frac{N+C}{C} \sum_{l=0}^{M-1} h_{HP}[l] \phi_q[n-l] \\ \overline{\epsilon}[n] &= \frac{N+C}{C} \sum_{l=0}^{M-1} h_{HP}[l] \epsilon[n-l] \end{aligned} \tag{82}$$

It is important to recognize that $h_{HP}[n]$ contains zeros at all nonzero multiples of $N+C$ samples. Therefore, $\overline{\phi}_q[n] = \phi_q[n]$ for $n \equiv q$. Also, $\epsilon[n]$ is white Gaussian noise with mean and variance

$$\begin{aligned} \mathrm{E}\left[\epsilon[l]\right] &= 0 && (83) \\ \mathrm{E}\left[\epsilon[l]\epsilon[r]\right] &= \sigma_\epsilon^2 \delta[l-r] && (84) \end{aligned}$$

Thus:

$$\begin{aligned} \mathrm{E}\left[\bar{\epsilon}[n]\right] &= 0 \\ \mathrm{Cov}\left[\bar{\epsilon}[n], \bar{\epsilon}[m]\right] &= \left(\frac{N+C}{C}\right)^2 \sum_{l=0}^{M-1} h_{HP}[n-l] h_{HP}[m-l] \sigma_\epsilon^2 \\ &= \left(\frac{N+C}{C}\right)^2 h_{HP}[n-m]\sigma_\epsilon^2 && (85) \end{aligned}$$

where the last line follows because $h_{HP}[n] = h_{HP}[-n]$ and $h_{HP}[n] * h_{HP}[n] = h_{HP}[n]$, where '$*$' denotes circular convolution with period M. These equations imply that

$$\mathrm{E}\left[\hat{\phi}_k[n] \middle| H_q, \underline{\phi}_q\right] = \overline{\phi}_q[n] u_k[n] \qquad (86)$$

and

$$\begin{aligned} \mathrm{Cov}\left[\hat{\phi}_{k_1}[n], \hat{\phi}_{k_2}[m] \middle| H_q, \underline{\phi}_q\right] &= \mathrm{Cov}\left[\bar{\epsilon}[n], \bar{\epsilon}[m] \middle| H_q, \underline{\phi}_q\right] u_{k_1}[n] u_{k_2}[m] \\ &= \left(\frac{N+C}{C}\right)^2 h_{HP}[n-m]\sigma_\epsilon^2 u_{k_1}[n] u_{k_2}[m] && (87) \end{aligned}$$

Now for the mean and variance of the approximate, unfiltered signal estimate, which is formed by subtracting the fault estimate from the observations, (65). Assume that the fault H_q has been diagnosed properly. Then

$$\begin{aligned} \hat{s}_A[n] &= z[n] - \hat{\phi}_q[n] \\ &= s[n] + \phi_q[n] + \epsilon[n] - \left(\overline{\phi}_q[n] + \bar{\epsilon}[n]\right) u_q[n] \end{aligned}$$

$$= \quad s[n] + \epsilon[n] - \bar{\epsilon}[n]u_q[n] \tag{88}$$

Thus:

$$\mathrm{E}\left[\hat{s}_A[n] | \, H_q, \underline{\phi}_q\right] = s[n] \tag{89}$$

and

$$\begin{aligned}
\mathrm{Cov}\left[\hat{s}_A[n], \hat{s}_A[m] | \, H_q, \underline{\phi}_q\right] &= \mathrm{E}\left[\left(\epsilon[n] - \bar{\epsilon}[n]u_q[n]\right)\left(\epsilon[m] - \bar{\epsilon}[m]u_q[m]\right) | \, H_q, \underline{\phi}_q\right] \\
&= \sigma_\epsilon^2 \left\{ \delta[n-m] - \left(\frac{N+C}{C}\right) h_{HP}[n-m]u_q[n] - \left(\frac{N+C}{C}\right) h_{HP}[m-n]u_q[m] \right. \\
&\quad \left. + \left(\frac{N+C}{C}\right)^2 h_{HP}[n-m]u_q[n]u_q[m] \right\}
\end{aligned} \tag{90}$$

In particular, since $h_{HP}[0] = C/(N+C)$,

$$\mathrm{Var}\left[\hat{s}_A[n] | \, H_q, \underline{\phi}_q\right] = \begin{cases} \frac{N}{C}\sigma_\epsilon^2 & \text{for } n \equiv q \\ \sigma_\epsilon^2 & \text{else} \end{cases} \tag{91}$$

From this we see that all samples of $\hat{s}_A[n]$ have the proper average value and that the corrected samples have $\frac{N}{C}$ times as much variance as the samples from the working converters. Also, the average variance in $\hat{s}_A[n]$ is,

$$\frac{1}{M}\sum_{n=0}^{M-1} \mathrm{Var}\left[\hat{s}_A[n] | \, H_k, \underline{\phi}_k\right] = \left[1 + \frac{N-C}{C(N+C)}\right]\sigma_\epsilon^2. \tag{92}$$

The ideal estimate under hypothesis H_q is formed by low pass filtering $\hat{s}_A[n]$,

$$\hat{s}[n] = \sum_{r=0}^{M-1} h_{LP}[n-r]\hat{s}_A[r] \tag{93}$$

Substituting (88) into this equation and recalling that $s[n]$ is low pass, yields,

$$\hat{s}[n] = s[n] + \sum_{r=0}^{M-1} h_{LP}[n-r]\left(\epsilon[r] - \bar{\epsilon}[r]u_q[r]\right). \tag{94}$$

Thus,

$$\mathrm{E}\left[\hat{s}[n] \mid H_q, \underline{\phi}_q\right] = s[n] \tag{95}$$

and we can compute the variance as follows,

$$\mathrm{Var}\left[\hat{s}[n] \mid H_q, \underline{\phi}_q\right] = \sum_{r=0}^{M-1}\sum_{t=0}^{M-1} h_{LP}[n-r]h_{LP}[n-t]\mathrm{Cov}\left[\hat{s}_A[r], \hat{s}_A[t] \mid H_q, \underline{\phi}_q\right] \tag{96}$$

$$\begin{aligned} = & \sum_{r=0}^{M-1} h_{LP}^2[n-r]\sigma_\epsilon^2 - 2\left(\frac{N+C}{C}\right)\sum_{r\equiv q}\sum_{t=0}^{M-1} h_{LP}[n-r]h_{LP}[n-t]h_{HP}[r-t]\sigma_\epsilon^2 \\ & + \left(\frac{N+C}{C}\right)^2 \sum_{r\equiv q}\sum_{t\equiv q} h_{LP}[n-r]h_{LP}[n-t]h_{HP}[r-t]\sigma_\epsilon^2. \end{aligned}$$

This can be simplified in a few steps. Note that $h_{LP}[n] * h_{HP}[n] = 0$, and also:

$$h_{HP}[n(N+C)] = \begin{cases} \frac{C}{N+C} & \text{for } n = 0 \\ 0 & \text{else.} \end{cases} \tag{97}$$

Thus,

$$\mathrm{Var}\left[\hat{s}[n] \mid H_q, \underline{\phi}_q\right] = \sum_{r=0}^{M-1} h_{LP}^2[n-r]\sigma_\epsilon^2 + \left(\frac{N+C}{C}\right)\sum_{r\equiv q} h_{LP}^2[n-r]\sigma_\epsilon^2. \tag{98}$$

After a little more algebra, we find that

$$\mathrm{Var}\left[\hat{s}[n] \mid H_q, \underline{\phi}_q\right] = \begin{cases} \frac{N}{C}\sigma_\epsilon^2 & \text{for } n \equiv q \\ \frac{N}{N+C}\sigma_\epsilon^2 + \frac{C}{N+C}S^2(n-q)\sigma_\epsilon^2 & \text{else} \end{cases} \tag{99}$$

where $S(l)$ is the circular sinc function with period $N+C$ in (56). The variance is thus highest for those signal samples, $n \equiv q$, from the faulty converter which have been interpolated from the other converter samples, $\frac{N}{C}\sigma_\epsilon^2$. All other signal samples have variance below σ_ϵ^2, with lowest variance occurring for signal samples from converters far from the faulty one, about $\frac{N\sigma_\epsilon^2}{N+C}$. In fact, we can show that signal estimate samples for the faulty converter are unaffected by the low pass filtering operation. To see this, note that $H_{LP}(\omega_r) = 1 - H_{HP}(\omega_r)$, and thus $h_{LP}[n] = \delta[n] - h_{HP}[n]$. Substituting gives:

$$\begin{aligned} \hat{s}[n] &= s_A[n] - \sum_{l=0}^{M-1} h_{HP}[n-l]s_A[l] \\ &= s_A[n] - \sum_{l=0}^{M-1} h_{HP}[n-l]\left(z[l] - \hat{\phi}_q[l]\right) \end{aligned} \tag{100}$$

For $n \equiv q$:

$$\begin{aligned} \hat{s}[n] &= s_A[n] - \left(z_H[n] - \sum_{l\equiv q} h_{HP}[n-l]\hat{\phi}_q[l]\right) \\ &= s_A[n] - \left(z_H[n] - h_{HP}[0]\left(\frac{N+C}{C}\right) z_H[n]\right) \\ &= s_A[n] \end{aligned} \tag{101}$$

The average variance of the ideal estimator is,

$$\begin{aligned} \frac{1}{M}\sum_{n=0}^{M-1} \mathrm{Var}\left[\hat{s}[n] \mid H_q, \underline{\phi}_q\right] &= \frac{N}{N+C}\sigma_\epsilon^2 + \left(\frac{N+C}{C}\right)\sum_{r\equiv q}\frac{1}{M}\sum_{n=0}^{M-1} h_{LP}^2[n-r]\sigma_\epsilon^2 \\ &= \frac{N(C+1)}{(N+C)C}\sigma_\epsilon^2. \end{aligned} \tag{102}$$

The statistics for $\hat{s}_A[n]$ and $\hat{s}[n]$ can also be derived for the case when all converters are working properly and hypothesis H_* is chosen. The mean and variance of the unfiltered signal estimator are:

$$\mathrm{E}\left[\hat{s}_A[n] | H_*\right] = s[n] \tag{103}$$

$$\mathrm{Var}\left[\hat{s}_A[n] | H_*\right] = \sigma_\epsilon^2. \tag{104}$$

The mean and variance of the filtered signal estimator are:

$$\mathrm{E}\left[\hat{s}[n] | H_*\right] = s[n] \tag{105}$$

$$\mathrm{Var}\left[\hat{s}[n] | H_*\right] = \sum_{l=0}^{M-1} h_{LP}^2[n-l]\sigma_\epsilon^2 = \frac{N}{N+C}\sigma_\epsilon^2. \tag{106}$$

B Ideal Likelihoods - Mean and Covariance

In this section we derive the formulas of section 3.5. First, define $\tau_l[n]$ as a shifted version of $h_{HP}[n]$ with all but every $(N+C)^{\underline{th}}$ sample set to zero:

$$\tau_l[n] = \begin{cases} h_{HP}[n+l] & \text{for } n \equiv 0 \\ 0 & \text{else.} \end{cases} \tag{107}$$

Then

$$\tau_l[n] = h_{HP}[n+l]\frac{1}{N+C}\sum_{r=0}^{N+C-1} e^{-\frac{j2\pi r}{N+C}n} \tag{108}$$

and Fourier transforming we obtain:

$$\tau_l(\omega_m) = \sum_{n=0}^{M-1} \tau_l[n]e^{-j\omega_m n} \tag{109}$$

$$= \frac{1}{N+C}\sum_{r=0}^{N+C-1} H_{HP}\left(\omega_m + \frac{2\pi r}{N+C}\right)e^{j\left(\omega_m + \frac{2\pi r}{N+C}\right)l}. \tag{110}$$

Now $\tau_l(\omega_m)$ is periodic with period $2\pi/(N+C)$. For frequencies in the range $\pi N/(N+C) \leq \omega_m < \pi(N+2)/(N+C)$ we can calculate:

$$\begin{aligned} \tau_l(\omega_m) &= \frac{1}{N+C}\sum_{r=0}^{C-1} 1 \cdot e^{j\left(\omega_m + \frac{2\pi r}{N+C}\right)l} \\ &= \frac{C}{N+C} S(l) e^{j\left(\omega_m + \frac{\pi(C-1)}{N+C}\right)l} \end{aligned} \tag{111}$$

where $S(l)$ is the circular sinc function defined in (56). A formula valid for all ω_m can then be derived by exploiting the periodicity of $\tau_l(\omega_m)$:

$$\tau_l(\omega_m) = \frac{C}{N+C} S(l) e^{j\left(\omega_m + \frac{\pi(C-1-2p_m)}{N+C}\right)l} \tag{112}$$

where p_m is the integer:

$$p_m = \left\lfloor m\left(\frac{N+C}{M}\right) - N/2 \right\rfloor \tag{113}$$

and where $\lfloor x \rfloor$ represents the largest integer no greater than x. Note that the phase of $\tau_l(\omega_m)$ is a sawtooth ranging $\pm\frac{\pi l}{N+C}$ about πl in steps of $\frac{2\pi l}{M}$, while the magnitude is constant.

Now to compute the statistics of the L_k'. Assume that fault H_q has occurred with value $\phi_q[n]$. Combining (32) with (81), and using (85):

$$\begin{aligned} \mathrm{E}\left[L_k' \middle| H_q, \underline{\phi}_q\right] &= \sum_{n\equiv k} \mathrm{E}\left[\hat{\phi}_k^2[n] \middle| H_q, \underline{\phi}_q\right] \\ &= \sum_{n\equiv k} \overline{\phi}_q^2[n] + \sum_{n\equiv k} \mathrm{E}\left[\overline{\epsilon}^2[n] \middle| H_q, \underline{\phi}_q\right] \\ &= \sum_{n\equiv k} \overline{\phi}_q^2[n] + \left(\frac{M}{N+C}\right)\left(\frac{N+C}{C}\right)\sigma_\epsilon^2 \end{aligned} \tag{114}$$

The first term above can be evaluated by using (82), recognizing that $h_{HP}[n] =$

$h_{HP}[-n]$, and substituting $\tau_{k-l}[n]$ for $h_{HP}[n+k-l]$ and $\tau_{l-p}[-n]$ for $h_{HP}[l-p-n]$:

$$
\begin{aligned}
\sum_{n\equiv k} \overline{\phi}_q^2[n] &= \left(\frac{N+C}{C}\right)^2 \sum_{n\equiv k}\sum_{l\equiv q}\sum_{p\equiv q} h_{HP}[n-l]h_{HP}[n-p]\phi_q[l]\phi_q[p] \\
&= \left(\frac{N+C}{C}\right)^2 \sum_{l\equiv q}\sum_{p\equiv q} \phi_q[l]\phi_q[p]\left[\sum_{n\equiv 0} h_{HP}[k+n-l]h_{HP}[k+n-p]\right] \\
&= \left(\frac{N+C}{C}\right)^2 \sum_{l\equiv q}\sum_{p\equiv q} \phi_q[l]\phi_q[p]\left[\sum_{n=0}^{M-1} \tau_{k-l}[n]\tau_{p-k}[-n]\right] \qquad (115)
\end{aligned}
$$

Apply Parseval's theorem and work in the frequency domain:

$$
\begin{aligned}
\sum_{n\equiv k} \overline{\phi}_q^2[n] &= \left(\frac{N+C}{C}\right)^2 \sum_{l\equiv q}\sum_{p\equiv q} \phi_q[l]\phi_q[p]\left[\frac{1}{M}\sum_{m=0}^{M-1} \tau_{k-l}(\omega_m)\tau_{p-k}(\omega_m)\right] \qquad (116) \\
&= \sum_{l\equiv q}\sum_{p\equiv q} \phi_q[l]\phi_q[p]S(k-l)S(p-k)\left[\frac{1}{M}\sum_{m=0}^{M-1} e^{j\left(\omega_m+\frac{\pi(C-1-2pm)}{N+C}\right)(p-l)}\right]
\end{aligned}
$$

Recall that the phase of the exponential in (116) is a sawtooth with range $\pm\pi(p-l)/(N+C)$. Therefore:

$$
\frac{1}{M}\sum_{m=0}^{M-1} e^{j\left(\omega_m+\frac{\pi(C-1-2pm)}{N+C}\right)(p-l)} = \begin{cases} 1 & \text{if } p=l \\ 0 & \text{if } p-l=\pm(N+C), \pm 2(N+C), \ldots \end{cases} \qquad (117)
$$

Also recognizing that $S(l)$ is symmetric and periodic with period $N+C$, equation (116) reduces to,

$$
\sum_{n\equiv k} \overline{\phi}_q^2[n] = S^2(q-k)\sum_{l\equiv q} \phi_q^2[l] \qquad (118)
$$

Combining (118) and (114) and using some algebra gives (51). For the no fault case, H_*, set $\mathrm{FNR}_q = 0$ to get (45).

We now turn our attention to the covariance of two likelihoods, L'_{k_1} and L'_{k_2}, under hypothesis H_q and fault $\phi_q[n]$. Substituting (32) gives:

$$\begin{aligned}\operatorname{Cov}\left[L'_{k_1}, L'_{k_2}\middle| H_q, \underline{\phi}_q\right] &= \sum_{n_1 \equiv k} \sum_{n_2 \equiv k} \operatorname{Cov}\left[\hat{\phi}^2_{k_1}[n_1], \hat{\phi}^2_{k_2}[n_2]\middle| H_q, \underline{\phi}_q\right] \\ &= \sum_{n_1 \equiv k} \sum_{n_2 \equiv k} \operatorname{Cov}\left[\left(\overline{\phi}_q[n_1] + \overline{\epsilon}[n_1]\right)^2, \left(\overline{\phi}_q[n_2] + \overline{\epsilon}[n_2]\right)^2 \middle| H_q, \underline{\phi}_q\right] \end{aligned} \tag{119}$$

Now suppose a, b, c, and d are zero mean Gaussian random variables. Then it is well known that $E\left[abc\right] = 0$ and

$$E\left[abcd\right] = E\left[ab\right]E\left[cd\right] + E\left[ac\right]E\left[bd\right] + E\left[ad\right]E\left[bc\right] \tag{120}$$

Using this in (119), plus the fact that $\operatorname{Cov}\left[a,b\right] = E\left[ab\right] - E[a]E[b]$, expanding terms and applying a lot of algebra gives:

$$\begin{aligned}\operatorname{Cov}\left[L'_{k_1}, L'_{k_2}\middle| H_q, \underline{\phi}_q\right] &= \sum_{n_1 \equiv k} \sum_{n_2 \equiv k} \left\{2\left(\frac{N+C}{C}\right)^4 h^2_{HP}[n_1 - n_2]\sigma^4_\epsilon \right. \\ &\qquad \left. + 4\overline{\phi}_q[n_1]\overline{\phi}_q[n_2]\left(\frac{N+C}{C}\right)^2 h_{HP}[n_1 - n_2]\sigma^2_\epsilon\right\}\end{aligned} \tag{121}$$

Substituting (82) gives:

$$\begin{aligned}&= \left(\frac{N+C}{C}\right)^4 \sum_{n_1 \equiv k_1} \sum_{n_2 \equiv k_2} \left\{h^2_{HP}\left[n_1 - n_2\right] 2\sigma^4_\epsilon \right. \\ &\qquad \left. + \sum_{l_1 \equiv q} \sum_{l_2 \equiv q} h_{HP}\left[n_1 - l_1\right] h_{HP}\left[n_2 - l_2\right] h_{HP}\left[n_1 - n_2\right] 4\sigma^2_\epsilon \phi_q\left[l_1\right] \phi_q\left[l_2\right]\right\}\end{aligned} \tag{122}$$

Now substitute (107) for $h_{HP}[n]$ and adjust the summations,

$$
\begin{aligned}
&= \left(\frac{N+C}{C}\right)^4 \sum_{n_1\equiv 0} \sum_{n_2\equiv 0} \tau_{k_1-k_2}^2 [n_1 - n_2] 2\sigma_\epsilon^4 \\
&\quad + \left(\frac{N+C}{C}\right)^4 \sum_{l_1\equiv 0} \sum_{l_2\equiv 0} \left[\sum_{n_1\equiv 0} \sum_{n_2\equiv 0} \tau_{q-k_1} [l_1 - n_1] \tau_{k_1-k_2} [n_1 - n_2] \tau_{k_2-q} [n_2 - l_2] \right] \\
&\qquad \cdot \phi_q [l_1 + q] \phi_q [l_2 + q] 4\sigma_\epsilon^2 .
\end{aligned}
\tag{123}
$$

Because $\tau_l[n]$ is periodic with period M, we can further reduce the first term of the above equation,

$$
\begin{aligned}
&= \left(\frac{N+C}{C}\right)^4 \left(\frac{M}{N+C}\right) \sum_{n\equiv 0} \tau_{k_1-k_2}^2 [n] 2\sigma_\epsilon^4 \\
&\quad + \left(\frac{N+C}{C}\right)^4 \sum_{l_1\equiv 0} \sum_{l_2\equiv 0} \left[\sum_{n_1\equiv 0} \sum_{n_2\equiv 0} \tau_{q-k_1} [l_1 - n_1] \tau_{k_1-k_2} [n_1 - n_2] \tau_{k_2-q} [n_2 - l_2] \right] \\
&\qquad \cdot \phi_q [l_1 + q] \phi_q [l_2 + q] 4\sigma_\epsilon^2
\end{aligned}
\tag{124}
$$

$$
\begin{aligned}
&= \left(\frac{N+C}{C}\right)^4 \left(\frac{M}{N+C}\right) \frac{1}{M} \sum_{m=0}^{M-1} \left|\tau_{k_1-k_2}(\omega_m)\right|^2 2\sigma_\epsilon^4 \\
&\quad + \left(\frac{N+C}{C}\right)^4 \sum_{l_1\equiv 0} \sum_{l_2\equiv 0} \left[\frac{1}{M} \sum_{m=0}^{M-1} \tau_{q-k_1}(\omega_m) \tau_{k_1-k_2}(\omega_m) \tau_{k_2-q}(\omega_m) e^{j\omega_m (l_1 - l_2)} \right] \\
&\qquad \cdot \phi_q [l_1 + q] \phi_q [l_2 + q] 4\sigma_\epsilon^2
\end{aligned}
\tag{125}
$$

$$
\begin{aligned}
&\mathrm{Cov}\left[L'_{k_1}, L'_{k_2} \middle| H_q, \underline{\phi}_q \right] \qquad (126)\\
&= \frac{M}{C}\sigma_\epsilon^4 \left\{ 2\left(\frac{N+C}{C}\right) S^2(k_1 - k_2) + 4S(q-k_1) S(k_1-k_2) S(k_2-q) \mathrm{FNR}_q \right\}
\end{aligned}
$$

where $S(l)$ is the circular sinc function defined in (56), and FNR_q is the fault-to-quantization noise ratio defined in (52). Formula (51) results by substituting $k_1 =$

$k_2 = q$ and noting that $S(0) = 1$. For the no fault case, H_*, formula (45) results by setting $\mathrm{FNR}_q = 0$.

The probabilities P_F, P_D, and P_M follow from Gaussian statistics. For any Gaussian variable $p(x) = N(m, \sigma^2)$,

$$\begin{aligned} P(x \geq \gamma) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_\gamma^\infty \exp\left(-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}\right) dx \\ &= \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{\gamma-m}{\sqrt{2}\sigma}}^\infty \exp\left(-x^2\right) dx\right] \\ &= \frac{1}{2}\mathrm{erfc}\left(\frac{\gamma - m}{\sqrt{2}\sigma}\right) \end{aligned} \tag{127}$$

Now if the number of terms $M/(N+C)$ summed to form each L'_k is large, then L'_k is approximately Gaussian. Formulas (42) and (49) follow directly from (127). Formula (53) follows from the observation that $L'_q - L'_k$ is Gaussian, while the mean (54) and variance (55) come from:

$$\begin{aligned} \mathrm{E}\left[L'_q - L'_k \middle| H_q, \underline{\phi}_q\right] &= \mathrm{E}\left[L'_q \middle| H_q, \underline{\phi}_q\right] - \mathrm{E}\left[L'_k \middle| H_q, \underline{\phi}_q\right] \quad (128) \\ \mathrm{Var}\left[L'_q - L'_k \middle| H_q, \underline{\phi}_q\right] &= \mathrm{Var}\left[L'_q \middle| H_q, \underline{\phi}_q\right] - 2\mathrm{Cov}\left[L'_k, L'_q \middle| H_q, \underline{\phi}_q\right] + \mathrm{Var}\left[L'_k \middle| H_q, \underline{\phi}_q\right] \end{aligned}$$

C Finite Order Likelihoods

Suppose we replace the "correct" likelihood formulas with the windowed formula:

$$L'[n] = \sum_l w[l]\hat{\phi}_k^2[n - l(N+C)] \tag{129}$$

where $n \equiv k$. If we assume that the fault is continuous, the window is long, and the average energy in the fault is independent of time, then:

$$\begin{aligned}
\mathrm{E}\left[L'[n]\,\middle|H_q,\underline{\phi}_q\right] &= \sum_l w[l]\mathrm{E}\left[\hat{\phi}_k^2[n-l(N+C)]\,\middle|H_q,\underline{\phi}_q\right] \\
&\approx \left(\sum_l w[l]\right)\frac{N+C}{M}\sum_{r=0}^{M-1}\mathrm{E}\left[\hat{\phi}_k^2[r]\,\middle|H_q,\underline{\phi}_q\right] \\
&= \left(\sum_l w[l]\right)\frac{N+C}{M}\mathrm{E}\left[L_k'\,\middle|H_q,\underline{\phi}_q\right] \qquad (130)
\end{aligned}$$

This implies that the expected value of the windowed likelihood is the same as that for the ideal likelihood, except that the factor $M/(N+C)$ is replaced by $\sum_l w[l]$. For our rectangular window, $w[l] = 1$ for $l = 0, \ldots, K_{int} - 1$ and $= 0$ otherwise. For the IIR update, $w[l] = \alpha^l$ for $l \geq 0$ and $= 0$ otherwise. Therefore the integration factor $M/(N+C)$ in (45), (51), (54), and (74) is replaced by K_{int} for the rectangular window and by $1/(1-\alpha)$ for the IIR window.

Similarly, under the same assumptions

$$\begin{aligned}
\mathrm{Var}\left[L'[n]\,\middle|H_q,\underline{\phi}_q\right] &= \sum_l\sum_p w[l]w[p]\mathrm{Cov}\left[\hat{\phi}_k^2[n-l(N+C)],\hat{\phi}_k^2[n-p(N+C)]\,\middle|H_q,\underline{\phi}_q\right] \\
&= \sum_l w^2[l]\mathrm{Var}\left[\hat{\phi}_k^2[n-l(N+C)]\,\middle|H_q,\underline{\phi}_q\right] \\
&\approx \left(\sum_l w^2[l]\right)\frac{N+C}{M}\sum_{r=0}^{M-1}\mathrm{Var}\left[\hat{\phi}_k^2[r]\,\middle|H_q,\underline{\phi}_q\right] \\
&= \left(\sum_l w^2[l]\right)\frac{N+C}{M}\mathrm{Var}\left[L_k'\,\middle|H_q,\underline{\phi}_q\right] \qquad (131)
\end{aligned}$$

where the second line follows because equation (87) implies that samples $\hat{\phi}_k[n]$ separated by multiples of $N+C$ are statistically independent. Formula (131) implies that the integration factor $M/(N+C)$ in (45), (51), (55), and (74) is replaced by K_{int} for the rectangular window and by $1/(1-\alpha^2)$ for the IIR window.

For short windows, the fault energy can no longer be modeled as independent of time. Furthermore, the oscillations in the tails of the high pass filter cause energy from one converter to contribute to all the likelihoods. The combination of these effects causes P_F, P_D, and especially P_M to degrade rapidly as the integration length decreases.

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