Reduced-Basis Approximation of the Viscosity-Parametrized Incompressible Navier-Stokes Equation: Rigorous *A Posteriori* Error Bounds

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Abstract— **We present a technique for the** *rapid* **and** *reliable* **prediction of linear-functional outputs of elliptic partial differential equations with affine (or approximately affine) parameter dependence. The essential components are (*i*) rapidly uniformly convergent global reduced-basis approximations — Galerkin projection onto a space W_N spanned by solutions of the governing partial differential equation at N selected points in parameter space; (*ii*)** *a posteriori* **error estimation — relaxations of the residual equation that provide inexpensive yet sharp and rigorous bounds for the error in the outputs of interest; and (*iii*) offline/online computational procedures — stratagems which decouple the generation and projection stages of the approximation process. The operation count for the online stage — in which, given a new parameter value, we calculate the output of interest and associated error bound — depends only on N (typically very small) and the parametric complexity of the problem.**

In this paper we extend our methodology to the viscosity-parametrized incompressible Navier-Stokes equations. There are two critical new ingredients: first, the now-classical Brezzi-Rappaz-Raviart framework for (here, *a posteriori***) error analysis of approximations of nonlinear elliptic partial differential equations; and second, offline/online computational procedures for efficient calculation of the "constants" required by the Brezzi-Rappaz-Raviart theory — in particular, rigorous lower and upper bounds for the Babuška inf-sup stability and Sobolev "L^4-H^1" continuity factors, respectively. Numerical results for a simple square-cavity model problem confirm the rapid convergence of the reduced-basis approximation and the good effectivity of the associated** *a posteriori* **error bounds.**

Keywords— **reduced-basis,** *a posteriori* **error estimation, output bounds, incompressible Navier-Stokes, elliptic partial differential equations**

I. Introduction

The optimization, control, and characterization of an engineering component or system requires the prediction of certain "quantities of interest," or performance metrics, which we shall denote *outputs* — for example deflections, maximum stresses, maximum temperatures, heat transfer rates, flowrates, or lifts and drags. These outputs are typically expressed as functionals of field variables associated with a parametrized partial differential equation which describes the physical behavior of the component or system. The parameters, which we shall denote *inputs*, serve to identify a particular "configuration" of the component: these inputs may represent design variables, such as geometry — for example, in optimization studies; decision variables, such as actuator power — for example in real-time control applications; or characterization variables, such as physical properties — for example in inverse problems. We thus arrive at an implicit *input-output* relationship, evaluation of which demands solution of the underlying partial differential equation.

Our goal is the development of computational methods that permit *rapid* and *reliable* evaluation of this partial-differential-equation-induced input-output relationship *in the limit of many queries* — that is, in the (real-time) design, optimization, control, and characterization contexts. Our particular approach is based on the reduced-basis method, first introduced in the late 1970s for nonlinear structural analysis [1], [15], and subsequently developed more broadly in the 1980s and 1990s [2], [3], [6], [16], [17], [21]; extension to the incompressible Navier-Stokes equations is considered in [8], [9], [10], [16]. The reduced-basis method recognizes that the field variable is not, in fact, some arbitrary member of the infinite-dimensional solution space associated with the partial differential equation; rather, it resides, or "evolves," on a much lower-dimensional manifold induced by the parametric dependence.

The reduced-basis approach as earlier articulated is local in parameter space in both practice and theory. To wit, Lagrangian or Taylor approximation spaces for the low-dimensional manifold are typically defined relative to a particular parameter point; and the associated *a priori* convergence theory relies on asymptotic arguments in sufficiently small neighborhoods [6]. As a result, the computational improvements — relative to conventional (say) finite element approximation — are often quite modest [17]. Our work

[11], [12], [13], [19], [27], [18] differs from these earlier efforts in several important ways: first, we develop (in some cases, provably) *global* approximation spaces; second, we introduce rigorous *a posteriori error estimators*; and third, we exploit *offline/online* computational decompositions (see [2] for an earlier application of this strategy within the reduced-basis context). These three ingredients allow us — for the restricted but important class of "parameter-affine" (or approximately parameter-affine) problems — to reliably decouple the generation and projection stages of reduced-basis approximation, thereby effecting computational economies of several orders of magnitude.

In earlier work [18], [19], [27] we focus primarily on linear coercive partial differential equations; more recently [26] we address the noncoercive case (and certain simple nonlinearities); and in [25] we finally construct completely rigorous error bounds for a nonlinear problem — the viscous Burgers equation. There are two critical new ingredients in our treatment of the Burgers equation: first, the now-classical Brezzi-Rappaz-Raviart [4] framework for (here, *a posteriori*) error analysis of approximations of nonlinear elliptic partial differential equations; and second, offline/online computational procedures for efficient calculation of *all the constants* required by the Brezzi-Rappaz-Raviart theory — in particular, rigorous lower and upper bounds for the Babuška inf-sup stability and Sobolev "L^4-H^1" continuity factors, respectively.

In this paper we shall extend our formulation for the Burgers problem to treat the full incompressible Navier-Stokes equations. We consider in this paper the specific case in which the only parametric variation is the viscosity (inverse Reynolds number). We present in Section II the general problem statement; in Section III the reduced-basis approximation; in Section IV the associated *a posteriori* theory; in Section V a discussion of the offline/online computational complexity; and in Section 6 illustrative numerical results for a simple square-cavity model problem.

II. Problem Formulation

We define $Y \equiv \{v \in \tilde{Y} \,|\, \nabla \cdot v = 0\}$, $\tilde{Y} \equiv (H_0^1(\Omega))^2$, $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) \,|\, v|_{\partial\Omega} = 0\}$, and $H^1(\Omega) \equiv \{v \,|\, v \in L^2(\Omega), \nabla v \in (L^2(\Omega))^2\}$, where Ω is a suitably regular bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Here $L^p(\Omega)$, $1 \leq p \leq \infty$, is the space of measurable functions for which

$$\|v\|_{L^p(\Omega)} \equiv \left(\int_\Omega (v_i v_i)^{p/2} \right)^{1/p} , \tag{1}$$

is finite; note that repeated indices shall imply summation over the two spatial directions. We associate to Y (and $\tilde{Y}$) the inner product

$$(w, v)_Y = \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} , \tag{2}$$

and norm $\|w\|_Y = (w, w)_Y^{1/2}$. We denote the dual space of Y as Y', with corresponding duality pairing $\langle \cdot, \cdot \rangle$.

Given a $\nu \in \mathcal{D} \equiv [\nu^{\min}, \nu^{\max}] \subset \mathbb{R}_+^1$, $0 < \nu^{\min} < \nu^{\max}$, we look for $u(\nu) \in Y$ such that

$$G(u(\nu); \nu) = 0 , \tag{3}$$

where $G \colon Y \to Y'$ is the ν-parametrized C^1 mapping given by

$$\langle G(w; \nu), v \rangle \equiv \nu \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} - \int_\Omega \frac{\partial v_i}{\partial x_j} w_i w_j - \langle F, v \rangle,$$
$$\forall \, w, v \in Y , \tag{4}$$

and $F \in Y'$ is a given linear functional describing the imposed force. Note that, thanks to our divergence-free space, the pressure is eliminated.

The Frechet derivative of G at $(z; \nu)$ is given by

$$\langle DG(z; \nu)w, v \rangle \equiv \nu \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} - \int_\Omega \frac{\partial v_i}{\partial x_j} (z_i w_j + w_i z_j),$$
$$\forall \, w, v \in Y , \tag{5}$$

for any $z \in Y$. We further define

$$\rho \equiv \sup_{v \in Y} \frac{\|v\|_{L^4(\Omega)}}{\|v\|_Y} , \tag{6}$$

in terms of which we can bound the continuity constant,

$$\gamma(z; \nu) \equiv \|DG(z; \nu)\|_{Y,Y'} , \tag{7}$$

as $\gamma(z; \nu) \leq \nu + 2\rho^2 \|z\|_Y$. Note that ρ, and hence $\gamma(z; \nu)$, is finite thanks to the continuous embedding of $H^1(\Omega)$ in $L^4(\Omega)$ [20].

III. Reduced-Basis Formulation

We first introduce a nested set of parameter samples $S_1 \equiv \{\overline{\nu}^1 \in \mathcal{D}\} \subset \cdots \subset S_{N^{\max}} \equiv \{\overline{\nu}^1 \in \mathcal{D}, \ldots, \overline{\nu}^{N^{\max}} \in \mathcal{D}\}$. (The samples may be constructed adaptively based on the inexpensive *a posteriori* error estimators of Section IV [26].) Then, for any $N \leq N^{\max}$, we define the (divergence-free) reduced-basis approximation space $W_N \subset Y$ as

$$W_N \equiv \text{span}\{\zeta_n \equiv u(\overline{\nu}^n), \ 1 \leq n \leq N\} . \tag{8}$$

In practice, $u(\overline{\nu} \cdot)$ is replaced by a "truth" Galerkin approximation $u^\mathcal{N}(\overline{\nu} \cdot) \in Y^\mathcal{N} \subset Y$. We assume that the dimension of $Y^\mathcal{N}$, $\mathcal{N}$, is sufficiently large that $u^\mathcal{N}(\nu)$ may be effectively equated with $u(\nu)$; as we shall see, the online complexity is *independent* of $\mathcal{N}$.

Our reduced-basis approximation is then: Given a $\nu \in \mathcal{D}$, find $u_N(\nu) \in W_N$ such that

$$\langle G(u_N(\nu); \nu), v \rangle = 0 , \quad \forall v \in W_N . \tag{9}$$

We shall assume that $DG(u_{N^{\max}}(\nu); \nu)$ is an isomorphism; more quantitatively, we suppose that $\beta_{N^{\max}}(\nu) \geq \beta_0 > 0$, $\forall \nu \in \mathcal{D}$, where

$$\beta_N(\nu) \equiv \|DG(u_N(\nu); \nu)^{-1}\|_{Y',Y}^{-1}. \tag{10}$$

(In this paper we consider only Galerkin projection; in fact, Petrov-Galerkin methods can provide greater stability [22].)

IV. A Posteriori Error Estimation

Our *a posteriori* error estimators require a rigorous upper (respectively, lower) bound for the continuity constant ρ (respectively, the inf-sup "constant" $\beta_N(\nu)$). We now discuss the methods by which we evaluate these bounds; we then assemble the full *a posteriori* error estimators.

A. Continuity Constant

To construct an upper bound for ρ, $\tilde{\rho}$, we define

$$\tilde{\rho} \equiv \sup_{v \in \tilde{Y}} \frac{\|v\|_{L^4(\Omega)}}{\|v\|_Y} = \left[\inf_{v \in \tilde{Y}} \frac{\|v\|_Y^2}{\|v\|_{L^4(\Omega)}^2} \right]^{-1/2} ; \qquad (11)$$

since $Y \subset \tilde{Y}$, it follows that $\rho \leq \tilde{\rho}$. To evaluate $\tilde{\rho}$ we introduce the Euler-Lagrange equation [23], [24] associated with (11),

$$\int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial \psi_i^*}{\partial x_j} = \lambda^* \int_\Omega v_i \psi_i^* \psi_j^* \psi_j^* , \qquad \forall\, v \in Y , \quad (12)$$

$$\int_\Omega (\psi_i^* \psi_i^*)^2 = 1 ; \qquad (13)$$

we then set $\tilde{\rho} = (\lambda_{\min}^*)^{-1/2}$, where $(\lambda^*, \psi^*)_{\min} \in (\mathbb{R}_+, Y)$ is the ground state of the system. In actual practice, it is difficult to isolate the requisite lowest-energy state; we therefore employ a homotopy approach.

In particular, given $\alpha \in [0,1]$, we define $(\lambda(\alpha), \psi(\alpha)) \in (\mathbb{R}_+, Y)$ by

$$\int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial \psi_i(\alpha)}{\partial x_j} = \lambda(\alpha) \left(\alpha \int_\Omega v_i\, \psi_i(\alpha) \psi_j(\alpha) \psi_j(\alpha) \right.$$
$$\left. + (1-\alpha) \int_\Omega v_i\, \psi_i(\alpha) \right), \quad \forall\, v \in Y , \quad (14)$$

$$\alpha \int_\Omega (\psi_i \psi_i)^2 + (1-\alpha) \int_\Omega \psi_i \psi_i = 1 ; \qquad (15)$$

clearly, $(\lambda(0), \psi(0))_{\min}$ is the ground state of the standard (vector) Laplacian, while $(\lambda(1), \psi(1))_{\min} \equiv (\lambda^*, \psi^*)_{\min}$ is the ground state of the desired nonlinear system (12)-(13). We thus first set $\alpha = 0$ and find $(\lambda(0), \psi(0))_{\min}$ by standard eigenvalue solution methods; then, for $\alpha = i\Delta\alpha$, $i = 1, \ldots, 1/\Delta\alpha$, we find $(\lambda(\alpha), \psi(\alpha))_{\min}$ by Newton iteration with initialization $(\lambda(\alpha - \Delta\alpha), \psi(\alpha - \Delta\alpha))_{\min}$; finally, we evaluate $\tilde{\rho} = (\lambda_{\min}^*)^{-1/2} \equiv (\lambda_{\min}(1))^{-1/2}$.

Note, if we prefer, we may calculate ρ_{ref} for a reference domain Ω_{ref} (e.g., the unit square). Then, for any $\Omega \subset$ "$\sigma\Omega_{\mathrm{ref}}$" — here $\sigma > 0$ is a dilation factor — ρ may be bounded by $\sqrt{\sigma}\rho_{\mathrm{ref}}$. This embedding/scaling approach (applicable only to Dirichlet problems) eliminates the need for case-by-base evaluation of the Sobolev constant — albeit at some loss of sharpness.

B. Inf-Sup "Constant"

To construct a lower bound for $\beta_N(\nu)$, $\tilde{\beta}(\nu)$, we first introduce a parameter set $U_J \equiv \{\tilde{\nu}^1 \in \mathcal{D}, \ldots, \tilde{\nu}^J \in \mathcal{D}\}$, a distance

$$d(\nu, \tilde{\nu}; N) \equiv \frac{4}{3} \beta_{N^{\max}}^{-1}(\tilde{\nu}) \left[|\nu - \tilde{\nu}| + \right.$$
$$\left. 2\tilde{\rho}^2 \|u_N(\nu) - u_{N^{\max}}(\tilde{\nu})\|_Y \right] , \quad (16)$$

and a mapping $\mathcal{I}_N\nu \equiv \arg\min_{\tilde{\nu} \in U_J} d(\nu, \tilde{\nu}; N)$. We then define

$$\tilde{\beta}(\nu) \equiv \frac{1}{4} \beta_{N^{\max}}(\mathcal{I}_N\nu) , \qquad (17)$$

and prove
Lemma 1: The inf-sup approximation $\tilde{\beta}(\nu)$ satisfies

$$0 < \frac{1}{4} \beta_0 \leq \tilde{\beta}(\nu) \leq \beta_N(\nu), \quad \forall\, \nu \in \tilde{\mathcal{D}}_N(U_J) , \qquad (18)$$

where $\tilde{\mathcal{D}}_N(U_J) \equiv \{\nu \in \mathcal{D} \mid d(\nu, \mathcal{I}_N\nu; N) \leq 1\}$.
Proof We first observe from (6) and (11) that

$$\left| \int_\Omega \frac{\partial v_i}{\partial x_j} w_i z_j \right| \leq \|v\|_Y \|w\|_{L^4(\Omega)} \|z\|_{L^4(\Omega)}$$
$$\leq \tilde{\rho}^2 \|v\|_Y \|w\|_Y \|z\|_Y .$$

It then follows from (5) that, $\forall\, \nu, \tilde{\nu} \in \mathcal{D}$,

$$|\langle (DG(u_N(\nu); \nu) - DG(u_{N^{\max}}(\tilde{\nu}); \tilde{\nu}))w, v \rangle|$$
$$= \left| (\nu - \tilde{\nu}) \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} - \int_\Omega \frac{\partial v_i}{\partial x_j} (u_N(\nu) - u_{N^{\max}}(\tilde{\nu}))_i w_j \right.$$
$$\left. - \int_\Omega \frac{\partial v_i}{\partial x_j} w_i (u_N(\nu) - u_{N^{\max}}(\tilde{\nu}))_j \right|$$
$$\leq |\nu - \tilde{\nu}|\, \|w\|_Y\, \|v\|_Y$$
$$+ 2\tilde{\rho}^2 \|u_N(\nu) - u_{N^{\max}}(\tilde{\nu})\|_Y \|w\|_Y \|v\|_Y ; \quad (19)$$

hence, $\forall\, \nu \in \tilde{\mathcal{D}}_N(U_J)$,

$$\|DG(u_N(\nu); \nu) - DG(u_{N^{\max}}(\mathcal{I}_N\nu); \mathcal{I}_N\nu)\|_{Y,Y'}$$
$$\leq |\nu - \mathcal{I}_N\nu| + 2\tilde{\rho}^2 \|u_N(\nu) - u_{N^{\max}}(\mathcal{I}_N\nu)\|_Y$$
$$\leq \frac{3}{4} \beta_{N^{\max}}(\mathcal{I}_N\nu) , \qquad (20)$$

and

$$\|DG(u_{N^{\max}}(\mathcal{I}_N\nu); \mathcal{I}_N\nu)^{-1}$$
$$(DG(u_N(\nu); \nu) - DG(u_{N^{\max}}(\mathcal{I}_N\nu); \mathcal{I}_N\nu))\|_{Y,Y} \leq \frac{3}{4} . \quad (21)$$

We conclude [5], [14] that $DG(u_N(\nu); \nu)^{-1}$ exists for all $\nu \in \tilde{\mathcal{D}}_N(U_J)$. We now note [5] that, $\forall \nu \in \tilde{\mathcal{D}}_N(U_J)$,

$$\beta_N^{-1}(\nu) \equiv \|DG(u_N(\nu); \nu)^{-1}\|_{Y', Y}$$

$$= \|[DG(u_{N^{\max}}(\mathcal{I}_N\nu); \mathcal{I}_N\nu) + (DG(u_N(\nu); \nu)$$
$$- DG(u_{N^{\max}}(\mathcal{I}_N\nu); \mathcal{I}_N\nu))]^{-1}\|_{Y', Y}$$

$$= \|DG(u_{N^{\max}}(\mathcal{I}_N\nu); \mathcal{I}_N\nu)^{-1}$$
$$\left[I + DG(u_{N^{\max}}(\mathcal{I}_N\nu); \mathcal{I}_N\nu)^{-1}\right.$$
$$\left.(DG(u_N(\nu); \nu) - DG(u_{N^{\max}}(\mathcal{I}_N\nu); \mathcal{I}_N\nu))\right]^{-1}\|_{Y', Y}$$

$$\leq \|DG(u_N^{\max}(\mathcal{I}_N\nu); \mathcal{I}_N\nu)^{-1}\|_{Y', Y} \Big/$$
$$\left[1 - \|DG(u_N^{\max}(\mathcal{I}_N\nu); \mathcal{I}_N\nu)^{-1}(DG(u_N(\nu); \nu)\right.$$
$$\left.- DG(u_{N^{\max}}(\mathcal{I}_N\nu); \mathcal{I}_N\nu))\|_{Y, Y}\right]$$

$$\leq 4\beta_{N^{\max}}^{-1}(\mathcal{I}_N\nu) . \tag{22}$$

The desired result then follows from (17) and our assumption $\beta_{N^{\max}}(\nu) \geq \beta_0$, $\forall \nu \in \mathcal{D}$. $\qquad \square$

C. Error Bounds

To begin, we define

$$\varepsilon_N(\nu) \equiv \sup_{v \in Y} \frac{\langle G(u_N(\nu); \nu), v \rangle}{\|v\|_Y} , \tag{23}$$

which is simply the dual norm of the residual. Note it is crucial that we define the dual norm (and hence the inf-sup parameter) with respect to Y, and not $\tilde{Y}$, since the residual is of course small only with respect to divergence-free functions. The central result is a direct application of the Brezzi-Rappaz-Raviart framework [4], [5], [7], [9] for approximation of nonlinear problems: for $\varepsilon_N(\nu)$ sufficiently small, there exists a solution to (3) in a small neighborhood of $u_N(\nu)$.

More precisely, given $\tau_{\max} \in \,]0, 1[$, we define

$$\tau_N(\nu) \equiv \frac{4\tilde{\rho}^2 \varepsilon_N(\nu)}{\tilde{\beta}(\nu)^2} , \tag{24}$$

$$\Delta_N(\nu) \equiv \frac{\tilde{\beta}(\nu)}{2\tilde{\rho}^2}\left[1 - \sqrt{1 - \tau_N(\nu)}\right] , \tag{25}$$

$$\Upsilon_N(\nu) \equiv \frac{2\gamma(u_N(\nu); \nu)}{\tilde{\beta}(\nu)(1 - \tau_{\max})} , \tag{26}$$

and prove

Theorem 1: For ν in $\tilde{\mathcal{D}}_N(U_J)$, and $\tau_N(\nu) \leq \tau_{\max} < 1$, there exists a unique solution of (3), $u(\nu)$, in $\mathcal{B}\left(u_N(\nu), \frac{\tilde{\beta}(\nu)}{2\tilde{\rho}^2}\right)$. Furthermore,

$$\|u(\nu) - u_N(\nu)\|_Y \leq \Delta_N(\nu) , \tag{27}$$

and

$$\Delta_N(\nu) \leq \Upsilon_N(\nu) \|u(\nu) - u_N(\nu)\|_Y . \tag{28}$$

Here $\mathcal{B}(z, r) = \{w \in Y \mid \|w - z\|_Y < r\}$.

Proof The proof follows directly from Lemma 1 and the Brezzi-Rappaz-Raviart framework, in particular Theorem 2.1 of [5] (slightly specialized to the quadratic nonlinearity of interest here). Only the effectivity result requires some elaboration.

To derive the effectivity result, (28), we note that $e(\nu) \equiv u(\nu) - u_N(\nu)$ satisfies

$$\langle DG(u_N(\nu); \nu) \, e(\nu), v \rangle$$
$$= -\langle G(u_N(\nu); \nu), v \rangle + \int_\Omega \frac{\partial v_i}{\partial x_j}(e(\nu))_i (e(\nu))_j . \tag{29}$$

We now note from standard duality arguments that $\varepsilon_N(\nu) = \|\hat{e}(\nu)\|_Y$, where $\hat{e}(\nu) \in Y$ satisfies

$$(\hat{e}(\nu), v)_Y = -\langle G(u_N(\nu); \nu), v \rangle, \quad \forall v \in Y . \tag{30}$$

We next choose $v = \hat{e}(\nu)$ in (29) and apply continuity to obtain

$$\|\hat{e}(\nu)\|_Y \leq \gamma(u_N(\nu); \nu)\|e(\nu)\|_Y + \tilde{\rho}^2\|e(\nu)\|_Y^2 . \tag{31}$$

However, since $0 \leq \tau_N(\nu) < 1$, it follows that $\sqrt{1 - \tau_N(\nu)} \geq 1 - \tau_N(\nu)$ and hence

$$\Delta_N(\nu) \leq \frac{\tilde{\beta}(\nu)}{2\tilde{\rho}^2}\tau_N(\nu) = \frac{2\varepsilon_N(\nu)}{\tilde{\beta}(\nu)} . \tag{32}$$

Thus from $\|e(\nu)\|_Y \leq \Delta_N(\nu)$, (31), and (32),

$$\begin{aligned}
\Delta_N(\nu) &\leq \frac{2\gamma(u_N(\nu); \nu)}{\tilde{\beta}(\nu)}\|e(\nu)\|_Y \\
&\quad + \left(\frac{2\tilde{\rho}^2}{\tilde{\beta}(\nu)}\right)\left(\frac{2\varepsilon_N(\nu)}{\tilde{\beta}(\nu)}\right)\Delta_N(\nu) \\
&\leq \frac{2\gamma(u_N(\nu); \nu)}{\tilde{\beta}(\nu)}\|e(\nu)\|_Y + \tau_{\max}\Delta_N(\nu) ; \tag{33}
\end{aligned}$$

the desired result directly follows. $\qquad \square$

We re-iterate that the dual norm of the residual is small only with respect to functions in Y, the space of functions in $\tilde{Y}$ that are divergence-free. Thus, with the exception of $\tilde{\rho}$ (which we have explicitly and conservatively defined in terms of $\tilde{Y}$), all quantities must be calculated with respect to the divergence-free space Y. However, we shall see in Section V that this affects only the offline — and not the online — computational complexity; the online complexity remains independent of the dimension of Y.

V. COMPUTATIONAL COMPLEXITY: OFFLINE-ONLINE DECOMPOSITION

In actual practice, our interest is not in $u(\nu)$ *per se*, but rather in a (say) linear-functional output, $s(\nu) \equiv \langle L, u(\nu) \rangle$,

where L is a prescribed member of Y'. We wish to reliably evaluate $s(\nu)$ rapidly in the *limit of many queries* — as demanded in the (adaptive) design optimization and (real-time) control contexts. For "rapidly," we approximate $s(\nu)$ by $s_N(\nu) \equiv \langle L, u_N(\nu) \rangle$. For "reliably," we provide the *a posteriori* bound $\Delta_N^s(\nu) \equiv \|L\|_{Y'} \Delta_N(\nu)$; under the hypotheses of Theorem 1, $|s(\nu) - s_N(\nu)| \leq \Delta_N^s(\nu)$.

We now discuss the computational stratagem by which we efficiently evaluate $s_N(\nu)$ and $\Delta_N(\nu)$ (and hence $\Delta_N^s(\nu)$). The fundamental ingredient is an offline/online computational decomposition [2], [11], [19] that breaks the requisite calculations into two parts: an expensive *offline* stage performed once, and an inexpensive *online* stage performed many times — for each new evaluation $\nu \to s_N(\nu), \Delta_N(\nu)$. The complexity of the online stage will depend on N, which is typically small (see Section VI), but *not* on $\mathcal{N}$, which is typically large; we will thus realize marginal *real-time* — and, thanks to $\Delta_N^s(\nu)$, *reliable* — response.

A. Calculation of $u_N(\nu)$ and $s_N(\nu)$

We begin with the calculation of $u_N(\nu)$ and $s_N(\nu) \equiv \langle L, u_N(\nu) \rangle$. To obtain $u_N(\nu)$, we apply Newton iteration: Given $u_N^k(\nu) \in W_N$, find $\delta_N^k(\nu) \equiv u_N^{k+1}(\nu) - u_N^k(\nu) \in W_N$ such that

$$\langle DG(u_N^k(\nu); \nu)\, \delta_N^k(\nu), v \rangle$$
$$= -\langle G(u_N^k(\nu); \nu), v \rangle, \quad \forall\, v \in W_N . \quad (34)$$

This can be expressed in terms of our reduced-basis expansions

$$u_N^k(\nu) = \sum_{n=1}^{N} u_{N\,n}^k(\nu)\, \zeta_n , \quad \delta_N^k(\nu) = \sum_{n=1}^{N} \delta_{N\,n}^k(\nu)\, \zeta_n \quad (35)$$

as

$$\sum_{n=1}^{N} \left(\nu A_{N\,m,n} + \sum_{n'=1}^{N} u_{N\,n'}^k(\nu) B_{N\,m,n',n} \right) \delta_{N\,n}^k(\nu)$$
$$= F_m - \sum_{n=1}^{N} \left(\nu A_{N\,m,n} + \frac{1}{2} \sum_{n'=1}^{N} u_{N\,n'}^k(\nu) B_{N\,m,n',n} \right) u_{N\,n}^k(\nu),$$
$$1 \leq m \leq N , \quad (36)$$

where

$$A_{N\,m,n} \equiv \int_\Omega \frac{\partial (\zeta_n)_i}{\partial x_j} \frac{\partial (\zeta_m)_i}{\partial x_j}, \quad 1 \leq n, m \leq N , \quad (37)$$

$$B_{N\,m,n',n} \equiv -\int_\Omega \frac{\partial (\zeta_m)_i}{\partial x_j} \left((\zeta_n)_i (\zeta_{n'})_j + (\zeta_{n'})_i (\zeta_n)_j \right),$$
$$1 \leq n, n', m \leq N , \quad (38)$$

and $F_{N\,m} = \langle F, \zeta_m \rangle$, $1 \leq m \leq N$. Upon convergence, we evaluate our output as

$$s_N(\nu) = \sum_{n=1}^{N} u_{N\,n}(\nu)\, L_{N\,n} , \quad (39)$$

where $L_{N\,n} = \langle L, \zeta_n \rangle$, $1 \leq n \leq N$.

The offline/online decomposition is clear. In the *offline* stage, we form the parameter-independent quantities $A_{N^{\max}} \in \mathbb{R}^{(N^{\max})^2}$, $B_{N^{\max}} \in \mathbb{R}^{(N^{\max})^3}$, and $F_{N^{\max}} \in \mathbb{R}^{N^{\max}}$, $L_{N^{\max}} \in \mathbb{R}^{N^{\max}}$ — at cost bounded by $O((N^{\max})^3 \mathcal{N}^\bullet)$ (where $\bullet$ indicates a "solver-dependent" exponent greater than unity). In the *online* stage, for any given ν, we first construct and solve (36) — at cost (per Newton iteration) $O(N^3)$; and then, upon convergence, we evaluate $s_N(\nu)$ — at cost $O(N)$. The crucial observation is that, in the online stage, the complexity is *independent* of $\mathcal{N}$.

B. Calculation of $\Delta_N(\nu)$ and $\tau_N(\nu)$

We now turn to the *a posteriori* error bound, in particular the calculation of the quantities — $\Delta_N(\nu)$ and $\tau_N(\nu)$ — required by Theorem 1. The three critical computational tasks are the calculation of of $\tilde{\rho}$, the construction of $\tilde{\beta}(\nu)$, and the evaluation of $\varepsilon_N(\nu)$. We note that $\tilde{\rho}$ is computed (for a particular problem) only once — offline; the procedure is summarized in Section IV-A, and is not discussed further here.

We thus begin by considering the construction of $\tilde{\beta}(\nu)$ of (17): we must first find $\mathcal{I}_N \nu$ and verify $\nu \in \tilde{\mathcal{D}}_N(U_J)$; we can then evaluate $\frac{1}{4} \beta_{N^{\max}}(\mathcal{I}_N \nu)$. To determine $\mathcal{I}_N \nu$ we need only compare

$$d(\nu, \tilde{\nu}; N) = \frac{4}{3} \beta_{N^{\max}}^{-1}(\tilde{\nu}) \left(|\nu - \tilde{\nu}| \right.$$
$$\left. + 2\tilde{\rho}^2 \sum_{n=1}^{N^{\max}} \sum_{n'=1}^{N^{\max}} \sigma_{N\,n}(\nu, \tilde{\nu})\, \sigma_{N\,n'}(\nu, \tilde{\nu})\, A_{N^{\max}\,n,n'} \right) \quad (40)$$

at a few points $\tilde{\nu} \in U_J$ in the vicinity of ν; here $\sigma_{N\,n}(\nu, \tilde{\nu}) \equiv u_{N\,n}(\nu) - u_{N^{\max}\,n}(\tilde{\nu})$, $1 \leq n \leq N^{\max}$, and $A_{N^{\max}\,n,n'}$ is defined in (37). (Note that we set $u_{N\,n} = 0$, $N < n \leq N^{\max}$.) Once $\tilde{\nu}^i = \mathcal{I}_N \nu$ is obtained, we simply evaluate $\frac{1}{4} \beta_{N^{\max}}(\tilde{\nu}^i)$.

The offline/online decomposition is clear. In the *offline* stage we compute $u_{N^{\max}}(\tilde{\nu}^j)$, $1 \leq j \leq J$ — at cost $O(JN^3)$; and we tabulate $\beta_{N^{\max}}(\tilde{\nu}^j)$, $1 \leq j \leq J$ — at cost $O(JN^\bullet)$. In the *online* stage, for any given ν and $u_N(\nu)$, we evaluate $d(\nu, \tilde{\nu}^j; N)$ of (40) for $\tilde{\nu}^j \approx \nu$ to obtain $\tilde{\nu}^i = \mathcal{I}_N \nu$ — at cost $O((N^{\max})^2)$; then, assuming $d(\nu, \tilde{\nu}^i; N) \leq 1$, we "look up" $\beta_{N^{\max}}(\tilde{\nu}^i)$. (To minimize the risk that $d(\nu, \tilde{\nu}^i; N) > 1$, we choose U_J such that $\tilde{\mathcal{D}}_N(U_J) = \mathcal{D}$ for some suitably large N ($< N^{\max}$). This condition can be verified only by quasi-exhaustive evaluation of $d(\nu, \mathcal{I}_N \nu; N)$ over a fine grid of parameter values in $\mathcal{D}$; however, these offline evaluations can be effected very efficiently by repeated application of our $O((N^{\max})^2)$ "online" $d(\nu, \tilde{\nu}; N)$ procedure.)

We now consider the calculation of $\varepsilon_N(\nu) = \|\hat{e}(\nu)\|_Y$. We

recall from (30) that $\hat{e}(\nu) \in Y$ satisfies

$$(\hat{e}(\nu), v)_Y = \langle F, v \rangle - \sum_{n=1}^{N} \left(\nu \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial (\zeta_n)_i}{\partial x} \right) u_{N\,n}(\nu)$$

$$+ \sum_{n=1}^{N} \sum_{n'=1}^{N} \left(\int_\Omega \frac{\partial v_i}{\partial x_j} (\zeta_n)_i (\zeta_{n'})_j \right) u_{N\,n}(\nu)\, u_{N\,n'}(\nu),$$

$$\forall\, v \in Y\ . \quad (41)$$

It follows from linearity that

$$\hat{e}(\nu) = \hat{z}^0 + \nu \sum_{n=1}^{N} \hat{z}_n^1\, u_{N\,n}(\nu)$$

$$+ \sum_{n=1}^{N} \sum_{n'=1}^{N} \hat{z}_{nn'}^2\, u_{N\,n}(\nu)\, u_{N\,n'}(\nu)\ , \quad (42)$$

where

$$(\hat{z}^0, v)_Y = \langle F, v \rangle, \qquad \forall\, v \in Y\ , \quad (43)$$

$$(\hat{z}_n^1, v)_Y = -\int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial (\zeta_n)_i}{\partial x_j}, \qquad \forall\, v \in Y,$$

$$1 \le n \le N^{\mathrm{max}}\ , \quad (44)$$

$$(\hat{z}_{nn'}^2, v)_Y = \int_\Omega \frac{\partial v_i}{\partial x_j} (\zeta_n)_i (\zeta_{n'})_j, \qquad \forall\, v \in Y,$$

$$1 \le n, n' \le N^{\mathrm{max}}\ . \quad (45)$$

We thus obtain

$$\|\hat{e}(\nu)\|_Y^2 = (\hat{z}^0, \hat{z}^0)_Y$$

$$+ \sum_{n=1}^{N} u_{N\,n}(\nu) \big\{ 2\nu (\hat{z}^0, \hat{z}_n^1)_Y$$

$$+ \sum_{n'=1}^{N} u_{N\,n'}(\nu) \big\{ 2(\hat{z}^0, \hat{z}_{nn'}^2)_Y + \nu^2 (\hat{z}_n^1, \hat{z}_{n'}^1)_Y$$

$$+ \sum_{n''=1}^{N} u_{N\,n''}(\nu) \big\{ 2\nu (\hat{z}_n^1, \hat{z}_{n'n''}^2)_Y$$

$$+ \sum_{n'''=1}^{N} u_{N\,n'''}(\nu) \big\{ (\hat{z}_{nn'}^2, \hat{z}_{n''n'''}^2)_Y \big\} \big\} \big\} \big\}\ , (46)$$

which is a nested quadruple sum.

The offline/online decomposition is now clear. In the *offline* stage, we form $(\hat{z}^0, \hat{z}^0)_Y, \ldots$ — at dominant cost $O((N^{\mathrm{max}})^4 \mathcal{N}^\bullet)$. Note that $\hat{z}^0$, $\hat{z}_n^1$, and $\hat{z}_{nn'}^2$, $1 \le n, n' \le N^{\mathrm{max}}$, are in the divergence-free space Y; although there are ways to calculate the requisite quantities without forming
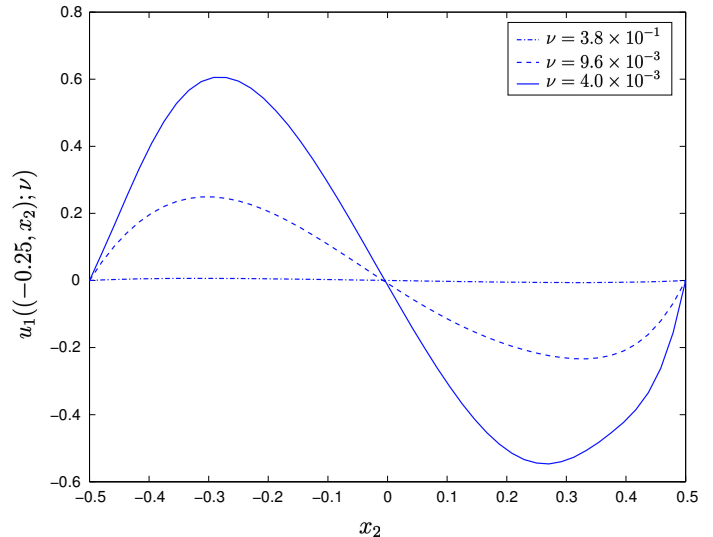

Fig. 1. Horizontal velocity $u_1((-0.25, x_2); \nu)$ as a function of the y-coordinate for $\nu = 3.8 \times 10^{-1}, 9.6 \times 10^{-3}$, and 4.0×10^{-3}.

the nullspace of the divergence operator, for our purposes here we choose the simpler option of direct construction of the nullspace. In the *online* stage, we simply evaluate (46) — at dominant cost (exploiting symmetries [26]) $N^4/4$. The N^4 scaling is steeper than desired, but not prohibitive for the small values of N typically required. In summary, we may compute not only $s_N(\nu)$, but also $\Delta_N(\nu)$, at *online* cost independent of $\mathcal{N}$.

Finally, we close this section by noting that the offline/online decomposition in fact applies more generally to any affine, or approximately affine, parameter (data, property, or geometry) dependence [19]. The latter, in turn, addresses a relatively large class of problems.

VI. Numerical Results

We consider the flow of a fluid of viscosity (inverse Reynolds number) $\nu \in \mathcal{D} \equiv [\nu^{\mathrm{min}} = 4.0 \times 10^{-3}, \nu^{\mathrm{max}} = 4.0 \times 10^{-1}]$ in a unit square $(x_1, x_2) \in \Omega \equiv]-\frac{1}{2}, \frac{1}{2}[\times]-\frac{1}{2}, \frac{1}{2}[$ with imposed force

$$\langle F, v \rangle = \int_\Omega x_1 v_2, \qquad \forall\, v \in Y\ ; \quad (47)$$

for this problem we obtain $\tilde{\rho} = 0.2852$. Standard proofs demonstrate that (3) admits a unique solution for ν sufficiently large; observations suggest that, in fact, (3) admits a unique solution *for all* $\nu \in \mathcal{D}$. We present in Figure 1 the horizontal velocity profiles $u_1((-0.25, x_2); \nu)$ as a function of x_2 for several values of ν (for a truth resolution $\mathcal{N} = 4,802$). Clearly, for $\nu = 4.0 \times 10^{-3}$ the nonlinear contributions are significant: the flow exhibits marked inertial effects.

We now present results for the case $N^{\mathrm{max}} = 10$. We present in Figures 2, 3, and 4 the stability factor $\beta_{N^{\mathrm{max}}}(\nu)$, the continuity factor $\gamma(u_{N^{\mathrm{max}}}(\nu); \nu)$, and the ratio
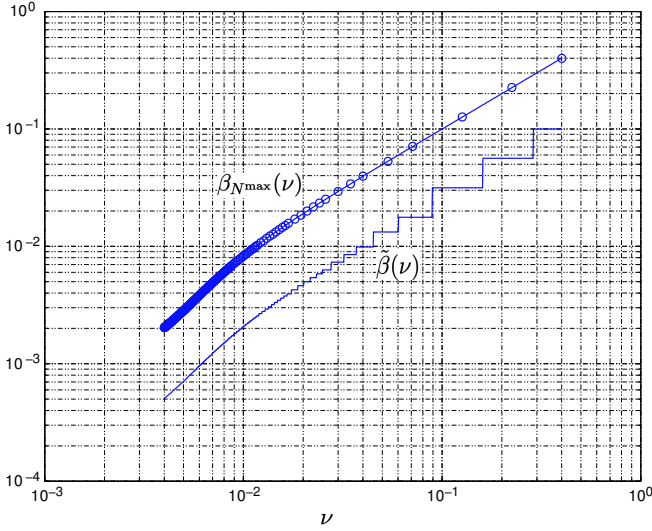
Fig. 2. The stability constant, $\beta_{N^{\max}}(\nu)$, and the lower bound for $\beta_N(\nu)$, $\tilde{\beta}(\nu)$, as a function of ν.
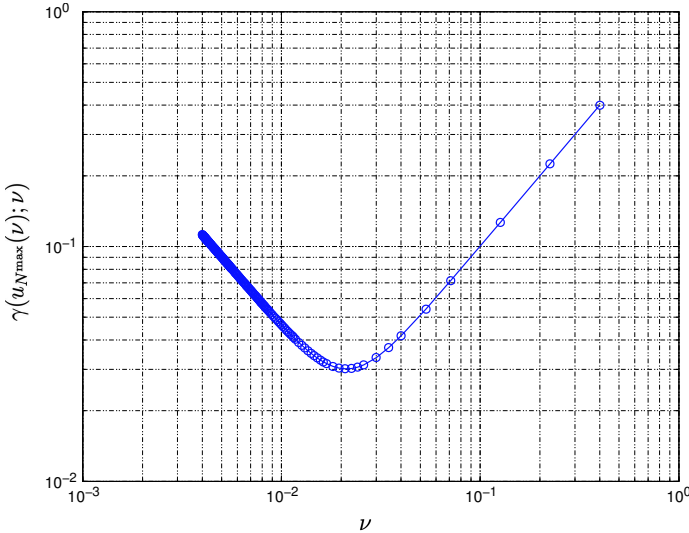

Fig. 3. The continuity factor $\gamma(u_{N^{\max}}(\nu); \nu)$ as a function of ν.

Fig. 4. The ratio $\gamma(u_{N^{\max}}(\nu); \nu)/\beta_{N^{\max}}(\nu)$ as a function of ν.

$\nu = 4.1 \times 10^{-3}$. We present in Table I the normalized error $\|e(\nu)\|_Y/\|u(\nu)\|_Y$ as a function of N for our three test values of ν. We observe that the error $e(\nu) \equiv u(\nu) - u_N(\nu)$ tends to zero (uniformly in ν) quite rapidly. Recall that $\| \cdot \|_Y$ is the H^1-norm, and hence measures the error in both the velocity and the velocity gradient.

	$\|e(\nu)\|_Y/\|u(\nu)\|_Y$		
N	$\nu = 1.0 \times 10^{-1}$	$\nu = 1.4 \times 10^{-2}$	$\nu = 4.1 \times 10^{-3}$
1	2.14×10^{-1}	1.94×10^{-1}	7.66×10^{-3}
2	6.34×10^{-4}	2.97×10^{-2}	7.52×10^{-3}
3	3.95×10^{-4}	1.51×10^{-2}	1.74×10^{-3}
4	1.52×10^{-4}	3.14×10^{-3}	1.28×10^{-3}
5	1.09×10^{-4}	2.00×10^{-3}	1.12×10^{-4}
6	1.27×10^{-5}	6.19×10^{-5}	8.99×10^{-5}
7	5.25×10^{-6}	1.92×10^{-5}	3.53×10^{-5}
8	3.77×10^{-6}	1.26×10^{-5}	1.19×10^{-6}
9	1.26×10^{-7}	1.14×10^{-6}	7.31×10^{-7}
10	9.26×10^{-8}	7.17×10^{-7}	1.61×10^{-7}

TABLE I
Normalized error in the reduced-basis approximation as a function of N.

We now turn to our error estimators. We present in Tables II, III, and IV $\tau_N(\nu)$, $\Delta_N(\nu)/\|u(\nu)\|_Y$ (the normalized error bound), and $\Delta_N(\nu)/\|e(\nu)\|_Y$ (the effectivity) as a function of N for $\nu = 1.0 \times 10^{-1}$, $\nu = 1.4 \times 10^{-2}$ and $\nu = 4.1 \times 10^{-3}$; here "*" indicates that $\nu \notin \tilde{\mathcal{D}}_N(U_J)$, and "−" indicates that $\tau_N(\nu) > 1$. For $\nu = 1.0 \times 10^{-1}$ we observe that $\tau_N(\nu) < 1$ for $N \geq 2$ — and hence we can provide a definitive error bound even for small N; that the relative er-

$\gamma(u_{N^{\max}}(\nu); \nu)/\beta_{N^{\max}}(\nu)$ as a function of ν; we also present in Figure 2 our lower bound for $\beta_N(\nu)$, $\tilde{\beta}(\nu)$. For the particular $U_{J=181}$ results presented here, $\tilde{\mathcal{D}}_N(U_J) = \mathcal{D}$ for all $N \geq 5$; thus $\tilde{\beta}(\nu) \leq \beta_N(\nu)$ for all $\nu \in \mathcal{D}$ for all $N \geq 5$. For future work we shall consider more efficient techniques such that $\tilde{\mathcal{D}}_N(U_J) = \mathcal{D}$ is achieved for smaller samples U_J. We further note from (26) and Figure 4 that, even at the lower ν, we expect effectivities — $\Delta_N(\nu)/\|e(\nu)\|_Y$ — no worse than $O(100)$; in fact, as we shall see, this bound is (pleasantly) pessimistic.

We begin with the reduced-basis prediction for $u(\nu)$, $u_N(\nu)$. In all the numerical examples, we shall consider three test values of ν: $\nu = 1.0 \times 10^{-1}, \nu = 1.4 \times 10^{-2}$, and
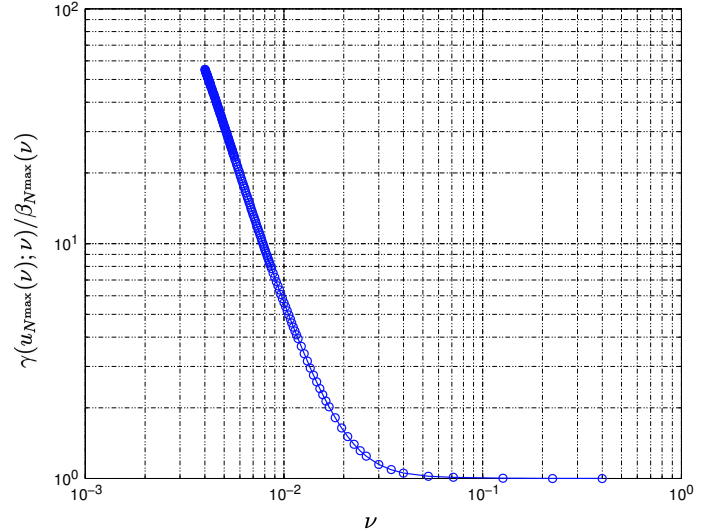
ror bound $\Delta_N(\nu)/\|u(\nu)\|_Y$ tends to zero rapidly; and that the effectivity is very good. For this value of ν, the "uniqueness radius" $\tilde{\beta}(\nu)/(2\tilde{\rho}^2) = 0.19$ $(\nu = 1.0 \times 10^{-1})$ is comfortably large relative to $\|u(\nu = 1.0 \times 10^{-1})\|_Y$.

For $\nu = 1.4 \times 10^{-2}$ (respectively, $\nu = 4.1 \times 10^{-3}$) we observe that, for $N < 2$ (respectively, $N < 3$), $\nu \notin \tilde{\mathcal{D}}_N(U_J)$, and for $N < 6$ (respectively, $N < 8$), $\tau_N(\nu) > 1$ — and hence we can obtain rigorous error bounds only for very accurate reduced-basis approximations; that the relative error bound $\Delta_N(\nu)/\|u(\nu)\|_Y$ still tends to zero rapidly with N — our sample S_N is constructed to provide uniform convergence; and that the effectivity is much better than the theoretical upper bound. Note also that the "uniqueness radii," $\tilde{\beta}(\nu)/(2\tilde{\rho}^2) = 2.0 \times 10^{-2}$ $(\nu = 1.4 \times 10^{-2})$ and 3.2×10^{-3} $(\nu = 4.1 \times 10^{-3})$, are small relative to $\|u(\nu = 1.4 \times 10^{-2})\|_Y = 1.4$ and $\|u(\nu = 4.1 \times 10^{-3})\|_Y = 4.7$. The rapid convergence of the reduced-basis method is important not only in efficiently reducing the error, but also in efficiently satisfying $\tau_N(\nu) < 1$; accuracy is required both to predict *and* to certify.

N	$\tau_N(\nu)$	$\dfrac{\Delta_N(\nu)}{\|u(\nu)\|_Y}$	$\dfrac{\Delta_N(\nu)}{\|e(\nu)\|_Y}$
1	1.38×10^0	$-$	$-$
2	4.08×10^{-3}	2.04×10^{-3}	3.22
3	2.54×10^{-3}	1.27×10^{-3}	3.22
4	9.77×10^{-4}	4.88×10^{-4}	3.22
5	7.01×10^{-4}	3.50×10^{-4}	3.22
6	8.20×10^{-5}	4.10×10^{-5}	3.22
7	3.38×10^{-5}	1.69×10^{-5}	3.22
8	2.43×10^{-5}	1.21×10^{-5}	3.22
9	7.73×10^{-7}	3.86×10^{-7}	3.05
10	5.27×10^{-7}	2.63×10^{-7}	2.84

TABLE II

NUMERICAL RESULTS FOR $\nu = 1.0 \times 10^{-1}$; $1.0 \times 10^{-1} \in \tilde{\mathcal{D}}_N(U_J)$ FOR ALL N.

N	$\tau_N(\nu)$	$\dfrac{\Delta_N(\nu)}{\|u(\nu)\|_Y}$	$\dfrac{\Delta_N(\nu)}{\|e(\nu)\|_Y}$
1	1.15×10^2	$*-$	$*-$
2	1.82×10^1	$-$	$-$
3	9.08×10^0	$-$	$-$
4	1.86×10^0	$-$	$-$
5	1.18×10^0	$-$	$-$
6	3.61×10^{-2}	2.71×10^{-4}	4.38
7	1.12×10^{-2}	8.35×10^{-5}	4.34
8	7.34×10^{-3}	5.47×10^{-5}	4.33
9	6.72×10^{-4}	5.00×10^{-6}	4.39
10	4.24×10^{-4}	3.16×10^{-6}	4.40

TABLE III

NUMERICAL RESULTS FOR $\nu = 1.4 \times 10^{-2}$; FOR $N < 2$, $1.4 \times 10^{-2} \notin \tilde{\mathcal{D}}_N(U_J)$.

N	$\tau_N(\nu)$	$\dfrac{\Delta_N(\nu)}{\|u(\nu)\|_Y}$	$\dfrac{\Delta_N(\nu)}{\|e(\nu)\|_Y}$
1	2.85×10^2	$*-$	$*-$
2	4.69×10^2	$*-$	$*-$
3	7.88×10^1	$-$	$-$
4	6.08×10^1	$-$	$-$
5	4.56×10^0	$-$	$-$
6	3.41×10^0	$-$	$-$
7	1.22×10^0	$-$	$-$
8	4.01×10^{-2}	1.40×10^{-5}	11.83
9	2.87×10^{-2}	1.00×10^{-5}	13.69
10	7.46×10^{-3}	2.59×10^{-6}	16.07

TABLE IV

NUMERICAL RESULTS FOR $\nu = 4.1 \times 10^{-3}$; FOR $N < 3$, $4.1 \times 10^{-3} \notin \tilde{\mathcal{D}}_N(U_J)$.

Finally, we note that the incremental cost to evaluate $u_N(\nu)$ (and therefore $s_N(\nu)$) for any given new ν is very small: first, because N is very small — thanks to the good convergence properties of W_N, and the "stopping criterion" provided by $\Delta_N(\nu)$; and second, because (36) can be very rapidly assembled and inverted — thanks to the offline/online computational decomposition. For our example, the online computation time (on a Pentium® M 1.6MHz processor) is typically 10-60 ms; the resulting computational savings relative to standard (well-designed) finite-element approaches are significant, typically $O(10^3)$-$O(10^4)$.

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