**December, 1997**

LIDS- P 2407

**Research Supported By:**

NSF grant ECS-9410531
NSF grant ECS-9796099
NSF grant ECS-9796033
AFOSR grant F49620-96-1-0123

# Integral Quadratic Constraints for Systems with Rate Limiters

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LIDS-P-2407

# Integral Quadratic Constraints for Systems With Rate Limiters

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December 8, 1997

**Abstract**

A new set of Integral Quadratic Constraints (IQC) is derived for a class of "rate limiters", modelled as a series connections of saturation-like memoryless nonlinearities followed by integrators. The result, when used within the standard IQC framework, is expected to be widely useful in nonlinear system analysis. For example, it enables "discrimination" between "saturation-like" and "deadzone-like" nonlinearities and can be used to prove stability of systems with saturation in cases when replacing the saturation block by another memoryless nonlinearity with equivalent slope restrictions makes the whole system unstable. In particular, it is shown that the $L_2$ gain of a unity feedback system with a rate limiter in the forward loop cannot exceed $\sqrt{2}$.

In addition, a new, more flexible version of the general IQC analysis framework is presented, which relaxes the homotopy and boundedness conditions, and is more aligned with the language of the emerging IQC software.

**Key Words:** nonlinear systems, saturation, induced gain, integral quadratic constraints, Hamilton-Jacoby-Bellman inequality.

## 1 Introduction

The aim of this paper is to improve the existing techniques of stability and performance analysis of systems with *rate limiters*, i.e. systems involving saturation of an input to an integrator (see Figure 1.1). Two general questions

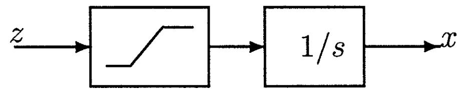


Figure 1.1: Rate limiter with ideal saturation

are to be answered:

- How to use the weighted small gain theorem and similar arguments in the case of a system that is not completely $L_2$ stable ?
- How to distinguish between the "saturation" and "deadzone" types of nonlinearities within the classical absolute stability framework (otherwise, when using Integral Quadratic Constraints = IQC) ?

The importance of the first question is based on the wide success of multiplier-based stability and performance analysis ("mu", scaled small gain, etc.) These IQC-based techniques provide low complexity/high accuracy results for systems that are represented as interconnections of $L_2$ bounded subsystems. However, such techniques usually experience serious difficulties when applied to critically stable systems.

On the other hand, the classical absolute stability was always weak at employing the "fine" differences between nonlinearities. For example, it is only natural to expect that replacing a saturation block $y \to \mathrm{sat}(y)$ by a deadzone block $y \to \mathrm{dzn}(y)$, where

$$\mathrm{sat}(y) = y/\max\{1, |y|\}, \quad \mathrm{dzn}(y) = y - \mathrm{sat}(y), \tag{1.1}$$

(see Figure 1.2) may change system behavior dramatically. However, it was not known how to represent the difference between the two nonlinearities within the multiplier analysis framework. For example, the criterion by Zames and Falb [4] for memoryless rate-bounded nonlinearities, will not make a distinction between $\phi(y) = \mathrm{sat}(y)$ and $\phi(y) = \mathrm{dzn}(y)$, because both have same derivative range $\dot{\phi}(y) \in [0, 1]$.

The main technical issue in this paper is validation of a set of IQC relating signals $z$ and $x$ in the system

$$x(t) = \int_0^t \phi(z(r))dr, \tag{1.2}$$

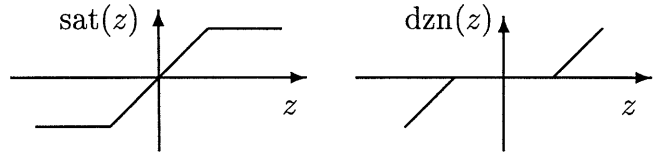


Figure 1.2: Saturation and deadzone

where $\phi$ is a "saturation-like" memoryless nonlinearity. Though system (1.2), because of its instability, does not fit well within the standard IQC analysis framework, the results originally derived for (1.2) can be easily transformed into a set of IQC for an "encapsulated" rate limiter system, defined by

$$\dot{x}(t) = \phi(v(t) - x(t)), \quad w(t) = x(t) + \phi(v(t) - x(t)), \quad x(0) = 0 \tag{1.3}$$

(see Figure 1.3). For example, it will follow from the main result that the gain "from $v$ to $w$" in system (1.3) does not exceed $\sqrt{2}$. For the special case of the ideal saturation $\phi(z) = \mathrm{sat}(z)$, we will thus recover the earlier result [2]. Note that while the gain is exactly $\sqrt{2}$ for $\phi(y) = \mathrm{sat}(y)$, replacing $\phi(y) = \mathrm{sat}(y)$ by its linearization at zero, $\phi(y) = y$, yields an identity system $w = v$ (the induced $L_2$ gain equals 1), while replacing $\phi(y) = \mathrm{sat}(y)$ by $\phi(y) = \mathrm{dzn}(y)$ results in an infinite $L_2$ gain.

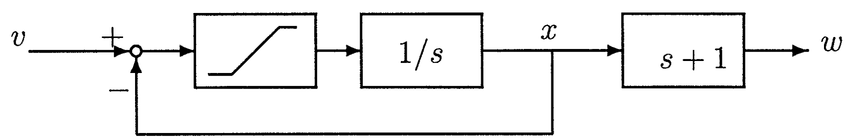


Figure 1.3: Encapsulation of a rate limiter

The IQC result has broad applications in the analysis of more complex systems (higher order, other nonlinearities, time-variance, uncertainty) that include the feedback interconnection (1.3) as a subsystem. Generally, the results of this paper can be applied to any system of the form

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = G_0 \begin{bmatrix} \Delta_1(v_1) \\ \phi(v_2) \end{bmatrix}, \quad G_0 = \begin{bmatrix} G_{11} & (s+1)(G_{12}-1)/s \\ G_{21} & (s+1)G_{22}/s \end{bmatrix}, \tag{1.4}$$

where $\phi$ is the same as in (1.3), $\Delta_1$ represents other nonlinearities/uncertainties in the system, and $G_{ij}$ are stable proper transfer matrices. While system (1.4) is not given in the standard IQC analysis format ($G_0$ is not stable), it can be reduced, via a simple feedback loop transformation, to a standard feedback interconnection of a stable LTI plant with a structured "uncertainty" which consists of blocks $\Delta_1$ and $\Delta_\phi$

$$\begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} = G \begin{bmatrix} \Delta_1(\bar{v}_1) \\ \Delta_\phi(\bar{v}_2) \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \tag{1.5}$$

where $\Delta_\phi$ is the operator $v \to w$ defined by (1.3) (see Figure 1.4).

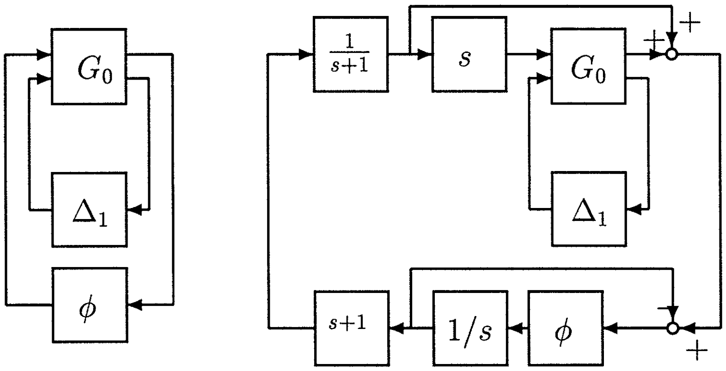


Figure 1.4: Loop transformation for encapsulation

It should be pointed out that the *finiteness* of the gain in system (1.3) follows from the more general result [3]. The main effort of this paper is concentrated on finding *minimal* gain bounds valid for saturation-like nonlinearities within a given sector.

# 2 IQC background

Technically speaking, the results of this paper do not rely on the theory of Integral Quadratic Constraints. However, it appears that they will be best used within the IQC framework. This section contains a brief presentation of the basics of IQC, which is somewhat different from the earlier description in [1]: some of the assumptions are relaxed, and a different general setup is used to align the theory with the emerging IQC software.

## 2.1 General setup

Integral Quadratic Constraints provide a simple, but often efficient way of analysing stability and performance of feedback interconnections of the form shown on Figure 2.5, where $f$ is the exogeneous disturbance, $e$ is the "in-

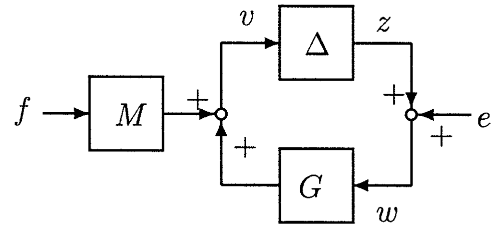


Figure 2.5: IQC analysis setup

terconnection noise", $M$ and $G$ are known stable LTI systems, $\Delta$ is the block representing nonlinear/uncertain/time-varying part of the system. The analysis is based on describing $\Delta$ as a relation between $v$ and $z$, using time-invariant integral quadratic inequalities $\sigma(v, z) \geq 0$. Such inequalities, which only have to be satisfied *under the assumption* that signals $f, w, z$ have finite energy, are called Integral Quadratic Constraints (IQC). As a rule, IQC for $\Delta$ are produced by forming any convex combination of "standard" IQC derived for "elementary" subsystems of $\Delta$. For *each* IQC describing $\Delta$, a simple frequency domain condition (which can also be written as a Linear Matrix Inequality (LMI) with respect to the "free" coefficients of the IQC) guarantees stability of the feedback interconnection. Simultaneously with stability, performance specifications represented in a quadratic inequality form $\sigma_0(w, f) \geq 0$ can be analyzed, subject to $e = 0$. Thus, stability and performance can be established by seacrhing through the set of all available IQC, trying to find one that proves stability. The search is equivalent to solving a system of LMI.

## 2.2 Notation and Terminology

*Signals* are elements of $L_{2e}^n$ – the set of locally square integrable functions $x : [0, \infty) \to \mathbf{R}^n$. The *energy* of a signal $x \in L_{2e}^n$ is defined by

$$\|x\|^2 = \int_0^\infty |x(t)|^2 dt. \tag{2.6}$$

$L_2^n$ denotes the set of signals $x \in L_{2e}^n$ of finite energy. For $x, y \in L_2^n$, the *scalar product* is defined by

$$\langle x, y \rangle = \int_0^\infty x(t)' y(t) dt. \tag{2.7}$$

When signal dimensions are obvious or irrelevant, the dimention index in $L_{2e}$ and $L_2$ will be dropped. The difference between spaces of pairs of vectors and spaces of concatenated vectors, such as the difference between $L_{2e}^n \times L_{2e}^m$ and $L_{2e}^{n+m}$ will be ignored. Two important operations on the signal spaces are *past projections*

$$(P_T v)(t) = \begin{cases} v(t) & \text{for} \quad t < T, \\ 0 & \text{for} \quad t \geq T. \end{cases} \tag{2.8}$$

and *causal LTI transformations* $G: \ w \to v$, defined by

$$v = \left(\frac{d}{dt}\right)^q (Cx + Dw), \ \dot{x} = Ax + Bw, \ x(0) = 0, \tag{2.9}$$

where $A, B, C, D$ are given matrices of appropriate size. (When $q > 0$, $G$ could be defined on a subset $\mathrm{Dom}(G)$ of $L_{2e}^n$ only.) The LTI transformation is called *bounded*, or *stable* if $A$ is a Hurwitz matrix and $q = 0$ (and hence $\mathrm{Dom}(G) = L_{2e}^n$. For convenience, $G$ will denote both the LTI operator and its transfer matrix

$$G(s) = s^q (D + C(sI - A)^{-1} B). \tag{2.10}$$

By a *time-invariant quadratic form* we mean any function $\sigma : \Omega \to \mathbf{R}$, ($\Omega \subset L_2^n$ is called the *domain* $\Omega = \mathrm{Dom}(\sigma)$ of $\sigma$), defined by

$$\sigma(g) = \sigma_H(g) = \langle f, Hf \rangle, \tag{2.11}$$

where $H$ is an LTI transformation defined on $\mathrm{Dom}(\sigma)$. When $H$ is bounded and $\mathrm{Dom}(\sigma) = L_2^n$, $\sigma$ is called a *bounded* time-invariant quadratic form.

A *system* is an operator $S : L_{2e}^n \to L_{2e}^m$. Multi-valued operators are allowed, (they are useful in describing systems with friction, hysteresis, etc.), in which case eqations such as $w = S(v)$ are understood as $w \in S(v)$. The non-commutative *distance* $\vec{d}(Q, S)$ between operators $Q$ and $S$ shows how well the output of $S$ can be approximated by the output of $Q$. It is defined by

$$\vec{d}(Q, S) = \inf\{r : \inf_{q \in Q(v)} \|P_T(s - q)\| \leq r \|P_T v\| \ \forall \ v \in L_{2e}^n, s \in S(v), T \geq 0\}. \tag{2.12}$$

In particular, the *induced $L_2$ gain* $||S||$ of system $S$ is defined by

$$||S|| = \vec{d}(0, S) = \sup_{P_T v \neq 0, w \in S(v)} \frac{||P_T w||}{||P_T v||}. \tag{2.13}$$

A *homotopy* of systems is a family $S = S_\tau$ depending on a parameter $\tau \in [0, 1]$. A homotopy $S_\tau$ is called *almost continuous* if

1. for any $\tau \in [0, 1)$, $\delta > 0$ there exists $\mu \in (\tau, 1)$ such that $d(S_\tau, S_\nu) < \delta$ for any $\nu \in [\tau, \mu]$;

2. for any $s \in S_\tau(v)$, $T \geq 0$, $\tau(i) \in [0, 1]$ such that $\tau(i) \to \tau \in [0, 1]$ there exist $s_i \in S_{\tau(i)}(v)$ such that $||P_T(s_i - s)|| \to 0$.

System $S$ is called *stable* if $||S|| < \infty$, and *causal* if the past of the output does not depend on the future of the input, i.e. when $P_T S P_T = P_T S$, which means that $P_T w \in P_T S(P_T v)$ iff $P_T w \in P_T S(v)$ for any $w, v$.

## 2.3 Feedback: well-posedness, stability, performance

Let $G : L_{2e}^m \to L_{2e}^n$ and $M : L_{2e}^k \to L_{2e}^n$ be stable LTI systems. Let $\Delta : L_{2e}^n \to L_{2e}^m$ be a causal system. By a *feedback interconnection* $\mathcal{F}[M, G, \Delta]$ we mean the system

$$(e, f) \to w = \mathcal{F}[M, G, \Delta](e, f)$$

defined by the equations

$$w = \Delta(Gw + Mf) + e.. \tag{2.14}$$

Here $e \in L_2^m$ plays the role of "interconnection noise" (see Figure 2.5), and is taken into account in stability calculations only. The signal $f$, used to define performance of the closed-loop system, plays the role of an "external disturbance". Interconnection $\mathcal{F}[M, G, \Delta]$ is *well-posed* if the operator is well-defined (i.e. a solution $w$ of (2.14) exists for any pair $(e, f)$) and causal. The interconnection is *stable* if system $\mathcal{F}[M, G, \Delta]$ is well-posed and stable.

Well-posedness of a feedback interconnection is usually equivalent to existence and continuability of solutions of the underlying equations. Stability means that solution of the feedback equations is not large when the interconnection noise and the external disturbance are small. Let $\sigma_0 : L_2^m \times L_2^k \to \mathbf{R}$ be a time-invariant quadratic form such that

$$\sigma_0(w, 0) \geq 0 \;\; \forall\, w \in L_2^m. \tag{2.15}$$

A stable feedback interconnection $\mathcal{F}[M, G, \Delta]$ is said to *satisfy the performance criterion* $\sigma_0 \geq 0$ if

$$\sigma_0(w, f) \geq 0 \;\; \forall\, w = \Delta(Gw + Mf),\;\; f \in L_2^k. \tag{2.16}$$

As a rule, (2.16) is equivalent to some "induced $L_2$ gain bound" constraint.

## 2.4 System analysis using IQC

Let $\Delta : L_{2e}^n \to L_{2e}^m$ and $\sigma :\; \mathrm{Dom}(\sigma) \to \mathbf{R}$ be a system and a time-invariant quadratic form. We say that the Integral Constraint $\sigma \geq 0$ is *valid* for $\Delta$ if

$$\sigma(v, \Delta(v)) \geq 0 \;\; \forall\, (v, \Delta(v)) \in \mathrm{Dom}(\sigma). \tag{2.17}$$

**Theorem 2.1** *Let $G : L_{2e}^m \to L_{2e}^n$, $M : L_{2e}^k \to L_{2e}^n$, $\sigma : \mathrm{Dom}(\sigma) \to \mathbf{R}$, $\sigma_0 : L_2^m \times L_2^k \to \mathbf{R}$ and $\Delta = \Delta_\tau : L_{2e}^n \to L_{2e}^m$ be two stable LTI transformations, a time-invariant quadratic functional, a bounded time-invariant quadratic functional, and a homotopy of systems, such that*

1. *feedback interconnection $\mathcal{F}[M, G, \Delta_\tau]$ is stable for $\tau = 0$ and well-posed for all $\tau \in [0, 1]$;*

2. *systems*
$$\tilde{\Delta}_\tau :\;\; (w, f) \to \Delta_\tau(Gw + Mf)$$
*are bounded and form an almost continuous homotopy;*

3. *time invariant quadratic form*
$$\tilde{\sigma}(z, w, f) = \sigma(Gw + Mf, z)$$
*is bounded, and the IQC $\sigma \geq 0$ is valid for $\Delta$ for all $\tau \in [0, 1]$;*

4. *$\sigma_0 : L_2^m \times L_2^k \to \mathbf{R}$ is a bounded time-invariant quadratic form such that $\sigma_0(w, 0) \leq 0$ for all $w \in L_2^m$.*

*Then $\mathcal{F}[M, G, \Delta_1]$ is stable and satisfies the performance criterion $\sigma_0 \geq 0$ if there exists $\epsilon > 0$ such that*

$$\sigma_0(w, f) - \sigma(Gw + Mf, w) \geq \epsilon \|w\|^2 \;\; \forall\, w \in L_2^m,\;\; f \in L_2^k. \tag{2.18}$$

Theorem 2.1 is proven in the Appendix.

As a rule, Theorem 2.1 is applied in the situation when $w = \Delta(v)$ is a block-diagonal composition $\Delta = \mathrm{diag}(\Delta_i)$ of several nonlinear/time-varying/uncertain blocks $w_i = \Delta_i(v_i)$, where $w_i = c_i^w w$, $v_i = c_i^v v$ are components of $w$ and $v$ respectively. The set of IQC describing $\Delta$ (generally, the more IQC the better) is formed as the set of all convex combinations of IQC describing the individual blocks

$$\Delta_i : \quad \sigma_{ij}(v_i, \Delta_i(v_i)) \geq 0,$$

where the second index $j$ may be ranging over an infinite set. Thus, the general form of $\sigma$ is

$$\sigma(v, w) = \sum_{i,j} x_{ij}\sigma_{ij}(c_i^v v, c_i^w w), \quad x_{ij} \geq 0.$$

A typical performance constraint is the induced $L_2$ norm bound in the channel $f \to z_0$, where $z_0 = G_0 w + M_0 f$ is a stable LTI transformation of $w, f$. Thus $\sigma_0$ is defined as

$$\sigma_0 = \gamma^2 \|f\|^2 - \|G_0 w + M_0 f\|^2,$$

and the combined stability/performance condition becomes the *existence* of $x_{ij} \geq 0$ such that

$$\gamma^2 \|f\|^2 - \|G_0 w + M_0 f\|^2 - \sum_{i,j} x_{ij}\sigma_{ij}(c_i^v v, c_i^w w) \geq \epsilon \|w\|^2, \quad x_{ij} \geq 0.$$

In most cases, using the Kalman-Popov-Yakubovich lemma, this can be rewritten as a finite system of Linear Matrix Inequalities, and then solved efficiently with simultaneous minimization of induced gain bound $\gamma$.

# 3 Main Result

The results of this section hold for a large class of *semiconcave* functions $\phi$. The definition of a semiconcave function summarizes those features of the ideal saturation nonlinearity which are essential in proving the $\sqrt{2}$-gain result and its generalizations.

**Definition** A monotonically non-decreasing odd function $\phi : \mathbf{R} \to \mathbf{R}$ is called *semiconcave* if $\phi(z) = \phi(z)/z$ is monotonically non-increasing over the interval $z \in (0, \infty)$.The set of all semiconcave functions will be denoted by $\mathcal{SC}$.

It is easy to see that a semiconcave function $\tilde{\phi} = \phi(z)$ is differentiable at $z = 0$ iff $\phi(z)/z$ is bounded on the interval $(0, \infty)$, in which case

$$\dot{\phi}(0) = \sup_{z>0} \frac{\phi(z)}{z}. \tag{3.19}$$

For convenience, we will consider (3.19) as a definition of $\dot{\phi}(0)$ in the case when the right side in (3.19) is infinity. Note also that $\dot{\phi}(0) = 0$ would imply $\phi \equiv 0$, in which case system (1.2) is trivial.

**Theorem 3.2** *Let $\phi$ be a semiconcave function with $\dot{\phi}(0) = K$, $0 < K < \infty$, $b \in \mathbf{R}$. The inequality*

$$\int_0^\infty \left\{2|z + bx|^2 - |z|^2 - z\phi(z)/K\right\} dt \geq 0 \tag{3.20}$$

*holds for any $x, z \in L_2$ satisfying relation (1.2). Moreover, if $b \geq 0$ then*

$$\int_0^T \left\{2|z + bx|^2 - |z|^2 - z\phi(z)/K\right\} dt \geq 0 \tag{3.21}$$

*for all $T \geq 0$, $z, x \in L_{2e}$ satisfying relation (1.2).*

A proof of Theorem 3.2 is given in the Appendix.

In a certain sense, it can be shown that Theorem 3.2 completes a description of the "extremal points" of the convex cone of all IQC of the form

$$\int_0^\infty \bar{\sigma}(x(t), z(t), \phi(z(t)))dt \geq 0, \tag{3.22}$$

where

$$\bar{\sigma}(x, z, u) = \begin{bmatrix} x \\ x \\ u \end{bmatrix}' \Sigma \begin{bmatrix} x \\ x \\ u \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{12} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{13} & \Sigma_{23} & \Sigma_{33} \end{bmatrix}, \ \Sigma_{33} \leq 0, \tag{3.23}$$

which are satisfied for any $x, z \in L_2$ satisfying relation (1.2). One corollary of Theorem 3.2 is the following "complete" description of all memoryless IQC relating $z, x$ and $\phi(z)$.

**Theorem 3.3** *Let $\bar{\sigma}$ be the quadratic form in (3.23), $K \in (0, \infty)$. The following conditions are equivalent.*

1. *Inequality (3.22) holds for any $x, z \in L_2$ satisfying relation (1.2) and for any semiconcave $\phi$ such that $\dot{\phi}(0) \leq K$.*

2. *The inequalities*

$$\left[ \begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & 2\Sigma_{22} \end{array} \right] \geq 0, \quad \Sigma_{22} + 2K\Sigma_{23} + K^2\Sigma_{33} \geq 0 \tag{3.24}$$

   *hold.*

A proof of Theorem 3.3 is given in the Appendix.

Inequality (3.20) can be considered as a family of IQC describing the relation between $z$ and $\phi(z)$. Since the corresponding $\sigma_L$ is defined by an unstable system $L$ (contains a pure integrator), it is difficult to use (3.20) directly in IQC analysis of systems involving semiconcave nonlinearities. To resolve this problem, (3.20) is re-written as a set of IQC describing the "encapsulated" rate limiter from Figure 3.6.

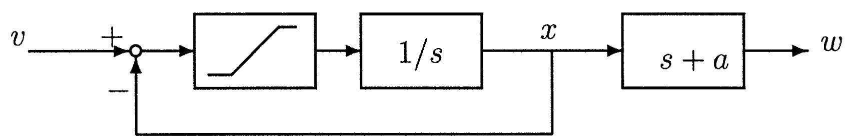


Figure 3.6: Encapsulation of a rate limiter

The following result is a direct implication of Theorems 3.2 and 3.3.

**Theorem 3.4** *Let $\phi$ be a semiconcave function with $\dot{\phi}(0) = K$, $0 < K < \infty$, $b \in \mathbf{R}$, $a > 0$. Define $\Delta_\phi^a$ as a system $v \to w$, where*

$$\dot{x} = \phi(v - x), \quad w = ax + \phi(v - x), \quad x(0) = 0. \tag{3.25}$$

*The system is stable, and the inequality*

$$\int_0^\infty \bar{\sigma}(x, v - x, w - ax) \geq 0 \tag{3.26}$$

*holds for any $v \in L_2$ for any $\bar{\sigma}$ defined by (3.23),(3.24). In particular, the induced $L_2$ gain in the channel $v \to w$ does not exceed* $\max\{K, \sqrt{2}a\}$.

**Proof** of Theorem 3.4. For $b = 1$, $z = v - x$, inequality (3.21) implies $\|P_T(v-x)\| \leq 2\|P_T v\|$. Hence system $\Delta_\phi^a$ is stable. In particular, $x, w \in L_2$ whenever $v \in L_2$. Now, proving an upper bound $\|\Delta_\phi^a\| \leq \gamma$ is equivalent to proving the IQC of the form

$$0 \leq \gamma^2\|v\|^2 - \|w\|^2 = \gamma^2\|x + (v-x)\|^2 - \|ax + (w - ax)\|^2,$$

which, according to Theorem 3.4, refers us to checking conditions (3.24) for

$$\bar{\sigma}(x, z, u) = \gamma^2|x + z|^2 - |ax + u|^2, \text{ i.e } \Sigma = \begin{bmatrix} \gamma^2 - a^2 & \gamma^2 & -a \\ \gamma^2 & \gamma^2 & 0 \\ -a & 0 & -1 \end{bmatrix}.$$

The first inequality in (3.24) yields $\gamma^2 \geq 2a^2$, while the second yields $\gamma^2 \geq K^2$, which results in the upper bound $\gamma = \max\{K, \sqrt{2}a\}$. ■

Note that $x = G_a w$ where $G_a(s) = 1/(s+a)$, i.e. (3.26) are true IQC relating $v$ and $w$. If $\phi$ is not a semiconcave function, these IQC are generally not valid. For example, it is easy to see that replacing $\phi$ by dzn yields an unstable system $\Delta_\phi^a$ (infinite $L_2$ gain) for any $a \neq 0$.

There are other IQC known to describe the relation between $z$ and $\phi(z)$. The classical criterion by Zames and Falb [4] states that

$$\langle Kz - \phi(z), H_1 z\rangle \geq 0, \tag{3.27}$$

$$\langle H_2(Kz - \phi(z)), z\rangle \geq 0, \tag{3.28}$$

where $H_1, H_2$ are LTI systems such that

$$H_i(s) = D_i + C_i(sI - A_i)^{-1}B_i, \; D_i \geq \int_0^\infty \left|C_i e^{A_i t} B_i\right| dt, \tag{3.29}$$

and $\dot{\phi} \in [0, K]$. These IQC can be re-written as input/output relations for $w = \Delta_\phi^a v$:

$$\langle K(v-x) - (w - ax), H_1(v-x)\rangle \geq 0, \tag{3.30}$$

$$\langle H_2(K(v-x) - (w-ax)), v - x\rangle \geq 0, \tag{3.31}$$

where, as before, $x = G_a w$.

Similarly, the Popov IQC $\langle \dot{z}, \phi(z)\rangle \geq 0$ is valid, and can be used in the analysis, whenever $z$ is an output of a strictly proper subsystem. In terms of the encapsulated rate limiter block $w = \Delta_\phi^a$, it can be re-written as the IQC

$$\langle \dot{v} - w + ax, w - ax\rangle \geq 0, \tag{3.32}$$

which can be used when $z$ is an output of a strictly proper subsystem.

# 4 The homotopy

An important condition required in applications of IQC is existence of a *homotopy* $\Delta = \Delta_\tau$, $\tau \in [0, 1]$, which "connects" a given operator $\Delta = \Delta_1$ to a simpler operator $\Delta = \Delta_0$ (see Theorem 2.1). Existence of the homotopy is usually obvious in the case when the quadratic form $\sigma(v, w)$ defining the IQC $\sigma(v, \Delta(v)) \geq 0$ is concave with respect to $w$ and convex with respect to $v$, i.e. when $\sigma(0, w) \leq 0$ and $\sigma(v, 0) \geq 0$ for all $w, v$. Then one can simply set $\Delta_\tau = \tau\Delta$. While the "convexity" condition is usually satisfied (it simply means that the IQC under consideration will be valid for $\Delta = 0$), the other (concavity) condition is sometimes not valid. In particular, this is the case in (3.26) and (3.30) for some valid choices of $b, a, H_i$.

This section is concerned with providing the required homotopy, that will work for any choice of $b, a, H_i$. The most natural choice appears to be the homotopy between $\Delta_1 = \Delta_\phi^a$ and the LTI system $\Delta_0 = K(s+a)/(s+K)$, where $\Delta_\tau = \Delta_{\phi[\tau]}^a$,

$$\phi[\tau](z) = \tau\phi(z) + (1-\tau)Kz. \tag{4.33}$$

Since $\phi[\tau]$ is quasiconcave and satisfies the rate bound $0 \leq \dot{\phi}[\tau] \leq K$ for all $\tau \in [0, 1]$, this is a valid homotopy preserving all IQC.

**Theorem 4.5** *The homotopy between $\Delta_1 = \Delta_\phi^a$ and the LTI system $\Delta_0 = K(s+a)/(s+K)$, where $\Delta_\tau = \Delta_{\phi[\tau]}^a$, is almost continuous.*

Theorem 4.5 is proven in the Appendix. Note that it is still not known to the author whether the homotopy is *strictly* continuous, i.e. whether $||\Delta_\tau - \Delta_\mu|| \to 0$ as $\mu - \tau \to 0$ (the continuity at $\tau = 1$ is the only real concern here, the continuity at all other points being trivial). Luckily, the new formulation of the general IQC stability theorem allows to use the notion of "almost continmuity" instead of the strong continuity.

# 5 Appendix

In the appendix, formal proofs of the major statements of this paper are given.

## 5.1 Proof of Theorem 2.1

The proof is similar to that given in [1], except for the modifications needed to accomodate the relaxed conditions of continuity of the homotopy The IQC assumption 3 and condition (2.18) are used to show that there is a constant $C > 0$, independent of $\tau \in [0, 1]$, such that

$$\|P_T w\|^2 \leq C(\|P_T e\|^2 + \|P_T f\|^2\|) \quad \forall\, T \geq 0 \tag{5.34}$$

for any solution of (2.14) and for any $\tau \in [0, 1]$ such that the interconnection $\mathcal{F}[M, G, \Delta_\tau]$ is stable.

Then, part 2 of the definition of almost continuity (weak lower semicontinuity), together with the assumption of stability of $\mathcal{F}[M, G, \Delta_0]$, is used to show that there exists a *maximal* $\tau = \tau_* \in [0, 1]$ such that $\mathcal{F}[M, G, \Delta_\tau]$ is stable, and part 1 of the same definition (strong upper semicontinuity) is used to show that this maximal $\tau_*$ must be equal to 1.

As soon as stability of $\mathcal{F}[M, G, \Delta_\tau]$ is established, i.e. $w$ is guaranteed to be square integrable whenever $f$ and $e$ are, the performance inequality is obvious: for

$$w = \Delta(Gw + Mf) = \Delta(v),$$

we have

$$\sigma_0(w, f) \geq \sigma(Gw + Mf, w) = \sigma(v, \Delta(v)) \geq 0.$$

### 5.1.1 Uniform gain bound

Since $\sigma(Gw + Mf, z)$ is a bounded time-invariant quadratic form, for any $\epsilon_1 > 0$ there exists $C_1 > 0$ such that

$$|\sigma(Gw + Mf, z + e) - \sigma(Gw + Mf, z)| \leq \epsilon_1 \|w\|^2 + C_1(\|e\|^2 + \|f\|^2).$$

Similarly, since $\sigma_0(w, 0) \geq 0$, for any $\epsilon_2 > 0$ there exists $C_2 > 0$ such that

$$\sigma_0(w, f) \leq \epsilon_2 \|w\|^2 + C_2 \|f\|^2.$$

Hence, by choosing $\epsilon_1 + \epsilon_2 < \epsilon$, for any $w, f, e \in L_2$ satisfying (2.14) we have (with $z = w - e$):

$$\begin{aligned}
0 &\leq \sigma(Gw + Mf, w - e) \\
&\leq \sigma(Gw + Mf, w) + \epsilon_1 \|w\|^2 + C_1(\|e\|^2 + \|f\|^2) \\
&\leq \sigma_0(w, f) - \epsilon \|w\|^2 + \epsilon_1 \|w\|^2 + C_1(\|e\|^2 + \|f\|^2) \\
&\leq -(\epsilon - \epsilon_1 - \epsilon_2)\|w\|^2 + (C_1 + C_2)(\|e\|^2 + \|f\|^2).
\end{aligned}$$

Hence

$$\|w\|^2 \leq C(\|e\|^2 + \|f\|^2) \tag{5.35}$$

for any solution of (2.14) of finite energy. Now, if $\tau \in [0,1]$ is such that $\mathcal{F}[M,G,\Delta_\tau]$ is stable, $w \in L_2^m$ whenever $e \in L_2^m$ and $f \in L_2^k$. Moreover, by the well-posedness, the map $(e,f) \to w$ is causal, i.e. replacing $e$ by $P_T e$ and $f$ by $P_T f$ does not change $P_T w$. hence (5.35) implies

$$\begin{aligned} \|P_T \mathcal{F}[M,G,\Delta_\tau](e,f)\| &= \|P_T \mathcal{F}[M,G,\Delta_\tau](P_T e, P_T f)\| \\ &\leq \|\mathcal{F}[M,G,\Delta_\tau](P_T e, P_T f)\| \\ &\leq C(\|P_T e\|^2 + \|P_T f\|^2. \end{aligned}$$

### 5.1.2 Stability points $\tau$ form a closed subset of $[0,1]$

Let $\mathcal{S}$ be the set of $\tau \in [0,1]$ such that $\mathcal{F}[M,G,\Delta_\tau]$ is stable.

**Lemma 5.1** *$\mathcal{S}$ is a closed subset of* $[0,1]$.

**Proof** Let $\tau(i) \in \mathcal{S}$, $\tau(i) \to \tau$ as $i \to \infty$. Our objective is to show that $\tau \in \mathcal{S}$. Let $w = \mathcal{F}[M,G,\Delta_\tau](e,f)$, i.e.

$$z = w - e = \Delta_\tau(Gw + Mf).$$

By the assumption, for any $T > 0$ there exists a sequence

$$z_i : \ z_i = \Delta_{\tau(i)}(Gw + Mf), \quad \|P_T(z_i - z)\| \to 0.$$

Since $\tau(i) \in \mathcal{S}$ and

$$w = \Delta_{\tau(i)}(Gw + Mf) + (e + z - z_i),$$

the inequality

$$\|P_T w\|^2 \leq C(\|P_T f\|^2 + \|P_T(e + z - z_i)\|^2)$$

holds, where $C$ does not depend on $i$. As $i \to \infty$, this yields stability of $\mathcal{F}[M,G,\Delta_\tau]$. ■

### 5.1.3 Stability preserved under small increments of $\tau$

The following statement, similar to Lemma 5.1, is based on the second part of the definition of almost continuity.

**Lemma 5.2** *If $\tau \in \mathcal{S}$ and $\tau < 1$ then $\nu \in \mathcal{S}$ for some $\nu > \tau$.*

**Proof** By the definition of almost continuity, for any $\delta > 0$ there exists $\nu \in (\tau, 1]$ such that for any $T > 0$, $e \in L^m_{2e}$, $f \in L^k_{2e}$, and for any $z_\tau \in \Delta_\tau(Gw + Mf)$ there exists $z_\mu \in \Delta(Gw + Mf)$ such that

$$\|P_T(z_\tau - z_\mu)\| \leq \delta(\|P_T w\|^2 + \|P_T f\|^2).$$

Choose $\delta = (4C)^{-1}$, where $C$ is the constant in (5.34). Then for any $w = \mathcal{F}[M, G, \Delta_\mu](e, f)$ we have

$$\begin{aligned} w &= \Delta_\mu(Gw + Mf) + e \\ &= z_\mu + e \\ &= z_\tau + (e + z_\mu - z_\tau) \\ &= \Delta_\tau(Gw + Mf) + (e + z_\mu - z_\tau). \end{aligned}$$

Hence, by stability of $\mathcal{F}[M, G, \Delta_\tau]$,

$$\begin{aligned} \|P_T w\|^2 &= C(\|P_T f\|^2 + \|P_T(e + z_\mu - z_\tau)\|^2) \\ &\leq C(\|P_T f\|^2 + 2\|P_T e\|^2 + 2\|P_T(z_\mu - z_\tau)\|^2) \\ &\leq C(\|P_T f\|^2 + 2\|P_T e\|^2 + 2\delta(\|P_T w\|^2 + \|P_T f\|^2)) \\ &\leq 2\delta C\|P_T w\|^2 + 2C\|P_T e\|^2 + (C + 2\delta C)\|P_T f\|^2 \\ &\leq 0.5\|P_T w\|^2 + @C\|P_T e\|^2 + (C + 0.5)\|P_T f\|^2. \end{aligned}$$

Hence

$$\|P_T w\|^2 \leq (4C + 1)(\|P_T e\|^2 + \|P_T f\|^2),$$

which proves stability of $\mathcal{F}[M, G, \Delta_\mu]$. ∎

## 5.2 Proof of Theorem 3.2

The IQC (3.20) to be proven follow from existence of a solution $V : \mathbf{R} \to \mathbf{R}$ of the Bellman inequality

$$2|w + bx|^2 - |w|^2 - w\phi(w)/K \geq \dot{V}(x)\phi(w). \tag{5.36}$$

Moreover, (3.21) follows from the fact that, for $b \geq 0$, $V$ can be chosen in such a way that $V(x) \geq 0$ for any $x$.

In order to prove existence of $V$ from (5.36), we notice that it is equivalent to

$$\dot{V}(x) \in \left[\sup_{v>0} \frac{|v|^2 - 2|z-v|^2}{\phi(v)} + \frac{v}{K}, \inf_{u>0} \frac{2|z+u|^2 - |u|^2}{\phi(u)} - \frac{u}{K}\right], \tag{5.37}$$

where $z = bx$. The reduction from (5.36) to (5.37) is done by substituting $w = u > 0$ and $w = -v < 0$ into (5.36). A major part of the proof concentrates on showing that the interval in (5.37) is not empty. Then $V(x)$ can be defined by its derivative (which, subject to (5.37), can be chosen arbitrarily), and by the initial condition $V(0) = 0$.

Finally, (3.20) can be obtained by integrating (5.36) with $x = x(t)$, $w = w(t)$, from $t = 0$ to $t = \infty$ (in the case when $x, w \in L_2$), or, with the use of the inequality $V \geq 0$, from $t = 0$ to $t = T$ (in the case when $b \geq 0$).

### 5.2.1 Continuity of the upper limit

For $b \geq 0, u > 0$ let

$$q(u,z) = \frac{2|u+z|^2 - |u|^2}{\phi(u)} - \frac{u}{K} = u\left(\frac{u}{\phi(u)} - \frac{1}{K}\right) + 4\frac{u}{\phi(u)}z + 2\frac{z^2}{\phi(u)}. \tag{5.38}$$

Since $\phi(u) \leq Ku$, we conclude that $q \geq 0$. Define

$$p(z) = \inf_{u>0} q(u,z). \tag{5.39}$$

**Lemma 5.3** *$p : [0,\infty) \to [0,\infty)$ is continuous.*

**Proof** Since

$$0 \leq \frac{dq(u,z)}{dz} \leq 2\frac{q}{z},$$

$p$ is locally Lipschitz on $(0,\infty)$. Also, since

$$q(z,z) \leq 8z^2/\phi(z) \to 0 \quad \text{as} \quad z \to 0,$$

we conclude that $p(z) \to 0$ as $z \to 0$. ■

### 5.2.2 The main inequality

We extend the definition of $p$ by setting

$$p(-z) = -p(z) \quad \text{for} \quad z > 0.$$

The following is the main technical detail of the paper.

**Lemma 5.4** *The inequality*

$$2|w+z|^2 - |w|^2 - w\phi(w)/K \geq p(z)\phi(w) \tag{5.40}$$

*holds for any* $z, w \in \mathbf{R}$.

**Proof** Since both $p$ and $\phi$ are odd functions, it is sufficient to consider the case when $z \geq 0$. Consider the following possible locations of $w$.

When $w = 0$, the inequality is obvious.

When $w > 0$, (5.40) follows directly from the definition of $p$ in (5.38),(5.39).

When $w = -v < 0$, (5.40) is equivalent to

$$\frac{2|z+u|^2 - |u|^2}{\phi(u)} - \frac{u}{K} \geq \frac{|v|^2 - 2|z-v|^2}{\phi(v)} + \frac{v}{K} \quad \forall\ u, v > 0,\ z \geq 0. \tag{5.41}$$

An equivalent form of (5.41) is

$$\frac{2z^2+u^2}{\phi(u)} + \frac{2z^2+v^2}{\phi(v)} - \frac{u+v}{K} \geq 4z\left(\frac{v}{\phi(v)} - \frac{u}{\phi(u)}\right) \quad \forall\ u, v > 0,\ z \geq 0. \tag{5.42}$$

Since $\phi(u) \leq Ku$ and $\phi(v) \leq Kv$, the left side of (5.42) is always non-negative. Hence, since $v/\phi(v)$ is monotonically non-decreasing, (5.42) holds for $v \leq u$.

Now let $v > u > 0$. We need to consider two different situations:

Case 1: $\phi(v) \leq Ku$. Then, since $\phi(u) \leq \phi(v)$,

$$\frac{2|z+u|^2 - |u|^2}{\phi(u)} - \frac{u}{K} - \frac{|v|^2 - 2|z-v|^2}{\phi(v)} - \frac{v}{K} \geq$$

$$
\begin{aligned}
&\geq \frac{2|z+u|^2-|u|^2}{\phi(v)}-\frac{u}{K}-\frac{|v|^2-2|z-v|^2}{\phi(v)}-\frac{v}{K} \\
&= \frac{4z^2+4z(u-v)+v^2+u^2}{\phi(v)}-\frac{u+v}{K} \\
&= \frac{|2z-v|^2+4zu+u^2}{\phi(v)}-\frac{u+v}{K} \\
&\geq \frac{|2z-v|^2+4zu+u^2}{Ku}-\frac{u+v}{K} \\
&= \frac{|2z-v|^2+2(2z-v)u+vu}{Ku} \\
&\geq \frac{|2z-v|^2+2(2z-v)u+u^2}{Ku} \\
&= \frac{|2z-v+u|^2}{Ku} \geq 0.
\end{aligned}
$$

Case 2: $\phi(v) > Ku$. Then, since $\phi(u) \leq Ku$,

$$
\frac{2|z+u|^2-|u|^2}{\phi(u)}-\frac{u}{K}-\frac{|v|^2-2|z-v|^2}{\phi(v)}-\frac{v}{K} \geq
$$

$$
\begin{aligned}
&\geq \frac{2|z+u|^2-|u|^2}{Ku}-\frac{u}{K}-\frac{|v|^2-2|z-v|^2}{\phi(v)}-\frac{v}{K} \\
&= \frac{2z^2}{Ku}+\frac{4z}{K}+\frac{2|z-v|^2-|v|^2}{\phi(v)}-\frac{v}{K} \\
&\geq \frac{2z^2}{\phi(v)}+\frac{4z}{K}+\frac{2|z-v|^2-|v|^2}{\phi(v)}-\frac{v}{K} \\
&= \frac{|2z-v|^2}{\phi(v)}+\frac{4z}{K}-\frac{v}{K} \\
&\geq \frac{|2z-v|^2}{Kv}+\frac{4z}{K}-\frac{v}{K} \\
&= \frac{4z^2}{Kv} \geq 0.
\end{aligned}
$$

■

### 5.2.3 Finishing remarks

To finish the proof of Theorem 3.2, define

$$V(x) = \int_0^{bx} p(z)dz.$$

By Lemma 5.3, $V$ is continuously differentiable. By Lemma 5.4, $V$ satisfies the Bellman inequality (5.36). Finally, since $p \geq 0$, $V \geq 0$ when $b \geq 0$.

## 5.3 Proof of Theorem 3.3

To prove that 1 implies 2, we essentially have to show the necessity of the resulting HJB inequality (5.36). To show that 2 implies 1, we represent $\bar{\sigma}$ as a convex combination of quadratic forms for which inequality (3.22) is known to be satisfied. 113 12/03 iam@ariel.harvard ¡¡Muz, mne nado nachinat' delat' priglashenie, i, naskol'ko ja ponim.

### 5.3.1 Necessity

The objective of this subsection is to prove that the inequality

$$\phi(u)^{-1}\bar{\sigma}(x, u, \phi(u)) + \phi(v)^{-1}\bar{\sigma}(x, -v, -\phi(v)) \geq 0 \tag{5.43}$$

must be satisfied for all $u, v > 0$ and for all $x \in \mathbf{R}$. Assume that, to the contrary,

$$\phi(u_0)^{-1}\bar{\sigma}(x_0, u_0, \phi(u_0)) + \phi(v_0)^{-1}\bar{\sigma}(x_0, -v_0, -\phi(v_0)) < 0$$

for some $\phi, x_0, u_0, v_0$. Then there exists $\epsilon > 0$ such that

$$\phi(u_0)^{-1}\bar{\sigma}(x_1, u_0, \phi(u_0)) + \phi(v_0)^{-1}\bar{\sigma}(x_2, -v_0, -\phi(v_0)) < -\epsilon \tag{5.44}$$

for all $x_1, x_2$ such that $|x_1 - x_0| < \epsilon$ and $|x_2 - x_0| < \epsilon$. Let us construct functions $x, z$ in (1.2) such that the integral (3.22) is negative. Let $N$ be a large integer. Define $z(t)$ as follows:

$$z(t) = \begin{cases} \mathrm{sgn}(x_0) & \text{for } t \in [0, |x_0|/\phi(1)], \\ -\mathrm{sgn}(x_0) & \text{for } t \in [|x_0|/\phi(1) + 2 * N, 2 * |x_0|/\phi(1) + 2 * N], \\ u_0 & \text{for } t \in [|x_0|/\phi(1) + (2k-2)/N, |x_0|/\phi(1) + (2k-1)/N],\ k = 1, 2, ..., N^2, \\ -v_0 & \text{for } t \in [|x_0|/\phi(1) + (2k-1)/N, |x_0|/\phi(1) + 2k/N],\ k = 1, 2, ..., N^2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the resulting $x(t)$ first raises to $x_0$, then "oscillates" within a small neighborhood of $x_0$ $N^2$ times, and then returns back to zero. The part of the integral in (3.22) that corresponds to the first and the last time segments does not depend on $N$, while the part that corresponds to "oscillations around $x_0$ is less than $-N\epsilon$, due to inequality (5.44), which contradicts to the assumption.

Now, using (5.43), it is easy to derive (3.24). First, substituting

$$v_0 = 1/K,\ \phi(v_0) = 1,\ x_0 = Rx,\ u_0 = Ru,$$

where $R \to \infty$, yields

$$\bar{\sigma}(x, u, 0) + \bar{\sigma}(x, 0, 0) \geq 0,$$

which is equivalent to the first inequality in (3.24). Similarly, substituting

$$x_0 = 0,\ v_0 = u_0 = 1,\ \phi(u) = \phi(v) = K$$

yields

$$\bar{\sigma}(0, 1, K) \geq 0,$$

which is equivalent to the second inequality in (3.24).

### 5.3.2 Sufficiency

Represent $\bar{\sigma}$ in the form

$$\bar{\sigma}(x, z, u) = \bar{\sigma}_0(x, z, u) + \bar{\sigma}_1(x, u) + \bar{\sigma}_2(z, u),$$

where

$$\begin{aligned} \bar{\sigma}_0(x, z, u) &= \Sigma_{11}x^2 + 2\Sigma_{12}xz + 2\Sigma_{22}z^2 + 2\Sigma_{22}zu, \\ \bar{\sigma}_1(x, u) &= 2\Sigma_{13}xu, \\ \bar{\sigma}_2(z, u) &= -\Sigma_{22}z^2 + 2(\Sigma_{23} - \Sigma_{22}/K)zu + \Sigma_{33}u^2. \end{aligned}$$

Note that, because of the first inequality in (3.24), $\bar{\sigma}_0$ is a convex combination of quadratic forms

$$\bar{\sigma}_b(x, z, u) = 2(z + bx)^2 - z^2 - zu/K,$$

for which the non-negativity of the integral in (3.22) is proven by Theorem 3.2. The integral of $\bar{\sigma}_1$ is zero because $u = dx/dt$. Finally, because of the second inequality in (3.24),

$$\bar{\sigma}_2(z, u) \geq 0 \ \text{ for any } \ u = kz,\ k \in [0, K].$$

This completes the proof of sufficiency.

## 5.4 Proof of Theorem 4.5

For $v \in L_2$, $0 \le \tau_0 < \tau_1 \le 1$ let

$$w_i = \Delta^a_{\phi[\tau_i]}(v) = ax_i + \tau_i\phi(v - x_i) + (1 - \tau_i)K(v - x_i),$$

where $i = 0, 1$,

$$\dot{x}_i = \tau_i\phi(v - x_i) + (1 - \tau_i)K(v - x_i), \; x_i(0) = 0.$$

Since

$$\phi(v - x_1) - \phi(v - x_2) = q(t)(x_2 - x_1), \quad q(t) \in [0, K],$$

for

$$y = x_1 - x_2, \quad u = \phi(v - x_2) - K(v - x_1)$$

we have

$$\dot{y}(t) = -(1 - \tau_1 + q(t))y(t) + (\tau_1 - \tau_0)u(t). \tag{5.45}$$

Since $\|\Delta_\phi[\nu]\|$ is bounded for any $\nu \in [0, 1]$, there exists a constant $C$ such that $\|P_T u\| \le C\|P_T v\|$ for all $T \ge 0$. Our goal is to show that the solution $y$ of (5.45) cannot be large when $\tau_1 - \tau_0$ is small.

Indeed, when $\tau_1 < 1$, multiplying both sides of (5.45) by $y(t)$ yields

$$\dot{y}y \le -(1 - \tau)y^2 + (\tau_1 - \tau_0)uy.$$

Since $y(0) = 0$, and $\dot{y}y = 0.5(d/dt)y^2$, integration of the last inequality over the time interval $[0, T]$ yields

$$(1 - \tau_1)\|P_T y\|^2 \le (\tau_1 - \tau_0)\langle u, y\rangle,$$

which in turn implies that

$$\|P_T y\| \le (1 - \tau_1)^{-1}(\tau_1 - \tau_0)C\|P_T v\|. \tag{5.46}$$

Similarly, when $\tau_1 = 1$, multiplying both sides of (5.45) by the sign of $y(t)$ yields

$$d|y|/dt \le (\tau_1 - \tau_0)|u|,$$

which implies

$$\begin{aligned} |y(T)| &\le (\tau_1 - \tau_0)\int_0^T |u(t)|dt \\ &\le (\tau_1 - \tau_0)\sqrt{(}T)\|P_T u\|. \end{aligned}$$

The last inequality, together with (5.46), imply almost continuity of $\Delta^a_{\phi[\tau]}$.

## 6 Acknowledgement

This work was supported by NSF under grants ECS-9410531, ECS-9796099, and ECS-9796033, and by AFOSR under grant F49620-96-1-0123. The author is grateful to Ulf Jonsson for useful comments.

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