Robust $\mathcal{H}_2$ Analysis for Continuous Time Systems

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Abstract

This paper treats the Robust $\mathcal{H}_2$ Performance question of evaluating the rejection of continuous time white noise, in the worst case over structured uncertainty in the system.

A frequency domain convex condition for robust $\mathcal{H}_2$ analysis is presented, with analogous properties as in the discrete time case. In particular, necessary and sufficient results are obtained introducing a continuous time version of the methodology of set descriptions of white noise.

In addition, a state-space test in terms of Linear Matrix Inequalities is developed for the robust $\mathcal{H}_2$ problem in the case of constant scales. This test is exact for robust $\mathcal{H}_2$ analysis under time varying uncertainty, thus rendering the problem finite dimensional in the same situation in which robust $\mathcal{H}_\infty$ analysis is finite dimensional.

1 Introduction

The Robust $\mathcal{H}_2$ problem is rooted in the efforts in the 1970s to provide robustness guarantees to the LQG regulator, designed from the point of view of white noise rejection. The difficulties encountered in this combination of classical and modern control (see, e.g. [31, 7, 8, 9, 34]) led to a decreasing interest in white noise rejection as a performance criterion, and the prevalence of $\mathcal{H}_\infty$ control [38], which by means of the small gain theorem [39] is directly linked to robust stability guarantees. This prevalence has led to a mature theory of robust performance based on the $\mathcal{H}_\infty$

measure (see e.g. [10, 19, 37, 32, 18, 29]), and also the $\mathcal{L}_1$ measure (see [6]), to the point where these performance criteria are often viewed as synonymous to robust control.

While the $\mathcal{H}_\infty$ and μ [10, 19] frameworks are natural for robust stability issues, the $\mathcal{H}_\infty$ measure is not very satisfactory as a disturbance rejection criterion. $\mathcal{H}_\infty$ optimal control treats the disturbance as an adversary which will excite the worst possible frequency, and consequently yields allpass closed loop transfer functions, paying the price of increased sensitivity over a large bandwidth in order to reduce sensitivity at the worst frequency. Such closed loops would exhibit very poor performance under the broadband disturbances of most real-world applications. This problem can be alleviated by frequency weighting the $\mathcal{H}_\infty$ problem, but weight selection becomes a largely ad hoc procedure, a difficulty which arises mainly because one attempts turn the peak frequency response specification ($\mathcal{H}_\infty$) into a measure of the response to broadband noise, which is really an $\mathcal{H}_2$ specification in terms of the RMS value of the frequency response.

For these reasons, a continuing research effort has sought to reinstate the $\mathcal{H}_2$ performance criterion, incorporating to it the issues of robustness which were absent form LQG regulators. Among the many references we mention [3, 15, 40, 11, 28, 35, 27, 40, 14, 12], which have produced a state space theory with robust performance bounds and synthesis methods to optimize these bounds, bringing the $\mathcal{H}_2$ criterion back into the picture of robust control. Still, the $\mathcal{H}_\infty$ and μ frameworks retained some advantage, with the availability of convex frequency domain tests for robust performance, and the theoretical characterization of such conditions as necessary and sufficient robustness tests [32, 18, 29]. In [22, 23], we have succeeded in closing this gap by providing a frequency domain characterization for robust $\mathcal{H}_2$ performance in discrete time, with analogous properties as the $\mathcal{H}_\infty$ conditions. In fact this test can be viewed as a systematic way to do weighted $\mathcal{H}_\infty$ analysis, where the weight, as well as the multipliers corresponding to the system uncertainty, are obtained from a convex condition in the frequency domain.

The purpose of this paper is twofold. In the first place, we provide an extension of the results in [22, 23] to the continuous time case. Such extension is not trivial since it must handle the issue of infinite bandwidth of continuous time white noise, which usually gives substantial mathematical difficulties. To that effect, we provide a continuous time version of set characterizations of white noise in terms of constraints on the cumulative spectrum, extending the method of [21], from which necessary and sufficient conditions for robust $\mathcal{H}_2$ performance are derived. A second objective of this paper is to derive a *state space* characterization for robust $\mathcal{H}_2$ performance, to play the role of the conditions available in the $\mathcal{H}_\infty$ theory. In fact, for the case of time varying structured uncertainty a finite dimensional convex test is available for robust $\mathcal{H}_\infty$ performance [19, 32, 18] in terms of a Linear Matrix Inequality (LMI, see [4]). Based on our frequency domain conditions we are able to derive a corresponding LMI test in state space which is exact for robust $\mathcal{H}_2$ performance under structured time varying uncertainty.

The paper is organized as follows. The problem formulation and notation are established in Section 2. In Section 3 we present the frequency domain condition for robust $\mathcal{H}_2$ performance, and prove its sufficiency for the case of time invariant uncertainty. The method for white noise characterization is introduced in Section 4, and applied in Section 5 to prove necessary and sufficient theorems involving the aforementioned condition. Section 6 covers the state space characterizations, and the conclusions are given in Section 7. The Appendix contains a technical proof.

2 Problem Formulation

This paper considers $\mathcal{L}_2$ signal spaces: $\mathcal{L}_2^n(\mathbb{R})$ denotes the Hilbert space of square-integrable, $\mathbb{C}^n$-valued functions over the real numbers, which is isomorphic via the Fourier transform with the frequency domain space $\mathcal{L}_2^n(j\mathbb{R})$ of square integrable functions on the imaginary axis; the distinction is dropped from now on, denoting both spaces by $\mathcal{L}_2^n$, or $\mathcal{L}_2$ if the vector dimension is clear from context. $\mathcal{L}_c(\mathcal{L}_2)$ denotes the set of causal, linear, bounded operators on $\mathcal{L}_2$. An important subset is the class of linear time invariant (LTI) elements of $\mathcal{L}_c(\mathcal{L}_2)$, which commute with the T second delay operator λ_T for every $T > 0$. The LTI operators in $\mathcal{L}_c(\mathcal{L}_2)$ can be represented by a transfer function $H(s)$ in the space $\mathcal{H}_\infty$ of analytic, essentially bounded functions on the right half plane, with the norm

$$\|H\|_{\mathcal{L}_2 \to \mathcal{L}_2} = \|H\|_\infty = \operatorname*{ess\,sup}_{\omega \in \mathbb{R}} \bar{\sigma}\left(H(j\omega)\right).$$

We will also be interested in the space $\mathcal{H}_2$ of analytic functions on the right half plane such that by

$$\|H\|_2 := \int_{-\infty}^{\infty} \operatorname{trace}(H(j\omega)^* H(j\omega)) \frac{d\omega}{2\pi} \tag{1}$$

is finite. $\mathcal{RH}_\infty$ and $\mathcal{RH}_2$ denote the subsets of rational functions in $\mathcal{H}_\infty$ and $\mathcal{H}_2$, respectively. Properties of these spaces can be found in [13] and references therein.

The results in this paper refer to the uncertain system of Figure 1, denoted by (M, Δ).

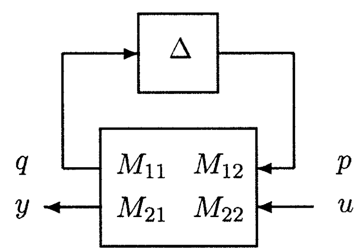

Figure 1: Uncertain system (M, Δ).

In Figure 1, the nominal map M is a finite dimensional LTI system in $\mathcal{RH}_\infty$. The perturbation Δ which represents system uncertainty is assumed to have spatial structure of block diagonal form

$$\Delta = \operatorname{diag}\left[\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L}, \Delta_{L+1}, \ldots, \Delta_{L+F}\right], \tag{2}$$

where the blocks are restricted to a class of dynamic operators, and normalized to size 1 in the $\mathcal{L}_2$-induced norm. For background and motivation on this setup, see [19]. The largest class of uncertainty considered here is the ball of structured linear time varying (LTV) perturbations

$$\mathbf{B}_{\mathbf{\Delta}_{\mathbf{LTV}}} = \{\Delta \in \mathcal{L}_c(\mathcal{L}_2) : \|\Delta\| \leq 1,\ \Delta = \operatorname{diag}\left[\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L}, \Delta_{L+1}, \ldots, \Delta_{L+F}\right]\}.$$

The uncertainty can also be restricted to be LTI, which gives the structured set $\mathbf{B}_{\mathbf{\Delta}_{\mathbf{LTI}}} = \{\Delta \in \mathbf{B}_{\mathbf{\Delta}_{\mathbf{LTV}}} : \lambda_T \Delta = \Delta \lambda_T,\ \forall T > 0\}$. Some recent work [29] has shown it is useful to introduce the mildly larger class of slowly varying operators, by defining for $\nu > 0$ the class

$$\mathbf{B}_{\mathbf{\Delta}^{\nu}} = \{\Delta \in \mathbf{B}_{\mathbf{\Delta}^{\mathbf{LTV}}} : \sup_{T>0} \frac{\|\lambda_T \Delta - \Delta \lambda_T\|}{T} \leq \nu\}$$

of operators with "rate of variation slower than ν". For $\nu = 0$ we recover $\mathbf{\Delta}^{\mathbf{LTI}}$, but some of the necessary conditions will hold for an arbitrarily small $\nu > 0$.

The system (M, Δ) is said to be robustly stable if M is stable, and if $I - \Delta M_{11}$ has an inverse in $\mathcal{L}_c(\mathcal{L}_2)$ for every $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$. When this holds, the closed loop map from u to y is well defined for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$ and given by the Linear Fractional Transformation (LFT)

$$\Delta \star M := M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}. \tag{3}$$

Given that the system is stable, a performance specification can be imposed; it has been customary in robust control to specify a disturbance rejection condition, where the exogenous inputs u are disturbances and the outputs y are error variables which must be kept small. One such specification is the requirement that the $\mathcal{L}_2$-induced norm of the closed loop be smaller than a prespecified amount (say 1). Since this corresponds to the $\mathcal{H}_\infty$ norm for the LTI case, we will say (with some abuse of language in the LTV case) that the system has robust $\mathcal{H}_\infty$ performance if it is robustly stable, and

$$\sup_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|\Delta \star M\|_{\mathcal{L}_2 \to \mathcal{L}_2} < 1.$$

As argued in Section 1, the $\mathcal{H}_\infty$ criterion for disturbance rejection is conservative in many situations, and a more appropriate alternative is to model u as a white noise disturbance. In the case of LTI systems, the response to white noise is measured by the $\mathcal{H}_2$ norm given in (1); correspondingly, if Δ is LTI we will say that the system (M, Δ) has robust $\mathcal{H}_2$ performance if it is robustly stable and

$$\sup_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|\Delta \star M\|_2 < 1.$$

We will also need to extend the notion of $\mathcal{H}_2$ performance to the case of LTV Δ; this is postponed until section Section 4.

2.1 Mathematical Preliminaries

The following mathematical facts are collected here for ease of reference (see, e.g., [17, 30, 33]).

First, we introduce the space $BV(\mathbb{R})$ of real-valued functions of bounded variation on $\mathbb{R}$. A function $\Psi(t)$ is of bounded variation if

$$TV(\Psi) := \sup \sum_{i=1}^{N} |\Psi(t_i) - \Psi(t_{i-1})| < \infty,$$

where the supremum is taken over N and the t_i, with $\infty < t_0 < \cdots < t_N < \infty$; $TV(\Psi)$ is the total variation of Ψ. An analogous definition applies to functions over the half-line $\mathbb{R}_+ = [0, \infty)$.

We introduce the Banach space $C_0(\mathbb{R}_+)$ of continuous, real-valued functions on $\mathbb{R}_+$ with limit 0 at infinity, with the norm $\|g\|_\infty := \sup_{t \in \mathbb{R}_+} |g(t)|$. It is a consequence of the Riesz representation theorem (see [30, 33]) that every functional in the dual space $C_0(\mathbb{R}_+)^*$ is characterized by a function $\Psi \in BV(\mathbb{R}_+)$ which operates in terms of the Stieltjes integral (see [33])

$$\Gamma_\Psi(g) = \int_0^\infty g(t) d\Psi(t).$$

Two useful properties of this integral are

$$\left| \int g(t) d\Psi(t) \right| \leq \|g\|_\infty TV(\Psi), \tag{4}$$

and the formula for integration by parts

$$\int_a^b g(t) d\Psi(t) = g(b)\Psi(b) - g(a)\Psi(a) - \int_a^b \Psi(t) dg(t). \tag{5}$$

Now consider the space $C(\mathbb{R}_+)$ of continuous, real-valued functions on $\mathbb{R}_+$ with a limit at infinity. This is also a Banach space with the norm $\| \cdot \|_\infty$, and is isomorphic to the direct sum $C_0(\mathbb{R}_+) \oplus \mathbb{R}$. Correspondingly, its dual is represented by $BV(\mathbb{R}_+) \oplus \mathbb{R}$, with functionals given by

$$\Gamma_{\Psi,\zeta}(g) = \int_0^\infty [g(t) - g(\infty)] d\Psi(t) + \zeta g(\infty). \tag{6}$$

In (6), Ψ is defined up to a constant, so we can choose the convention $\Psi(0) = -\zeta$. With this in place, we can integrate (6) by parts to obtain the equivalent formula

$$\Gamma_{\Psi,\zeta}(g) = g(0)\zeta - \int_0^\infty \Psi(t) dg(t). \tag{7}$$

Finally, a key element in the proofs of this paper is the following geometric version of the Hahn-Banach theorem, (see, e.g., [17]):

Theorem 1 *Let $\mathcal{K}_1$, $\mathcal{K}_2$ be disjoint convex sets in a real normed space V, such that $\mathcal{K}_2$ is open. Then there exists a bounded functional $\Gamma \in V^*$, $\Gamma \neq 0$, and a real number α such that*

$$\Gamma(k_1) \leq \alpha < \Gamma(k_2), \quad \textit{for all } k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2. \tag{8}$$

3 A Frequency Domain Condition for Robust $\mathcal{H}_2$ Performance

The objective is to provide a condition for robust $\mathcal{H}_2$ performance analysis for the system (Δ, M). For this purpose we introduce scaling matrices of the form

$$X = \text{diag}\left[X_1, \ldots, X_L,\, x_{L+1} I_{m_1}, \ldots, x_{L+F} I_{m_F}\right] \tag{9}$$

which commute with the elements in $\mathbf{\Delta}$. We will denote by $\mathbb{X}$ the set of positive definite, continuous scaling functions $X(\omega)$ with the structure (9).

Condition 1 *There exists $X(\omega) \in \mathbb{X}$, and a matrix function $Y(\omega) = Y^*(\omega) \in \mathbb{C}^{m \times m}$, such that*

$$M(j\omega)^* \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} < 0; \tag{10}$$

$$\int_{-\infty}^{\infty} \text{trace}(Y(\omega)) \frac{d\omega}{2\pi} < 1. \tag{11}$$

Remarks:

- This condition is very similar to the scaled-small gain conditions for robust $\mathcal{H}_\infty$ performance analysis [19]; in that case only (10) is imposed, with $Y(\omega)$ replaced by the identity matrix, which imposes that the worst-case gain of the system is less than 1 at every frequency.
- The addition of the "multiplier" $Y(\omega)$, subject to (11), allows the gain across frequency to vary, provide that the accumulated effect over frequency is less than 1; this integral across frequency provides a performance specification of the $\mathcal{H}_2$ type.
- An analogous condition was obtained in [22, 23] for the discrete-time case.
- To make Condition 1 precise, we interpret (10) to mean that there exists $\epsilon > 0$ such that

$$M(j\omega)^* \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} \leq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix} \quad \forall \omega \in \mathbb{R}. \tag{12}$$

  In this way the first block is made strictly negative definite, but a weaker bound is imposed on the second block to make it compatible with $Y \in \mathcal{L}_1(\mathbb{R})$, as needed for (11).

The theoretical properties of Condition 1 can be summarized as follows.

1. It is sufficient for robust $\mathcal{H}_2$ performance under $\Delta \in \mathbf{B}_{\mathbf{\Delta}_{\mathbf{LTI}}}$.
2. It is necessary and sufficient for robust $\mathcal{H}_2$ performance under $\Delta \in \mathbf{B}_{\mathbf{\Delta}^\nu}$ for small enough ν.
3. If X is imposed to be constant, it is necessary and sufficient for robust $\mathcal{H}_2$ performance under $\Delta \in \mathbf{B}_{\mathbf{\Delta}_{\mathbf{LTV}}}$.

The same three statements are true of the corresponding condition for $\mathcal{H}_\infty$ performance (see [19, 32, 18, 29]), which shows that Condition 1 is indeed the appropriate extension for the $\mathcal{H}_2$ case. These properties have already been established in *discrete* time in [22, 23].

We will now present the theorem for the LTI case, the other two are postponed to Section 5.

Theorem 2 *Suppose Condition 1 holds for matrix functions* $X(\omega)$,$Y(\omega)$. *If* $\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathbf{LTI}}$, *then the system is robustly stable and*

$$\sup_{\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathbf{LTI}}} \|\Delta \star M\|_2 < 1.$$

Proof: The first block of the inequality (12) gives $\|X(\omega)^{\frac{1}{2}} M_{11}(j\omega) X(\omega)^{-\frac{1}{2}}\|_\infty < 1$, which implies (see [19]) robust stability of the system under LTI perturbations. Furthermore, defining

$$\hat{M} = \begin{bmatrix} X^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \tag{13}$$

we have that

$$(\hat{M}^* \hat{M})(j\omega) - \begin{bmatrix} I & 0 \\ 0 & Y(\omega) \end{bmatrix} \leq 0 \quad \forall \omega \in \mathbb{R}. \tag{14}$$

Fix $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$, LTI. For any fixed frequency, since $\Delta(j\omega)$, $X^{\frac{1}{2}}(j\omega)$ commute, we can replace M by $\hat{M}$ in Figure 1, giving $\Delta(j\omega) \star M(j\omega) = \Delta(j\omega) \star \hat{M}(j\omega)$. Using (14), we have

$$|y(\omega)|^2 + |q(j\omega)|^2 \leq |p(j\omega)|^2 + u(j\omega)^* Y(\omega) u(j\omega), \tag{15}$$

where we use the signal denominations of Figure 1.

Since Δ is LTI, contractive we have $|p(j\omega)|^2 \leq |q(j\omega)|^2$, which leads to

$$u(j\omega)^* (\Delta \star M)(j\omega)^* (\Delta \star M)(j\omega) u(j\omega) = |y(\omega)|^2 \leq u(j\omega)^* Y(\omega) u(j\omega).$$

Since this holds for any $u(j\omega)$, we have

$$(\Delta \star M)(j\omega)^* (\Delta \star M)(j\omega) \leq Y(\omega)$$

across frequency. Computing the trace and integrating gives, using (11),

$$\|\Delta \star M\|_2^2 \leq \int_{-\infty}^{\infty} \mathrm{trace}(Y(\omega)) \frac{d\omega}{2\pi} < 1.$$

∎

The preceding proof is remarkably simple and analogous to the standard theory for $\mathcal{H}_\infty$ performance [19]. Also, the multiplier $Y(\omega)$ is readily seen as playing the role of a weighting function which modifies the $\mathcal{H}_\infty$-type condition (10) to yield $\mathcal{H}_2$ performance. While weights with this property are known to exist, so far no systematic method has been available to find them. In Condition 1, the weight Y and the scaling X for the uncertainty are obtained by a convex condition in terms of a Linear Matrix Inequality (LMI) across frequency.

This convexity indicates in principle tractability of computation, although in the general case of frequency varying scales $X(\omega)$ as in Theorem 2, the condition is infinite dimensional (see Section 6 for the constant scales case). This is the same as in the corresponding condition for robust $\mathcal{H}_\infty$ performance (i.e. the frequency domain μ upper bounds [19]), and correspondingly, similar methods can be used to obtain a finite dimensional approximation.

One such method is to grid the frequency axis, which provides no hard guarantees but is more closely related to engineering intuition. The advantage of this method is that the problem can be decoupled across frequency; at a fixed frequency ω_i, we solve the problem

Minimize trace(Y_i) , subject to

$$\begin{aligned} M(j\omega_i)^* \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} M(j\omega_i) - \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} &< 0, \\ X_i &> 0. \end{aligned}$$

Such minimization of a linear function subject to an LMI constraint can be computed efficiently by the tools in [4], being of the same dimensionality and similar complexity to the μ upper bound computation. Subsequently, the sum $\frac{1}{2\pi}\sum_{i=1}^{N} \text{trace}(Y_i)(\omega_i - \omega_{i-1})$, can be used to provide an approximation to the integral in (11).

The alternative method is to select a finite set of rational basis functions for $X(j\omega)$ and $Y(j\omega)$ restrict the search to the span of these functions. Condition (10) will then depend linearly on a finite number of unknowns, and negativity over frequency can be converted to a single LMI in state-space via the Positive Real Lemma (see, e.g. [4]). This procedure offers a guaranteed sufficient condition, but is computationally intensive since the problem is coupled. Once again the complexity is similar to the $\mathcal{H}_\infty$ performance case.

As an additional remark, we note that if the uncertainty includes real parametric blocks, Condition 1 can be tightened by the use of "G-scales" introduced analogously to the corresponding upper bounds for mixed μ [37], and the sufficient condition still follows along similar lines as Theorem 2.

4 White Noise Rejection in a Worst-Case Setting

The following two sections are dedicated to showing that Condition 1 has necessity properties which are analogous to the $\mathcal{H}_\infty$ case. These are more technical in nature and require characterizations of white noise which are compatible with a worst-case analysis for time varying systems.

White noise arises most commonly due to the accumulated effect of microscopic fluctuations, which typically produces a broadband statistical spectrum. As an example, the spectrum of thermal noise in electrical circuits is flat up to frequencies of the order of 10^{12} Hz (see [5]). When studying systems which operate at a much lower bandwidth, the standard abstraction is to assume that the spectrum is flat over the frequency interval $(-\infty, +\infty)$; this allows one to accurately predict, for example, the power of the output of an LTI filter which is driven by such noise.

The mathematical formalization of such an abstraction raises, however, theoretical difficulties, since white noise falls outside the standard theory of stationary random processes. The rigorous treatment of white noise requires the introduction of the Wiener process and stochastic calculus (see, e.g. [16]) to formalize the differential equations which arise in filtering theory. This theory was in fact applied by Stoorvogel [35] to prove sufficiency of a robust $\mathcal{H}_2$ performance test in the case of time varying systems. It is difficult, however, to obtain necessity results in this manner due to the combination of stochastic and worst-case analysis.

In this paper we obtain such necessity results by extending the methodology of set descriptions of white noise, developed in [20, 21, 22, 23] for the discrete time case. In particular, the necessary and sufficient conditions obtained in [22, 23] use sets of signals defined by constraints on the cumulative spectrum, a procedure inspired on the so-called Bartlett test [1] from time series analysis.

We now extend this method to continuous time. A deterministic formalization of the concept of signals with flat spectrum over infinite frequencies, is in fact no easier than in the stochastic case; [40] discusses these difficulties when attempting a formalization within the class of bounded power signals. The main idea of our formulation is to reverse the limiting process: instead of making the infinite bandwidth abstraction and then analyzing rejection, we propose to analyze first the response to signals with flat spectrum over a large bandwidth, and *then* take the limit over the bandwidth. This procedure is mathematically very well behaved, and is conceptually equally satisfactory in regard to the analysis of "realistic" white noise.

The development in this section is done for simplicity in the case of scalar noise. At the end of the section we will indicate the extension to vector-valued signals. Given a signal $u \in \mathcal{L}_2^1$, consider the cumulative spectrum

$$F_u(\beta) = \int_{-\beta}^{\beta} |u(j\omega)|^2 \frac{d\omega}{2\pi}. \tag{16}$$

The function $F_u(\cdot)$ has the following properties:

- F_u is continuous on $\beta \in [0, \infty)$.
- $F_u(0) = 0$, $\lim_{\beta\to\infty} F_u(\beta) = \|u\|_2^2$. Therefore $F_u \in C(\mathbb{R}_+)$ (cf. Section 2.1).
- F_u is monotone nondecreasing.

We wish to characterize the set of signals with flat (unit) spectrum over a bandwidth B. This can be done by imposing that the cumulative spectrum lies in the set

$$S_{\eta,B} := \left\{ g \in C(\mathbb{R}_+) : \min\left(\frac{\beta}{\pi} - \eta, \frac{B}{\pi} - \eta\right) \leq g(\beta) \leq \frac{\beta}{\pi} + \eta \right\}, \tag{17}$$

which is depicted in Figure 2. Correspondingly, we define the set of white signals with accuracy parameter η, bandwidth parameter B as

$$W_{\eta,B} := \{u \in \mathcal{L}_2 : F_u(\cdot) \in S_{\eta,B}\}. \tag{18}$$

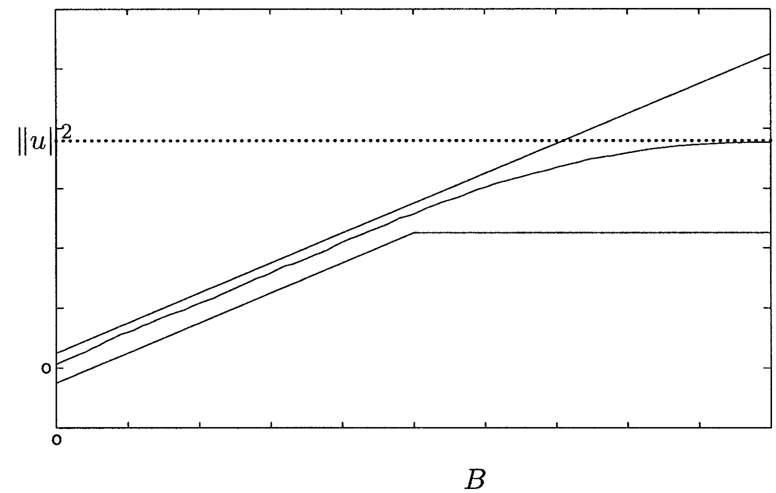

Figure 2: Constraints on the accumulated spectrum

Remarks:

- The functions in $S_{\eta,B}$ are approximately (up to accuracy η) linear in the bandwidth $|\omega| \leq B$, so signals in $W_{\eta,B}$ are approximately white over this bandwidth.
- For frequencies higher than B, the lower bound is relaxed, allowing the spectrum to taper off, which is necessary in the space $\mathcal{L}_2$ (and in realistic noise models). We still impose an upper bound which constrains the distribution of energy with frequency.
- The sets $S_{\eta,B}$ are nested: if $\eta \leq \eta'$, $B \geq B'$ then $S_{\eta,B} \subset S_{\eta',B'}$.

Consider a system $H \in \mathcal{L}_c(\mathcal{L}_2)$. Define

$$\|H\|_{W_{\eta,B}} := \sup_{u \in W_{\eta,B}} \|Hu\|. \tag{19}$$

In general, $\|H\|_{W_{\eta,B}}$ may be infinite (e.g. for $H \in \mathcal{RH}_\infty$ which is not strictly proper), but the class of systems where it is finite can be seen to be independent of η, B. We now identify a subclass of such systems by considering the family of frequency functions

$$\mathcal{F} = \{Y \in BV(\mathbb{R}) \,:\, \exists G \in \mathcal{L}_1(\mathbb{R}_+), G \text{ monotone decreasing, with } 0 \le Y(\omega) = Y(-\omega) \le G(|\omega|)\}\,. \tag{20}$$

Proposition 3 *Let $Y(\omega) \in \mathcal{F}$, with G the corresponding upper bound, and $u \in W_{\eta,B}$. Then*

$$\int_{-\infty}^{\infty} Y(\omega)|u(j\omega)|^2 \frac{d\omega}{2\pi} \le \int_{-\infty}^{\infty} Y(\omega)\frac{d\omega}{2\pi} + \eta[3G(B) + TV(Y)] + \frac{1}{\pi}\int_B^{\infty} G(\omega)d\omega. \tag{21}$$

Proof: An integration by parts gives

$$\int_{-B}^{B} Y(\omega)|u(j\omega)|^2 \frac{d\omega}{2\pi} = \int_0^B Y(\omega)(|u(j\omega)|^2 + |u(-j\omega)|^2)\frac{d\omega}{2\pi} = Y(B)F_u(B) - \int_0^B F_u(\omega)dY.$$

A similar calculation gives

$$\int_{-B}^{B} Y(\omega)\frac{d\omega}{2\pi} = \int_0^B Y(\omega)\frac{d\omega}{\pi} = Y(B)\frac{B}{\pi} - \int_0^B \frac{\omega}{\pi}dY,$$

from where

$$\begin{aligned}\int_{-B}^{B} Y(\omega)|u(j\omega)|^2 \frac{d\omega}{2\pi} &\le \int_{-B}^{B} Y(\omega)\frac{d\omega}{2\pi} + Y(B)\left|F_u(B) - \frac{B}{\pi}\right| + \left|\int_0^B (F_u(\omega) - \frac{\omega}{\pi})dY\right| \\ &\le \int_{-\infty}^{\infty} Y(\omega)\frac{d\omega}{2\pi} + \eta G(B) + \eta TV(Y).\end{aligned} \tag{22}$$

The last bound follows from (4) and $F_u \in S_{\eta,B}$. Now consider

$$\begin{aligned}\int_B^{\infty} Y(\omega)(|u(j\omega)|^2 + |u(-j\omega)|^2)\frac{d\omega}{2\pi} &\le \int_B^{\infty} G(\omega)(|u(j\omega)|^2 + |u(-j\omega)|^2)\frac{d\omega}{2\pi} \\ &= -F_u(B)G(B) - \int_B^{\infty} F_u(\omega)dG(\omega),\end{aligned} \tag{23}$$

where the last equality is an integration by parts. Since G is monotone decreasing, then $-dG$ is non-negative. Also, $F_u(\omega) \le \varphi(\omega) := \min(\frac{\omega}{\pi} + \eta, \|u\|_2^2)$, which implies

$$\begin{aligned}-\int_B^{\infty} F_u(\omega)dG(\omega) \le -\int_B^{\infty} \varphi(\omega)dG(\omega) &= \varphi(B)G(B) + \int_B^{\infty} G(\omega)d\varphi(\omega) \\ &\le \varphi(B)G(B) + \frac{1}{\pi}\int_B^{\infty} G(\omega)d\omega.\end{aligned}$$

Substitution into (23) gives

$$\int_B^\infty Y(\omega)(|u(j\omega)|^2 + |u(-j\omega)|^2)\frac{d\omega}{2\pi} \le [\varphi(B) - F(B)]G(B) + \frac{1}{\pi}\int_B^\infty G(\omega)d\omega$$
$$\le 2\eta G(B) + \frac{1}{\pi}\int_B^\infty G(\omega)d\omega. \quad (24)$$

Combining (22) and (24) gives (21).

■

Corollary 4 *For any $H \in \mathcal{RH}_\infty$, $\|H\|_2 \le \|H\|_{W_{\eta,B}}$. If in addition $|H(j\omega)|^2 \in \mathcal{F}$, then*

$$\lim_{\substack{\eta\to 0\\ B\to\infty}} \|H\|_{W_{\eta,B}} = \|H\|_2. \quad (25)$$

Proof:

For every $B' > B$, since the indicator function of $[-B', B']$ is in $W_{\eta,B}$, we conclude that $\int_{-B'}^{B'} |H(j\omega)|^2 \frac{d\omega}{2\pi} \le \|H\|^2_{W_{\eta,B}}$, which implies $\|H\|_2 \le \|H\|_{W_{\eta,B}}$.

Also, applying (21) to $Y(\omega) = |H(j\omega)|^2$ gives

$$\|H\|^2_{W_{\eta,B}} \le \|H\|_2^2 + \eta(3G(B) + TV(|H|^2)) + \frac{1}{\pi}\int_B^\infty G(\omega)d\omega,$$

which implies (25) by taking limit when $\eta\to 0$, $B\to\infty$ (recall $G \in \mathcal{L}_1(\mathbb{R}_+)$). ■

We have characterized a rich class of transfer functions in $\mathcal{H}_\infty \cap \mathcal{H}_2$, namely those with spectrum of bounded variation, and bounded by a monotone decreasing frequency function of $|\omega|$ which is in $\mathcal{L}_2$. These mild restrictions are satisfied, for example, by any transfer function in $\mathcal{RH}_2$. For such systems, the worst-case gain over $W_{\eta,B}$ is finite and converges, as $\eta\to 0$, $B\to\infty$ to the $\mathcal{H}_2$ norm, which provides an alternative way to motivate the standard $\mathcal{H}_2$ criterion.

In addition, this procedure can be transported directly to classes of LTV systems. In particular we will be interested in systems obtained by LFT from the configuration of Figure 1, with $M \in \mathcal{RH}_\infty$ and Δ possibly LTV. Assume that the transfer functions $M_{12}(j\omega)$, $M_{22}(j\omega)$ directly seen by the noise are in $\mathcal{RH}_2$, and that (Δ, M) is stable. Then writing (3) in the form

$$\Delta \star M = \begin{bmatrix} M_{21}\Delta(I - M_{11}\Delta)^{-1} & I \end{bmatrix} \begin{bmatrix} M_{21} \\ M_{22} \end{bmatrix}$$

it follows that

$$\|\Delta \star M\|_{W_{\eta,B}} \leq \left\| \left[\, M_{21}\Delta(I - M_{11}\Delta)^{-1} \;\; I \, \right] \right\|_{\mathcal{L}_2 \to \mathcal{L}_2} \left\| \begin{bmatrix} M_{21} \\ M_{22} \end{bmatrix} \right\|_{W_{\eta,B}} < \infty,$$

and we can define the $\mathcal{H}_2$ norm of such a system by

$$\|\Delta \star M\|_2 := \lim_{\substack{\eta \to 0 \\ B \to \infty}} \|\Delta \star M\|_{W_{\eta,B}}, \tag{26}$$

which naturally extends the LTI case, and captures the property which is interesting from the point of view of applications: rejection of signals with flat spectrum over a large bandwidth. In the next section, we develop necessary and sufficient conditions for robust $\mathcal{H}_2$ performance analysis of time varying systems in the sense of (26).

To conclude this section, we indicate how the previous definitions extend to the case of multivariable noise. For $u \in \mathcal{L}_2^m$, treated as a column vector, we define the cumulative spectrum

$$F_u(\beta) = \int_{-\beta}^{\beta} u(j\omega)u(j\omega)^* \frac{d\omega}{2\pi}, \tag{27}$$

which takes values in the space $C_H^{m\times m}(R_+)$ of continuous, hermitian matrix-valued functions on the positive reals, with a limit at infinity. Using the matrix norm $\|A\|_{\max} := \max_{i,j} |a_{ij}|$, introduce the sets

$$\begin{aligned} S_{\eta,B}^m &= \left\{ g \in C_H^{m\times m}(R_+) : \sup_{0\leq\beta\leq B} \left\| g(\beta) - \frac{\beta}{\pi} I \right\|_{\max} \leq \eta; \ \text{trace}(g(\beta)) \leq m\left(\frac{\beta}{\pi} + \eta\right) \ \forall \beta \geq B \right\}; \\ W_{\eta,B}^m &= \{ u \in \mathcal{L}_2^m : F_u \in S_{\eta,B}^m \}. \end{aligned}$$

The signals in $W_{\eta,B}^m$ have cumulative spectrum which approximates the one for ideal white noise up to a bandwidth B; note that a spectrum which is exactly equal to I is not possible within the space of $\mathcal{L}_2$ signals (it must be a rank one matrix at every frequency) but $W_{\eta,B}^m$ is non-empty for any $\eta > 0$. In addition a constraint is imposed on the increase of energy with frequency beyond B. Defining $\|H\|_{W_{\eta,B}^m}$ as in (19), the theory follows analogously to the scalar case. In particular (25) holds for LTI systems H such that the spectrum $H(j\omega)^*H(j\omega)$ falls in the class

$$\mathcal{F}^m = \left\{Y \in BV^{m\times m}(\mathbb{R}) : \exists G \in \mathcal{L}_1(\mathbb{R}_+), G \text{ monotone decreasing, with } 0 \le Y(\omega) = Y(-\omega)^* \le G(|\omega|)I\right\}, \tag{28}$$

where $BV^{m\times m}(\mathbb{R})$ is the set of hermitian matrix valued functions with entries of bounded variation.

5 Necessary and Sufficient Conditions

In this section we consider the system (M, Δ) of Figure 1, with M_{12}, M_{22} in $\mathcal{RH}_2$. We will say the system has robust $\mathcal{H}_2$ performance if there exists $\eta > 0$ and $B > 0$ such that

$$\sup_{\Delta\in\mathbf{B_\Delta}} \|\Delta \star M\|_{W_{\eta,B}} < 1. \tag{29}$$

This corresponds to the notion of $\mathcal{H}_2$ norm introduced in (26), requiring in addition that the limit be uniform across Δ. We now state two necessary and sufficient conditions for robust $\mathcal{H}_2$ performance based on Condition 1.

Theorem 5 *There exists $\nu > 0$ such that the system (M, Δ) has robust $\mathcal{H}_2$ performance for $\Delta \in \mathbf{B}_{\mathbf{\Delta}^\nu}$ iff there exists $X(\omega) \in \mathbb{X}$ of bounded variation and $Y(\omega) \in \mathcal{F}^m$, satisfying Condition 1.*

Theorem 6 *The system (M, Δ) has robust $\mathcal{H}_2$ performance for $\Delta \in \mathbf{B}_{\mathbf{\Delta}_{\mathbf{LTV}}}$, iff there exists a constant matrix $X \in \mathbb{X}$, and $Y(\omega) \in \mathcal{F}^m$, satisfying Condition 1.*

Discussion:

- The previous statements exactly characterize Condition 1, and are analogous to their counterparts for $\mathcal{H}_\infty$ performance. The constant scales condition is exact for LTV uncertainty, and Theorem 5 serves to argue, as in [29], that there is mild conservatism involved in using the frequency dependent scales condition for LTI uncertainty. This argument has of course only qualitative value, based on the point of view that arbitrarily slowly varying uncertainty is a modest augmentation, from an engineering standpoint, to LTI uncertainty. We are not claiming that there is a small quantitative gap between the worst-case $\mathcal{H}_2$ norms under these two assumptions; this gap has not been precisely quantified but examples can be given where it is non-negligible, and state space bounds such as those in [35, 12] may give a tighter result.

- The main appeal of our condition is that it provides a symmetric theory which summarizes tractable methods for robustness analysis. Setting the blocks in X to be either constant or frequency-varying selects between LTV or LTI (slowly-varying) uncertainty. Fixing $Y = I$ or allowing it to vary in frequency as in Condition 1, chooses between $\mathcal{H}_\infty$ or $\mathcal{H}_2$ performance.

- Also, combined LTV/LTI uncertainty structures can be studied by the corresponding combination of constant and frequency dependent X scales, and combinations of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance can be studied by including Y terms only for the signals which are assumed white. For any of these combinations, we can state the robust performance problem for which Condition 1 is necessary and sufficient, which can be proved by analogous methods as those described below.

In the sequel, we will provide a proof of Theorem 6, which illustrates the method required to handle robustness analysis over $W_{\eta,B}$, extending the ideas of [22, 23]. The proof of Theorem 5 involves some additional technicalities and will be reported in [26] due to space limitations.

5.1 Proof of Theorem 6

For simplicity, the proof will be described in detail for the case of scalar inputs $u \in \mathcal{L}_2^1$, and for uncertainty $\Delta = \text{diag}[\Delta_1, \ldots, \Delta_F]$ consisting only of full blocks. For the general case see the remarks at the end of the section.

[Sufficiency]: The first block of (12) gives $\|X^{\frac{1}{2}} M_{11}(\jmath\omega) X^{-\frac{1}{2}}\|_\infty < 1$, which implies [19, 32] robust stability of the system under LTV perturbations. Also, $X^{\frac{1}{2}}$ and Δ commute, so define $\hat{M}$ as in

(13), which verifies (14), and leads to (15), which can be integrated across frequency to give

$$\|y\|^2 + \|q\|^2 \le \|p\|^2 + \int_{-\infty}^{\infty} u(j\omega)^* Y(\omega) u(j\omega) \frac{d\omega}{2\pi}.$$

Since $\|\Delta\| \le 1$, then $\|p\| \le \|q\|$, leading (for scalar u) to

$$\|(\Delta \star M)u\|^2 = \|y\|^2 \le \int_{-\infty}^{\infty} Y(\omega)|u(j\omega)|^2 \frac{d\omega}{2\pi}.$$

Fix η, B; for $u \in W_{\eta,B}$, since $Y \in \mathcal{F}$ we invoke (21) to obtain

$$\|(\Delta \star M)u\|^2 \le \int_{-\infty}^{\infty} Y(\omega) \frac{d\omega}{2\pi} + \eta[3G(B) + TV(Y)] + \frac{1}{\pi} \int_{B}^{\infty} G(\omega) d\omega.$$

Since $\int_{-\infty}^{\infty} Y(\omega) \frac{d\omega}{2\pi} < 1$ from (11), the right hand side can be made less than 1 for small enough η, large enough B. This gives

$$\sup_{\Delta \in \mathbf{B_\Delta}} \|(\Delta \star M)\|_{W_{\eta,B}} < 1.$$

[Necessity]: The idea is to characterize the robust performance condition in terms of a set of quadratic constraints (see [18]), and then invoke a duality argument to obtain Condition 1.

Let $z = \mathrm{col}(z_1, \ldots, z_{F+1})$ be the vector of all inputs to the M system, where $z_1 \ldots z_F$ partition p in correspondence with the blocks $\Delta_1, \ldots, \Delta_F$, and $z_{F+1} = u$. Analogously $(Mz)_i, i = 1 \ldots F+1$ denotes the partition of the output of M. Introduce scalar valued quadratic functions of $z \in \mathcal{L}_2$,

$$\sigma_i(z) = \|(Mz)_i\|^2 - \|z_i\|^2, \;\; i = 1 \ldots F,$$

which are used to impose that the perturbation blocks Δ_i are contractive.

By hypothesis (29) holds for some η, B. We select γ such that

$$\sup_{\Delta \in \mathbf{B_\Delta}} \|(\Delta \star M)\|_{W_{\eta,B}} < \gamma < 1, \tag{30}$$

and introduce the additional (non-homogeneous) quadratic form

$$\sigma_{F+1}(z) = \|(Mz)_{F+1}\|^2 - \gamma^2,$$

which compares the norm of the output y with γ. To impose that $u = z_{F+1} \in W_{\eta,B}$, we define the function $\rho : \mathcal{L}_2 \mapsto C(\mathbb{R}_+)$,

$$\rho(z) = F_u(\cdot), \tag{31}$$

where F_u is given by (16). Therefore $u \in W_{\eta,B}$ if and only if $\rho(z) \in S_{\eta,B}$.

By considering the real Banach space $\mathbb{V} = \mathbb{R}^{n+1} \oplus C(\mathbb{R}_+)$, we can collect all these functions together in a quadratic map $\Lambda : \mathcal{L}_2 \mapsto \mathbb{V}$, given by

$$\Lambda(z) = (\sigma_1(z), \dots, \sigma_{F+1}(z), \rho(z)). \tag{32}$$

The robust performance condition (30) imposes restrictions on the values Λ can take. Assume there existed $z \in \mathcal{L}_2$ such that

$$\Lambda(z) \in \mathcal{K}_0 := \{(r_1, \dots, r_{F+1}, g),\ r_i \geq 0,\ g \in S_{\eta,B}\} \subset \mathbb{V}. \tag{33}$$

Applying $z = \mathrm{col}(p, u)$ to M we obtain the corresponding signals (q, y). Since $\|q_i\| \geq \|p_i\|$ for $i = 1 \dots F$, there exist contractive operators $\Delta_i : \mathcal{L}_2 \to \mathcal{L}_2$, such that $p_i = \Delta_i q_i$, for $i = 1 \dots F$. Setting $\Delta = \mathrm{diag}[\Delta_1, \dots, \Delta_F]$ results in $\Delta q = p$ consistently with Figure 1. Also, we have $\|y\| \geq \gamma$ from the inequality on $\sigma_{F+1}(z)$, and $u \in W_{\eta,B}$ from the constraint on $\rho(z)$. This means we have found $\Delta \in \mathbf{B}_{\mathbf{\Delta LTV}}$, $u \in W_{\eta,B}$ such that

$$\|(\Delta \star M)u\| \geq \gamma,$$

violating (30). This reasoning leads to the following statement.

Proposition 7 *If (30) holds, there exists $\epsilon > 0$ such that $\nabla \cap \mathcal{K}_\epsilon = \emptyset$, where $\nabla = \{\Lambda(z) : z \in \mathcal{L}_2\} \subset V$ is the range of Λ, and*

$$\mathcal{K}_\epsilon := \{(r_1, \dots, r_{F+1}, g) :\ r_i > -\epsilon^2,\ g \in S^o_{\eta,B}\} \subset \mathbb{V},$$

with $S^o_{\eta,B}$ the interior of $S_{\eta,B}$ in the space $C(\mathbb{R})$.

To obtain this result the preceding argument must be strengthened in two ways. First, we must impose the causality of Δ which was not considered in the construction above. Second, we have replaced the set $\mathcal{K}_0$ by the slightly different set $\mathcal{K}_\epsilon$. These details are quite involved, but completely analogous to those of the discrete time case of [23, 25]; for this reason they are omitted here.

We have reduced robust $\mathcal{H}_2$ performance to a geometric separation condition in the space $\mathbb{V}$. To bring in the Hahn-Banach theorem, we note that $\mathcal{K}_\epsilon$ is open and convex in $\mathbb{V}$, and that

Proposition 8 *The closure $\bar{\nabla}$ of ∇ is convex in $\mathbb{V}$.*

Proposition 8 is proved in the Appendix. By choosing $\mathcal{K}_1 = \bar{\nabla}$, $\mathcal{K}_2 = \mathcal{K}_\epsilon$, we are in a position to apply Theorem 1, and obtain the corresponding $\Gamma \in \mathbb{V}^*$, $\Gamma \neq 0$, $\alpha \in \mathbb{R}$, satisfying (8).

The structure of $\mathbb{V}$ and the Riesz representation theorem imply that Γ can be represented by $(x_1, \ldots, x_{F+1}, \Gamma_{\Psi,\zeta})$, where $x_i \in \mathbb{R}$, and $\Gamma_{\Psi,\zeta} \in BV(\mathbb{R}_+) \oplus \mathbb{R}$. Then (8) gives

$$\sum_{i=1}^{F+1} x_i \sigma_i(z) + \Gamma_{\Psi,\zeta}(\rho(z)) \leq \alpha < \sum_{i=1}^{F+1} x_i r_i + \Gamma_{\Psi,\zeta}(g) \quad \forall\, z \in \mathcal{L}_2, r_i > -\epsilon^2, g \in S^o_{\eta,B}. \tag{34}$$

We now apply (7) to describe the action of $\Gamma_{\Psi,\zeta}$. Note that $[\rho(z)](0) = 0$, and $d[\rho(z)](\omega) = (|u(j\omega)|^2 + |u(-j\omega)|^2)\frac{d\omega}{2\pi}$, which gives

$$\Gamma_{\Psi,\zeta}(\rho(z)) = -\int_0^\infty \Psi(\omega)(|u(j\omega)|^2 + |u(-j\omega)|^2)\frac{d\omega}{2\pi} = -\int_{-\infty}^\infty Y(\omega)|u(j\omega)|^2 \frac{d\omega}{2\pi}, \tag{35}$$

with $Y(\omega) := \Psi(|\omega|) \in BV(\mathbb{R})$. Therefore (34) is rewritten as

$$\sum_{i=1}^{F+1} x_i \sigma_i(z) - \int_{-\infty}^\infty Y(\omega)|u(j\omega)|^2 \frac{d\omega}{2\pi} \;\leq\; \alpha, \quad \forall z \in \mathcal{L}_2; \tag{36}$$

$$\sum_{i=1}^{F+1} x_i r_i + g(0)\zeta - \int_0^\infty \Psi(\omega) dg(\omega) \;>\; \alpha, \quad \text{for } \begin{cases} r_i > -\epsilon^2 \\ g \in S^o_{\eta,B} \end{cases}. \tag{37}$$

These fundamental relations will lead to the desired Condition 1, through a sequence of steps.

1. $x_i \geq 0$. This follows from (37).

2. $Y \geq 0$. To see this, select $z_i = 0,\ i = 1 \ldots F$. Then $\sigma_i(z) \geq 0, i = 1 \ldots F$, so (36) gives

$$-\gamma^2 x_{F+1} - \alpha \leq \sum_{i=1}^{F+1} x_i \sigma_i(z) - \alpha \leq \int_{-\infty}^{\infty} Y(\omega)|u(j\omega)|^2 \frac{d\omega}{2\pi}.$$

   Since $u = z_{F+1}$ is arbitrary in $\mathcal{L}_2$, we must have $Y \geq 0$.

3. $\alpha < 0$, $Y \in \mathcal{L}_1(\mathbb{R})$, and $\int_{-\infty}^{\infty} Y(j\omega)\frac{d\omega}{2\pi} \leq -\alpha$. Consider $g_B(\omega) = \frac{1}{\pi}\min(\omega, B)$ which is in $S^o_{\eta,B}$, and $r_i = 0$; then (37) gives

$$0 \leq \int_{-B}^{B} Y(\omega)\frac{d\omega}{2\pi} < -\alpha.$$

   This yields the desired conditions.

4. $x_{F+1} > 0$. If it were 0, applying (36) with $z = 0$ would yield $0 \leq \alpha$, a contradiction. From here we conclude that x_i, Y and α can be renormalized so that $x_{F+1} = 1$, preserving (36-37) and the conclusions of the previous steps.

5. Setting $x_{F+1} = 1$ in (36), and using definition of σ_i gives

$$\left\langle \left(M^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} M - \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right) z, z \right\rangle \leq \alpha + \gamma^2 \quad \forall z \in \mathcal{L}_2,$$

   where $X := \text{diag}[x_1 I, \ldots, x_F I] \geq 0$ commutes with Δ. This implies that

$$M(j\omega)^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X & 0 \\ 0 & Y(\omega) \end{bmatrix} \leq 0 \quad \forall \omega \in \mathbb{R}, \tag{38}$$

   and also that $0 \leq \alpha + \gamma^2$, so $-\alpha \leq \gamma^2$ which implies from Step 3 that

$$\int_{-\infty}^{\infty} Y(j\omega)\frac{d\omega}{2\pi} \leq \gamma^2 < 1. \tag{39}$$

   Equations (38-39) are "almost" what is required. To complete the proof, we must ensure $X > 0$, tighten (38) to (12), and ensure that $Y(\omega)$ is in the class $\mathcal{F}$. This is considered next.

6. Since the system is robustly stable under structured LTV perturbations, there exists [32, 18] $X_0 \in \mathbb{X}$ such that $M_{11}(j\omega)^* X_0 M_{11}(j\omega) - X_0 \leq -2I \ \ \forall\, \omega$. Next, define

$$Y_0 = M_{12}^* X_0 M_{12} + M_{12}^* X_0 M_{11} \left[I - M_{11}^* X_0 M_{11} + X_0\right]^{-1} M_{11}^* X_0 M_{12}.$$

which is a rational function in $\mathcal{L}_1(j\mathbb{R})$ since $[I - M_{11}^* X_0 M_{11} + X_0]^{-1} \in \mathcal{RL}_\infty$, $M_{12} \in \mathcal{RH}_2$, and $M_{11} \in \mathcal{RH}_\infty$. By a Schur complement operation, we have

$$M(j\omega)^* \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix} M(j\omega) - \begin{bmatrix} X_0 & 0 \\ 0 & Y_0(\omega) \end{bmatrix} \leq \begin{bmatrix} -I & 0 \\ 0 & 0 \end{bmatrix} \quad \forall \omega \in \mathbb{R}, \tag{40}$$

which implies from (38) that

$$M(j\omega)^* \begin{bmatrix} \bar{X} & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} \bar{X} & 0 \\ 0 & \bar{Y}(\omega) \end{bmatrix} \leq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix} \quad \forall \omega \in \mathbb{R}, \tag{41}$$

for $\bar{X} = X + \epsilon X_0 > 0$, $\bar{Y} = Y + \epsilon Y_0 \geq 0$ and any $\epsilon > 0$. For small enough ϵ, we can obtain from (39)

$$\int_{-\infty}^{\infty} \bar{Y}(\omega) \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} Y(\omega) \frac{d\omega}{2\pi} + \epsilon \int_{-\infty}^{\infty} Y_0(\omega) \frac{d\omega}{2\pi} < 1. \tag{42}$$

As noted before, $Y \in BV(\mathbb{R})$; since Y_0 is rational and has no poles poles on the imaginary axis, we also have $Y_0 \in BV(\mathbb{R})$ and consequently $\bar{Y} \in BV(\mathbb{R})$. The class $\mathcal{F}$ imposes the additional requirement that $\bar{Y}(|\omega|) \leq G(\omega)$ for a *monotonic* $G \in \mathcal{L}_1(\mathbb{R}_+)$. Since $\bar{Y}$ is bounded this requirement only refers to the behavior of $\bar{Y}$ as $\omega \to \infty$. Noting that (41) imposes that $\bar{Y}$ is lower bounded by a *rational*, strictly proper function of frequency (therefore monotonic at high frequency), we can modify $\bar{Y}$ at very high frequencies to be in $\mathcal{F}$ while still satisfying (41) and (42).

■

We conclude this section by commenting briefly on the various extensions to the above proof.

Multivariable noise $u \in W_{\eta,B}^m$.

For the sufficiency proof, the bound (21) is extended to

$$\int_{-\infty}^{\infty} u^* Y u \frac{d\omega}{2\pi} \leq \int_{-\infty}^{\infty} \text{trace}(Y) \frac{d\omega}{2\pi} + \eta[2mG(B) + \sum_{i,j=1}^{m} (|Y_{i,j}(B) + TV(Y_{i,j}))] + \frac{m}{\pi} \int_B^{\infty} G(\omega) d\omega,$$

for $Y \in \mathcal{F}^m$. For the necessity, we use $\rho(z) = F_u(\beta)$ given by (27), taking values in $C_H^{m\times m}(R_+)$. The functionals on this space have the form

$$\Gamma_{\Psi,\zeta}(g) = \sum_{i,j} \left(\int_0^{\infty} [g_{i,j}(t) - g_{i,j}(\infty)] d\Psi_{i,j}(t) + \zeta_{i,j} g_{i,j}(\infty) \right),$$

where Ψ is a hermitian-matrix valued function on $\mathbb{R}_+$, with entries of bounded variation, and ζ is a hermitian matrix. The proof follows in a similar way.

δI perturbations in Δ.

If the i-th block of Δ is δI_{r_i}, then the scalar quadratic function σ_i must be replaced (see [25]) by a matrix-valued function

$$\Sigma_i(z) = \int_{-\infty}^{\infty} [(Mz)_i(j\omega)(Mz)_i^*(j\omega) - z_i(j\omega)z_i^*(j\omega)] \frac{d\omega}{2\pi},$$

which takes values in the space of hermitian $r_i \times r_i$ matrices. The functionals in this space are of the form $\Gamma_{X_i}(A) = \text{trace}(X_i A)$, where X_i is a full, hermitian matrix. The argument then proceeds in a similar fashion, X_i becoming a sub-block of the scaling matrix X.

Slowly varying perturbations

The preceding proof can be extended to obtain Theorem 5. For the necessity side, one must replace σ_i, $i = 1 \dots F$ by a function-valued quadratic map $\varphi_i : \mathcal{L}_2 \mapsto C_0(\mathbb{R})$, of the form

$$[\varphi_i(z)](\beta) = \int_{\beta}^{\beta+h} (|(Mz)_i|^2 - |z_i|^2)\frac{d\omega}{2\pi}.$$

Constraints on φ_i for a fixed $h > 0$ provide a characterization of $\Delta \in \mathbf{B}_{\mathbf{\Delta}^\nu}$, and the duality argument extends. Details on this procedure will be provided in [26].

6 State-space Conditions for Robust $\mathcal{H}_2$ Performance

In Section 3 we remarked that Condition 1 with frequency dependent $X(\omega)$ was an infinite dimensional test. For the case of constant X scales (i.e. LTV uncertainty), we will show in this section that the problem can be reduced to a finite dimensional convex test in state space, analogously to what happens in the case of $\mathcal{H}_\infty$ performance.

For this purpose we select a state-space realization for M,

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

The assumption $M_{12}, M_{22} \in \mathcal{RH}_2$ implies that D_{12} and D_{22} are 0. To simplify the formulas we will also assume that D_{11} and D_{21} are 0 (M is strictly proper), although this second restriction is not essential and can be removed. Correspondingly we use

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{array}\right]. \tag{43}$$

It is assumed that the realization (43) is minimal and that A is Hurwitz, of state dimension n.

6.1 Review of state-space tests for Robust $\mathcal{H}_\infty$ Performance

Given a system M with state-space realization (43), where A is Hurwitz and (C, A) observable, the condition $\|M\|_\infty < 1$ can be characterized in terms of the solutions to the algebraic Ricatti equation

$$AP + PA' + BB' + PC'CP = 0. \tag{44}$$

More precisely, $\|M\|_\infty < 1$ if and only if there exists $P_- \geq 0$ solving (44), and $A + P_-C'C$ Hurwitz. P_- is called the stabilizing solution of (44). This equivalence is the so-called Bounded Real Lemma (see, e.g. [41]), and can alternatively be stated in terms of LMIs: A is Hurwitz and $\|M\|_\infty < 1$ if and only if there exists a solution $P > 0$ to

$$\begin{bmatrix} AP + PA' + BB' & PC' \\ CP & -I \end{bmatrix} < 0. \tag{45}$$

An advantage of this second characterization is that it can be easily combined with the scaling matrices which arise in structured robust performance conditions. In particular, the condition for robust $\mathcal{H}_\infty$ performance of (M, Δ) under structured LTV uncertainty is given [19, 32, 18] by the existence of $X \in \mathbb{X}$ such that

$$\left\| \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} X^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \right\| < 1.$$

Incorporating the scaling X into the state space realization and using (45) gives the LMI condition

$$\begin{bmatrix} AP + PA' + B_1XB_1' + B_2B_2' & PC' \\ CP & -\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} < 0, \quad P > 0, \quad X \in \mathbb{X}, \tag{46}$$

which is a finite dimensional convex test, necessary and sufficient for robust $\mathcal{H}_\infty$ performance under structured LTV uncertainty.

6.2 State-Space Conditions for an Auxiliary Mixed $\mathcal{H}_\infty/\mathcal{H}_2$ Problem

The objective is to obtain a finite dimensional test for the analogous situation of robust $\mathcal{H}_2$ performance under structured LTV uncertainty. We know from Section 5 that this corresponds to Condition 1 under constant scales X; however, the presence of the variable $Y(\omega)$ seems to indicate that Condition 1 remains infinite dimensional. In the following we will show that in fact the condition can be reduced to a finite dimensional test in state space.

For this purpose we begin by analyzing the test obtained from Condition 1 by setting $X = I$, which specifies the existence of $Y(j\omega) = Y(j\omega)^* \in \mathbb{C}^{m\times m}$ such that

$$M(j\omega)^* M(j\omega) - \begin{bmatrix} I & 0 \\ 0 & Y(j\omega) \end{bmatrix} < 0, \tag{47}$$

$$\int_{-\infty}^{\infty} \text{trace}(Y(j\omega)) \frac{d\omega}{2\pi} < 1. \tag{48}$$

This pair of conditions allows in fact an interpretation in terms of a mixed $\mathcal{H}_\infty/\mathcal{H}_2$ performance problem (with no uncertainty). If the system $M = [M_1 \; M_2]$ has the input u partitioned into the vectors u_1, $u_2 \in \mathbb{C}^m$, with u_1 an arbitrary $\mathcal{L}_2$ signal, and u_2 white noise, (47-48) can be interpreted as a test for the rejection of such signals. We will not expand on this interpretation here (see [23] for the discrete time version), but remark that such mixed performance problem was among those considered in [40] (see also [28]), although [40, 11] concentrated mostly on a solving a different problem with causality restrictions between u_1 and u_2. For a comparison between these problems see [40, 23].

The following theorem provides a state-space characterization of the conditions (47-48).

Theorem 9 *Let* $M = [M_1 \; M_2] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & 0 & 0 \end{array}\right]$, *A Hurwitz,* (C, A) *observable. The following are equivalent:*

1. $\exists\, Y(j\omega) = Y(j\omega)^* \in \mathbb{C}^{m\times m}$ *such that (47-48) hold.*

2. $\|M_1\|_\infty < 1$, *and* $\|N^{-1}M_2\|_2 < 1$, *where* N *is the spectral factor satisfying* $N \in \mathcal{RH}_\infty$, $N^{-1} \in \mathcal{RH}_\infty$, *and* $I - M_1M_1^* = NN^*$.

3. The algebraic Ricatti equation

$$AP + PA' + B_1B_1' + PC'CP = 0 \tag{49}$$

admits a stabilizing solution $P_- \geq 0$, *and defining* Z *to satisfy the Lyapunov equation*

$$(A + P_-C'C)'Z + Z(A + P_-C'C) + C'C = 0, \tag{50}$$

we have

$$\text{trace}(B_2'ZB_2) < 1. \tag{51}$$

Proof:

($1 \Longrightarrow 2$) We first rewrite (47) in terms of the partition of M (the convention (12) is in place), and obtain

$$\begin{bmatrix} M_1^*M_1 - I & M_1^*M_2 \\ M_2^*M_1 & M_2^*M_2 - Y \end{bmatrix} \leq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix} \quad \forall\, \omega \in \mathbb{R}.$$

from the first block we conclude that $\|M_1\|_\infty < 1$. Also, a Schur complement operation yields

$$Y(\omega) \geq M_2^* \left[I + M_1(I - M_1^*M_1)^{-1}M_1^*\right] M_2 = M_2^*(I - M_1M_1^*)^{-1}M_2 = (N^{-1}M_2)^*(N^{-1}M_2), \tag{52}$$

with N satisfying $I - M_1M_1^* = NN^*$. Substitution of (52) into (48) gives

$$\|N^{-1}M_2\|_2^2 = \int_{-\infty}^{\infty} \text{trace}[(N^{-1}M_2)(j\omega)^*(N^{-1}M_2)(j\omega)]\frac{d\omega}{2\pi} < 1.$$

($2 \Longrightarrow 1$) The previous argument can be easily reversed to yield the converse implication.

($2 \Longleftrightarrow 3$) As mentioned in Section 6.1, the condition $\|M_1\|_\infty < 1$ is equivalent to the solvability of (49). Furthermore, the same Ricatti equation provides a way of computing the spectral factorization which defines N: it is shown in [41] that if P_- is the stabilizing solution of (49), then

$$N = \left[\begin{array}{c|c} A & -P_-C' \\ \hline C & I \end{array}\right] \in \mathcal{RH}_\infty$$

satisfies $I - M_1(j\omega)M_1(j\omega)^* = N(j\omega)N(j\omega)^*$ for all ω, with

$$N^{-1} = \left[\begin{array}{c|c} A + P_-C'C & P_-C' \\ \hline C & I \end{array}\right] \in \mathcal{RH}_\infty.$$

Composing with $M_2 = \left[\begin{array}{c|c} A & B_2 \\ \hline C & 0 \end{array}\right]$ gives

$$N^{-1}M_2 = \left[\begin{array}{cc|c} A & 0 & B_2 \\ P_-C'C & A + P_-C'C & 0 \\ \hline C & C & 0 \end{array}\right].$$

This last realization is non-minimal, and a similarity transformation $T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$ allows us to eliminate unobservable states and reduce it to

$$N^{-1}M_2 = \left[\begin{array}{c|c} A + P_-C'C & B_2 \\ \hline C & 0 \end{array}\right].$$

Now the computation of the $\mathcal{H}_2$ norm of $N^{-1}M_2$ can readily be performed by solving the Lyapunov equation (50) and computing $\text{trace}(B_2'ZB_2)$. Therefore $\|N^{-1}M_2\|_2 < 1$ is equivalent to (51).

■

The state-space conditions of the previous theorem can be computed by a three-stage procedure: first solve (49) for the stabilizing solution, then substitute in (50) and solve the equation for Z, finally compute (51).

We are interested, however, in combining this test with a search for the scaling $X \in \mathbb{X}$ for robust $\mathcal{H}_2$ performance. For this reason we seek a single "one-shot" test for the combination of (49-51), to play the role of (45) in the $\mathcal{H}_\infty$ case. The key observation is that there is an explicit formula for the solution to the Lyapunov equation (50). In fact, from the theory of the algebraic Ricatti equation developed extensively in [36] (see also [41]), we extract the following[1]:

- Given that $\|M_1\|_\infty < 1$, (C, A) observable there also exists an antistabilizing solution P_+ to (49) such that $-(A + P_+C'C)$ is Hurwitz. Furthermore $P_- < P_+$ and we have

$$P_- \le P \le P_+ \tag{53}$$

[1]These references usually work with the dual equations and controllability assumptions, which are easily converted to our case with (C, A) observable.

for every P satisfying (49) (or the LMI (45) with B_1 replacing B).

- The matrix $Z = (P_+ - P_-)^{-1}$ satisfies the Lyapunov equation (50).

Note that, given $P_+ - P_- > 0$, the last observation follows by direct substitution into (50), using the fact that both P_- and P_+ satisfy (49).

We are now in a position to write an LMI test for the mixed performance problem.

Theorem 10 *The conditions of Theorem 9 are satisfied if and only if there exist hermitian $n \times n$ matrices P_-, P_+, Z, satisfying*

$$P_- > 0, \quad (54)$$

$$\begin{bmatrix} AP_- + P_-A' + B_1B_1' & P_-C' \\ CP_- & -I \end{bmatrix} < 0, \quad (55)$$

$$\begin{bmatrix} AP_+ + P_+A' + B_1B_1' & P_+C' \\ CP_+ & -I \end{bmatrix} < 0, \quad (56)$$

$$\begin{bmatrix} Z & I \\ I & P_+ - P_- \end{bmatrix} > 0, \quad (57)$$

$$\mathrm{trace}(B_2'ZB_2) < 1. \quad (58)$$

Proof: For convenience we use the notation

$$\Upsilon(P) := \begin{bmatrix} AP + PA' + B_1B_1' & PC' \\ CP & -I \end{bmatrix}.$$

Note that Υ is an affine map, so

$$\Upsilon(\alpha P_1 + (1-\alpha)P_2) = \alpha\Upsilon(P_1) + (1-\alpha)\Upsilon(P_2). \quad (59)$$

[Necessity] Let $P_-^0 < P_+^0$ be the stabilizing and antistabilizing solutions of (49), and $Z^0 = (P_+^0 - P_-^0)^{-1}$ be the corresponding solution to (50), satisfying $\mathrm{trace}(B_2'Z^0B_2) < 1$. It follows from (49) that $\Upsilon(P_-^0) \leq 0$, $\Upsilon(P_+^0) \leq 0$. Also, from (45) we know there exists $P^{str} > 0$ satisfying the strict inequality $\Upsilon(P^{str}) < 0$. Now pick $\alpha \in (0,1)$, and define

$$\begin{aligned} P_-(\alpha) &= \alpha P^{str} + (1-\alpha)P_-^0, \\ P_+(\alpha) &= \alpha P^{str} + (1-\alpha)P_+^0, \\ Z(\alpha) &= [P_+(\alpha) - P_-(\alpha)]^{-1} + \alpha I. \end{aligned}$$

By construction, for any $\alpha \in (0,1)$ we have $P_+ > P_- > 0$, $\Upsilon(P_+) < 0$ and $\Upsilon(P_-) < 0$ (using (59)). Also $Z > (P_+ - P_-)^{-1}$, which implies by Schur complement that (57) holds. Finally, $Z(\alpha) \to Z^0$ as $\alpha \to 0+$, so (58) holds for sufficiently small α.

[Sufficiency]: (54) and (55) imply by (45) that A is Hurwitz, and $\|M_1\|_\infty < 1$. This means we can introduce P_-^0, P_+^0 , Z^0 as before. From (53) and (57) we know that

$$P_-^0 \leq P_- < P_+ \leq P_+^0.$$

This implies that $P_+^0 - P_-^0 > P_+ - P_- > 0$, so we have

$$Z > (P_+ - P_-)^{-1} > (P_+^0 - P_-^0)^{-1} = Z^0,$$

using (57). Now (58) implies $\mathrm{trace}(B_2' Z^0 B_2) < 1$, so the conditions of Theorem 9 hold.

■

6.3 State-space LMIs for Robust $\mathcal{H}_2$ Performance

Using the characterization of Theorem 10, we are finally in a position to state the main result of this section, which is an exact state-space condition for robust $\mathcal{H}_2$ performance in the case of structured LTV uncertainty.

Theorem 11 *The system (M, Δ) has robust $\mathcal{H}_2$ performance for $\Delta \in \mathbf{B}_{\mathbf{\Delta}_{\mathrm{LTV}}}$, if and only if there exist $X \in \mathbb{X}$, and hermitian $n \times n$ matrices P_-, P_+, Z satisfying*

$$P_- > 0, \tag{60}$$

$$\begin{bmatrix} AP_- + P_-A' + B_1XB_1' & P_-C' \\ CP_- & -\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} < 0, \tag{61}$$

$$\begin{bmatrix} AP_+ + P_+A' + B_1XB_1' & P_+C' \\ CP_+ & -\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} < 0, \tag{62}$$

$$\begin{bmatrix} Z & I \\ I & P_+ - P_- \end{bmatrix} > 0, \tag{63}$$

$$\mathrm{trace}(B_2'ZB_2) < 1. \tag{64}$$

Proof: From Theorem 6 robust $\mathcal{H}_2$ performance is equivalent to Condition 1 with constant scales; for convenience we write this condition replacing X by X^{-1} which is also in $\mathbb{X}$:

$$M(j\omega)^* \begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X^{-1} & 0 \\ 0 & Y(\omega) \end{bmatrix} < 0,$$

$$\int_{-\infty}^{\infty} \mathrm{trace}(Y(\omega)) \frac{d\omega}{2\pi} < 1.$$

Defining

$$\hat{M} = \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} X^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1X^{\frac{1}{2}} & B_2 \\ \hline \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} C & 0 & 0 \end{array}\right],$$

$\hat{M}$ satisfies the conditions (47-48). Applying Theorem 10, this is equivalent to the LMIs

$$P_- > 0,$$

$$\begin{bmatrix} AP_- + P_-A' + B_1XB_1' & P_-C' \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} CP_- & -I \end{bmatrix} < 0, \tag{65}$$

$$\begin{bmatrix} AP_+ + P_+A' + B_1XB_1' & P_+C' \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} CP_+ & -I \end{bmatrix} < 0, \tag{66}$$

$$\begin{bmatrix} Z & I \\ I & P_+ - P_- \end{bmatrix} > 0,$$
$$\text{trace}(B_2' Z B_2) < 1.$$

Multiplying (65) and (66) on the left and right by $\begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} X^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix}$, these equations are equivalent to (60-64). ■

The previous result provides an exact, finite dimensional LMI condition for robust $\mathcal{H}_2$ performance under LTV perturbations, which plays the same role as (46) for $\mathcal{H}_\infty$ performance. Although (60-64) is somewhat more expensive to compute, the condition is of the same nature since it is exact and its dimensionality depends only on the original system. The LMIs (60-64) can be evaluated by the standard methods of [4].

As a final remark, we note that analogous state-space tests can be developed for the discrete time problem studied in [22, 23], by replicating the previous steps in a routine way.

7 Conclusions

This paper provides a general method for robust $\mathcal{H}_2$ analysis in continuous time, complementing the results in [23] for the discrete time case, and extending them to include state space characterizations. The resulting theory completely parallels that of the $\mathcal{H}_\infty$ performance measure, and the conditions involve a comparable computational cost, with the advantage that the rejection of broadband noise is directly addressed. It is expected that these tools will be incorporated into current practical uses of the $\mathcal{H}_\infty$ and μ frameworks [2], alleviating the effort of weight selection in these methods, since the performance weights can now be obtained directly from Condition 1.

While these tests give an exact treatment of LTV uncertainty, and a mildly conservative method for the LTI case, there is probably more room for improvement in the case of parametric uncertainty. The use of "G-scales" in Condition 1 suggested in Section 3 helps impose that the perturbations are real, but can still be very conservative. This is well known in the standard mixed μ theory [37], and here the conservatism can be greater since the $\mathcal{H}_2$ condition depends on all frequencies (not just the worst), and these tests do not impose that the perturbation is constant across frequency (see the discussion in [24]). This potential conservatism emphasizes the need for lower bounds on robust performance, extending those of the μ theory [19, 37], which remain open for future research.

Finally, we have the question of synthesis. It is shown in [23] that "D-K" iteration methods for synthesis extend directly to the case of $\mathcal{H}_2$ performance, and the same happens for the continuous time case considered here. These iterations are based on $\mathcal{H}_\infty$ optimal control, using the weight $Y(\omega)$ obtained from Condition 1 to shape the synthesis. The availability of state-space conditions for the mixed performance problem (47-48) raises the possibility of state-space synthesis for this problem which is slightly different than the one considered in [11], and could lead to an alternative procedure for robust $\mathcal{H}_2$ synthesis. These issues, as well as the practical behavior of these iterations, are the main open directions for further research in this problem.

Appendix: Proof of Proposition 8

Consider a convex combination $\Lambda_0 = \alpha\Lambda(z) + (1-\alpha)\Lambda(f)$ of two points in ∇. Let $z^k = \sqrt{\alpha}z + \sqrt{1-\alpha}\lambda_k f$, where λ_k is the k-second delay, k is an integer. We have

$$\|z_i^k\|^2 \overset{k\to\infty}{\longrightarrow} \alpha\|z_i\|^2 + (1-\alpha)\|f_i\|^2, \tag{67}$$

$$\|(Mz^k)_i\|^2 \overset{k\to\infty}{\longrightarrow} \alpha\|(Mz)_i\|^2 + (1-\alpha)\|(Mf)_i\|^2. \tag{68}$$

This follows from the fact that $\langle z, \lambda_k f\rangle \overset{k\to\infty}{\longrightarrow} 0$ for any functions $z, f \in \mathcal{L}_2$. For (68) we use $M\lambda_k = \lambda_k M$ from the time invariance of M. (67-68) imply that

$$\sigma_i(z^k) \overset{k\to\infty}{\longrightarrow} \alpha\sigma_i(z) + (1-\alpha)\sigma_i(f) \quad i = 1 \dots F+1. \tag{69}$$

We now show that

$$\rho(z^k) \overset{k\to\infty}{\longrightarrow} \alpha\rho(z) + (1-\alpha)\rho(f), \tag{70}$$

with convergence in the sense of $C(\mathbb{R}_+)$. From (31) and (16) it follows that

$$\left[\rho(z^k) - \alpha\rho(z) - (1-\alpha)\rho(f)\right](\beta) = 2\sqrt{\alpha(1-\alpha)}Re\int_{-\beta}^{\beta} z_{F+1}^*(j\omega)e^{j\omega k}f_{F+1}(j\omega)\frac{d\omega}{2\pi}.$$

Introduce

$$\varphi_k(\beta) := \int_{-\beta}^{\beta} z_{F+1}^*(j\omega)e^{j\omega k}f_{F+1}(j\omega)\frac{d\omega}{2\pi} = \left\langle 1_{[-\beta,\beta]}z_{F+1}, \lambda_k f_{F+1}\right\rangle. \tag{71}$$

We must show that $\varphi_k(\beta)$ converges to 0 uniformly over $\beta \in \mathbb{R}_+$. Pointwise convergence at a fixed β (including ∞) follows from (71). If it were not uniform over $\mathbb{R}_+$, we could find a subsequence k_j and points $\beta_{k_j} \in \mathbb{R}_+$ with $|\varphi_{k_j}(\beta_{k_j})| \geq \epsilon$. By taking a further subsequence we can assume that $\beta_{k_j} \stackrel{j\to\infty}{\longrightarrow} \beta_0$, where β_0 may be infinity. Now we write

$$\varphi_{k_j}(\beta_{k_j}) = \varphi_{k_j}(\beta_0) + \int_{\beta_0}^{\beta_{k_j}} z_{F+1}^*(\omega)\ e^{j\omega k_j}f_{F+1}(\omega)\frac{d\omega}{2\pi} + \int_{-\beta_{k_j}}^{-\beta_0} z_{F+1}^*(\omega)\ e^{j\omega k_j}f_{F+1}(\omega)\frac{d\omega}{2\pi},$$

which implies that

$$0 < \epsilon \leq |\varphi_{k_j}(\beta_{k_j})| \leq |\varphi_{k_j}(\beta_0)| + \left|\int_{\beta_0}^{\beta_{k_j}} |z_{F+1}||f_{F+1}|\frac{d\omega}{2\pi}\right| + \left|\int_{-\beta_{k_j}}^{-\beta_0} |z_{F+1}||f_{F+1}|\frac{d\omega}{2\pi}\right|. \tag{72}$$

The right hand side of (72) converges to 0 from the pointwise convergence of φ_{k_j}, and the fact that $\beta_{k_j} \stackrel{j\to\infty}{\longrightarrow} \beta_0$, (since $|z_{F+1}||f_{F+1}| \in \mathcal{L}_1(\mathbb{R})$). This is a contradiction, so we have shown (70), which together with (69) implies that $\Lambda(z^k) \stackrel{k\to\infty}{\longrightarrow} \alpha\Lambda(z) + (1-\alpha)\Lambda(f)$ in the topology of V.

Since $\Lambda(z^k) \in \nabla$, we conclude that $co(\nabla) \subset \bar{\nabla}$, where $co(\nabla)$ is the convex hull of ∇ and $\bar{\nabla}$ its closure. This implies that $co(\bar{\nabla}) \subset \overline{co(\nabla)} \subset \bar{\nabla}$, so $\bar{\nabla}$ is convex. ∎

Acknowledgements

This research was initiated during my stay at Caltech; I am indebted to John Doyle who encouraged me to seek a finite dimensional state-space solution, and provided useful suggestions. Most of the work was conducted at the Laboratory for Information and Decision Systems at MIT. I am grateful to Professors Munzer Dahleh and Sanjoy Mitter for their support, under grants ARO DAAL03-92-G-0115, Draper Lab DL-H-484752 and a grant from Siemens AG.

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