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LQG Control with Communication Constraints*

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Dedicated to Tom Kailath on the occasion of his sixtieth birthday.

Abstract

The average cost control problem for linear stochastic systems with Gaussian noise and quadratic cost is considered in the presence of communication constraints. The latter take the form of finite alphabet codewords, being transmitted to the controller with ensuing delay and distortion. It is shown that if instead of the state observations an associated "innovations process" is encoded and transmitted, then the separation principle holds, leading to an optimal control linear in state estimate. An associated "off-line" optimization problem for code length selection is formulated. Some possible extensions are also pointed out.

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LQG control, separation principle, communication constraints, average cost control, optimum code length.

1 Introduction

Most traditional analyses of control systems presuppose that the observation vector is available in its entirety to the controller at each decision epoch. In many real engineering systems, however, the situation is different. What the controller sees will not often be the original observation vector from the sensor, but a quantized version of it transmitted over a communication channel with accompanying transmission delays and distortion, subject to bit rate constraints. This calls for control systems analysis that explicitly accounts for such communication constraints. This problem has attracted some attention in recent years, see, e.g. [3, 5, 7, 8, 9]. For related work on multirate control of sampled-data systems, see [6] and the references therein. The aim of this work is to show that the classical Linear-Quadratic-Gaussian (LQG) problem does admit a rather clean treatment in this framework, with the proviso that it is not the state or the observation vector that is encoded and transmitted, but an associated process we dub the 'innovations' process by slight abuse of terminology. In fact, a 'separation principle' holds and this will be the main result of this exercise.

There are two key features of our formulation that make this work. The first is the choice of 'innovations process' alluded to above in place of the observation process as the signal to be quantized and encoded. Unlike the latter, the former is an i.i.d. Gaussian sequence with statistics independent of control. This allows us to use a fixed optimal vector quantizer for which extensive analysis is available for the Gaussian case [4]. Secondly, the least squares estimation at the output end of the channel can now be based only on the current channel output and does not have to remember the past outputs, as it ideally should, if the observations were to be encoded directly. This makes the estimation scheme at the controller end completely transparent. These observations will become self-evident as we proceed.

The second key feature is the centroid property of the optimal vector quantizer, which allows us to interpret the quantized random variable as the conditional expectation of the original random variable given an appropriate sub-σ-field. This interpretation nicely fits in with the least squares estimation scheme we use.

The paper is organized as follows. The next section describes the problem formulation in detail. Section 3 derives the optimal controller. Section 4 describes the associated optimal code-length selection problem. Section 5 sketches some possible extensions.

2 Preliminaries

Consider the control system

$$X_{k+1} = AX_k + Bu_k + v_k\,, \quad k \geq 0\,, \tag{1}$$

where

i. $\{X_k\}$ is an R^d-valued 'state' process, X_0 prescribed,

ii. $\{u_k\}$ is an R^m-valued control process,

iii. $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times m}$,

iv. $\{v_k\}$ is i.i.d. $N(0, Q)$ noise, that is, normally distributed, zero-mean with covariance Q

v. the following 'nonanticipativity' condition holds: $\{v_j, j \geq k\}$ is independent of $\{x_j, u_j, v_{j-1}, j \leq k\}$ for all $k \geq 0$.

Let $G \in \mathbb{R}^{d\times d}$, $F \in \mathbb{R}^{m\times m}$ be prescribed positive semidefinite matrices. Our control problem is to minimize

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} E\left[X_k^T G X_k + u_k^T F u_k\right]$$

over $\{u_k\}$ as above, subject to the communication mechanism described below. Before getting into the details thereof, we lay down the following assumptions:

A1. The pair (A, B) is controllable.

A2. The pair $(A, G^{\frac{1}{2}})$ is observable.

A3. $||A||_2 \triangleq \lambda_{\max}(A^T A) < 1$.

The above control problem is well-posed under (A1)–(A2) [2, pp. 228–229]. (A3) will be used later. We come now to the encoding and communication mechanism.

Fix an integer $M \geq 1$, the 'code length.' Also let $N \geq 1$ be another integer, the 'communication delay' given by $N = \psi(M)$ for some prescribed nondecreasing map $\psi : \mathrm{N} \to \mathrm{N}$. (Typically, $\psi(n) = [n/r] + 1$ where $[\cdot]$ represents integer part and $r > 0$ is the transmission rate in bits per second.) For $k \geq 0$, let

$$X_{(k+1)N} = A^N X_{kN} + \sum_{i=0}^{N-1} A^{N-i} B u_{kN+i} + \tilde{v}_{k+1},$$

where $\tilde{v}_{k+1} = \zeta_{(k+1)N}$ for

$$\zeta_{kN+i} = \sum_{j=0}^{i-1} A^{i-j-1} v_{kN+j}\,, \quad 0 < i \leq N\,.$$

Then $\{\tilde{v}_k\}$ are i.i.d. $N(0, \tilde{Q}_N)$ where

$$\tilde{Q}_i = \sum_{j=0}^{i-1} A^{i-j-1} Q (A^T)^{i-j}\,, \quad 0 < i \leq N\,.$$

We call $\{\tilde{v}_k\}$ the innovations process by abuse of terminology.

At time kN, $k \geq 0$, start transmitting M-bit encoding of $\tilde{v}_k$. The transmission is complete at time $(k+1)N$.

Let $\{a_1, \ldots, a_\ell\}$ denote the range of the vector quantizer, assumed to satisfy the usual optimality conditions [4, Section 11.2]. Let $\{A_1, \ldots, A_\ell\}$ denote the finite partition of R^d generated by the vector quantizer, such that A_i gets mapped to a_i, $1 \leq i \leq \ell$. Let ζ_k denote the σ-field generated by the events $\{\tilde{v}_k \in A_i\}$, $i \leq i \leq \ell$. Then the centroid condition of optimal vector quantizer [4, p. 352] translates into

$$\hat{v}_k = E[\tilde{v}_k / \zeta_k]\,, \quad k \geq 0\,.$$

Letting $p_i = P(\tilde{v}_k \in A_i)$, $1 \leq i \leq \ell$, it is clear that

$$E[\hat{v}_k] = \sum_i p_i a_i = 0\,.$$

We assume a memoryless channel that maps a_i to a_j with probability $q(i,j)$, $1 \leq i$, $j \leq \ell$. Let v'_k be the output of the channel to input $\hat{v}_k$. Then assuming that the channel noise is independent of $\{\tilde{v}_k\}$, the LMS estimate of $\tilde{v}_k$ at time k is $\overline{v}_k = E[\tilde{v}_k/v'_k] = E[\hat{v}_k/v'_k]$ calculated as follows using the Bayes rule:

$$\overline{v}_k = \sum_j \sum_i p_i q(i,j) \Big(\sum_s p_s q(s,j)\Big)^{-1} a_i \quad I_{\{v'_k = a_j\}} \,.$$

Clearly, $E[\overline{v}_k] = 0$ and

$$E \triangleq \operatorname{cov}(\overline{v}_k) = \sum_{i,j,m} p_i p_m q(i,j) q(m,j) \Big(\sum_s p_s q(s,j)\Big)^{-1} a_i a_m^T \,.$$

The controller thus receives $\overline{v}_k$ at time $(k+1)N$, $k \geq 0$, and has to optimize the control system based on this information. The next section studies this control problem.

3 The optimal controller

With the aim of formulating a 'separated control problem,' we first study the evolution of $\hat{X}_k$ = the LMS estimate of X_k, $k \leq 0$. At time $(k+1)N$, $\hat{X}_{(k+1)N}$ is obtained as follows:

Step 1. Update $\hat{X}_{kN}$ to

$$\begin{aligned} \tilde{X}_{kN} &= \hat{X}_{kN} + \overline{v}_k \\ &= A^N \tilde{X}_{(k-1)N} + \sum_{i=0}^{N-1} A^{N-i} B u_{(k-1)N+i} + \overline{v}_k \,. \end{aligned}$$

Step 2. Set $\hat{X}_{(k+1)N} = A^N \tilde{X}_{kN} + \sum_{i=0}^{N-1} A^{N-i} B u_{kN+i}$.
For times $kN+i$, $0 < i < N$, $k \geq 0$, we have

Step 3.

$$\begin{aligned} \hat{X}_{kN+i} &= A^i \hat{X}_{kN} + \sum_{j=0}^{i-1} A^{i-j-1} B u_{kN+j} \\ \Big(&= A \hat{X}_{kN+i-1} + B u_{kN+i-1} \Big) \,. \end{aligned}$$

Let $e_k = X_k - \hat{X}_k$ denote the estimation error and $R_k = \operatorname{cov}(e_k)$, $k \geq 0$. Then the evolution of $\{R_k\}$ is described by: $R_N = \tilde{Q}_N$ and for $k \geq 1$,

$$R_{kN+i} = \begin{cases} A R_{kN+i-1} A^T + Q \,, & 0 < i < N \\ A^N R^{kN} (A^N)^T + \Big(A^N E (A^N)^T - A^N \tilde{Q}_N (A^N)^T\Big) + \tilde{Q}_N \,, & i = N \end{cases}$$

In particular, the evolution of $\{R_k\}$ is deterministic and independent of $\{u_k\}$. Combining this with the observation that

$$E[X_k^T G X_k] = E[\hat{X}_k^T G \hat{X}_k] + tr[GR_k], \tag{2}$$

we can consider the following 'separated' control problem: Minimize

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} E\left[\hat{X}_k^T G \hat{X}_k + u_k^T F u_k\right] \tag{3}$$

where $\{\hat{X}_k\}$ evolves according to Step 1–Step 3 above. This evolution can be rewritten as

$$\hat{X}_{k+1} = A\hat{X}_K + Bu_k + w_k\,, \quad k \geq 0\,,$$

where $\{w_k\}$ is a zero mean noise sequence given by

$$w_k = \begin{cases} 0 & k \notin \{iN, \quad i \geq 0\} \\ A^N \overline{v}_{i-1} & k = iN, \quad i \leq 0 \end{cases}$$

Now one can mimic the usual arguments for LQG control with minor modifications, to obtain the optimal controller. We sketch them below, closely following the treatment of [2, p. 228–229].

Under our assumptions A1–A2, the unique positive semidefinite solution $\{K_n\}$ to the Riccati equation:

$$K_{k+1} = A^T\left(K_k - K_k B(B^T K_k B + F)^{-1} B^T K_k\right) A + G, \quad K_0 = 0\,,$$

converges as $k \to \infty$ to the unique positive semidefinite solution K of the algebraic Riccati equation

$$K = A^T\left(K - KB(B^T KB + F)^{-1} B^T K\right) A + G\,.$$

The optimal value of the n-stage costs [1, p. 130–132]

$$\frac{1}{n} E\left[\sum_{k=0}^{n-1}\left(\hat{X}_k^T G \hat{X}_k + u_k^T F u_k\right)\right]$$

is seen to equal

$$\frac{1}{n}\left(X_0^T K_n X_0 + \sum_{k=0}^{n-1} E[w_k^T K_k w_k]\right)\,,$$

and tends to

$$\begin{aligned}\lambda &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} E[w_k^T K_k w_k] \\ &= \frac{1}{N}\operatorname{tr}\left[(A^T)^N K A^N E\right] .\end{aligned}$$

Also, the N-stage optimal feedback policy in its initial stages tends to the stationary feedback policy

$$\mu(x) = -(B^T K B + R)^{-1} B^T K A x . \tag{4}$$

Using the definitions of λ, K and $\mu(\cdot)$, it is easy to show that

$$\begin{aligned}x^T K x &= \min_u \left[x^T G x + u^T F x + (Ax + Bu)^T K (Ax + Bu)\right] , \\ x^T K x + N\lambda &= \\ &\min_u \Big[x^T G x + u^T F x + E\Big[(Ax + Bu + w_{kN})^T K (Ax + Bu + w_{kN})\Big] , \\ &\qquad k \geq 0 ,\end{aligned}$$

with the minimum in both cases being attained by $u = \mu(x)$. Now one can mimic the standard dynamic programming arguments of [1, p. 191–192] [2, p. 229] to deduce that under arbitrary admissible control $\{u_k\}$, the cost (3) is greater than or equal to λ, with equality if $u_k = \mu(\hat{X}_k)$, $k \geq 0$. We summarize these findings in the following:

Theorem 1. *The optimal feedback controller for the separated control problem (and hence for the original control problem) is given by*

$$u_k = \mu(\hat{X}_k) , \quad k \geq 0 ,$$

for $\mu(\cdot)$ defined by (4).

4 Optimal Codelength

Consider the matrix equation

$$\overline{R} = A^N \overline{R} (A^N)^T + C(N) \tag{5}$$

where $C(N) = A^N E(A^N)^T + \tilde{Q}_N - A^N \tilde{Q}_N (A^N)^T$. Under A3, (5) has a unique symmetric positive definite solution $\overline{R}$ given explicitly by

$$\overline{R} = \sum_{i=0}^{\infty} A^{Ni} C(N) (A^T)^{Ni} . \tag{6}$$

As $k \to \infty$, $R_{kN} \to \overline{R}$, as can be easily verified.

Given (2) and the dynamics of $\{R_k\}$, the total cost for the original control problem is

$$J(N) \triangleq \lambda(N) + \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{tr}\left[G A^i \overline{R} (A^T)^i\right] + \mathrm{tr}[GQ] , \tag{7}$$

where λ is written as $\lambda(N)$ to make its dependence on N explicit. Recalling that $N = \psi(M)$, the choice of optimal M would be obtained by minimizing $M \to J(\psi(M))$ over $M \in \mathrm{N}$. The expression (7) has a complicated dependence on N, but this optimization problem is 'off-line.' Also, the following could be used to advantage for any computational scheme:

i. To go from N to $N+1$, the quantities $A^N K (A^T)^N$, $A^N E (A^T)^N$, $\tilde{Q}_N$ can be updated by pre- and post-multiplying by A, A^T resp. Similarly, $A^N \tilde{Q}_N (A^T)^N$ is updated by pre- and post-multiplying by A^2 and $(A^T)^2$ respectively. This takes care of the updates of $\lambda(N)$ and $C(N)$.

ii. Updating $\overline{R} = \overline{R}(N)$ poses a harder problem. One possible approximation scheme is to use for each N the recursion

$$\tilde{R}_{k+1} = A^N \tilde{R}_k (A^T)^N + C(N) \, , k \geq 1 \, ,$$

till a suitable stopping criterion is satisfied and use the resultant matrix $\tilde{R}(N)$ as an approximation for $R(N)$. Repeat the iteration for $N+1$ with $C(N)$ replaced by $C(N+1)$, using $\tilde{R}(N)$ as the initial guess. Our assumption A3 ensures good convergence behavior of the above recursion.

iii. Invoking assumption A3 again, the summations in 6 and 7 could simply be approximated by a finite summation with a fixed number of terms for moderately large values of N.

In conclusion, observe that in our model, shorter codes correspond to low resolution and low delays, while longer codes mean higher resolution, but longer delays. Thus the above optimization problem captures the tradeoff between delay and accuracy.

5 Extensions and open issues

i. Partial observations: Suppose we do not observe $\{X_k\}$, but an accompanying r-dimensional observation process $\{Y_k\}$ described by

$$Y_k = HX_k + \eta_k\ , \quad k \geq 0\ ,$$

where $H \in \mathbb{R}^{d \times r}$ and $\{\eta_k\}$ are i.i.d. $N(0, S)$, independent of $\{v_k\}$. Suppose the pair (A, H) is observable and the pair $(A, Q^{1/2})$ controllable. Then we can carry through the foregoing analysis for this problem with just one change: replace (1) by the steady state Kalman filter [4, R. M. Gray].

ii. Coding issues: The preceding section considered optimization over code length, not over codes. Also, the entire analysis ignores the fact that if the encoding is sequentially transmitted and received, each bit carries its own information. The situation is particularly transparent for tree codes [4, Ch. 15]. These are based on successive refinements of partitions, to which we can assign a corresponding increasing family of σ-fields along the lines of ζ (which then corresponds to the largest σ-field in this chain). If the 'centroid rule' is observed at each stage of refinement, the first m bits of an M-bit codeword ($m \leq M$) would correspond to the conditional expectation with respect to the m-th σ-field in the chain. Thus the controller receives a succession of conditional expectations over finer and finer σ-fields during each N-interval. The updates rules of Step 1–Step 3 of Section 3 can be easily modified to accommodate this situation and the rest of the analysis is similar to that above. This would, in fact, seem to make a case for using tree codes in control applications. The situation for other coding schemes, however, is complicated. Also, the problem becomes considerably harder if we consider variable length codes. Finally, we have used a simple model for the channel. More complex situations need to be analyzed.

iii. Distributed control: If several observations are being recorded, encoded and transmitted in a distributed manner, with or without synchronism, the problem would appear to be much more difficult and at the same time, much more interesting for applications. We hope to address this in subsequent work.

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