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Analysis of Robust $\mathbf{H}_2$ Performance Using Multiplier Theory

Eric Feron*

Abstract

In this paper, the problem of determining the worst-case $\mathbf{H}_2$ performance of a control system subject to linear time-invariant uncertainties is considered. A set of upper bounds on the performance is derived, based on the theory of stability multipliers and the solution of an original optimal control problem. The numerical issues raised by the resulting computational problems are discussed: in particular, newly developed interior-point convex optimization methods, combined with Linear Matrix Inequalities apply very well to the fast and accurate solution of these problems. The new results compare favorably with prior ones. The method can be extended to other types of perturbations.

Keywords: $\mathbf{H}_2$ performance, stability multipliers, convex optimization, linear matrix inequalities.

AMS subject classification: 93B40, 93D05, 93D09, 93D10, 93D25.

Abbreviated title: Analysis of robust $\mathbf{H}_2$ performance

Introduction

Among all performance indices known to control engineering, the $\mathbf{H}_2$ performance index holds a special place for historical and practical reasons. The historical reasons are that minimizing the $\mathbf{H}_2$ norm of a linear control system via feedback, better known as the LQG problem, is among the first optimal control problems to have been solved analytically (for an extensive presentation and bibliography, see [1]). The practical reason is that this problem can be solved using reliable and fast computational procedures [2, 21, 11].

It is however well-known that the performance of the LQG-optimal controller can be very sensitive to perturbations on the nominal system [12]. In view of this fact, devising analysis and synthesis tools that will respectively evaluate and minimize worst-case $\mathbf{H}_2$ norms of control systems is especially relevant.

In this paper, we consider the following specific problem: given a linear control system perturbed by linear, time-invariant perturbations, what is its worst-case $\mathbf{H}_2$ norm? This question has remained open until recently when some attempts have been made at its solution. Packard and Doyle [27], and Bernstein and Haddad [4, 5, 6] are among the first to consider the problem of robust $\mathbf{H}_2$ performance in the face of dynamic and parametric uncertainty. Stoorvogel [37, 38], Petersen, Rotea and McFarlane [30, 31] find bounds on the worst-case $\mathbf{H}_2$ norm of a system subject to norm-bounded, noncausal, possibly nonlinear and time-varying uncertainties. Peres, Geromel and Souza [28, 29] find upper bounds on the $\mathbf{H}_2$ norm of linear, time-varying and uncertain LTI systems based on quadratic Lyapunov functions. The book and the papers

*Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA 02139 email: feron@mit.edu fax: (617) 253-4196. This work was presented at the 33rd IEEE Conference on Decision and Control. It was supported by the Army Research Office under Grant ARO DAAL03-92-G0115 (Center for Intelligent Control Systems), NSF under Grant ECS-9409715 and the Charles Stark Draper Career Development Chair at MIT.

by Boyd, El Ghaoui, Feron, and Balakrishnan [9, 16, 7, 15] show that the computation of all these bounds on $\mathbf{H}_2$ performance can be reduced to convex optimization problems involving linear matrix inequalities, which can be solved via efficient convex optimization techniques. In [9, 15], attempts are made to refine the upper bounds on $\mathbf{H}_2$ performance when dealing with particular classes of perturbations such as static nonlinearities and parametric uncertainties, using Lur'e Lyapunov functions and causal multipliers. Other attempts at obtaining reliable upper bounds on robust $\mathbf{H}_2$ performance include the recent paper by Paganini, Doyle and D'Andrea [14].

In this paper, we propose to extend the results presented in [9, 15] by using noncausal multipliers to evaluate the worst-case $\mathbf{H}_2$ norm of linear systems perturbed by linear, time-invariant perturbations. Using noncausal multipliers is a well-known technique to determine the stability of uncertain systems (see [10, 39] and references therein), and has proven to yield effective computational procedures [36, 34]. We believe this paper is the first attempt to use them to determine robust $\mathbf{H}_2$-performance of linear systems subject to linear perturbations. It is organized as follows:

The first part is devoted to a few definitions and notations. In particular, we recall the notions of boundedness, positivity and passivity of operators.

In the second part, we formulate the robust $\mathbf{H}_2$ analysis problem and sketch our line of attack to get upper bounds on worst-case $\mathbf{H}_2$ performance. We present a new upper bound on robust $\mathbf{H}_2$ performance, based on the use of certain dynamic Lagrange multipliers.

In the third part of this paper, we present a way to compute the upper bound on robust performance using convex optimization and linear matrix inequalities. In particular, we exhibit convenient linear families of finite-dimensional multipliers to perform this computation.

In the fourth part, we discuss the obtained results: in particular, we study conditions for the obtained upper bound to be finite. We also study special cases and show they correspond to results having already appeared in the literature. A numerical example is provided that illustrates the usefulness of dynamic multipliers to determine accurate upper bounds on robust performance.

1 Notation

In this paper, $\mathbf{R}$ (resp. $\mathbf{C}$) denotes the set of real (resp. complex) numbers. $\mathbf{R}_+$ denotes the set of nonnegative real numbers. $\mathbf{R}^{n\times p}$ (resp. $\mathbf{C}^{n\times p}$) is the vector space of $n \times p$ real (resp. complex) matrices. $\mathbf{R}^{n\times 1}$ ($\mathbf{C}^{n\times 1}$) is abbreviated $\mathbf{R}^n$ ($\mathbf{C}^n$). For the random variable x, $\mathbf{E}\,x$ denotes the expected value of x. For any matrix X, X^T denotes its transpose and X^* denotes its complex conjugate transposed. The identity matrix is noted I. If X is invertible, then its inverse is noted X^{-1}. When X is not invertible, its Moore-Penrose inverse is noted $X^\dagger$. When $X \in \mathbf{R}^{n\times n}$, $\mathbf{Tr}\,X$ denotes the trace of X. A square matrix X is said stable if all its eigenvalues lie in the open left complex half-plane. Given a set of matrices $X_1 \in \mathbf{R}^{n_1\times p_1}, \ldots, X_N \in \mathbf{R}^{n_N\times p_N}$, and defining $n = \sum_{i=1}^N n_i$, $p = \sum_{i=1}^N p_i$, $\mathbf{diag}(X_1, \ldots, X_N)$ denotes the $n \times p$ matrix

$$\begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_N \end{bmatrix}.$$

Note that the X_is need not be square. From time to time, when no ambiguity is possible, $\mathbf{diag}(X_1, \ldots, X_N)$ is noted $\mathbf{diag}_{i=1}^{L}(X_i)$, or $\mathbf{diag}_i(X_i)$.

For any two matrices X and $Y \in \mathbf{R}^{n\times n}$, the inequality $X \leq Y$ means that X and Y are symmetric, and that the difference $Y - X$ is positive. The inequality $X < Y$ means that X and Y are symmetric, and that the difference $Y - X$ is positive-definite.

$\mathbf{L}_2(\mathbf{R})$ denotes the Hilbert space of functions h mapping $\mathbf{R}$ into $\mathbf{R}^{n\times p}$ which satisfy

$$\int_{-\infty}^{\infty} \mathbf{Tr}\, h(t)^T h(t) dt < \infty.$$

It is equipped with the standard scalar product

$$\forall g, h \in \mathbf{L}_2(\mathbf{R}), \quad < g, h > \triangleq \int_{-\infty}^{\infty} \mathbf{Tr}\, g(t)^T h(t) dt,$$

and for all $h \in \mathbf{L}_2(\mathbf{R})$, the Euclidean norm $< h, h >^{1/2}$ of h is noted $\|h\|_2$. $\mathbf{L}_2(\mathbf{R}_+)$ denotes the subspace of $\mathbf{L}_2(\mathbf{R})$ made of the functions h satisfying $h(t) = 0$ when $t < 0$. $\mathbf{L}_{2e}$ denotes the space of functions h mapping $\mathbf{R}$ into $\mathbf{R}^{n \times p}$ satisfying $h(t) = 0$ for $t < 0$ and

$$\forall t \geq 0, \quad \int_{-\infty}^{t} \mathbf{Tr}\, h(t)^T h(t) dt < \infty.$$

Following the usage of Francis [18], we suppress the dependence of these spaces on the integers n and p.

For a given operator Δ and a function p mapping $\mathbf{R}$ into $\mathbf{R}^n$, $(\Delta p)(t)$ denotes the value taken by the image function Δp at time t. An operator Δ is said *causal* if for any function p and any time t, $(\Delta p)(t)$ only depends on the past values of p up to time t. It is said *anticausal* if $(\Delta p)(t)$ only depends on the future values of p from time t. In any other case, Δ is said *noncausal*.

Given a set of operators $\Delta_1, \ldots, \Delta_L$, $\mathbf{diag}(\Delta_1, \ldots \Delta_L)$ stands for the operator which maps the function taking the value $\begin{bmatrix} u_1(t)^T & \ldots & u_L(t)^T \end{bmatrix}^T$ at time t to the function taking the value $\begin{bmatrix} (\Delta_1 u_1)(t)^T & \ldots & (\Delta_L u_L)(t)^T \end{bmatrix}^T$ at time t.

Let H map $\mathbf{L}_2(\mathbf{R})$ into $\mathbf{L}_2(\mathbf{R})$ and be linear. The *adjoint* of H, noted H^*, is the unique linear operator satisfying:

$$< x, Hy > = < H^* x, y >, \quad \forall x, y \in \mathbf{L}_2(\mathbf{R}).$$

Defining s as the usual Laplace variable, the transfer function of H is noted $H(s)$ whenever it exists.

Let us now introduce the notions of finite-gain, positivity and passivity that we will use throughout this paper.

Definition 1.1 ([32]) *An operator F mapping $\mathbf{L}_2(\mathbf{R})$ into $\mathbf{L}_2(\mathbf{R})$ has finite gain (or, equivalently, is bounded) if there exists a positive δ such that for any $u \in \mathbf{L}_2$,*

$$\|Fu\|_2 \leq \delta \|u\|_2 .$$

The smallest such δ is called the gain of F and noted $\|F\|_\infty$.

Definition 1.2 ([10]) *A linear operator G mapping $\mathbf{L}_2(\mathbf{R})$ into $\mathbf{L}_2(\mathbf{R})$ is said positive if for any $u \in \mathbf{L}_2(\mathbf{R})$,*

$$< Gu, u > \geq 0.$$

G is said strictly positive if there exists $\delta > 0$ such that for any $u \in \mathbf{L}_2(\mathbf{R})$,

$$< Gu, u > \geq \delta \|u\|_2^2 .$$

Definition 1.3 ([10]) *A linear, causal operator G mapping $\mathbf{L}_{2e}$ into $\mathbf{L}_{2e}$ is said passive if for any $u \in \mathbf{L}_{2e}$ and $T \geq 0$,*

$$\int_0^T (Gu)^T u \, dt \geq 0.$$

2 Problem statement and line of attack

Consider the system

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + B_p p(t) + B_w w(t), \quad x(0) = x_0, \\ q(t) &= C_q x(t) + D_{qp} p(t), \\ z(t) &= C_z x(t), \\ p(t) &= (\Delta q)(t), \end{aligned} \tag{1}$$

where $x : \mathbf{R} \to \mathbf{R}^n$, $p : \mathbf{R} \to \mathbf{R}^{n_p}$, $q : \mathbf{R} \to \mathbf{R}^{n_p}$, $z : \mathbf{R} \to \mathbf{R}^{n_z}$, and $w : \mathbf{R} \to \mathbf{R}^{n_w}$ and all quantities are equal to 0 for $t < 0$. Assume that the matrix A is stable. Δ is a perturbation that satisfies the following set of assumptions:

$$\begin{aligned}
&\Delta = \mathbf{diag}(\Delta_1, \ldots, \Delta_{n_p}),\\
&\forall u \in \mathbf{L}_2(\mathbf{R}),\ \ (\Delta_i u)(t) = \int_0^\infty \delta_i(\tau) u(t-\tau)\, d\tau,\\
&\int_0^\infty |\delta_i(\tau)|\, d\tau < \infty,\\
&\Delta_i \text{ is passive },\quad i = 1, \ldots, n_p.
\end{aligned} \tag{2}$$

In the literature, the passivity assumption on Δ is often replaced by a finite-gain assumption [13]. Standard loop-transformations allow us to move almost freely from one framework to the other (see [10, p. 215] and the end of this paper for a more detailed discussion).

Much of the existing literature is devoted to studying the robust stability of the system (1) against the uncertainty Δ, in the following sense: Assume $w = 0$ and any initial condition x_0; then the signals x, p, q and z belong to $\mathbf{L}_2(\mathbf{R}_+)$.

In this paper, we assume the system (1) to be robustlty stable, and we are interested in evaluating its worst-case $\mathbf{H}_2$ performance against the uncertainty Δ: Let

$$H_\Delta(s) = H_{zw}(s) + H_{zp}(s)\Delta(s)(I - H_{qp}(s)\Delta(s))^{-1} H_{qw}(s),$$

where

$$\begin{aligned}
H_{zw}(s) &= C_z(sI - A)^{-1} B_w,\\
H_{zp}(s) &= C_z(sI - A)^{-1} B_p,\\
H_{qp}(s) &= C_q(sI - A)^{-1} B_p + D_{qp},\\
H_{qw}(s) &= C_q(sI - A)^{-1} B_w.
\end{aligned} \tag{3}$$

The $\mathbf{H}_2$ norm of the system (1) is defined as

$$\|H_\Delta\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{Tr}\, H_\Delta(j\omega)^* H_\Delta(j\omega)\, d\omega \right)^{1/2}.$$

Equivalently, using Parseval's theorem, $\|H_\Delta\|_2$ may also be expressed as $\|h_\Delta\|_2$, where h_Δ is the impulse matrix of H_Δ. In the subsequent developments of this paper, it will also be very convenient to express it as

$$\|H_\Delta\|_2 = \left(\mathbf{E}\, \|z\|_2^2 \right)^{1/2}, \tag{4}$$

where z is the output of the system (1) with the following assumptions: the input w is identically 0 and the initial condition x_0 is equal to $B_w u$, where u is a random variable satisfying $\mathbf{E}\, uu^T = I$. (The expectation appearing in (4) is therefore to be taken with respect to u.)

The robust $\mathbf{H}_2$ analysis problem is to compute the worst-case $\mathbf{H}_2$ norm of the system (1) over all possible values of Δ that satisfy (2). This computation is in general quite a complicated problem. Thus, we propose to replace it by the computation of upper bounds on robust $\mathbf{H}_2$ norm, using a technique similar to the classical technique of Lagrange multipliers: Consider any family $\mathcal{M}$ of operators M mapping $\mathbf{L}_2(\mathbf{R})$ into $\mathbf{L}_2(\mathbf{R})$ such that the operator $M^*\Delta$ is positive for any Δ satisfying (2). The following lemma gives us an *upper bound* on the worst-case $\mathbf{H}_2$ norm of the system (1).

Lemma 2.1 *We have the inequality:*

$$\max_\Delta \mathbf{E}\, \|z\|_2^2 \le \min_{M \in \mathcal{M}} \mathbf{E} \max_{\tilde{p} \in \mathbf{L}_2(\mathbf{R}_+)} \|\tilde{z}\|_2^2 + 2 < \tilde{p}, M\tilde{q} >, \tag{5}$$

where $\tilde{p}$, $\tilde{q}$ and $\tilde{z}$ are the inputs and outputs of the system

$$
\begin{aligned}
\frac{d}{dt}\tilde{x}(t) &= A\tilde{x}(t) + B_p\tilde{p}(t), \;\; \tilde{x}(0) = B_w u, \\
\tilde{q}(t) &= C_q\tilde{x}(t) + D_{qp}\tilde{p}(t) \\
\tilde{z}(t) &= C_z\tilde{x}(t)
\end{aligned}
\tag{6}
$$

where all variables belong to $\mathbf{L}_2(\mathbf{R}_+)$, and u is a random variable satisfying $\mathbf{E}\, uu^T = I$.

Proof: Consider the system (1). For any $M \in \mathcal{M}$, we have $< p, Mq >=< \Delta q, Mq >=< M^*\Delta q, q >\geq 0$, since $M^*\Delta$ is positive. Therefore, for any Δ satisfying (2), any initial condition x_0 and any $M \in \mathcal{M}$, we have $\|z\|_2^2 \leq \|z\|_2^2 + 2 < p, Mq >$. Since $p \in \mathbf{L}_2(\mathbf{R}_+)$, we furthermore have

$$
\|z\|_2^2 + 2 < p, Mq > \leq \max_{\tilde{p}\in\mathbf{L}_2(\mathbf{R}_+)} \|\tilde{z}\|_2^2 + 2 < \tilde{p}, M\tilde{q} >,
$$

where the right-hand side of the inequality may be infinite. Taking expected values (with respect to the random variable u) on both sides of these inequalities, we conclude that

$$
\mathbf{E}\,\|z\|_2^2 \leq \mathbf{E} \max_{\tilde{p}\in\mathbf{L}_2(\mathbf{R}_+)} \|\tilde{z}\|_2^2 + 2 < \tilde{p}, M\tilde{q} > .
$$

This ends the proof of our lemma. ∎

Note that the multiplier M can indeed be seen as a *Lagrange multiplier.* Such an approach is not unlike the one encountered in the papers by Yakubovich [41, 17, 42] and Megretsky [22, 23, 24], where it is named $\mathcal{S}$-procedure. In the remainder of this paper, we will show that a suitable choice of the family of multipliers $\mathcal{M}$ allows for the right-hand side of (5) to be easily computed.

3 Upper bound computation via linear families of finite-dimensional multipliers

3.1 Linear families of finite-dimensional operators

Following an idea arising in [8, 33, 9], we consider finite-dimensional, noncausal operators $M \in \mathcal{M}$, where

$$
\mathcal{M} = \left\{
\begin{array}{l}
M = \mathbf{diag}(M_1, \ldots, M_{n_p}), \\
M_i(s) = m_{i0} + \sum_{j=1}^{N} \frac{m_{ij}}{(s+1)^j} + \frac{m_{ij}}{(-s+1)^j}, \\
M_i(j\omega) \geq 0, \;\; \forall \omega \in \mathbf{R}, \\
m_{i,j} \in \mathbf{R}, \;\; 1 \leq i \leq n_p, \;\; 0 \leq j \leq N
\end{array}
\right\}.
$$

(We refer the reader to [18] for a complete discussion of the representation of noncausal operators via transfer functions with unstable poles.) Thus, the set $\mathcal{M}$ is parameterized by the real numbers m_{ij}, $i = 1, \ldots, n_p$, $j = 0, \ldots, N$. Each transfer function $M_i(s)$ may alternatively be written as $M_i(s) = C_{M_i}(sI - A_{M_i})^{-1}B_{M_i} + D_{M_i}$, where

$$
\begin{aligned}
&A_{M_i} = \begin{bmatrix} A_{\mathrm{ca}i} & 0 \\ 0 & -A_{\mathrm{ac}i} \end{bmatrix}, \quad B_{M_i} = \begin{bmatrix} B_{\mathrm{ca}i} \\ -B_{\mathrm{ac}i} \end{bmatrix}, \\
&C_{M_i} = \begin{bmatrix} C_{\mathrm{ca}i} & C_{\mathrm{ac}i} \end{bmatrix}, \qquad D_{M_i} = m_{i0},
\end{aligned}
\tag{7}
$$

and

$$
A_{\mathrm{ca}i} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 1 \\ 0 & \cdots & \cdots & 0 & -1 \end{bmatrix}, \quad A_{\mathrm{ac}i} = A_{\mathrm{ca}i}, \quad A_{\mathrm{ca}i} \in \mathbf{R}^{N\times N},
$$
$$
B_{\mathrm{ca}i} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T, \qquad B_{\mathrm{ac}i} = B_{\mathrm{ca}i},
$$
$$
C_{\mathrm{ca}i} = \begin{bmatrix} m_{iN} & \cdots & m_{i1} \end{bmatrix}, \qquad C_{\mathrm{ac}i} = C_{\mathrm{ca}i}, \quad i = 1, \ldots, n_p. \tag{8}
$$

Likewise, the transfer function $M(s)$ may also be written as $M(s) = C_M(sI - A_M)^{-1}B_M + D_M$, with

$$
A_M = \begin{bmatrix} A_{\mathrm{ca}} & 0 \\ 0 & -A_{\mathrm{ac}} \end{bmatrix}, \quad B_M = \begin{bmatrix} B_{\mathrm{ca}} \\ -B_{\mathrm{ac}} \end{bmatrix},
$$
$$
C_M = \begin{bmatrix} C_{\mathrm{ca}} & C_{\mathrm{ac}} \end{bmatrix}, \qquad D_M = \mathbf{diag}_i(m_{i0}), \tag{9}
$$

and

$$
\begin{aligned}
A_{\mathrm{ca}} &= \mathbf{diag}_i(A_{\mathrm{ca}i}), & A_{\mathrm{ac}} &= \mathbf{diag}_i(A_{\mathrm{ac}i}), \\
B_{\mathrm{ca}} &= \mathbf{diag}_i(B_{\mathrm{ca}i}), & B_{\mathrm{ac}} &= \mathbf{diag}_i(B_{\mathrm{ac}i}), \\
C_{\mathrm{ca}} &= \mathbf{diag}_i(C_{\mathrm{ca}i}), & C_{\mathrm{ac}} &= \mathbf{diag}_i(C_{\mathrm{ac}i}), \\
i &= 1, \ldots, n_p.
\end{aligned} \tag{10}
$$

To check $\mathcal{M}$ is indeed an admissible set, we must check that for any $M \in \mathcal{M}$ and any Δ satisfying (2), $M^*\Delta$ is positive. Since $M^*\Delta$ is a diagonal operator, we just need to check that $M_i^*\Delta_i$ is positive for $i = 1, \ldots, n_p$. From [10, p. 174], passivity of Δ_i is equivalent to the inequality

$$
\Delta_i(j\omega)^* + \Delta_i(j\omega) \geq 0, \quad \forall\omega \in \mathbf{R}. \tag{11}
$$

By hypothesis, $M_i(j\omega)$ is real and nonnegative. Thus, the inequality (11) implies

$$
\Delta_i(j\omega)^* M_i(j\omega) + M_i(j\omega)\Delta_i(j\omega) \geq 0. \tag{12}
$$

Thus, positivity of $M_i^*\Delta_i$ holds, since by Parseval's theorem, we have

$$
\begin{aligned}
& < u, M^*\Delta_i u > \\
= \; & \frac{1}{2\pi}\int_{-\infty}^{\infty} u(j\omega)^*(\Delta_i(j\omega)^* M_i(j\omega) + M_i(j\omega)\Delta_i(j\omega))u(j\omega)d\omega, \quad \forall u \in \mathbf{L}_2(\mathbf{R}).
\end{aligned}
$$

The inequality $M_i(j\omega) + M_i(j\omega)^* \geq 0$ for all $\omega \in \mathbf{R}$ can be expressed in a convenient form via a straightforward application of Theorems 3 and 4 of Willems [40]:

Lemma 3.1 *The inequality*

$$
M_i(j\omega) + M_i(j\omega)^* \geq 0, \quad \forall\omega \in \mathbf{R}
$$

is satisfied if and only if there exists a symmetric matrix P_i satisfying

$$
\begin{bmatrix} A_{Mi}^T P_i + P_i A_{Mi} & P_i B_{Mi} - C_{Mi}^T \\ B_{Mi}^T P_i - C_{Mi} & -(D_{Mi} + D_{Mi}^T) \end{bmatrix} \leq 0. \tag{13}
$$

Note that this lemma requires controllability of (A_{M_i}, B_{M_i}) to hold and that this assumption is indeed satisfied.

When $\tilde{q} \in \mathbf{L}_2(\mathbf{R}_+)$, a simple state-space representation of $M\tilde{q}$ can be given that will be useful in the subsequent developments in this paper:

Lemma 3.2 *For any $\tilde{q} \in \mathbf{L}_2(\mathbf{R}_+)$, we can write $M\tilde{q}$ as the output of the system*

$$\begin{aligned} \frac{d}{dt} x_M &= A_M x_M + B_M \tilde{q}, \;\; x_M(0) = \left[\begin{array}{cc} 0 & x_{\mathrm{ac}0}^T \end{array} \right]^T, \\ M\tilde{q} &= C_M x_M + D_M \tilde{q}, \end{aligned} \tag{14}$$

where $x_{\mathrm{ac}0}$ is the unique initial condition such that $\lim_{t\to\infty} x_M = 0$, given by

$$x_{\mathrm{ac}0} = \int_0^\infty e^{A_{\mathrm{ac}}\tau} B_{\mathrm{ac}} \tilde{q}(\tau) d\tau. \tag{15}$$

A proof of this lemma may be found in the appendix.

3.2 Upper bound computation

Having identified an appropriate family of multipliers $\mathcal{M}$, we can now describe a numerical implementation of the upper bound on worst-case $\mathbf{H}_2$ norm given in Lemma 2.1.

We first proceed to compute $\mathbf{E} \max_{\tilde{p} \in \mathbf{L}_2(\mathbf{R}_+)} \|\tilde{z}\|^2 + < \tilde{p}, M\tilde{q} >$ for a given operator M, where $\tilde{p}$, $\tilde{q}$, and $\tilde{z}$ satisfy (6) and $\tilde{x}(0) = B_w u$, where $\mathbf{E}\, uu^T = I$. Introduce the augmented system

$$\begin{aligned} \frac{d}{dt} \bar{x}(t) &= A_{MH} \bar{x}(t) + B_{MH} \tilde{p}(t), \;\; \bar{x}(0) = \left[\begin{array}{ccc} \tilde{x}(0)^T & 0 & x_{\mathrm{ac}0}^T \end{array} \right]^T, \\ (M\tilde{q})(t) &= C_{MH} \bar{x}(t) + D_{MH} \tilde{p}(t), \\ \tilde{z}(t) &= C_{MHz} \bar{x}(t) \end{aligned} \tag{16}$$

where $A_{MH}, B_{MH}, C_{MH}, D_{MH}, C_{MHz}$ are given by

$$A_{MH} = \left[\begin{array}{cc} A & 0 \\ B_M C_q & A_M \end{array} \right], \quad B_{MH} = \left[\begin{array}{c} B_p \\ B_M D_{qp} \end{array} \right],$$

$$C_{MH} = \left[\begin{array}{cc} D_M C_q & C_M \end{array} \right], \quad D_{MH} = D_M D_{qp},$$

$$C_{MHz} = \left[\begin{array}{ccc} C_z & 0 & 0 \end{array} \right],$$

and $x_{\mathrm{ac}0}$ is given by (15). From Lemma 3.2, computing $\max_{\tilde{p} \in \mathbf{L}_2(\mathbf{R}_+)} \|\tilde{z}\|_2^2 + 2 < \tilde{p}, M\tilde{q} >$ is equivalent to computing

$$\max_{\tilde{p} \in \mathbf{L}_2(\mathbf{R}_+)} \int_0^\infty \left[\begin{array}{c} \bar{x}(t) \\ \tilde{p}(t) \end{array} \right]^T \left[\begin{array}{cc} C_{MHz}^T C_{MHz} & C_{MH}^T \\ C_{MH} & D_{MH} + D_{MH}^T \end{array} \right] \left[\begin{array}{c} \bar{x}(t) \\ \tilde{p}(t) \end{array} \right] dt, \tag{17}$$

where $\tilde{p}$ and $\bar{x}$ satisfy (16). If the initial condition $\bar{x}(0)$ was constant, the solution to this quadratic optimal control problem could be obtained by standard methods such as the ones described in [40]. Unfortunately, this is not the case, because the noncausal multiplier M is involved, which makes $\bar{x}(0)$ depend on $\tilde{p}$ through the relation (15). In fact, assuming (A_{MH}, B_{MH}) is controllable (we will make this technical assumption from now on), $x_{\mathrm{ac}0}$ spans all of $\mathbf{R}^{Nn_p}$ as $\tilde{p}$ spans all of $\mathbf{L}_2(\mathbf{R}_+)$. Therefore, in order to compute (17) subject to the constraints (16) and (15), we propose the following 2-step strategy:

(i) Fix $x_{\mathrm{ac}0}$. Compute (17) subject to the constraints (16) and

$$\int_0^\infty e^{A_{\mathrm{ac}}\tau} B_{\mathrm{ac}} \tilde{q}(\tau) d\tau = x_{\mathrm{ac}0}.$$

(ii) Maximize the resulting solution over $x_{\mathrm{ac}0} \in \mathbf{R}^{Nn_p}$.

From Lemma 3.2, the step (i) is equivalent to computing (17) subject to the constraints (16) and $\lim_{t\to\infty} \bar{x}(t) = 0$. This is a well-known problem, whose solution is given, for example, by Willems:

Lemma 3.3 ([40], Theorem 3) *Assume that (A_{MH}, B_{MH}) is controllable. The value of (17) subject to the constraints (16) and $\lim_{t\to\infty} \bar{x}(t) = 0$ is finite if and only if there exists a symmetric matrix P satisfying the matrix inequality*

$$\begin{bmatrix} A_{MH}^T P + P A_{MH} + C_{MHz}^T C_{MHz} & P B_{MH} + C_{MH}^T \\ B_{MH}^T P + C_{MH} & D_{MH} + D_{MH}^T \end{bmatrix} \leq 0. \tag{18}$$

It is then given by $\bar{x}(0)^T P^- \bar{x}(0)$, where P^- is the smallest (in the sense of the partial ordering of symmetric matrices) among all matrices P satisfying (18).

In particular, we see that P^- is independent from the initial condition $\bar{x}(0)$. Therefore, the step (ii) is simply done by maximizing

$$\begin{bmatrix} x_0^T & 0 & x_{\mathrm{ac}0}^T \end{bmatrix} P^- \begin{bmatrix} x_0^T & 0 & x_{\mathrm{ac}0}^T \end{bmatrix}^T$$

over $x_{\mathrm{ac}0}$. Partitioning P^- as

$$P^- = \begin{bmatrix} P_{11}^- & P_{12}^- & P_{13}^- \\ P_{12}^{-T} & P_{22}^- & P_{23}^- \\ P_{13}^{-T} & P_{23}^{-T} & P_{33}^- \end{bmatrix},$$

this problem is equivalent to maximizing

$$\phi(x_{\mathrm{ac}0}) = x_0^T P_{11}^- x_0 + 2 x_0^T P_{13}^- x_{\mathrm{ac}0} + x_{\mathrm{ac}0}^T P_{33}^- x_{\mathrm{ac}0}.$$

The function ϕ is quadratic in $x_{\mathrm{ac}0}$. Thus, it has a maximum if and only if $P_{33}^- \leq 0$ and it has a stationary point $x_{\mathrm{ac}0}$, solution to the equation $P_{13}^{-T} x_0 = -P_{33}^- x_{\mathrm{ac}0}$. In this case, the maximum value of ϕ is given by

$$x_0^T (P_{11}^- - P_{13}^- P_{33}^{-\dagger} P_{13}^{-T}) x_0, \tag{19}$$

Assume now that $x_0 = B_w u$, where u is a random variable satisfying $\mathbf{E}\, u u^T = I$. Then

$$\mathbf{E} \max_{\tilde{p} \in \mathbf{L}_2(\mathbf{R}_+)} \|\tilde{z}\|_2^2 + 2 < \tilde{p}, M\tilde{q} > \tag{20}$$

is finite if and only if

$$\begin{aligned} & P_{33}^- \leq 0, \\ & \forall u \in \mathbf{R}^{n_w}, \ \exists x_{\mathrm{ac}0} \text{ such that } P_{13}^{-T} B_w u = -P_{33}^- x_{\mathrm{ac}0} \end{aligned} \tag{21}$$

Then, from (19), the value of (20) is

$$\begin{aligned} & \mathbf{E}\, u^T B_w^T (P_{11}^- - P_{13}^- P_{33}^{-\dagger} P_{13}^{-T}) B_w u \\ = \ & \mathbf{Tr}\, B_w^T (P_{11}^- - P_{13}^- P_{33}^{-\dagger} P_{13}^{-T}) B_w. \end{aligned} \tag{22}$$

It is possible to write the second condition in (21) in a more compact manner, by remarking that it is equivalent to the requirement that $P_{13}^{-T} B_w$ lie in the range of P_{33}^-, or, equivalently, in the nullspace of $I - P_{33}^- P_{33}^{-\dagger} = I - P_{33}^{-\dagger} P_{33}^-$. Thus, this condition may also be written $(I - P_{33}^{-\dagger} P_{33}^-) P_{13}^{-T} B_w = 0$.

Introducing the symmetric matrix $\Gamma \in \mathbf{R}^{n_w \times n_w}$ as a slack variable, we can also write the value of (22) together with the condition (21) as the minimum value of $\mathbf{Tr}\,\Gamma$ subject to the conditions

$$\begin{aligned} &B_w^T(P_{11}^- - P_{13}^- P_{33}^{-\dagger} P_{13}^{-T})B_w \leq \Gamma, \\ &P_{33}^- \leq 0, \\ &(I - P_{33}^{-\dagger} P_{33}^-)P_{13}^{-T} B_w = 0. \end{aligned} \tag{23}$$

Using Schur complements (see [9, p. 28] for details), it is also the minimum value of $\mathbf{Tr}\,\Gamma$ subject to the single constraint

$$\begin{bmatrix} B_w^T P_{11}^- B_w - \Gamma & B_w^T P_{13}^- \\ P_{13}^{-T} B_w & P_{33}^- \end{bmatrix} \leq 0,$$

We now remark that for a general symmetric matrix P partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix},$$

the matrix

$$X(P) = \begin{bmatrix} B_w^T P_{11} B_w & B_w^T P_{13} \\ P_{13}^T B_w & P_{33} \end{bmatrix}$$

varies monotonically with P, meaning that if $P_1 \leq P_2$, then $X(P_1) \leq X(P_2)$. Therefore, given the definition of P^- in Lemma 3.3, we may compute the value of $\mathbf{E}\max_{\tilde{p}\in\mathbf{L}_2(\mathbf{R}_+)} \|\tilde{z}\|_2^2 + 2 < \tilde{p}, M\tilde{q} >$ by minimizing $\mathbf{Tr}\,\Gamma$ over the variables P and Γ subject to the matrix constraints (18) and

$$\begin{bmatrix} B_w^T P B_w - \Gamma & B_w^T P_{13} \\ P_{13}^T B_w & P_{33} \end{bmatrix} \leq 0. \tag{24}$$

Thus the value of $\min_{M\in\mathcal{M}} \mathbf{E}\max_{\tilde{p}\in\mathbf{L}_2} \|\tilde{z}\|_2^2 + 2 < \tilde{p}, M\tilde{q} >$ is obtained by minimizing $\mathbf{Tr}\,\Gamma$ over the variables P, Γ and $M \in \mathcal{M}$ subject to the matrix constraints (18) and (24). Remarking that $M \in \mathcal{M}$ if and only if the inequality (13) holds, we can now summarize the computation of the upper bound on robust $\mathbf{H}_2$ performance in the following Theorem:

Theorem 3.1 *Consider the system (6). The quantity*

$$\min_{M\in\mathcal{M}} \mathbf{E} \max_{\tilde{p}\in\mathbf{L}_2(\mathbf{R}_+)} \|\tilde{z}\|_2^2 + 2 < \tilde{p}, M\tilde{q} >,$$

where $\tilde{z}$, $\tilde{p}$ and $\tilde{q}$ satisfy (6) is computed as the minimum of $\mathbf{Tr}\,\Gamma$ *over the variables Γ, P, $P_1, \ldots, P_{n_p}$, m_{ij}, $i = 1, \ldots, n_p$, $j = 0, \ldots, N$ satisfying the constraints*

$$\begin{bmatrix} A_{Mi}^T P_i + P_i A_{Mi} & P_i B_{Mi} - C_{Mi}^T \\ B_{Mi}^T P_i - C_{Mi} & -(D_{Mi} + D_{Mi}^T) \end{bmatrix} \leq 0, \quad i = 1, \ldots, n_p, \tag{25}$$

$$\begin{bmatrix} A_{MH}^T P + P A_{MH} + C_{MHz}^T C_{MHz} & P B_{MH} + C_{MH}^T \\ B_{MH}^T P + C_{MH} & D_{MH} + D_{MH}^T \end{bmatrix} \leq 0, \tag{26}$$

and

$$\begin{bmatrix} B_w^T P_{11} B_w - \gamma & B_w^T P_{13} \\ P_{13}^T B_w & P_{33} \end{bmatrix} \leq 0, \tag{27}$$

where

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix} \tag{28}$$

has been partitioned conformally with the dimensions of A, A_{ca} and A_{ac} (where A_{ca} and A_{ac} are given by (10)).

4 Discussion

In this section, we discuss the main result of this paper and compare it with some previous approaches.

4.1 Computational issues

We see that Theorem 3.1 gives us *effective means* to compute the upper bound on the worst-case $\mathbf{H}_2$ norm of the system (1): indeed, we have to minimize the linear objective $\mathbf{Tr}\,\Gamma$ over the variables Γ, P, $P_1, \ldots, P_{n_0}$, m_{ij}, $i = 1, \ldots, n_p$, $j = 0, \ldots, N$, which appear *linearly* in the matrix constraints (25)–(27). In particular, new interior-point convex optimization algorithms will solve this problem very efficiently [26, 9]. Note also that the size of the optimization problem grows with the dimension N of the family of multipliers used. Note finally that the solution of the optimization problem in Theorem 3.1 via interior-point methods requires that all soft inequality signs of the form $\leq$ appearing in the constraints (25)–(27) be replaced by strict inequality signs. This does not present significant problems in most practical cases. (For a detailed discussion, we refer the reader to [9, § 2.5]).

4.2 When is the obtained bound finite?

Theorem 3.1 provides an upper bound for the worst-case $\mathbf{H}_2$ norm of the system (1). However, it does not guarantee that this upper bounds is finite. Thus, it is interesting to examine cases for which this bound is guaranteed to be finite.

One such case arises when there exists $M \in \mathcal{M}$ such that $-MH_{qp}$ is strictly positive, where H_{qp} is the operator whose transfer function is given in (3). Then there exists a positive δ such that

$$\forall p \in \mathbf{L}_2(\mathbf{R}), \quad < p, MH_{qp}p > \leq -\delta < p, p > .$$

Define the impulse matrices

$$h_{zw}(t) = \begin{cases} C_z e^{At} B_w \text{ for } t \geq 0, \\ 0 \text{ otherwise} \end{cases}$$

and

$$h_{qw}(t) = \begin{cases} C_q e^{At} B_w \text{ for } t \geq 0, \\ 0 \text{ otherwise} \end{cases} .$$

Then, for any $\lambda > 0$, any initial condition $\tilde{x}(0) = B_w u$ and any $\tilde{p} \in \mathbf{L}_2(\mathbf{R}_+)$ in the system (6), we have

$$\begin{aligned} & \|\tilde{z}\|_2^2 + 2\lambda < \tilde{p}, M\tilde{q} > \\ = \; & \|h_{zw}u + H_{zp}\tilde{p}\|_2^2 + 2\lambda < \tilde{p}, MH_{qp}\tilde{p} + Mh_{qw}u > \\ \leq \; & \|h_{zw}u\|_2^2 + 2(\|h_{zw}u\|_2 \|H_{zp}\|_\infty + \lambda \|h_{qw}u\|_2 \|M\|_\infty) \|\tilde{p}\|_2 + (\|H_{zp}\|_\infty^2 - 2\lambda\delta) \|\tilde{p}\|_2^2 . \end{aligned}$$

Choosing $\lambda = (\|H_{zp}\|_\infty^2 + 1)/2\delta$, we therefore have

$$\begin{aligned} & \|\tilde{z}\|_2^2 + 2\lambda < \tilde{p}, M\tilde{q} > \\ \leq \; & \|h_{zw}u\|_2^2 + 2(\|h_{zw}u\|_2 \|H_{zp}\|_\infty + \lambda \|h_{qw}u\|_2 \|M\|_\infty) \|\tilde{p}\|_2 - \|\tilde{p}\|_2^2 \\ \leq \; & \|h_{zw}u\|_2^2 + (\|h_{zw}u\|_2 \|H_{zp}\|_\infty + \lambda \|h_{qw}u\|_2 \|M\|_\infty)^2 \\ \leq \; & \|h_{zw}u\|_2^2 + 2(\|h_{zw}u\|_2^2 \|H_{zp}\|_\infty^2 + \lambda^2 \|h_{qw}u\|_2^2 \|M\|_\infty^2). \end{aligned}$$

(Note that the quantities $||M||_\infty$ and $||H_{zp}||_\infty$ are well-defined, since the corresponding transfer functions have no poles on the imaginary axis.) Taking expected values with respect to the random variable u, we have

$$\begin{array}{rl} & \mathbf{E}\max_{\tilde{p}\in\mathbf{L}_2(\mathbf{R}_+)} ||\tilde{z}||_2^2 + 2\lambda < \tilde{p}, M\tilde{q} > \\ \leq & ||h_{zw}||_2^2 + 2(||h_{zw}||_2^2\,||H_{zp}||_\infty^2 + \lambda^2\,||h_{qw}||_2^2\,||M||_\infty^2). \end{array}$$

Noting that $M \in \mathcal{M}$ implies $\lambda M \in \mathcal{M}$, we conclude our upper bound is finite.

It is interesting to remark that the strict positivity of $-MH_{qp}$ is one of the conditions used in the classical theory of stability multipliers to prove stability of the system (1) as described in [10, p. 203]. Thus, whenever stability of the system (1) can be proved via stability multipliers, then we can provide finite bounds on its worst-case $\mathbf{H}_2$ performance. Note that numerical methods involving linear matrix inequalities to prove robust stability of the system (1) using linear families of finite-dimensional multipliers may be found in [3, 35].

4.3 Special cases and comparison with earlier results

In this section, we investigate what happens when considering special cases of the system (1) and of the multiplier M.

Consider first the case when the system (1) is perfectly known, that is: $B_p = 0, C_q = 0, D_{qp} = 0$. The optimization problem in Theorem 3.1 is solved by choosing $m_{ij} = 0$ for all i and j, $P_i = 0$ for all i,

$$P = \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where P_{11} satisfies $A^T P_{11} + P_{11}A + C_z^T C_z = 0$, and $\Gamma = B_w^T P B_w$. P_{11} is the *observability Gramian* and the obtained bound is then exact.

Second, consider the case when $N = 0$ and $n_p = 1$, that is, when the multiplier $M = m_{10} = m$ is simply a nonnegative scalar. Then, applying Theorem 3.1 leads to the computation of

$$\min_{m\geq 0} \mathbf{E} \max_{p\in\mathbf{L}_2(\mathbf{R}_+)} ||\tilde{z}||^2 + 2m < \tilde{p}, \tilde{q} > \tag{29}$$

via convex programming and linear matrix inequalities.

This special case in which scalar, memoryless multipliers are used has already appeared in the literature, for example in the papers by Stoorvogel [37, 38], although in a different format: in these papers, the problem under consideration is to compute the worst-case $\mathbf{H}_2$ norm of the system

$$\begin{aligned} \frac{d}{dt}x(t) &= \hat{A}x(t) + \hat{B}_p\hat{p}(t) + B_w w(t), \quad x(0) = x_0, \\ \hat{q}(t) &= \hat{C}_q x(t) + \hat{D}_{qp}\hat{p}(t), \\ z(t) &= C_z x(t), \end{aligned} \tag{30}$$

when $\hat{p}$ and $\hat{q} \in \mathbf{L}_2(\mathbf{R}_+)$ are subject to the constraint

$$||\hat{p}||_2^2 \leq ||\hat{q}||_2^2. \tag{31}$$

Stoorvogel obtains an upper bound on the worst-case $\mathbf{H}_2$ norm for this system by relaxing the constraint (31) and by computing

$$\min_{m\geq 0} \mathbf{E} \max_{\hat{p}\in\mathbf{L}_2(\mathbf{R}_+)} ||z||_2^2 + 2m(||\hat{q}||_2^2 - ||\hat{p}||_2^2), \tag{32}$$

where z, $\hat{p}$, and $\hat{q}$ satisfy (30), subject to the boundary conditions $x(0) = B_w u$, $w(t) = 0$ and u is a random variable satisfying $\mathbf{E}\,uu^T = I$. This formulation can be cast in our framework the following way: Introduce the scattering variables $q = (\hat{p}+\hat{q})/2$ and $p = -(\hat{p}-\hat{q})/2$. Then, following the same reasoning as in [10, p. 215], the system (30) and the constraint (31) can also be written as

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + B_p p(t) + B_w w(t), \quad x(0) = x_0, \\ q(t) &= C_q x(t) + D_{qp} p(t), \\ z(t) &= C_z x(t), \end{aligned} \tag{33}$$

and

$$< p, q > \geq 0, \tag{34}$$

where A, B_p, C_q and D_{qp} are determined from $\hat{A}$, $\hat{B}_p$, $\hat{C}_q$ and $\hat{D}_{qp}$ via the relations

$$\begin{aligned} &A = \hat{A} + \hat{B}_p(I - \hat{D}_{qp})^{-1}\hat{C}_q, \quad && B_p = -2\hat{B}_p(I - \hat{D}_{qp})^{-1} \\ &C_q = (I - \hat{D}_{qp})^{-1}C_q && D_{qp} = -(I + \hat{D}_{qp})(I - \hat{D}_{qp})^{-1}. \end{aligned} \tag{35}$$

(These relations are valid if and only if $I - \hat{D}_{qp}$ is invertible.) The upper bound (32) can then be written as

$$\min_{m\geq 0} \mathbf{E} \max_{p\in \mathbf{L}_2(\mathbf{R}_+)} \|z\|_2^2 + 2m < p, q >, \tag{36}$$

where z, p, and q satisfy (33), subject to the boundary conditions $x(0) = B_w u$, $w(t) = 0$ and u is a random variable satisfying $\mathbf{E}\, uu^T = I$. But then the quantity (36) is the same as our special case (29). Thus, the problem considered in [37, 38] involves perturbations that are more general than ours (since Δ passive and $p = \Delta q$ implies $< p, q > \geq 0$ but the converse may not hold), but the resulting bounds on $\mathbf{H}_2$ performance are always larger than ours, because the approach taken in [37, 38] is in fact a special case of our approach. Similar comments may be made about the results presented in [30, 9, 15, 20].

5 Example

In this section, we present an example to illustrate the developed method, and compare it with earlier results. We consider the system (1) with

$$A = \begin{bmatrix} -0.1 & -0.1 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2 & -0.2 & 0 \\ 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \\ 0 \\ 0.001 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 11.98 & 0 & 0.01 & 0 & 20 \end{bmatrix}, \qquad D_{qp} = -12,$$

$$C_z = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

and the perturbation Δ is any passive, linear, time-invariant and single-input, single-output system. It is easy to check (via a Nyquist plot, for example) that the transfer function $-H(s) = -C_q(sI - A)^{-1}B_p - D_{qp}$ has positive dissipation, such that by application of the Passivity Theorem [10], the system (1) is stable. Using the software described in [19, 25], we have plotted the bounds on the square of its $\mathbf{H}_2$ norm as a function of N, ($2N$ is the order of the noncausal multiplier which is used). Thus, $N = 0$ corresponds to the use of simple constant-gain, memoryless multipliers. As can be seen, the use of dynamic, noncausal multipliers improves the estimate on the square of the worst-case $\mathbf{H}_2$ norm by a factor of 5. Note also that the best upper bound converges to a steady state value quite fast with the size of the multiplier. This result is indeed obtained at the expense of increased computations.

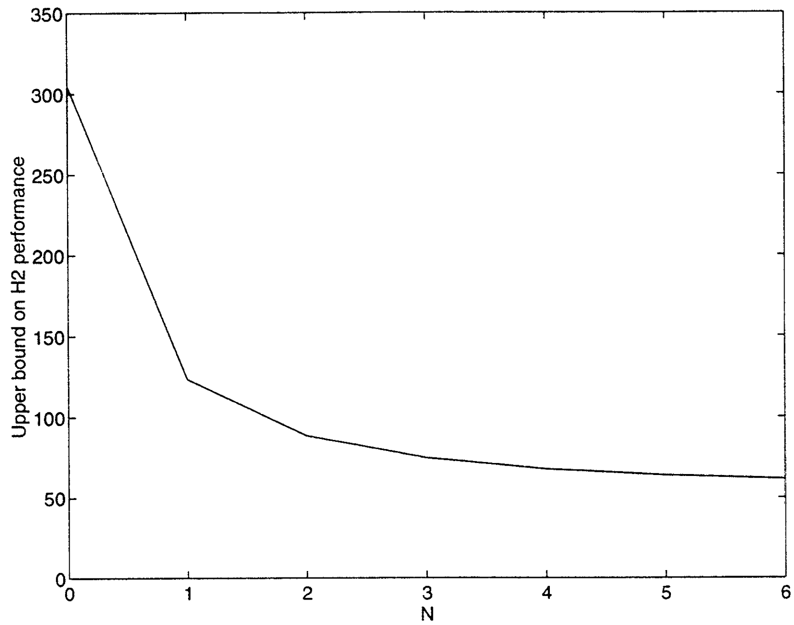

Figure 1: Upper bound on worst performance as a function of multiplier order

6 Conclusion and extensions

In this paper, we have considered the problem of determining an upper bound for the worst-case $\mathbf{H}_2$ norm of linear systems subject to linear time-invariant uncertainties, by extending the theory of stability multipliers to handle $\mathbf{H}_2$ performance.

We have shown this bound appears as the solution of a convex optimization problem involving linear matrix inequalities. Thus there exist algorithms that will compute it fast and accurately.

We have shown this bound is always sharper than the ones devised earlier for larger classes of uncertainties. One example shows that this improvement can be significant.

This work can be extended in many directions. For example, the theory of stability multipliers has proven to be effective not only on linear time-invariant perturbations, but also other classes of uncertainties, including memoryless, sector-bounded and monotonic nonlinearities, or constant, unknown linear gains (parametric uncertainties). Thus, this paper could be easily extended to these cases (with the restriction that $\mathbf{H}_2$ norms of nonlinear systems require careful definition). The set of allowable multipliers $\mathcal{M}$ would then be different.

Appendix: Proof of Lemma 3.2

Using inverse Fourier transforms, it is easy to show that for any $\tilde{q} \in \mathbf{L}_2(\mathbf{R})$, we have

$$(M\tilde{q})(t) = \int_{-\infty}^{t} C_{\rm ca} e^{A_{\rm ca}(t-\tau)} B_{\rm ca}\tilde{q}(\tau)d\tau + D_M\tilde{q}(t) + \int_{t}^{\infty} C_{\rm ac} e^{A_{\rm ac}(\tau-t)} B_{\rm ac}\tilde{q}(\tau)d\tau. \tag{37}$$

Thus, $(M\tilde{q})(t)$ is the sum of three parts: the first integral accounts for the causal part of H, the midterm represents a possible feedthrough term, and the second integral accounts for the anticausal part of M (this is the reason for using the subscripts 'ca' and 'ac', which stand for 'causal' and 'anticausal' respectively).

When $\tilde{q} \in \mathbf{L}_2(\mathbf{R}_+)$, the first integral in (37) is easily computed as $r_{\rm ca}(t)$, the output of the system

$$\begin{aligned} \frac{d}{dt}x_{\rm ca}(t) &= A_{\rm ca}x_{\rm ca}(t) + B_{\rm ca}\tilde{q}(t), \quad x_{\rm ca}(0) = 0, \\ r_{\rm ca}(t) &= C_{\rm ca}x_{\rm ca}(t). \end{aligned}$$

The second integral can be transformed the following way:

$$\begin{aligned}
& \int_t^\infty C_{\mathrm{ac}} e^{A_{\mathrm{ac}}(\tau - t)} B_{\mathrm{ac}} \tilde{q}(\tau) d\tau \\
= & \int_0^\infty C_{\mathrm{ac}} e^{A_{\mathrm{ac}}(\tau - t)} B_{\mathrm{ac}} \tilde{q}(\tau) d\tau - \int_0^t C_{\mathrm{ac}} e^{A_{\mathrm{ac}}(\tau - t)} B_{\mathrm{ac}} \tilde{q}(\tau) d\tau \\
= & \; C_{\mathrm{ac}} \left[e^{-A_{\mathrm{ac}} t} \int_0^\infty e^{A_{\mathrm{ac}} \tau} B_{\mathrm{ac}} \tilde{q}(\tau) d\tau - \int_0^t e^{-A_{\mathrm{ac}}(t-\tau)} B_{\mathrm{ac}} \tilde{q}(\tau) d\tau \right].
\end{aligned}$$

Therefore, defining

$$x_{\mathrm{ac}0} = \int_0^\infty e^{A_{\mathrm{ac}} \tau} B_{\mathrm{ac}} \tilde{q}(\tau) d\tau,$$

($x_{\mathrm{ac}0}$ is always defined and finite, since $\tilde{q} \in \mathbf{L}_2(\mathbf{R}_+)$ and A_{ac} is stable), the second integral term in (37) can be written as $r_{\mathrm{ac}}(t)$, the output of the system

$$\begin{aligned}
\frac{d}{dt} x_{\mathrm{ac}}(t) &= -A_{\mathrm{ac}} x_{\mathrm{ac}}(t) - B_{\mathrm{ac}} \tilde{q}(t), \quad x_{\mathrm{ac}}(0) = x_{\mathrm{ac}0}, \\
r_{\mathrm{ac}}(t) &= C_{\mathrm{ac}} x_{\mathrm{ac}}(t).
\end{aligned}$$

Thus the expression (14) for $(M\tilde{q})(t)$. The fact that $\lim_{t\to\infty} x_{\mathrm{ca}}(t) = 0$ is a direct consequence of the fact that A_{ca} is stable; the fact that $\lim_{t\to\infty} x_{\mathrm{ac}}(t) = 0$ follows from the identity

$$x_{\mathrm{ac}}(t) = \int_t^\infty e^{A_{\mathrm{ac}}(\tau - t)} B_{\mathrm{ac}} \tilde{q}(\tau) d\tau.$$

Conversely, consider the system (14), and let $x_{\mathrm{ac}0}$ be such that $\lim_{t\to\infty} x_{\mathrm{ac}}(t) = 0$. Then

$$x_{\mathrm{ac}}(t) = e^{-A_{\mathrm{ac}} t} x_{\mathrm{ac}0} - \int_0^t e^{-A_{\mathrm{ac}}(t-\tau)} \tilde{q}(\tau) d\tau.$$

If $\lim_{t\to\infty} x_{\mathrm{ac}}(t) = 0$, then $\lim_{t\to\infty} e^{A_{\mathrm{ac}} t} x_{\mathrm{ac}}(t) = 0$. Therefore, from the above equality, we must have

$$\lim_{t\to\infty} \int_0^t e^{A_{\mathrm{ac}} \tau} \tilde{q}(\tau) d\tau = x_{\mathrm{ac}0},$$

which proves $x_{\mathrm{ac}0}$ is indeed unique. ∎

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