January, 1995

LIDS- P 2283

Research Supported By:

ARO grant DAAL03-92-G-0115
AFOSR grant F49620-92-J-0002
NSF grant MIP-9015281

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LIDS-P-2283
January 1995

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"Permission to publish this abstract separately is granted"

EDICS: SP 3.5 and SP 3.8

Abstract

Orthogonal projection on vector subspaces arizes in many applied fields. The common assumption about the orthogonal complementary subspace is that it is spanned by white noise components. We extend some previously derived closed form recursive expansion formulas of such operators to an arbitrary correlated noise field. The simplicity of the results is not only insightful but potentially very powerful for many applications.

*The work of the first and third authors is partially supported by grants from ARO (DAAL03-92-G-0115), AFOSR (F49620-92-J-0002), and NSF (MIP-9015281).

1 Introduction

The analysis of a variety of applied problems invokes finite dimensional operators (matrices) on account of their compactness and their algebraic simplicity. The analysis of these operators, as well as the analysis of the more general infinite dimensional operators, have been for the most part, carried out in a deterministic setting.

In spectral analysis, in contrast, powerful tools which are based on eigenanalysis and which have emerged over the last fifteen years, are fundamentally statistical. They use the eigen factorization of the covariance function of a stationary random process together with the orthogonality of two complementary vector subspaces in a Hilbert space, to uncover hidden harmonics [1, 2, 3].

The spectral analysis techniques which are of interest here, are parametric and assume a well defined structure of the underlying spectrum of the operator, to result in an improved resolution. An error term is induced when estimating the covariance matrix, and results in a deviation of estimated parameters. The theoretical prediction of this deviation relies on a perturbation analysis of the projection operators onto corresponding invariant subspaces [4].

In array processing, various analyses have been carried out, and the most frequenlty encountered being that which assumes an additive white gaussian error term. A number of specific eigen-based estimation algorithms were considered, and their properties derived [5, 6, 4, 7, 8, 9, 10]. In [4, 7], an approach based on the perturbation of the projection operators onto the invariant subspaces, was proposed. This technique was also statistical and by avoiding the use of specific individual eigenvectors, resulted in greater simplification and efficiency, and provided much more insight than was previously possible. The perturbation study was, however, restricted to a white noise error term, and scenarios deviating from such an assumption (e.g. array sensors exhibiting some mutual coupling), would require a more general noise field model.

In this correspondence, we extend some of the previous perturbation results [4] and generalize the noise field model to that of a correlated one.

Section 2 covers the relevant background and the statement of the problem. We give our main result in Section 3 and conlude with some remarks in Section 4.

2 Background

2.1 Formulation

We let the space of observations $\Xi \in \mathcal{C}^L$ be a Hilbert space endowed with an inner product which provides the usual norm,

$$\| \mathbf{u} \| = < \mathbf{u}, \mathbf{u} >^{1/2} .$$

An observed vector $\mathbf{x}(t) \in \Xi$ is written as,

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad t = 1, \cdots, T \tag{1}$$

where $\mathbf{A} \in \mathcal{C}^{L \times 1}$ and restricted to the space of Vandermonde matrices, $\mathbf{s}(t) \in \mathcal{C}^{D \times 1}$, and $\mathbf{n}(\mathrm{t}) \in \mathcal{C}^{\mathrm{L} \times 1}$ respectively represent the signal and noise vectors. The observed process $\mathbf{x}(t)$ is assumed to be a normal random process $N(0, \mathbf{R})$, and circular (i.e. $\mathrm{E}\{\mathbf{x}(t)\mathbf{x}(t)^T\} = 0$, $\mathrm{E}\{\mathbf{x}(t)\mathbf{x}(t)^H\} = \mathbf{R}$), where $(\cdot)^T, (\cdot)^H$ respectively denote transposition and conjugate transposition, i.e. Hermitian transpose). The invariant subspaces of Ξ, can be obtained by way of an eigen decomposition of the Hermitian covariance matrix $\mathbf{R}$,

$$\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H + \mathbf{V}\mathbf{\Gamma}\mathbf{V}^H = \mathbf{A}\mathbf{P}\mathbf{A}^H + \mathbf{N}, \tag{2}$$

with $\mathbf{P} = \mathrm{E}\{\mathbf{s}(t)\mathbf{s}(t)^H\}$. Note that $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_D\}$ and $\mathbf{\Gamma} = \mathrm{diag}\{\gamma_{D+1}, \cdots, \gamma_L\}$, are real and positive, and it is assumed that $\lambda_i \neq \lambda_j, \forall i \neq j$ and $\gamma_{i'} \neq \gamma_{j'} \forall i' \neq j'$. In signal analysis, the subspace spanned by the columns of $\mathbf{U}$ is referred to as the *signal subspace*, and that spanned by the columns of $\mathbf{V}$ is the *noise subspace.* Denoting $\mathbf{S} = \mathbf{A}\mathbf{P}\mathbf{A}^H$ and $\mathbf{N} = \mathbf{V}\mathbf{\Gamma}\mathbf{V}^H$, we point out that these matrices are not observable. It is, nevertheless, possible in some cases (additive white noise), to isolate the two subspaces by using the physics of a given problem.[1] In what follows, we carry out an analysis of projectors onto signal and noise subspaces, or $\mathbf{S}$, $\mathbf{N}$.

[1] Since the eigenvalues represent the power of the components present in the process, in some sense, the variances of $\mathbf{n}(t)$ (in practical cases, this represents noise) will exhibit a smaller magnitude than that of their counterpart which are the variances of the components of $\mathbf{s}(t)$ (signal in practice).

2.2 Problem statement

As forementioned, one commonly makes use of an estimate $\widehat{\mathbf{R}}$ of a covariance matrix $\mathbf{R}$. By denoting all nonexact expressions by a $\hat{}$, we can write, with $\mathbf{\Delta R} \in \mathcal{C}^{L \times L}$, and $\| \mathbf{\Delta R} \|$ small, relative to $\mathbf{R}$,

$$\widehat{\mathbf{R}} = \mathbf{R} + \mathbf{\Delta R}, \tag{3}$$

and defining a projection operator on the column space of $\mathbf{S}$ (respectively $\mathbf{N}$) as,

$$\begin{aligned} \widehat{\mathbf{\Pi}} &= \widehat{\mathbf{U}}\widehat{\mathbf{U}}^H, \\ \widehat{\mathbf{\Pi}}^{\perp} &= \widehat{\mathbf{V}}\widehat{\mathbf{V}}^H, \end{aligned} \tag{4}$$

we proceed to derive the resulting perturbation term $\mathbf{\Delta\Pi}$ of $\widehat{\mathbf{\Pi}}$ (respec. $\mathbf{\Delta\Pi}^{\perp}$ of $\widehat{\mathbf{\Pi}}^{\perp}$) by performing a series expansion of $\widehat{\mathbf{\Pi}}$ (respec. $\widehat{\mathbf{\Pi}}^{\perp}$) around $\mathbf{\Pi}$,

$$\widehat{\mathbf{\Pi}} = \mathbf{\Pi} + \mathbf{\Delta\Pi} = \mathbf{\Pi} + \delta\mathbf{\Pi} \cdots \delta^n\mathbf{\Pi} + \cdots, \tag{5}$$

where $\delta^n\mathbf{\Pi}$ denotes the n^{th} order term of the expansion of $\widetilde{\mathbf{\Pi}}$ with respect to $\mathbf{\Delta R}$.

Given that the sum of two orthogonal projection operators is identity $\mathbf{I}$, we make the following, *Claim*: $\mathbf{\Delta\Pi} = -\mathbf{\Delta\Pi}^{\perp}$.

Proof:

$$\widehat{\mathbf{\Pi}}^{\perp} = (\mathbf{I} - \widehat{\mathbf{\Pi}}) = (\mathbf{I} - \mathbf{\Pi}) - \mathbf{\Delta\Pi} = \mathbf{\Pi}^{\perp} - \mathbf{\Delta\Pi} = \mathbf{\Pi}^{\perp} + \mathbf{\Delta\Pi}^{\perp}$$

■

The analysis in the next section also makes extensive use of the following properties characteristic of projection operators:

Properties **P1** Idempotence: $\widehat{\mathbf{\Pi}}\widehat{\mathbf{\Pi}} = \widehat{\mathbf{\Pi}}$, **P2** Commutativity: $\widehat{\mathbf{\Pi}}\widehat{\mathbf{R}} = \widehat{\mathbf{R}}\widehat{\mathbf{\Pi}}$
P3 $\delta^n\mathbf{\Pi}$ is hermitian, as a result of $\widehat{\mathbf{\Pi}}$ being hermitian

3 Perturbation Analysis

3.1 General Noise Field

Assuming the perturbation model given in Eq. 3, our goal is to evaluate the induced error on the operators $\widetilde{\mathbf{\Pi}}$ and $\widetilde{\mathbf{\Pi}}^{\perp}$. The above properties will turned out very useful in carrying out the following analysis. Letting $\odot$ denote the Hadamard product, we state the following,

Theorem 1 *The expansion terms of* $\widehat{\mathbf{\Pi}}$ *and* $\widehat{\mathbf{\Pi}}^{\perp}$ *resulting from a perturbation term* $\mathbf{\Delta R}$ *of a Hermitian matrix* $\mathbf{R}$ *are related by the following recurrence:*

$$\begin{aligned}
\delta\mathbf{\Pi} &= \mathbf{U}\,\mathbf{\Sigma}\mathbf{V}^H + \mathbf{V}\mathbf{\Sigma}^H\mathbf{U}^H \\
\delta^n\mathbf{\Pi} &= \mathbf{U}\,\Sigma_{n-1}\mathbf{V}^H + \mathbf{V}\,\mathbf{\Sigma}_{n-1}^H\mathbf{U}^H \\
&\quad -\mathbf{\Pi}\,(\nabla\mathbf{\Pi}^T)\,(\mathbf{J}\nabla\mathbf{\Pi})^H\mathbf{\Pi} + \mathbf{\Pi}^{\perp}\,(\Delta\mathbf{\Pi}^T)\,(\mathbf{J}\Delta\mathbf{\Pi})^H\,\mathbf{V}^H\mathbf{\Pi}^{\perp},
\end{aligned}$$

where $\nabla\mathbf{\Pi} = [\delta\mathbf{\Pi}, \delta^2\mathbf{\Pi}, \cdots, \delta^{n-1}\mathbf{\Pi}]^T$, *and* $n > 1$
the matrix $\mathbf{\Sigma} = \mathbf{U}^H\mathbf{\Delta R V} \odot \mathbf{H}$ *and*
$\Sigma_{n-1} = \left(\mathbf{U}^H\delta^{n-1}\mathbf{\Pi}\mathbf{\Delta R V} - \mathbf{U}^H\mathbf{\Delta R}\delta^{n-1}\mathbf{\Pi V}\right) \odot \mathbf{H}$ *with* $\mathbf{H} = [H_{ij}] = [\frac{1}{\lambda_i - \gamma_j}]$,
and $\mathbf{J}$ *is the* $(n-1)L \times (n-1)L$ *block exchange matrix.*

Proof: First note that we can write,

$$\begin{aligned}
\delta^n\mathbf{\Pi} &= \left(\mathbf{\Pi} + \mathbf{\Pi}^{\perp}\right)(\delta^n\mathbf{\Pi})\left(\mathbf{\Pi} + \mathbf{\Pi}^{\perp}\right) \\
&= \mathbf{\Pi}\delta^n\mathbf{\Pi}\mathbf{\Pi} + \mathbf{\Pi}^{\perp}\delta^n\mathbf{\Pi}\mathbf{\Pi}^{\perp} + \mathbf{\Pi}\delta^n\mathbf{\Pi}\mathbf{\Pi}^{\perp} + \mathbf{\Pi}^{\perp}\delta^n\mathbf{\Pi}\mathbf{\Pi},
\end{aligned} \tag{6}$$

where the first two terms of Eq. (6) are obtained by the idempotence property, while the other two terms are derived using the commutativity property. Using the idempotence property $\widehat{\mathbf{\Pi}}\widehat{\mathbf{\Pi}} = \widehat{\mathbf{\Pi}}$, we obtain

$$(\mathbf{\Pi} + \delta\mathbf{\Pi} + \cdots)(\mathbf{\Pi} + \delta\mathbf{\Pi} + \cdots) = \mathbf{\Pi} + (\delta\mathbf{\Pi})\mathbf{\Pi} + \mathbf{\Pi}(\delta\mathbf{\Pi}) + \cdots,$$

which, by equating terms of corresponding first order, results in,

$$\delta\mathbf{\Pi} = \mathbf{\Pi}(\delta\mathbf{\Pi}) + (\delta\mathbf{\Pi})\mathbf{\Pi}. \tag{7}$$

Premultiplying and postmultiplying Equation (7) respectively by $\mathbf{\Pi}$ leads to,

$$\begin{aligned} \mathbf{\Pi}(\delta\mathbf{\Pi})\mathbf{\Pi} &= \mathbf{\Pi}(\delta\mathbf{\Pi})\mathbf{\Pi} + \mathbf{\Pi}(\delta\mathbf{\Pi})\mathbf{\Pi} \;\; \text{or,} \\ \mathbf{\Pi}(\delta\mathbf{\Pi})\mathbf{\Pi} &= 0. \end{aligned} \tag{8}$$

We can similarly show that, $\mathbf{\Pi}^{\perp}(\delta\mathbf{\Pi})\mathbf{\Pi}^{\perp} = 0$. To evaluate the expressions of the expansion terms, we call upon the commuting property, $\widehat{\mathbf{\Pi}}\widehat{\mathbf{R}} = \widehat{\mathbf{R}}\widehat{\mathbf{\Pi}}$, to derive the following,

$$\begin{aligned} (\mathbf{\Pi} + \delta\mathbf{\Pi})(\mathbf{R} + \mathbf{\Delta R}) &= (\mathbf{R} + \mathbf{\Delta R})(\mathbf{\Pi} + \delta\mathbf{\Pi}) \\ (\delta\mathbf{\Pi})\mathbf{R} + \mathbf{\Pi}\mathbf{\Delta R} &= \mathbf{\Delta R}\mathbf{\Pi} + \mathbf{R}(\delta\mathbf{\Pi}). \end{aligned} \tag{9}$$

The expression of $\delta\mathbf{\Pi}$ in terms of $\mathbf{\Pi}$, $\mathbf{\Pi}^{\perp}$ and $\mathbf{\Delta R}$, can now be arrived at by a simple algebraic premultiplication by $\mathbf{U}^H$ and postmultiplication by $\mathbf{V}$:

$$\begin{aligned} \mathbf{U}^H(\delta\mathbf{\Pi})\mathbf{R} + \mathbf{U}^H\mathbf{\Delta R} &= \mathbf{U}^H\mathbf{\Delta R}\mathbf{\Pi} + \mathbf{\Lambda}\mathbf{U}^H(\delta\mathbf{\Pi}), \;\; \text{and} \\ \mathbf{U}^H(\delta\mathbf{\Pi})\mathbf{V}\mathbf{\Gamma} + \mathbf{U}^H\mathbf{\Delta R}\mathbf{V} &= \mathbf{\Lambda}\mathbf{U}^H(\delta\mathbf{\Pi})\mathbf{V}, \end{aligned} \tag{10}$$

where the fact that $\mathbf{RU} = \mathbf{U\Lambda}$ and $\mathbf{RV} = \mathbf{V\Gamma}$ was used. The latter equation can hence be rewritten as,

$$\mathbf{\Lambda}\mathbf{U}^H(\delta\mathbf{\Pi})\mathbf{V} - \mathbf{U}^H(\delta\mathbf{\Pi})\mathbf{V}\mathbf{\Gamma} = \mathbf{U}^H\mathbf{\Delta R}\mathbf{V}, \tag{11}$$

which in turn, results in,

$$\text{Vec}\{\mathbf{U}^H(\delta\mathbf{\Pi})\mathbf{V}\} = [(-\mathbf{\Gamma}) \oplus \mathbf{\Lambda}]^{-1}\text{Vec}\{\mathbf{U}^H\mathbf{\Delta R}\mathbf{V}\}, \tag{12}$$

where we used the fact that $\mathbf{AX} + \mathbf{XB} = \mathbf{C}$, if $\mathbf{A}$ is $(n \times n)$, $\mathbf{B}$, $(m \times m)$ and $\mathbf{X}$ is $(n \times m)$, has the following solution: $(\mathbf{B}^T \oplus \mathbf{A})\text{Vec}\{\mathbf{X}\} = \text{Vec}\{\mathbf{C}\}$, with $\oplus$ denoting a direct sum, and $\text{Vec}(\cdot)$ is an operator which concatenates columns of a matrix into a vector. This can also be more simply expressed as

$$\mathbf{U}^H(\delta\mathbf{\Pi})\mathbf{V} = \mathbf{U}^H\mathbf{\Delta R}\mathbf{V} \odot \mathbf{H},$$

where $\mathbf{H} = [H_{ij}] = [\frac{1}{\lambda_i - \gamma_j}]$. The derivation for the second and higher order terms is given in the appendix. ∎

3.2 White Noise Field

It is clear that another choice of complementary linear manifold (e.g. different noise structure such as white noise with a scaled identity for a covariance matrix) will lead to the scenario in [4] and restated here as a corollary:

Corollary 1 *The previous expansion of* $\widehat{\mathbf{\Pi}}$ *and* $\widehat{\mathbf{\Pi}}^{\perp}$*, for a white noise case, i.e.* $\mathbf{R} = \mathbf{S} + \sigma^2\mathbf{I}$ *simplifies to the following:*

$$
\begin{aligned}
\delta\mathbf{\Pi} &= \mathbf{\Pi}^{\perp}\,\Delta\mathbf{R}\,\mathbf{S}^{\sharp} + \mathbf{S}^{\sharp}\,\Delta\mathbf{R}\,\mathbf{\Pi}^{\perp}, \\
\delta^n\mathbf{\Pi} &= -\mathbf{\Pi}^{\perp}\,(\delta^{n-1}\mathbf{\Pi})\,\Delta\mathbf{R}\,\mathbf{S}^{\sharp} + \mathbf{\Pi}^{\perp}\Delta\mathbf{R}\,(\delta^{n-1}\mathbf{\Pi})\,\mathbf{S}^{\sharp} \\
&\quad -\mathbf{S}^{\sharp}\,\Delta\mathbf{R}\,(\delta^{n-1}\mathbf{\Pi})\,\mathbf{\Pi}^{\perp} + \mathbf{S}^{\sharp}\,(\delta^{n-1}\mathbf{\Pi})\,\Delta\mathbf{R}\,\mathbf{\Pi}^{\perp} \\
&\quad -\mathbf{\Pi}\,(\nabla\mathbf{\Pi}^T)\,(\mathbf{J}\nabla\mathbf{\Pi})\,\mathbf{\Pi} + \mathbf{\Pi}^{\perp}\,(\nabla\mathbf{\Pi}^T)\,(\mathbf{J}\nabla\mathbf{\Pi})\,\mathbf{\Pi}^{\perp}\, n > 1.
\end{aligned}
$$

where $\nabla\mathbf{\Pi} = [\delta\mathbf{\Pi}, \delta^2\mathbf{\Pi}, \cdots, \delta^{n-1}\mathbf{\Pi}]^T$, $\mathbf{J}$ *is the* $(n-1)L \times (n-1)L$ *block exchange matrix, and* $\mathbf{S}^{\sharp}$ *is the pseudo inverse of* $\mathbf{S}$.

Proof:(Direct application of theorem and details are in [4].) ■

4 Conclusion

We have generalized some previous closed form expressions for the expansion of eigen projection operators of Hermitian matrices. Recurrence formulae have derived, allowing one to obtain fairly easily an expansion term of any order, thereby facilitating their application to any subspace-based problem in spectral analysis.

Appendix

Second and higher order perturbation terms

Equating the n^{th} order terms of $\widehat{\mathbf{\Pi}}\widehat{\mathbf{\Pi}} = \widehat{\mathbf{\Pi}}$ yields:

$$\sum_{j=1}^{n-1} \delta^{n-j}\mathbf{\Pi}\delta^{j}\mathbf{\Pi} + \mathbf{\Pi}\delta^{n}\mathbf{\Pi} + \delta^{n}\mathbf{\Pi}\mathbf{\Pi} = \delta^{n}\mathbf{\Pi}, \tag{13}$$

which, by using the idempotence property, leads to

$$\left\{\begin{array}{rcl} \mathbf{\Pi}\delta^n\mathbf{\Pi}\mathbf{\Pi} & = & -\mathbf{\Pi}\left(\sum_{j=1}^{n-1}\delta^{n-j}\mathbf{\Pi}\delta^j\mathbf{\Pi}\right)\mathbf{\Pi} \\ \mathbf{\Pi}^\perp\delta^n\mathbf{\Pi}\mathbf{\Pi}^\perp & = & \mathbf{\Pi}^\perp\left(\sum_{j=0}^{n-1}\delta^{n-j}\mathbf{\Pi}\delta^j\mathbf{\Pi}\right)\mathbf{\Pi}^\perp. \end{array}\right. \tag{14}$$

Let us consider only the n^{th} order terms in the following equation,

$$\begin{aligned} \widehat{\mathbf{\Pi}}\widehat{\mathbf{R}} &= \widehat{\mathbf{R}}\widehat{\mathbf{\Pi}}, \text{ thus,} \\ \delta^{n-1}\mathbf{\Pi}\mathbf{\Delta}\mathbf{R} + \delta^n\mathbf{\Pi}\mathbf{R} &= \mathbf{\Delta}\mathbf{R}\delta^{n-1}\mathbf{\Pi} + \mathbf{R}\delta^n\mathbf{\Pi} \end{aligned} \tag{15}$$

Premultiplying the above equation by $\mathbf{U}^H$ and postmultiplying by $\mathbf{V}$ results in,

$$-\mathbf{U}^H\delta^n\mathbf{\Pi}\mathbf{V}\mathbf{\Gamma} + \mathbf{\Lambda}\mathbf{U}^H\delta^n\mathbf{\Pi}\mathbf{V} = \mathbf{U}^H\delta\mathbf{\Pi}^{n-1}\mathbf{\Delta}\mathbf{R}\mathbf{V} - \mathbf{U}^H\mathbf{\Delta}\mathbf{R}\delta^{n-1}\mathbf{\Pi}\mathbf{V}$$

which as previously stated, has the following solution,

$$\begin{aligned} \mathrm{Vec}\left\{\mathbf{U}^H\delta^n\mathbf{\Pi}\mathbf{V}\right\} &= \\ &(\mathbf{\Lambda}\oplus-\mathbf{\Gamma})^{-1}\mathrm{Vec}\left\{\mathbf{U}^H\mathbf{\Delta}\mathbf{R}\delta^{n-1}\mathbf{\Pi}\mathbf{V}\right\} \\ &-\mathbf{U}^H\delta^{n-1}\mathbf{\Pi}\mathbf{\Delta}\mathbf{R}\mathbf{V} \\ \mathbf{U}\delta^n\mathbf{\Pi}\mathbf{V} &= \\ &\left(-\mathbf{U}^H\mathbf{\Delta}\mathbf{R}\delta^{n-1}\mathbf{\Pi}\mathbf{V} + \mathbf{U}^H\delta^{n-1}\mathbf{\Pi}\mathbf{\Delta}\mathbf{R}\mathbf{V}\right)\odot\mathbf{H}, \end{aligned} \tag{16}$$

from which it is easy to obtain the first two terms of Eq. 8. ■

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