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On the Number of Wavelengths and Switches in All-Optical Networks

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***Abstract*—We consider optical networks using wavelength division multiplexing, where the path a signal takes is determined by the network switches, the wavelength of the signal, and the location the signal originated. Therefore, a signal is routed through a combination of circuit switching and wavelength routing (assigning it a wavelength). We present a bound on the minimum number of wavelengths needed based on the connectivity requirements of the users and the number of switching states. In addition, we present a lower bound on the number of switching states in a network using a combination of circuit switching, wavelength routing, and frequency changing. The bounds hold for all networks with switches, wavelength routing, and wavelength changing devices. Several examples are presented including a network with near optimal wavelength re-use.**

I. Introduction

We consider all-optical networks (AONs) using wavelength division multiplexing, circuit switching and wavelength routing (λ-routing for short). An advantage of λ-routing is spatial re-use of wavelengths. We show that there is a limit to the possible amount of wavelength re-use.

In a λ-routing network, the path a signal takes is a function of the the wavelength of the signal and the location of the signal transmitter. If the signal paths are under control of the network, e.g. through the use of switches or dynamic wavelength routing devices, we say that the network is *configurable.* Otherwise we say the network is *passive* or *fixed.* In a configurable network, the wavelength paths can be modified to suit the current traffic demands. The recently proposed Linear Lightwave Network (LLN) [1] is a configurable λ-routing network. In a passive network, the path is only a function of the wavelength and signal origin.

Since we are allowing the use of wavelength conversion within the network, a signal launched from a transmitter may arrive at a receiver on a different wavelength. In fact, a signal launched from a transmitter may arrive at a variety of receivers on many different wavelengths and/or arrive at a receiver on several different wavelengths.

In networks where the number of active users far exceeds the number of available wavelengths,[1] it will be necessary to simultaneously assign many transmitters the same wavelength. Since two signals using the same wavelength cannot travel over the same fiber simultaneously, collisions within the network need to be prevented. In addition, we must insure that signals do not collide at any intended receiver. That is, if receiver m is listening to wavelength λ at time t, we must insure that only one signal assigned to λ at time t arrives at receiver m. If two or more arrive, we say there is *contention.* In this paper we present bounds, some of which are shown to be tight, on the number of wavelengths required under various user connectivity requirements. The lower bounds allow the possibility of rearranging the wavelengths assigned to active sessions to accommodate new session requests. None of the constructions require rearranging.

In section II, we formalize the network model and user connectivity requirements. In section III, we analyze two special types of passive λ-routing networks. We show that for these special cases, the number of wavelengths must be on the order of the number of active sessions for a broad class of user connectivity requirements.

We then relax the wavelength routing restrictions used in section III and consider λ-routing networks with arbitrary topology, wavelength changers, and switches. A lower bound on the number of wavelengths is presented in section IV.A. Section IV.B discusses the influence of fixed wavelength changing on the bound. In section IV.C, we show that the bound can be very tight and that near optimal wavelength re-use is possible without wavelength conversion.

Section V is devoted to *connectors,* i.e. networks that can establish arbitrary one-to-one connections between users. For passive networks, we use the bound to show that the number of wavelengths must be on the order of the square root of the number of users. In addition, we show that $8\sqrt{M \log_2 M}$ wavelengths are sufficient, where M is the number of users.[2] Furthermore, we show that for a fixed number of wavelengths, λ-routing cannot change the order of growth of the number of required switches from

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[1] Bandwidth is a scarce commodity!

[2] The sufficiency of $O(\sqrt{M \log M})$ wavelengths, where $O(g)$ means some function bounded above by a constant times g, has been independently discovered by Alok Aggarwal, et. al. [2].

that of a conventional circuit switched network with one wavelength. However, it may be possible to combine wavelength routing and circuit switching to reduce the required number of switches by a constant factor. For instance, with $M^{1/4}$ wavelengths, it may be possible to reduce the required number of switches by a factor of 2. However, to reduce the number of switches by a factor of 10, at least $M^{9/20}$ wavelengths are required. Further reductions require the number of wavelengths to grow at a rate which rapidly approaches $\sqrt{M}$.

In section VI, we extend our bounds to *partial connectors*, i.e. networks that can establish arbitrary one-to-one connections between users, up to a fixed number of sessions. We show that the number of wavelengths must be on the order of the square root of the number of active sessions. The results are briefly summarized in section VII.

II. Network Model

We first consider networks with M_t transmitters, M_r receivers, and F wavelengths. Each transmitter (receiver) is connected to one outgoing (incoming) fiber. To model wavelength changing, we define an *origin-destination* channel, or OD channel, as an ordered pair of channels and use the notation $f{:}f'$, where f and f' are wavelengths, to represent an OD channel. We say that transmitter n is connected to receiver m on OD channel $f{:}f'$ if a signal launched from n on f is received at m on f'. If a transmitter or receiver is tuned to wavelength f, we say that it is *assigned* f. Note that there is no assumed relationship between the OD channels connecting transmitter n to receiver m and the OD channels connecting transmitter m to receiver n. Using the OD channel terminology, the connectivity of a λ-routing network can be fully described by the set $\mathcal{H} = \{H_\psi | \psi \in \Psi\}$, where $H_\psi(n,m)$ is the set of OD channels connecting transmitter n to receiver m in *switching state* ψ, and Ψ are the switching states of the network. A switching state represents the joint state of all devices within the network, i.e. switches, wavelength routers, and wavelength changers.

In networks without wavelength changing, $f{:}f' \in H_\psi(n,m)$ implies that $f = f'$. In this case, we use the obvious short hand notation of f for $f{:}f'$. If $|\Psi| = 1$, the network is passive. Also, if $F = 1$, then then the network is a conventional circuit switched network.

As an example, consider the 2 transmitter, 3 receiver, passive λ-routing network shown in Fig. 1. The λ-nodes selectively route the signals from the transmitters to the receivers based on wavelength only. The π-node represents a fixed wavelength converter which changes wavelength Red to wavelength Green. Since the network is passive, the wavelength changing of the π-node and the wavelength routing of the λ-nodes cannot be reconfigured. Thus if there is a path from transmitter n to receiver m on wavelength λ, there is always such a path even if there is no traffic between n and m. Paths for three wavelengths, R, G, and B are also shown in Fig. 1. The only freedom, after the network topology and wavelength paths have been determined, is in the tuning of the transmitters and receivers. This network can support any matching of the transmitters to two of the receivers (with no multi-casting) except the matching $T = \{(1,2),(2,3)\}$. To see that this traffic cannot be supported, notice that transmitters 1 and 2 must both be assigned Red. Since there is a path from 2 to 2 on Red, the two sessions collide at receiver 2. This need not happen for any other pair of session provided wavelengths are assigned properly. The connection matrix of the network is given by

$$H = \begin{bmatrix} \{G{:}G, R{:}R\} & \{R{:}R\} & \{B{:}B\} \\ \{G{:}G\} & \{R{:}R\} & \{R{:}G\} \end{bmatrix} \quad (1)$$

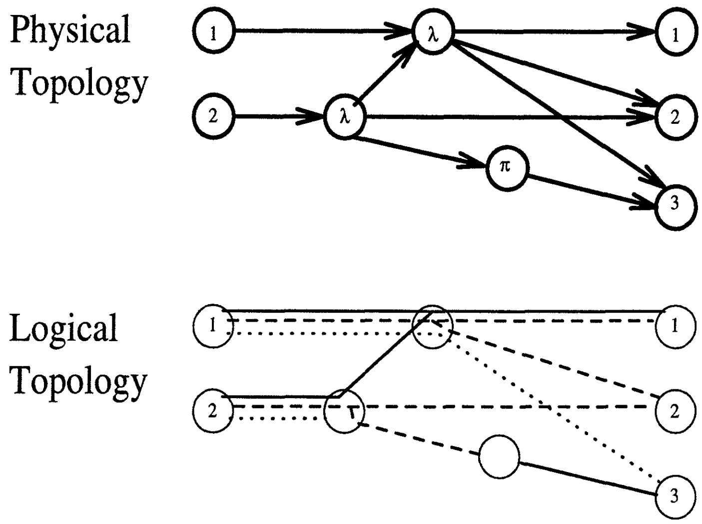

Fig. 1. Example of a λ-routing network, Solid = G , Dashed = R , Dotted = B.

For the remainder of this paper, we consider networks with M users where each user has one transmitter and one receiver, i.e. $M = M_t = M_r$. The techniques and results can be easily extended to the case where $M_t \neq M_r$.

Before deriving our bounds, we need to carefully describe the connectivity requirements of the users. A *session* (n,m) is defined to be an ordered pairing of a transmitter n to a receiver m. A *traffic* T is a set of simultaneous sessions. If $(n,m) \in T$, we say that (n,m), n, and m are *active* in T. If two sessions are active in the same traffic T, they are said to be *concurrent* in T. The *traffic set* $\mathcal{T}$ is the set of allowable traffics states. For our purposes, $\mathcal{T}$ fully describes the connectivity requirements of the users. An important example is the *Permutation Traffic Set*, i.e. the traffic set containing all one-to-one complete matchings of transmitters to receivers. A network which supports permutation routing is called a connector. Note that the Permutation Traffic Set for M users has $M!$ elements, each element being a traffic. Another important example is the *ρ-Permutation Traffic Set*, defined to be the set of all one-to-one matchings of transmitters to receivers with at most ρM active sessions. A network which supports ρ-permutation routing is called a partial connector.

III. Broadcast and Simple Networks

A *broadcast* AON with F wavelengths is one in which each transmitter is connected to each receiver on all wavelengths, i.e. $H(n,m) = \{1, 2, ..., F\}$, for all (n,m). Clearly, to do ρ-permutation routing, exactly ρM wavelengths are needed in a broadcast network.

A *simple* AON is defined to be a passive AON where pairs of users are connected by at most one OD channel, i.e. $|H(n,m)| \leq 1$. Simple networks have the practical advantage that the OD channel used by a session is not a function of the other active sessions. Let $F_s(M,\rho)$ be the minimum number of wavelengths to do ρ-permutation routing with a simple AON. In this section, we show

$$\left\lceil \frac{M+1}{2} \right\rceil \leq F_s(M,\rho) \leq \left\lceil \frac{M}{2} \right\rceil + 2 \qquad (2)$$

for all $\rho M \geq 2$. Notice that the required number of wavelengths in independent of ρ, for $\rho M \geq 2$. If $\rho M = 1$, then a broadcast network supports the traffic set and exactly one wavelength is required. Also if $\rho \leq .5$, (2) shows that a broadcast network is more wavelength efficient than a simple wavelength routing network.

Theorem 1 *Let $F_s(M,\rho)$ be the minimum number of wavelengths to do ρ-permutation routing with a simple AON. Then,*

$$\left\lceil \frac{M+1}{2} \right\rceil \leq F_s(M,\rho) \leq \left\lceil \frac{M}{2} \right\rceil + 2 \qquad (3)$$

Proof: Let H be any simple AON that supports ρ-permutation routing, $\rho M \geq 2$. Then clearly H must support $\frac{2}{M}$-permutation routing. That is, H must support the following traffic set,

$$\mathcal{T}_2 \stackrel{\text{def}}{=} \{(n,m),(x,y) \mid n \neq x, m \neq y\} \qquad (4)$$

We will show that at least $\lceil \frac{M+1}{2} \rceil$ wavelengths are required to avoid contention.

Let $f(n,m) : f'(n,m)$ be the OD channel in $H(n,m)$. We refer to $f(n,m)$ as the o-color and $f'(n,m)$ as the d-color of (n,m). Now define $d_r(n,m)$ to be the number of entries in row n with o-color $f(n,m)$. Note that $d_r(n,m) \geq 1$ since $H(n,m)$ contains o-color $f(n,m)$. Similarly define $d_c(n,m) \geq 1$ to be the number of entries in column m with d-color $f'(n,m)$. Note also that $d_r(n,m) \leq M$ and $d_c(n,m) \leq M$ since a color can be used at most M times in a row or column.

Notice that if $d_c(n,m) \geq 2$ and $d_r(n,m) \geq 2$, then there exists an m' such that $f' = f'(n,m') = f'(n,m)$ and similarly there exists an n' such that $f = f(n,m) = f(n',m)$. It therefore follows that there is contention at receiver m' in traffic $\{(n',m),(n,m')\}$ as the reader can easily verify. Therefore,

$$\frac{1}{d_r(n,m)} + \frac{1}{d_c(n,m)} \geq 1 + \frac{1}{M} \qquad (5)$$

for all (n,m). Now summing over all (n,m) gives

$$\sum_{(n,m)} \frac{1}{d_r(n,m)} + \frac{1}{d_c(n,m)} \geq M^2 \left(1 + \frac{1}{M}\right). \qquad (6)$$

TABLE I.
10 user AON with 7 wavelengths.

6	2	3	4	5	7	**1**	**1**	**1**	**1**
1	6	3	4	5	2	7	2	2	2
1	2	6	4	5	3	3	7	3	3
1	2	3	6	5	4	4	4	7	4
1	2	3	4	6	5	5	5	5	7
7	**1**	**1**	**1**	**1**	6	2	3	4	5
2	7	2	2	2	**1**	6	3	4	5
3	3	7	3	3	**1**	2	6	3	4
4	4	4	7	4	**1**	2	3	6	5
5	5	5	5	7	**1**	2	3	4	6

To finish the proof, we need only count the left hand side of (6) in a different way. Notice that for all n,

$$\sum_{m=1}^{M} \frac{1}{d_r(n,m)} = \sum_{f=1}^{F} \left(\sum_{\substack{m, \\ H(n,m)=f}} \frac{1}{d_r(n,m)} \right) \leq F \qquad (7)$$

since the inner sum is 1 if f is used in row n and 0 otherwise. Similarly, the number of d-colors used in column m is

$$\sum_{n=1}^{M} \frac{1}{d_c(n,m)} \leq F \qquad (8)$$

Thus, the left hand side of (6) is at most $2FM$. The fact that F must be integer completes the proof of the lower bound.

The following construction, which uses $\lceil \frac{M}{2} + 2 \rceil$ wavelengths without wavelength changing is adapted from [3]. The construction supports ρ-permutation routing for all $0 \leq \rho \leq 1$. Specifically for a $2M \times 2M$ square, call the rows $x_1, x_2, ...x_M, y_1, y_2, ...y_M$ and the columns $u_1, u_2, ...u_M, v_1, v_2, ...v_M$. The coloring is as follows,

$$\begin{array}{lll}
f(x_i,u_j) = f'(x_i,u_j) & = j & \text{for } i = 1,2,...M, j \neq i \\
f(x_i,v_j) = f'(x_i,v_j) & = i & \text{for } i = 1,2,...M, j \neq i \\
f(y_i,u_j) = f'(y_i,u_j) & = i & \text{for } i = 1,2,...M, j \neq i \\
f(y_i,v_j) = f'(y_i,v_j) & = j & \text{for } i = 1,2,...M, j \neq i
\end{array}$$

$$\begin{array}{lll}
f(x_i,u_i) = f'(x_i,u_i) & = M+1 & \text{for } i = 1,2,...M \\
f(x_i,v_i) = f'(x_i,v_i) & = M+2 & \text{for } i = 1,2,...M \\
f(y_i,u_i) = f'(y_i,u_i) & = M+2 & \text{for } i = 1,2,...M \\
f(y_i,v_i) = f'(y_i,v_i) & = M+1 & \text{for } i = 1,2,...M
\end{array}$$

A 7 wavelength, 10 user example is shown in Table I. Wavelength 1 is printed in boldface to help the reader discern the pattern.

□

In the remaining sections, we will relax the restriction $|H(n,m)| = 1$. Before proceeding, we note 3 extensions of this work. First, for simple AONs without wavelength changing, the minimum number of wavelengths is exactly $\lceil \frac{M}{2} + 2 \rceil$. The slight improvement to the lower bound is shown by relating the problem of supporting $\mathcal{T}_2$ in a simple network without wavelength changing to a previously solved graph coloring problem [3]. The added complexity is not worth the improved results for our purposes so we omit the details. Second, this problem can be extended to the case where the number of transmitters is not equal to the number of receivers [4,5]. Third, the results of this section hold for passive AONs that are not simple, but assign a fixed OD channel to each session. In this case $f(n,m):f'(n,m)$ represents the OD channel used by session (n,m) and as is the case for simple networks, the session (n,m) is always routed through the same path in the network. This type of routing is called *oblivious* routing [6].

IV. Lower Bound

In this section, we only concern ourselves with traffics that are one-to-one complete matchings of transmitters to receivers, i.e. $|T| = M$ and each transmitter (receiver) is in exactly one session in each allowable traffic. Extensions of the bound presented in this section to traffics with multicasting and/or with less than M active sessions can be found in [7,5]. Section VI of this paper discusses the special case of ρ-permutation routing.

The organization of this section is as follows. In section IV.A, we present the lower bound. Section IV.B discusses the influence of wavelength changing on this bound. The example in section IV.C shows that this bound can be tight.

A. Lower Bound

For any allowable traffic, an OD channel must be assigned to each active session. Two concurrent sessions *collide* if they arrive at the same receiver on the same wavelength. Such a collision may not be fatal; it is fatal only if the collision occurs at one of the intended receivers. If two sessions have a fatal collision, we say they *contend.*

Let $F(\mathcal{T},S)$ be the minimum number of wavelengths needed to support all the traffics in $\mathcal{T}$ without contention for any λ-routing network with $|\Psi| = S$ states. Our main result, proved in this section, is that

$$F(\mathcal{T},S) \;\geq\; \left(\frac{|\mathcal{T}|}{S}\right)^{\frac{1}{2M}} \tag{9}$$

For example, the permutation traffic set has $|\mathcal{T}| = M!$ traffics. Therefore, applying the bound and using $M! \geq M^M e^{-M}$, at least $\sqrt{M/e}$ wavelengths are needed to arbitrarily interconnect M users in any passive ($S = 1$) λ-routing network.

Define a *tuning state*, $\mathbf{v}$, as the $2 \times M$ matrix

$$\mathbf{v} \;=\; \begin{bmatrix} f_{o,1} & f_{o,2} & \cdots & f_{o,M} \\ f_{d,1} & f_{d,2} & \cdots & f_{d,M} \end{bmatrix} \tag{10}$$

where $f_{o,n}$ ($f_{d,m}$) is the wavelength assigned to transmitter n (receiver m) in this tuning state. Let v_{in} and v_{out} be the first and second row of $\mathbf{v}$, respectively. v_{in} and v_{out} are known as the transmitting and receiving tuning states, respectively.

A tuning state $\mathbf{v}$, together with a switching state ψ, defines a *network state.* If $\mathcal{H}$ is a λ-routing network, we say that a network state is *feasible* for traffic T if all active sessions (n,m) in T can be assigned OD channel $f_{o,n}:f_{d,m}$ without contention when the network is in switching state ψ. Network state feasibility is formally defined below.

Definition 1 Network State Feasibility
Let $\mathcal{H} = \{H_\psi \mid \psi \in \Psi\}$ be a λ-routing network, $\mathbf{v}$ be a tuning state, and ψ a switching state. Then the network state defined by $\mathbf{v}$ and ψ is feasible for a traffic T if

(C1) Active sessions are connected, i.e.
$\forall (n,m) \in T\,,\;\; f_{o,n}:f_{d,m} \in H_\psi(n,m)$.

(C2) Concurrent sessions do not contend, i.e.
$f_{o,n}:f_{d,y} \notin H_\psi(n,y)$ *and* $f_{o,x}:f_{d,m} \notin H_\psi(x,m)$ *if* $\{(n,m),(x,y)\} \subseteq T$.

Condition (2) says that for all pairs of concurrent sessions (n,m) and (x,y), there is not a path from n to y on $f_{o,n}:f_{d,y}$ and similarly there is not a path from x to m on $f_{o,x}:f_{d,m}$.

We say that a network $\mathcal{H}$ *supports* traffic set $\mathcal{T}$ without contention if for each allowable traffic, there is a feasible network state. We will show below, in lemma 3, that a network state can be feasible for at most one traffic. Therefore, the number of network states must be at least the number of traffics. Bounding the number of network states will then prove (9). Following the proof of (9), we will discuss some further interpretations of this result with an emphasis on the role wavelength changers play in defining feasible network tuning states. Afterwards, we will presents some examples.

We now formally prove our bound.

Theorem 2 Lower Bound
Let $F(\mathcal{T},S)$ be the minimum number of wavelengths needed to support $\mathcal{T}$ without contention for any λ-routing network with S states. Then,

$$F(\mathcal{T},S) \geq \left(\frac{|\mathcal{T}|}{S}\right)^{\frac{1}{2M}} \tag{11}$$

which implies that for any λ-routing network with F wavelengths,

$$S \;\geq\; |\mathcal{T}|F^{-2M} \tag{12}$$

where $|\mathcal{T}|$ is the number of traffics states in $\mathcal{T}$.

Proof: Part I: $S = 1$
First consider an arbitrary passive λ-routing network specified by a connection matrix H. Since H is assumed to be able to support $\mathcal{T}$, it must be true that there is a feasible tuning state for each $T \in \mathcal{T}$. We will show in the lemma below that for a passive network, conditions (C1) and (C2) imply that a tuning state can be feasible for at most one traffic. Therefore, in a passive network, the number of tuning states must be

greater than the number of allowable traffics. Since the number of tuning states is F^{2M}, this completes the proof for passive networks.

Part II: $S \geq 1$

Let $\mathcal{T}_\psi$ be the traffics supported by the network in switching state ψ. Then since $\mathcal{T} = \bigcup_{\psi\in\Psi} \mathcal{T}_\psi$, there exists a switching state that supports at least $|\mathcal{T}|/S$ traffics. Now apply Part I in this switching state. This proves (11) for arbitrary S. Solving (11) for S gives (12). □

Lemma 3 *In a passive network, a tuning state can be feasible for at most one traffic.*

Proof: Consider an arbitrary passive λ-routing network specified by a connection matrix H. Let $V(T)$ be the set of feasible tuning vectors for a traffic T. That is $V(T)$ are those vectors $\mathbf{v}$ that satisfy conditions (C1) and (C2).

Suppose $\mathbf{v}$ were feasible for two traffics. Specifically, suppose $\mathbf{v} \in V(T)$ and $\mathbf{v} \in V(T')$. Since $T \neq T'$, there must be at least one receiver m matched to n in T, i.e. $(n,m) \in T$, and not matched to n in T', i.e. $(n,m) \notin T'$. Since m is active in T', there must be an $x \neq n$ such that $(x,m) \in T'$. From condition (C1) applied in T, there must be a path from n to m on $f_{o,n} : f_{d,m}$, i.e. $f_{o,n} : f_{d,m} \in H(n,m)$. Also from condition (C1) applied in T' there must be a path from x to m on $f_{o,x} : f_{d,m}$, i.e. $f_{o,x} : f_{d,m} \in H(x,m)$. Now since x is active in T, there must be a y such that $(x,y) \in T$. Therefore condition (2) is violated since $\{(n,m),(x,y)\} \subseteq T$ and there is a path from x to m on OD channel $f_{o,x} : f_{d,m}$. This is a contradiction since we assume $\mathbf{v}$ was feasible for T. □

B. Discussion

Let # network states = # switching states × # tuning states. According to Theorem 2, the total number of network states must be large enough to assign a unique network state to each traffic in the traffic set. That is,

$$\text{\# switching states} \times \text{\# tuning states} \geq \text{\# traffics}$$

is a necessary condition to avoid contention for any λ-routing network. Since there are F^{2M} tuning states, this reduces to

$$\text{\# switching states} \times F^{2M} \geq \text{\# traffics}$$

Now consider networks without wavelength changing. There is a temptation to apply the following erroneous reasoning. The number of traffics is $|\mathcal{T}|$. There are M active sessions and each session is assigned one of the F wavelengths. Therefore, it must be true that $S \times F^M \geq |\mathcal{T}|$ for networks without wavelength changing. This is not true. We will show by example in section IV.C that this argument can vastly overestimate the required number of wavelengths for networks without wavelength changing. The above reasoning fails because it counts the possible number of tuning states for any traffic T, not the total number of tuning states. That is, it is true that for any T, $|V(T)| \leq F^M$ for networks without wavelength changing and $|V(T)| \leq F^{2M}$ for networks with wavelength changing. However, the total number of tuning states for networks without wavelength changing is much greater than F^M. The correct version of Theorem 2 for networks without wavelength changing is given below in Theorem 4. The improvement is negligible; we include it to emphasize that the number of feasible tuning states in a network without wavelength changing is not limited to F^M and to spare the ambitious reader from repeating the argument.

Theorem 4 uses the fact that in a network without wavelength changing, $f{:}f' \in H_\psi(n,m)$ implies $f = f'$. Combining this with condition (C1), we see that in order for $\mathbf{v}$ to be feasible for a traffic, $f_{o,n} = f_{d,m}$ for all active sessions (n,m). This implies that in any traffic the number of receivers tuned to wavelength f must equal the number of transmitters tuned to wavelength f. Therefore, in networks without wavelength changing, the receiving tuning state must be a permutation of the transmitting tuning state. Therefore, the possible number of tuning states is much less than F^{2M}. However, it is also much larger than F^M. In fact, so much larger that (11) will barely be affected.

Theorem 4 Lower Bound for Networks without Wavelength Changing

Let $F'(\mathcal{T},S)$ be the minimum number of wavelengths needed to support $\mathcal{T}$ without contention for any λ-routing network without wavelength changing. Let S be the number of switching states. Then,

$$F'(\mathcal{T},S) \geq (1+\epsilon)\left(\frac{|\mathcal{T}|}{S}\right)^{\frac{1}{2M}} \tag{13}$$

where $|\mathcal{T}|$ is the number of traffics in $\mathcal{T}$ and ϵ goes to 0 faster than $\frac{\ln\sqrt{M}}{\sqrt{M}}$.

Proof: In Appendix. □

The preceding theorem does not say that wavelength changing cannot help, only that the absence of wavelength changing will not significantly change the lower bound derived earlier.

C. Near Optimal Wavelength Reuse

Now we present an example to make 2 important points. First, $S \times F^M \geq |\mathcal{T}|$ is not a necessary condition to avoid contention in networks without wavelength changing. Second, the bound derived in Theorem 2 can be extremely tight, even in the absence of wavelength changing.

In order to describe the traffic set, group the transmitters into disjoint sets, T-Groups, of size $\sqrt{M}$. Number the T-Groups from 1 to $\sqrt{M}$ and let $I(n)$ be the T-Group of transmitter n. Similarly group the receivers into R-Groups and let $1 \leq J(m) \leq \sqrt{M}$ be the R-Group of receiver m. Now define $\mathcal{T}_{SB}$ to be the set of all traffics without multicasting with exactly one active session between each [T-Group,R-Group] pair. A typical traffic is shown in Fig. 2. It is straightforward to show that

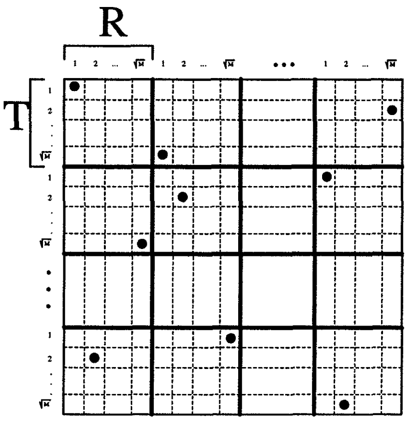

Fig. 2. Typical Traffic of $\mathcal{T}_{SB}$.

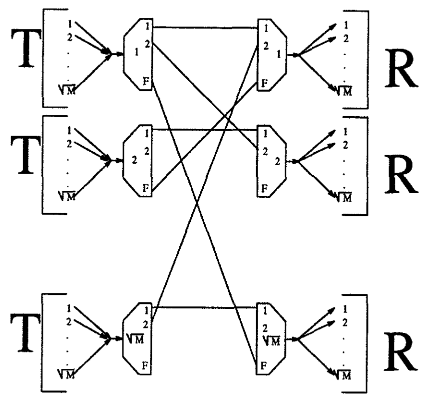

Fig. 3. Implementation of $\mathcal{T}_{SB}$.

$|\mathcal{T}_{SB}| = (\sqrt{M}!)^{2\sqrt{M}} \approx e^{-M} M!$. Applying Theorem 2 with $S = 1$ to these traffics, the required number of wavelengths is bounded below by

$$F \geq \frac{\sqrt{M}}{e} \tag{14}$$

for any passive λ-routing network. To see that the lower bound is very tight, consider the non-wavelength changing passive network shown in Fig. 3. The network uses a total of $\sqrt{M}$ wavelengths and can support $\mathcal{T}_{SB}$. Numbering the wavelengths by the integers from 1 to $F = \sqrt{M}$, the connection matrix is defined by

$$H(n,m) = (I(n) - J(m)) \bmod \sqrt{M} + 1 \tag{15}$$

If we had used the incorrect bound $F^M \geq |\mathcal{T}_{SB}|$ for networks without wavelength changing, we would have predicted that at least $\frac{M}{e^2}$ wavelengths were needed. Note also that H is a simple network (see section III) that uses $\sqrt{M}$ wavelengths. Theorem 1 does not apply in this case since $\mathcal{T}_2 \not\subseteq \mathcal{T}_{SB}$. Therefore, even though H can support $|\mathcal{T}_{SB}|$ and $|\mathcal{T}_{SB}|$ is quite large, it cannot support many traffics with only two sessions. For instance, H cannot support two simultaneously sessions between a T-Group and an R-Group, even if these two sessions are the only active sessions in the entire network.

The internal network connecting the T-Groups to the R-Groups shown in Fig. 3 is called a WDM cross-connect [8]. WDM cross-connects are examples of Latin Routers which can be implemented in a distributed fashion using a Unique Path Multi-stage Interconnection Network (UPMIN). In the UPMIN design, each stage consists of identical generalized Mach Zehnder interferometers of size $M^{\frac{1}{4}} \times M^{\frac{1}{4}}$ [9].

V. Permutation Routing

Because of its importance, we now discuss permutation routing. Recall that permutation routing was defined as the set of all complete matchings of transmitters and receivers and that a network which supports permutation routing is called a connector. Let $F(M,S)$ be the minimum number of wavelengths to do permutation routing

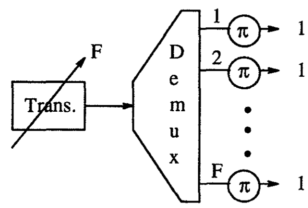

Fig. 4. An all-optical implementation of a $1 \times F$ switch.

over all λ-routing networks with S switching states. We show in section V.A that

$$8 \cdot S^{-\frac{1}{2M}} \sqrt{M \log_2 M} \geq F(M,S) \geq S^{-\frac{1}{2M}} \sqrt{\frac{M}{e}} \tag{16}$$

For $S = 1$, $8\sqrt{M \log M}$ wavelengths suffice, significantly less than the best previously reported upper bound of $\lceil \frac{M}{2} + 2 \rceil$ wavelengths [7].[3] However, the proof in V.A uses fixed wavelength changers whereas the construction of section III does not. Proofs of the sufficiency of $\Theta\left(\sqrt{M \log M}\right)$ wavelengths without wavelength changers can be found in [10,2].

A. Determining the Minimum Number of Wavelengths

In this section, we will prove (16). The lower bound follows immediately from Theorem 2 and the inequality $M! \geq M^M e^{-M}$. The upper bound is derived using a transformation and known results in switching networks. The remainder of this section discusses the upper bound.

Consider Fig. 4. Here a tunable transmitter is followed by a demultiplexer which separates the F wavelengths each onto a separate fiber. Each demultiplexer output is then followed by a wavelength changer which takes any input wavelength and converts it to wavelength 1. All-to-one wavelength changing devices have been demonstrated, but currently their use is limited to signals using amplitude modulation [11,12,13]. The device in Fig. 4 is functionally equivalent to an $1 \times F$ switch, the state of the switch

[3] This is the construction discussed in section III.

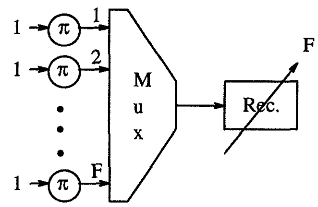

Fig. 5. An all-optical implementation of an $F \times 1$ switch.

being determined by the wavelength of the transmitter. Similarly, the device in Fig. 5 is functionally equivalent to a $F \times 1$ switch, where here the i^{th} wavelength changer converts wavelength 1 to wavelength i. Now consider a network with M transmitters, M receivers, and F wavelengths. Use the constructions above to build an $1 \times F$ switch after each transmitter and a $F \times 1$ switch before each receiver. By connecting outputs of the $1 \times F$ switches to inputs of the $F \times 1$ switches (not necessarily in a one-to-one fashion), a network with two switching stages can be built. Such a network is an example of a depth 2 interconnection network [14]. Here, however we have the added restriction that the switches be of size no more than F. [14, Theorem 3] proves the existence of depth 2 connectors using $\log_2 S = O(M^{3/2}\sqrt{\log_2 M})$ switching crosspoints. It is easy to verify that the proof uses switches of size no more than $8\left\lceil \sqrt{M \log_2 M}\, \right\rceil$. When $S = 1$, the upper bound in (16) follows.

To show the case where $S > 1$, we replace the $1 \times F$ switches and the $F \times 1$ switches by $1 \times Fd$ and $Fd \times 1$ switches, where $d = \left\lfloor S^{\frac{1}{2M}} \right\rfloor$. The construction of the switches is straightforward. For instance, a $1 \times Fd$ switch is shown in Fig. 6. A network built from these switches has less than S states and the number of wavelengths used is the size of the switches times d. Therefore,

$$F(M, d^{2M}) \cdot d \leq 8 \left\lceil \sqrt{M \log_2 M} \, \right\rceil . \tag{17}$$

The upper bound in (16) now follows by dividing both sides by d and ignoring the integer constraints. We have just proved the following theorem,

Theorem 5 *Let $F(M, S)$ be the minimum number of wavelengths to do permutation routing over all λ-routing networks with S states. Then*

$$8 \cdot \left\lfloor S^{-\frac{1}{2M}} \right\rfloor \left\lceil \sqrt{M \log_2 M} \right\rceil \geq F(M,S) \geq S^{-\frac{1}{2M}} \sqrt{\frac{M}{e}} \tag{18}$$

The upper bound in (18) can be written

$$\frac{F(M,S)}{\sqrt{\log M}} \leq \Theta\left(S^{-\frac{1}{2M}} \sqrt{M}\right) \tag{19}$$

Recently, Aggarwal, et. al. [2] have improved this to

$$\frac{F(M,S)}{\sqrt{1 + \log F(M,S)}} \leq \Theta\left(S^{-\frac{1}{2M}} \sqrt{M}\right) \tag{20}$$

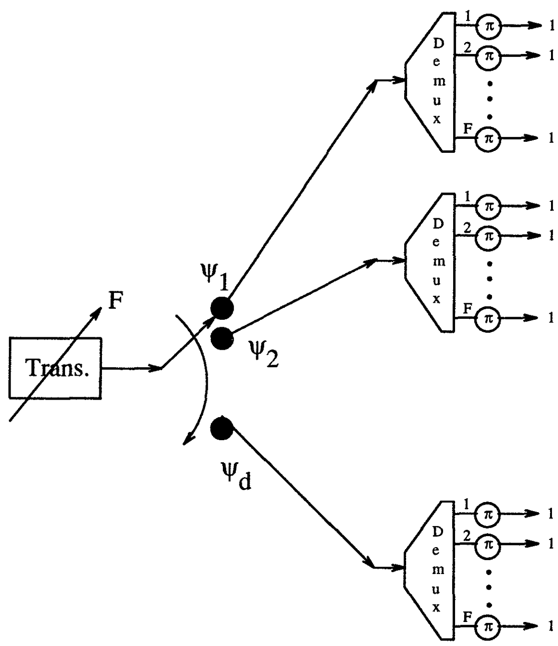

Fig. 6. An all-optical implementation of a $1 \times Fd$ switch using a $1xd$ switch. ψ_i is the i^{th} state of the switch.

For passive networks, the upper bound presented there and in (18) are both $\Theta\left(\sqrt{M \log M}\right)$ wavelengths.

It should be noted that although [14] proves the existence of 2-stage switching networks with switch size $\Theta(\sqrt{M \log M})$, explicit constructions with this switch size are not currently known.

B. Determining the Minimum Number of Switches

Since for $F < \sqrt{M/e}$, it is impossible to build a connector without switches, we now consider the issue of whether using a combination of switching and wavelength routing can significantly reduce the number of switches required when $F < \sqrt{M/e}$. Let $S(M, F)$ be the minimum number of states to do permutation routing over all λ-routing networks with F wavelengths so that $\log_2 S(M, F)$ is the minimum required number of 2×2 switches. Then,

$$\log_2 S(M,F) \geq M \log_2 M - 2M \log_2 F - 1.44M \tag{21}$$

When $F = 1$, (21) agrees with the well known formula for the minimum complexity of an $M \times M$ non-blocking switch [15]. Define the *λ-routing gain* to be the possible reduction in switches due to λ-routing. That is, define G to be

$$G = \frac{\log_2 S(M,1)}{\log_2 S(M,F)} \tag{22}$$

It is easy to see that a gain of 2 may be achieved if the number of wavelength grows at a rate of $M^{1/4}$. However, to achieve a gain of 10, F must grow like $M^{9/20}$. Larger gains require F to grow at a rate rapidly approaching $\sqrt{M}$.

Consider the following variation: either the transmitters or the receivers are tunable to one of F wavelengths, but not both. In this case, the number of tuning states is no more than F^M. Applying lemma 3, at least $\Theta\left(M \log \frac{M}{F}\right)$ switches are needed to build a non-blocking connector. Pieris and Sasaki have shown that this bound is in fact achievable [16].

VI. ρ-Permutation Routing

In section V we showed that $\sqrt{M/e}$ wavelengths are required for networks supporting M arbitrary sessions without multicasting. Here we extend those results to networks supporting $\rho M < M$ arbitrary sessions without multicasting. We show that $\sqrt{\rho M/e}$ wavelengths are required in this case.

Recall that the ρ-permutation traffic set is the set of all one-to-one matchings of transmitters to receivers with no more than ρM active sessions. Let $F(M, \rho, S)$ be the minimum number of wavelengths to do ρ-permutation routing over all λ-routing networks with S states. Also, let $S(M, \rho, F)$ be the minimum number of states to do ρ-permutation routing over all λ-routing networks with F wavelengths. Then,

$$F(M,\rho,S) \geq S^{-\frac{1}{2\rho M}}\sqrt{\frac{\rho M}{e}} \quad (23)$$

$$\log_2 S(M,\rho,F) \geq \rho M \log_2\left(\frac{\rho M}{F^2}\right) - 1.44\rho M \quad (24)$$

since to do ρ-permutation routing over M users, it is necessary to do permutation routing over ρM users. The lower bounds now follow from sections V.A and V.B. Therefore, passive AONs require $\sqrt{\rho M/e}$ wavelengths to do ρ-permutation routing, as claimed.

VII. Conclusions

In this paper we considered networks using a combination of wavelength routing, wavelength changing, and circuit switching. A general bound on the number of wavelengths was presented. The bound holds for all AONs and can be tight, even in the absence of wavelength converters. For ρ-permutation routing on M users, at least $\sqrt{\rho M/e}$ wavelengths are needed to avoid contention. For permutation routing, there exists passive AONs with no more than $8\sqrt{M \log_2 M}$ wavelengths.

For configurable networks, (21) shows that WDM combined with wavelength routing and wavelength changing cannot change the order of growth of the number of switches. However it may be possible to reduce the number of switches by a factor. For instance, with $\approx 10^8$ users and 1000 wavelengths, wavelength routing could possibly reduce the number of switches by about a factor of 5. Therefore, even in very large networks, wavelength routing may reduced hardware cost and switching control complexity.

Appendix

Theorem 6 Lower Bound for Networks without Wavelength Changing:

Let $F'(\mathcal{T}, S)$ be the minimum number of wavelengths needed to support $\mathcal{T}$ without contention for any λ-routing network without wavelength changing. Let S be the number of switching states. Then,

$$F'(\mathcal{T},S) \geq (1+\epsilon)\left(\frac{|\mathcal{T}|}{S}\right)^{\frac{1}{2M}} \quad (25)$$

where $|\mathcal{T}|$ is the number of traffics in $\mathcal{T}$ and ϵ goes to 0 faster than $\frac{\ln\sqrt{M}}{\sqrt{M}}$.

Proof: Let V be the number of tuning states where v_{out} is a permutation of v_{in}. Also, let $V(\mathbf{k})$, $\mathbf{k} = [k_1, k_2, ..., k_F]$, be the number of transmitting tuning states with k_i transmitters using wavelength i. Obviously, $V(\mathbf{k})$ is also the number of receiving tuning states with k_i receivers using wavelength i. In addition, each receiving state counted in $V(\mathbf{k})$ is a permutation of any v_{in} with k_i transmitters using wavelength i. Therefore

$$V = \sum_{\sum_{i=1}^{F} k_i = M} V^2(k_1, k_2, ..., k_F) \quad (26)$$

$$= \sum_{\sum_{i=1}^{F} k_i = M} \left(\frac{M!}{k_1!k_2!...k_F!}\right)^2 \quad (27)$$

Using the Central Limit Theorem, we approximate the multinomial distribution by an $F-1$ dimensional Gaussian density with mean $\frac{M}{F}$ and variance $M\frac{F-1}{F^2}$ in each dimension.[4] V can then be approximated by

$$V \approx F^{2M}\left[\frac{\left(\sqrt{F}\right)^F}{\left(\sqrt{4\pi M}\right)^{F-1}}\right] \quad (28)$$

Since the number of traffics can be no more than the number of network states, $V \times S \geq |\mathcal{T}|$. Let F^* be the smallest F such that $V \times S \geq |\mathcal{T}|$. Using the Gaussian approximation for V and relaxing the integer constraints,

$$\ln F^* = \frac{\ln|\mathcal{T}| - \ln S}{2M} + \frac{F^*}{4M}\ln\frac{4\pi M}{F^*} - \frac{1}{4M}\ln 4\pi M \quad (29)$$

Now, define α such that $F^* = \alpha F_o$, where $\ln F_o = \frac{\ln|\mathcal{T}| - \ln S}{2M}$. Since F_o is the lower bound on F for networks with channel changing, $\alpha \geq 1$. Thus (29) reduces to

$$\ln\alpha = \frac{\alpha F_o}{4M}\ln\frac{4\pi M}{\alpha F_o} - \frac{1}{4M}\ln 4\pi M \quad (30)$$

Note that the right hand side of (30) is an increasing function of both α and F_o. Since F^* is no more than M (since $V \geq F^M$ and $|\mathcal{T}| \leq M!$), α is no more than M/F_o. Using $\alpha \leq \frac{M}{F_o}$ to bound the right hand side of

[4] It is also possible to lower bound the multinomial to reach the same conclusion.

(30) shows that $\ln\alpha \leq .25\ln 4\pi$. Using $\alpha \leq (4\pi)^{\frac{1}{4}}$, we get

$$\ln\alpha \leq \frac{(4\pi)^{\frac{1}{4}}F_o}{4M}\ln\frac{(4\pi)^{3/4}M}{F_o} - \frac{1}{4M}\ln 4\pi M \qquad (31)$$

Since the right hand side of (31) is increasing with F_o, we can replace F_o by its largest possible value. F_o is increasing with $\frac{|T|}{S}$ and $\frac{|T|}{S} \leq M!$. Therefore, $F_o \leq (M!)^{1/2M}$. Using Sterling's approximation gives us our final bound on $\ln\alpha$,

$$\ln\alpha \leq \frac{(4\pi)^{\frac{1}{4}}(2\pi M)^{\frac{1}{4M}}}{4\sqrt{Me}}\ln\left(\frac{(4\pi)^{\frac{3}{4}}\sqrt{Me}}{(2\pi M)^{\frac{1}{4M}}}\right) - \frac{\ln 4\pi M}{4M}$$

Since $M^{1/M} = 1 + O(\frac{\ln M}{M})$, $\ln\alpha = O(\frac{\ln\sqrt{M}}{\sqrt{M}})$. Let $\epsilon = \alpha - 1$, so $\epsilon = O(\frac{\ln\sqrt{M}}{\sqrt{M}})$. This completes the proof. □

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