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Why You Can't Build an Arbiter*

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Abstract

I give a formal proof that you can't build an arbiter. The proof method gives bounds which show that at best one can arbitrate between two signals arriving ϵ seconds apart in time $O(\ln(1/\epsilon))$. I construct a simple, idealized device which achieves this optimal performance.

1 Introduction

In this note, I formally prove that there is no solution to the arbiter problem. The assumptions made are that the arbiter is deterministic, time-invariant, and constructed of Lipschitz continuous components. That is, it has repeatable behavior and finite gain. This explicit formulation and proof in terms of ODEs appears to be new and yields bounds on achievable performance.

The paper is organized as follows. Section 2 begins with the arbiter's definition and technical specifications. For the most part, the exposition follows that in [23]. It ends with a theorem, which I've called the Asynchronous Arbiter Theorem, stating that one cannot build a device meeting these specifications. In Section 3, we pause to review some of the considerable literature. Section 4 contains a detailed model of the arbiter, as a set of differential equations with inputs and outputs, and includes all assumptions made about the device. It also contains the proof of the Asynchronous Arbiter Theorem and some discussion.

In Section 5, I give equations for a device which arbitrates two signals arriving ϵ seconds apart in time $O(\ln(1/\epsilon))$. The proof in Section 4 shows that this is optimal (under the assumptions). The device is idealized in the sense that it assumes one can start a differential equation at a precise initial state. Section 6 briefly summarizes and discusses the paper's results. Some technical lemmas are collected in the Appendix.

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2 The Problem

The definition and technical specifications of the arbiter below are adapted from [23].

An *arbiter* is a device that can be used to decide the winner of two-person races. It is housed in a box with two input buttons, labeled B_0 and B_1, and two output lines, W_0 and W_1, that can each be either 0 or 1. For ease of exposition, let the vectors

$$B = (B_0, B_1), \qquad W = (W_0, W_1)$$

denote the button states and outputs, respectively. There is also a reset button. After pressing the reset button, the output should be $(1,0)$ if B_0 is pressed before B_1; it should be $(0,1)$ if B_1 is pressed before B_0.

Let T_i denote the time that button B_i is pressed. Basically, the function of the arbiter is to make a binary choice based on the value of the continuous variable $T_0 - T_1$. If the difference is negative, the output should be $(1,0)$; if it is positive, the output should be $(0,1)$. Upon reset, the output is set to $(0,0)$.

Now, here are the arbiter's technical specifications:

S1. The reset button causes the output to become $(0,0)$, perhaps after waiting for some specified time, where it remains until one or both buttons are pressed.

S2. The pressing of either or both buttons causes, after an interval of at most T_d units, the output to be either $(0,1)$ or $(1,0)$; the output level persists until the next reset input.

S3. If B_0 is pressed T_a seconds or more before B_1 is pressed, then the output will be $(1,0)$, indicating that B_0 was pressed first. Similarly, if B_1 is pressed T_a seconds or more before B_0 is pressed, then the output will be $(0,1)$, indicating that B_1 was pressed first.

S4. If B_0 and B_1 are pressed within T_a seconds of each other, then the output is either $(1,0)$ or $(0,1)$—one does not care which—after the T_d-second interval.

Problem 1 (Asynchronous arbiter problem.) *Construct an absolutely reliable arbiter that meets the specifications $S1$–$S4$.*

In the next section, I prove that there is no solution to Problem 1:

Theorem 2 (Asynchronous arbiter theorem.) *For no choice of the values for T_a and T_d is it possible to construct an absolutely reliable arbiter that meets the specifications $S1$–$S4$.*

3 Previous Work

The arbiter problem or mutual exclusion problem is a cult problem in computer science. The problem also appears in building synchronizers and latches. See [19, 5] and their references for a history of these so-called "glitch" problems. See [2] for the equivalence of these and other digital circuit problems.

The problem has generated a lot of literature and many proofs that one cannot be built. Still, there are those who are pretty sure you can't build one, but believe that there must be some trick by which they could build one.

Many of the proofs available in the literature come down to an intermediate value argument. More precisely, to the topological fact that one cannot have continuous maps with a connected domain and a disconnected range [16]. (For briefness, we will refer to this below as the *connectedness law.*) The works [9, 14, 10, 11] fall into this category. However, researchers still question the way the modeling of inputs. For example, [11] responds to [14] by saying that "[Marino] requires to be included in the input function space, functions with unbounded or undefined second derivatives. Without these functions, his theorem cannot guarantee that metastable behavior can occur." They then go on to prove that metastable behavior is still unavoidable if one restricts the inputs to be more "practical." However, even this does not satisfy the critics. Some assert that one can add some kind of "metastability detector" to get around these problems [18].

In [13], it is shown using index theory that metastable states must exist in an R-S latch with state space $\mathbb{R}^2$. He correctly asserts that such states are at the heart of the arbiter problem. But later, he says that the only way to avoid this is to assure that all pulses are "long enough; the only way this seems possible is by feedback links which report the occurrence of transitions to the sender" [13, p. 79]. However, if one had access to a device which could report transitions, then one could trivially solve the arbiter problem!

Many people take it for granted that the connectedness law itself is a proof of the inability to build arbiters. For example, the "proof" in [23, p. 94] simply says

> In order to understand why the arbiter is unrealizable, we must recognize that we have introduced continuous variables—the times at which the buttons are pressed—into our discrete model. Let t_0 and t_1 be the times at which B_0 and B_1, respectively, are pressed. Then the function of the of the arbiter is to make a binary choice (W_0 or W_1) based on the value of the continuously variable $t_0 - t_1$: If the difference is negative, W_0 is to be 1 and W_1 is to be 0; otherwise W_0 is to be 0 and W_1 is to be 1.

This is not much more than restatement of the problem. Moreover, one cannot invoke the connectedness law (which is not stated in [23]) so easily. One must prove that the map given by the device is indeed continuous before one makes such an appeal. This is not trivial since one has, at heart, a "discrete model." One must explicitly rule out the possibility that this "discrete model" introduces any discontinuities. Indeed, in [3] we easily show that some reasonable models for describing the interaction of discrete and continuous variables [4, 21, 1, 17] can implement reliable arbiters.

In [15], they show that arbiters cannot act correctly even in unbounded time. This can be related to our result in Section 5 by the fact that one cannot start the differential equation at exactly zero. Thus there will be some input signals which cause the device to directly hit a metastable equilibrium point, staying there for all future time. Indeed, if the signals arrive at exactly the same time our simple device will still stay at the origin for unbounded time. Similarly, this trivial statement can be deduced whenever metastable states exist. While these statements are true mathematically under their assumptions, one wonders about the practicality of such statements, given that real devices are subject to a small amount of noise.

Practically, also, one should note that the problem does not arise in synchronous situations, since the input space is then disconnected. One can achieve arbitrarily low synchronization failure rates in these cases [20].

4 The Proof

First, a more detailed model of the arbiter is required. The assumptions are as follows:

A1. The arbiter's components are physical devices having finite gain: satisfying Lipschitz continuity.

A2. The arbiter is of finite size.

A3. The arbiter's behavior is experimentally repeatable. That is, it is time-invariant and deterministic.

The arbiter will be modeled as a system of ordinary differential equations with inputs and outputs as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t), B(t)), \qquad x(0) \in h^{-1}((0,0)) \\ W(t) &= h(x(t)) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $W(t) \in \mathbb{R}^2$, $B(t) \in \{0,1\}^2$, with $B(\cdot)$ left-continuous. Each $f(\cdot, B)$, $B \in \{0,1\}^2$, is a globally Lipschitz map. Thus, each vector field $f(\cdot, B)$ defines a global flow $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n : (t, x) \mapsto \phi_{(B_1,B_2)}(t)(x)$, where $\{\phi_{(B_1,B_2)}(t) | t \in \mathbb{R}\}$ is the group of C^1 diffeomorphisms of $\mathbb{R}^n$ with infinitesimal generator $f(\cdot, B_1, B_2)$. Further, $h : \mathbb{R}^n \to \mathbb{R}^2$ is globally Lipschitz continuous, with constant L_W, from norm $|\cdot|$ on $\mathbb{R}^n$ to the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^2$.

Note $L_W > 0$ since h is not a constant, having $(0,0)$, $(1,0)$, and $(0,1)$ in its image. Note also that the action of the reset button is unmodeled; it is not necessary to the proof.

Next, some notation. Let $|\cdot|$ denote both an arbitrary norm on $\mathbb{R}^n$ and absolute value in $\mathbb{R}$, the meaning being clear from usage. For ease of exposition, let F_t, G_t, H_t denote the fundamental solutions $\phi_{(0,1)}(t)$, $\phi_{(1,0)}(t)$, and $\phi_{(1,1)}(t)$, respectively. Also let $L > 0$ denote a positive, finite bound of the maximum of the four Lipschitz constants corresponding to each of the $f(\cdot, B)$.

Now, I am ready to settle the arbiter problem.

Proof [of Theorem 2] The proof is by contradiction, assuming an arbiter can be built which satisfies the above assumptions.

Assume that the arbiter has been reset, is in state $x(0) = x_0$ at time $t = 0$ with $h[x_0] = (0,0)$, and that one of the buttons is pressed at time $t = 0$.[1] Also, assume that the reset button is not pressed until some time $T_R \gg T_a + T_d$. The behavior of the device from $t = 0$ to $t = T_R$ is completely determined by which button was pressed first and at what time the second button is pressed (if ever). Therefore, let $x_\rho(t)$ denote the solution at time t of Equation (1) starting at time $t = 0$ at state $x(0) = x_0$ with fixed parameter $\rho \stackrel{\text{def}}{=} T_0 - T_1$. Thus, ρ represents the difference between the times when B_0 and B_1 are pressed. If B_0 is pressed but B_1 is never pressed, set $\rho = \infty$. If B_1 is pressed but B_0 is never pressed, set $\rho = -\infty$.

The arbiter specifications require that for $T_a + T_d \le t \le T_R$,

$$h[x_\rho(t)] = \begin{cases} (1,0), & \rho \le -T_a \\ (0,1), & \rho \ge T_a \\ (1,0) \text{ or } (0,1), & \text{otherwise} \end{cases}$$

[1]This is without loss of generality as the arbiter equations are time-invariant.

These specifications and Lemma 4[2] are such that for any $\delta > 0$, one can find $-T_a \leq \sigma < \tau \leq T_a$, with $\tau - \sigma < \delta$, and with one of $h[x_\sigma(T_a + T_d)]$, $h[x_\tau(T_a + T_d)]$ equal to $(1, 0)$ and the other equal to $(0, 1)$.

Now, pick $\delta < \min\{T_a, T_d, 1/L\}$ and

$$c = \max\{|f(x_0, 1, 0)|, |f(x_0, 0, 1)|, |f(x_0, 1, 1)|\}$$

Note $c > 0$, for otherwise $h[x_\sigma(t)] = h[x_\tau(t)] = h[x_0]$ for all $0 \leq t \leq T_R$, a contradiction. Finally, for ease of notation, define $X_t = x_\sigma(t)$ and $Y_t = x_\tau(t)$. Note that $X_0 = Y_0 = x_0$.

The proof splits into three cases:

1. $0 \leq \sigma < \tau \leq T_a$.
2. $-T_a \leq \sigma < \tau \leq 0$.
3. $-T_a < \sigma < 0 < \tau < T_a$.

Case 1. In this case,

$$\begin{aligned} X_t &= \begin{cases} F_t(x_0), & 0 \leq t \leq \sigma \\ H_{t-\sigma}(F_\sigma(x_0)), & \sigma \leq t \leq T_R \end{cases} \\ Y_t &= \begin{cases} F_t(x_0), & 0 \leq t \leq \tau \\ H_{t-\tau}(F_\tau(x_0)), & \tau \leq t \leq T_R \end{cases} \end{aligned}$$

Thus, $X_\sigma = Y_\sigma$. Now, by Corollary 6

$$|Y_\tau - Y_\sigma| \leq cL^{-1}(e^{L\tau} - e^{L\sigma})$$

Thus, Lemma 7 gives

$$\begin{aligned} |X_{\sigma+T_d} - Y_{\tau+T_d}| &\leq cL^{-1}(e^{L(\tau-\sigma)} - 1)e^{L\sigma}e^{LT_d} \\ &\leq cL^{-1}(e^{L\delta} - 1)e^{L(T_a+T_d)} \\ &\leq c\delta(e - 1)e^{L(T_a+T_d)} \end{aligned}$$

where the last line follows from $L\delta < 1$. But,

$$\sqrt{2} = \|h[X_{\sigma+T_d}] - h[Y_{\tau+T_d}]\| \leq L_W |X_{\sigma+T_d} - Y_{\tau+T_d}|$$

yielding

$$K_1 \stackrel{\text{def}}{=} \frac{\sqrt{2}}{cL_W(e-1)e^{L(T_a+T_d)}} \leq \delta \tag{2}$$

Case 2. The argument is similar to Case 1 and yields the same inequality on δ.

Case 3. In this case,

$$\begin{aligned} X_t &= \begin{cases} G_t(x_0), & 0 \leq t \leq |\sigma| \\ H_{t-|\sigma|}(G_{|\sigma|}(x_0)), & |\sigma| \leq t \leq T_R \end{cases} \\ Y_t &= \begin{cases} F_t(x_0), & 0 \leq t \leq \tau \\ H_{t-\tau}(F_\tau(x_0)), & \tau \leq t \leq T_R \end{cases} \end{aligned}$$

[2]Referenced lemmas and corollaries appear in the Appendix.

Note that $\max\{|\sigma|, |\tau|\} \le \delta$. This and Lemma 5 give

$$\left|X_{|\sigma|} - Y_\tau\right| \le \left|X_{|\sigma|} - x_0\right| + |Y_\tau - x_0| \le 2cL^{-1}(e^{L\delta} - 1)$$

Thus, Lemma 7 gives

$$\begin{aligned}\left|X_{|\sigma|+T_d} - Y_{\tau+T_d}\right| &\le 2cL^{-1}(e^{L\delta} - 1)e^{LT_d} \\ &\le 2ce^{LT_d}(e-1)\delta\end{aligned}$$

where the last line follows from $L\delta < 1$ and, again,

$$\sqrt{2} = \left\|h\left[X_{|\sigma|+T_d}\right] - h\left[Y_{\tau+T_d}\right]\right\| \le L_W \left|X_{|\sigma|+T_d} - Y_{\tau+T_d}\right|$$

so

$$K_3 \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}cL_W(e-1)e^{LT_d}} \le \delta \tag{3}$$

Thus, choosing $\delta < \min\{T_a, T_d, 1/L, K_1, K_3\}$ would have achieved a contradiction in all three cases. □

The basic argument used above is that one cannot have a continuous map from a connected space (*e.g.*, $\mathbb{R}$ containing ρ) to a disconnected space (*e.g.*, $\{(1,0), (0,1)\}$) [16]. Thus, the impossibility remains if one assumes a nondeterministic arbiter acting as follows: Produce an intermediate output of $(1, 0)$ or $(0, 1)$ as before, along with the new possibility of (a, b), some other distinguishable state, meaning "I don't know," "flip a coin," or "invoke metastability detector."

I have just demonstrated explicitly the continuity of the system of switched differential equations modeling the arbiter. But the explicit proof yields bounds on arbiter behavior. Specifically, while the proof prohibits the construction of an arbiter with $T_d = O(1)$, it does not prohibit an arbitration device with $T_d = O(\ln(1/\rho))$. I construct such a device in the next section.

Note that the inputs B are assumed to be ideal in the sense that they switch from 0 to 1 instantaneously. Imposing continuity assumptions on B as signals in $[0,1]^2$ would lead to a similar result.

5 An Optimal Arbitration Device

The inequalities (2) and (3) derived in the last section show that it is impossible to build an arbiter with constant T_d. However, these inequalities do not prohibit one's building a device with $T_d = O(\ln(1/\delta))$. In particular, plugging in

$$T_d = L^{-1}\ln(K/\delta)$$

where

$$K \ge \max\left\{\frac{\sqrt{2}}{cL_W(e-1)e^{LT_a}}, \frac{1}{cL_W(e-1)\sqrt{2}}\right\}$$

satisfies them, showing the possibility of a device which arbitrates between two signals arriving within time ϵ of each other in time $T_d = O(\ln(1/\epsilon))$. Such a device exists.

The device is given by the following equations

$$\begin{aligned} \dot{x}(t) &= f(x(t), B(t)), \qquad x(0) = 0 \\ W(t) &= h(x(t)) \end{aligned}$$

where $x(t) \in \mathbb{R}$ and

$$\begin{aligned} f(x,0,0) &= 0 \\ f(x,0,1) &= -1 \\ f(x,1,0) &= 1 \\ f(x,1,1) &= Lx \end{aligned}$$

and

$$h(x) = \begin{cases} (0,1), & x \leq -1 \\ (0, 2|x|-1), & -1 \leq x \leq -1/2 \\ (0,0), & -1/2 \leq x \leq 1/2 \\ (2x-1, 0), & 1/2 \leq x \leq 1 \\ (1,0), & 1 \leq x \end{cases}$$

Theorem 3 *The device described above can arbitrate between two signals arriving within time* $0 < \epsilon$ *of each other in time* $T_d = O(\ln(1/\epsilon))$.

Proof Assume that B_0 is pressed at time $t = 0$ and B_1 is pressed at time $t = \epsilon$. Then

$$x(t) = \begin{cases} t, & 0 \leq t \leq \epsilon \\ e^{L(t-\epsilon)}\epsilon, & \epsilon \leq t \end{cases}$$

A simple calculation shows that $x(t) \geq 1$ for

$$t \geq \max\{L^{-1}\ln(1/\epsilon), 0\} + 1$$

The argument for B_1 pressed before B_0 is similar. □

Note that the device has no reset button. The assumption that one can somehow set $x(0) = 0$ is implicit. Clearly, this is a device that is not immune to noise.

6 Conclusions

In this note, I have formally proven that there is no solution to the arbiter problem. The assumptions made were that the arbiter is deterministic, time-invariant, and constructed of Lipschitz continuous components. That is, it has repeatable behavior and finite gain.

The same arguments hold even if one is to produce an intermediate output of $(1,0)$ or $(0,1)$ as before, along with the new possibility of (a,b), some other distinguishable state, meaning "I don't know," "flip a coin," or "invoke metastability detector." Thus the proof disallows practical devices using ternary logic [22].

Moreover, the proof does not depend on the fact that the inputs B_i are assumed to be ideal in the sense that they switch from 0 to 1 instantaneously. It holds in spite of this. Imposing continuity assumptions on B as signals in $[0,1]^2$ leads to a similar result.

The explicit formulation and solution of the arbiter problem in terms of ODEs appears to be new and yields bounds on achievable performance in terms of "circuit gain." Thus, the result should be useful to circuit designers.

In [7, 12], the authors give ODEs for CMOS synchronization circuits; [7] studies the linearized equations, while [12] studies the full nonlinear equations. It would be interesting to compare this practical characterization of metastability against our theoretical bounds. This is the subject of future research.

Acknowledgements

In his course at MIT John Wyatt [24] presented the arbiter problem and the challenge to produce an ODE-based model and proof. This report is the answer to that challenge.

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Appendix

Lemma 4 *If X is a connected metric space, Y is a discrete topological space with two points, and $f : X \to Y$ is surjective, then for every $\delta > 0$ one can find $x, z \in X$ such that $d(x, z) < \delta$ and $f(x) \neq f(z)$.*

Proof Assume the contrary. Then for all $x \in X$, $f(B_\delta(x)) = \{f(x)\} \subset V$, where $B_\delta(x)$ denotes the ball of radius δ about x and V is any open set about $f(x)$ in Y. Thus, f is continuous [16]. But f continuous and X connected implies $f(X) = Y$ is connected [16], a contradiction. □

Lemma 5 *Suppose*

$$\dot{x}(t) = f(x(t))$$

with f globally Lipschitz continuous in x with constant $L_f \geq 0$. Then, for any L such that $L \geq L_f$ and $L > 0$, and any $t_2 \geq t_1$,

$$|x_{t_2} - x_{t_1}| \leq |f(x_{t_1})| L^{-1}(e^{L(t_2-t_1)} - 1)$$

Proof Note that for $t \geq t_1$,

$$x_t - x_{t_1} = \int_{t_1}^{t} f(x_{t_1}) ds + \int_{t_1}^{t} [f(x_s) - f(x_{t_1})] ds$$

So that

$$\begin{aligned} |x_t - x_{t_1}| &\leq \int_{t_1}^{t} |f(x_{t_1})| ds + \int_{t_1}^{t} |f(x_s) - f(x_{t_1})| ds \\ &\leq (t - t_1)|f(x_{t_1})| + \int_{t_1}^{t} L|x_s - x_{t_1}| ds \end{aligned}$$

Now, substituting $\tau = t - t_1$ and $\sigma = s - t_1$, this becomes

$$|x_{\tau+t_1} - x_{t_1}| \leq \tau |f(x_{t_1})| + \int_0^{\tau} L|x_{\sigma+t_1} - x_{t_1}| d\sigma$$

Finally, defining $u(\tau) = |x_{\tau+t_1} - x_{t_1}|$, this becomes

$$u(\tau) \leq \tau|f(x_{t_1})| + \int_0^\tau Lu(\sigma)d\sigma$$

The result now follows from the well-known Bellman-Gronwall inequality [6, p. 252].

□

Corollary 6 *Under the same assumptions plus the fact that the system was in state x_{t_0} at time $t_0 \leq t_1 \leq t_2$,*

$$|x_{t_2} - x_{t_1}| \leq |f(x_{t_0})|L^{-1}(e^{L(t_2-t_0)} - e^{L(t_1-t_0)})$$

Proof Note that Lipschitz continuity gives

$$|f(x_{t_1})| \leq L|x_{t_1} - x_{t_0}| + |f(x_{t_0})|$$

But, the lemma gives in turn

$$|x_{t_1} - x_{t_0}| \leq |f(x_{t_0})|L^{-1}(e^{L(t_1-t_0)} - 1)$$

So that the result follows. □

The following lemma is well-known (see, *e.g.*, [8, p. 169]).

Lemma 7 *Let $y(t), z(t)$ be solutions to*

$$\dot{x} = f(x)$$

where f has global Lipschitz constant $L \geq 0$. Then for all $t \geq t_0$,

$$|y(t) - z(t)| \leq |y(t_0) - z(t_0)|e^{L(t-t_0)}$$

□

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