

COMPLEXITY THEORETIC ASPECTS OF PROBLEMS IN CONTROL THEORY[†]

John N. Tsitsiklis
Center for Intelligent Control Systems
and
Laboratory for Information and Decision Systems
M.I.T.
Cambridge, MA 02139

ABSTRACT. We introduce some of the concepts of computational complexity theory. We then survey available complexity-theoretic results for some of the main problems of control theory.

INTRODUCTION.

Our subject is motivated by asking what it means “to have a solution to a problem.” The answer to this question has been changing with time: for example, ancient Greeks required a constructive procedure using ruler and compass, classical mathematicians required closed form formulas (maybe involving special functions), etc. The theory of computation, as developed during the last fifty years considers a problem to be solved if an algorithm is provided that can compute its solution. There do exist problems that are provably unsolvable, such as the halting problem or Diophantine equations. Even problems that are solvable in principle, as most problems in control theory are, can be of varying difficulty if one considers the required computational resources, such as running time. Complexity theory is the branch of computer science that deals with the classification of problems in terms of their computational difficulty. In this paper, we define some concepts from complexity theory and provide an overview of existing complexity results for several problems in control theory.

COMPLEXITY THEORY.

Mainstream complexity theory deals with discrete problems; that is, problems whose instances can be encoded in a finite binary string and are therefore suitable input to a digital computer. *Problems* (such as matrix inversion) have *instances* (any particular square matrix defines an instance of matrix inversion). Different instances of the same problem are, in general, of different *sizes*, where “size” means the number of bits used in a natural encoding of that instance. We say that a problem is *polynomial time solvable*, or for short, belongs to P , if there exists an algorithm and some integer k such that the time it takes this algorithm to solve any instance of size n has order of magnitude $O(n^k)$. Some classical problems in P are linear programming, matrix inversion, the shortest path problem in graphs, etc. Problems in P are often viewed as “well-solved”.

There is another class of problems, called NP (for nondeterministic polynomial time), that

[†] Research supported by the ARO under grant DAAL03-92-G-0115 (*Center for Intelligent Control Systems*).

contains all problems that can be “transformed” or reformulated as integer programming problems. While P is a subset of NP , there is no known polynomial time algorithm for integer programming and it is generally conjectured that $P \neq NP$. If this conjecture is true, then integer programming is not solvable in polynomial time and the same is true for those problems in NP which are the “hardest”; such problems are called NP -complete. More generally, we will say that a problem is NP -hard if it is at least as hard as integer programming.

Proving that a problem is NP -hard is viewed as evidence that the problem is difficult. Assuming the validity of the conjecture $P \neq NP$, NP -hard problems do not have polynomial time algorithms. More generally, NP -hardness often reflects a certain absence of structure which limits the nature of theoretical results that can be established. In practical terms, NP -hardness usually means that a problem should be approached differently: instead of trying to develop an algorithm which is provably efficient all of the time, effort should be concentrated on easier special cases or on heuristics that work acceptably most of the time; usually, this is to be determined by extensive experimentation rather than by theoretical means.

DECENTRALIZED DECISION MAKING.

Witsenhausen’s problem [W68] is the simplest conceivable generalization of linear quadratic Gaussian control (LQG) to a nonclassical information pattern (decentralized control). The solution to this problem is still unknown despite persistent efforts. An explanation is provided by the fact that this problem, suitably discretized, is NP -hard [PT86]. The point here is not that we might wish to solve Witsenhausen’s problem computationally; rather, NP -hardness is an indication that the problem is fundamentally more difficult — and less structured — than its centralized analog (LQG).

The above result is not fully satisfactory because it does not rule out the possibility that NP -hardness is only a consequence of the problem discretization, not of the inherent difficulty of the original problem. While it seems difficult to establish a complexity result for the original (continuous) version of this problem, results similar to NP -hardness have been established for a related problem which we discuss below.

In the *team decision* problem [MR72], we are given two random variables y_1, y_2 , with known joint probability distribution, and a cost function $c : \mathfrak{R}^4 \mapsto \mathfrak{R}$. Agent i ($i = 1, 2$) observes the value of y_i and makes a decision u_i according to a rule $u_i = \gamma_i(y_i)$. A cost of $c(y_1, y_2, u_1, u_2)$ is then incurred. The problem is to find rules γ_1 and γ_2 , so as to minimize the expectation of the cost. This problem is NP -hard for the case where the y_i s have finite range [PT82,TA85]. It remains NP -hard even for a special case that arises in decentralized detection [TA85].

In the continuous version of the problem, we take the random variables y_1 and y_2 to be uniformly distributed on $[0, 1]$. The function c is assumed to be Lipschitz continuous with known Lipschitz constant. Such a function cannot be represented by a binary string, as required by digital computers, and for this reason, we need a suitably adapted model of computation: we assume that a digital computer obtains information on the function c by submitting queries to an “oracle”; a

typical query consists of a rational (hence finitely describable) vector in the domain of c , together with an integer k ; the oracle returns the k most significant bits of the value of c at that point. Finally, there is an accuracy parameter ϵ : instead of looking for an optimal solution, we only desire a solution whose expected cost comes within ϵ of the optimal. The computational cost or running time under this model of computation can be viewed as consisting of two parts: a) the number of queries (information cost) and b) the time spent on actual computations (combinatorial cost).

For the continuous version of the team decision problem, the cost of information is only $O(1/\epsilon^4)$; it suffices to learn the value of the function c at points on a grid with spacing ϵ and any smaller number of queries is insufficient. On the other hand, assuming that $P \neq NP$, there exists no algorithm that solves the problem with accuracy ϵ in time polynomial in $1/\epsilon$ [PT86].

Two remarks are in order:

- a) For many problems whose complexity has been studied within this framework, the information cost and the combinatorial cost are comparable. Examples can be found in numerical integration, numerical integration of PDEs, nonlinear programming, etc. In contrast, we have here an exponential gap between the two types of costs.
- b) Problems with closed form solutions often have complexity which is polynomial in the logarithm of $1/\epsilon$. Even problems like PDEs have complexity of the order of $1/\epsilon^k$, where the exponent k depends on the dimension of the problem and the smoothness of the data. In this respect, the team decision problem is significantly harder than most PDE problems.

MARKOV DECISION THEORY.

Consider a controlled Markov chain $x(t)$ that takes values in the finite set $\{1, \dots, n\}$. We are given its transition probabilities $p(x(t+1) | x(t), u(t))$, where $u(t)$ is the control applied at time t . The cost per stage is of the form $c(x(t), u(t))$ and there is a discount factor $\alpha \in (0, 1)$. The objective is to minimize the infinite horizon expected cost

$$V(i) = E\left[\sum_{t=0}^{\infty} \alpha^t c(x(t), u(t)) \mid x(0) = i\right]$$

over all causal control policies. The key to solving this problem is Bellman's equation:

$$V(i) = \min_u [c(i, u) + \alpha \sum_j p(j | i, u) V(j)]$$

While there are several algorithms for solving Bellman's equation (e.g., policy iteration, successive approximation, etc.), none of the algorithms of this type is known to run in polynomial time. On the other hand, this problem is polynomially solvable because it can be transformed to linear programming [B87].

The problem becomes more difficult if the control is to be chosen on the basis of imperfect information. Suppose that at each time t , we observe $y(t) = h(x(t))$, where h is a known function. We restrict to policies in which the current decision $u(t)$ is determined by a rule of the form $u(t) = F(y(0), y(1), \dots, y(t), t)$. If we let $p(i, t) = \Pr(x(t) = i \mid \text{past history})$, the problem can

be reformulated as a perfect information problem with state vector $p(t) = (p(1,t), \dots, p(n,t))$. The cost-to-go function $V(p(t))$ is known to be piecewise linear in p and this leads to a finite algorithm [SS73], at least for the case of finite-horizon problems. Unfortunately, the number of linear “faces” can increase exponentially with the time horizon, and so does the required algorithm. Are more efficient algorithms likely to exist? The answer is probably negative because the problem is NP-hard [PT87].

There are analogs of these results that apply to the problem of supervisory control of discrete-event systems, as formulated by Ramadge and Wonham [RW87]. These problems are similar to the problems of Markov decision theory except that the transition probabilities are not given (or may not exist) and the problem consists of finding feedback laws that are guaranteed to avoid certain undesirable states. While the perfect information problem was known to be polynomial, the corresponding imperfect information problem, as well as the corresponding problem of decentralized control, are NP-hard [T88].

The Markov decision problem has also been studied extensively for the case of continuous state spaces. In a simple version of the problem, we may assume that the state and the control take values in $[0, 1]$. Bellman’s equation becomes

$$V(x) = \min_u [c(x, u) + \alpha \int_0^1 p(y | x, u) V(y) dy]$$

Let us assume that the functions c and p have bounded first derivatives. If we wish to solve Bellman’s equation with accuracy ϵ , it is not hard to show that $O(1/\epsilon^3)$ “oracle queries” suffice; it turns out that this many queries are also necessary [CT89]. The natural iterative method for this problem (successive approximation) has computational complexity $O((1/\epsilon^3) \log(1/\epsilon))$: the cost per iteration is $O(1/\epsilon^3)$ and $\log(1/\epsilon)$ iterations are needed to get within ϵ of the solution. In fact, the logarithmic gap between the lower bound of $O(1/\epsilon^3)$ and the performance of successive approximation can be closed. It turns out that a “one-way multigrid” method solves the problem with a total of $O(1/\epsilon^3)$ arithmetic operations and is therefore an optimal algorithm [CT91]. The key idea is that most iterations are performed on coarse grids and are therefore relatively inexpensive.

OTHER RESULTS.

We mention briefly some more problems in control theory for which complexity theoretic results are available.

Nonlinear controllability: The question here is whether it is possible to generalize Kalman’s controllability conditions to nonlinear systems. Consider a bilinear system of the form

$$\frac{dx}{dt} = (A + \sum_{i=1}^m u_i G_i)x + Bu, \quad x \in M,$$

where M is a manifold defined as the zero set of a given set of polynomials. It turns out that deciding whether such a system is controllable (i.e., whether every state can be reached from any other state)

is NP -hard [S88]. As a result, whatever necessary and sufficient conditions for controllability are found will be computationally very difficult to test (unless $P = NP$).

Robust control: Consider the linear system $dx/dt = Ax$ and assume that $A = A_0 + \alpha_1 A_1 + \dots + \alpha_k A_k$, where each α_i is an unknown parameter that lies in $[0, 1]$. We are interested in properties of the plant that are valid for all choices of $\alpha_1, \dots, \alpha_k$. It turns out that many interesting problems within this framework are unlikely to have computationally efficient solutions. For example, deciding whether A is guaranteed to be nonsingular or stable is NP -hard [PR92,N92]. As a corollary of this result, computing the structured singular value μ of a linear system is also an NP -hard problem [BYDM92].

Simultaneous stabilization: Let there be given matrices A_1, \dots, A_k and B . The problem is whether there exists a matrix K such that $A_i - BK$ is stable for all i . This problem can be also shown to be NP -hard.

Control of queueing systems: Consider the standard problem of controlling a closed queueing network with several servers and several customer classes. (Control involves routing decisions for customers that complete service and sequencing decisions; the latter deal with choosing which customer to serve next at any given server with several customers in queue.) The objective is to find a control policy that minimizes the expected weighted sum of queue lengths. This problem is NP -hard even in the special case where the service time distributions are exponential [PT93]. In addition, its computational complexity is exponential for the case of deterministic service times [PT93]. Note that the latter result is stronger than any of the results mentioned earlier in this paper in that it does not rely on the validity of the conjecture $P \neq NP$.

CLOSING COMMENTS.

Complexity theory can provide new insights to control problems. However, any results obtained have to be interpreted with caution. Proving that a problem is NP -hard does not mean that the problem is hopelessly intractable or that work on it should be abandoned. Rather, a different line of attack may be called for.

REFERENCES.

- [B87] D. P. Bertsekas, *Dynamic Programming and Stochastic Control*, Prentice Hall, 1987.
- [BYDM92] R. D. Braatz, P. M. Young, J. C. Doyle and M. Morari, "Computational Complexity of μ Calculation", preprint, 1992.
- [CT89] C.-S. Chow and J.N. Tsitsiklis, "The Complexity of Dynamic Programming", *Journal of Complexity*, Vol. 5, 1989, pp. 466-488.
- [CT91] C.-S. Chow and J.N. Tsitsiklis, "An Optimal One-Way Multigrid Algorithm for Discrete-Time Stochastic Control", *IEEE Transactions on Automatic Control*, Vol. AC-36, No. 8, 1991, pp. 898-914.
- [MR72] J. Marschak and R. Radner, *The Economic Theory of Teams*, Yale Univ. Press, 1972.
- [N92] A. S. Nemirovsky, "Several NP -hard Problems Arising in Robust Stability Analysis", preprint, 1992.

- [PT82] C.H. Papadimitriou and J.N. Tsitsiklis, “On the Complexity of Designing Distributed Protocols”, *Information and Control*, Vol. 53, No. 3, 1982, pp. 211–218.
- [PT86] C.H. Papadimitriou and J.N. Tsitsiklis, “Intractable Problems in Control Theory”, *SIAM Journal on Control and Optimization*, Vol. 24, No. 4, 1986, pp. 639–654.
- [PT87] C.H. Papadimitriou and J.N. Tsitsiklis, “The Complexity of Markov Decision Processes”, *Mathematics of Operations Research*, Vol. 12, No. 3, 1987, pp. 441–450.
- [PT93] C.H. Papadimitriou and J.N. Tsitsiklis, “The Complexity of Networks of Queues,” in preparation.
- [PR92] S. Poljak and J. Rohn, “Checking Robust Nonsingularity is NP-hard”, preprint.
- [RW87] P. J. Ramadge and W. M. Wonham, “Supervisory Control of a Class of Discrete-Event Processes”, *SIAM J. Control and Optimization*, Vol. 25, 1987, pp. 206–230.
- [S88] E. Sontag, “Controllability is Harder to Decide than Accessibility” *SIAM J. Control and Optimization*, Vol. 26, 1988, pp. 1106–1118.
- [SS73] R. D. Smallwood and E. J. Sondik, “The Optimal Control of Partially Observable Markov Decision Processes over a Finite Horizon”, *Operations Research*, Vol. 11, 1973, pp. 1971–1088.
- [T88] J.N. Tsitsiklis, “On the Control of Discrete Event Dynamical Systems”, *Mathematics of Control, Signals and Systems*, Vol. 2, No. 2, 1989, pp. 95–107.
- [TA85] J.N. Tsitsiklis and M. Athans, “On the Complexity of Decentralized Decision Making and Detection Problems”, *IEEE Transactions on Automatic Control*, Vol. 30, No. 5, 1985, pp. 440–446.
- [W68] H. S. Witsenhausen, “A Counterexample in Stochastic Optimum Control,” *SIAM J. Control and Optimization*, Vol. 6, pp. 138–147.