Area and Length Preserving Geometric Invariant Scale-Spaces

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Abstract

In this paper, area preserving multi-scale representations of planar curves are described. This allows *smoothing without shrinkage* at the same time preserving all the scale-space properties. The representations are obtained deforming the curve via geometric heat flows while simultaneously magnifying the plane by a homethety which keeps the enclosed area constant. When the Euclidean geometric heat flow is used, the resulting representation is Euclidean invariant, and similarly it is affine invariant when the affine one is used. The flows are geometrically intrinsic to the curve, and exactly satisfy all the basic requirements of scale-space representations. In the case of the Euclidean heat flow, it is completely local as well. The same approach is used to define length preserving geometric flows. A similarity (scale) invariant geometric heat flow is studied as well in this work. The geometric scale-spaces are implemented using an efficient numerical algorithm, and their smoothing property is demonstrated by examples as well.

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1 Introduction

Multi-scale representations and smoothing of signals have been studied now for several years since the basic work of Witkin [56] (see for example [4, 9, 13, 23, 25, 28, 29, 30, 31, 32, 36, 37, 42, 58]). In this work we deal with multi-scale representations of closed planar curves, that is, the boundaries of bounded planar shapes. We show how to derive a smoothing operation which is sometimes local, and which satisfies all the standard properties of scale-spaces *without shrinkage*.

A curve may be described as the trajectory of a point moving in the plane. Formally, we define a curve $\mathcal{C}(\cdot)$ as the map

$$\mathcal{C}(p) : S^1 \to \mathbf{R}^2.$$

$\mathcal{C}$ can be written using Cartesian coordinates, i.e., $\mathcal{C}(p) = [x(p), y(p)]^T$, where $x(\cdot)$ and $y(\cdot)$ are maps from S^1 to $\mathbf{R}$. The parametrization p determines the traveling velocity on the curve. We assume throughout this paper that all of our mappings are sufficiently smooth, so that all the relevant derivatives may be defined. We should add that these results may be generalized to non-smooth curves based on the theory of viscosity solutions. We also assume that our curves have no self-intersections, i.e., are embedded. Moreover, since the curve $\mathcal{C}(p)$ is assumed to be closed, the functions $x(p)$ and $y(p)$ are periodic.

A multi-scale representation of a given curve is obtained by deforming the curve with a "filter" $\mathcal{K}(p,t)$, where $t \in [0, \infty)$ stands for the filter scale (larger values of t correspond to signals at coarser resolutions). Therefore, given an initial planar curve $\mathcal{C}_0$, a family of processed curves $\mathcal{C}(t)$ is obtained via

$$\mathcal{C}(t) := \Omega_{\mathcal{K}(t)}[\mathcal{C}_0], \tag{1}$$

where $\Omega_{\mathcal{K}(t)}[\cdot]$ represents the action of the filter $\mathcal{K}(t)$. Note that multi-scale representations can be defined for multi-dimensional functions as well.

Not every kernel can be used in defining a scale-space. Indeed, several conditions need to be imposed on the operator $\Omega_{\mathcal{K}}$. For example, the initial curve must be recovered for $t \to 0$, i.e., $\mathcal{C}_0 = \lim_{t\to 0} \mathcal{C}(t)$. The most important requirement is the *causality criterion*, which states that no "information" is created when moving from fine to coarse scales. Information is usually represented by zero crossings of the Laplacian (or of the curvature) or extrema of the signal. The causality property is related to the *semi-group* property, which states that the signal $\mathcal{C}(t_2)$ can be obtained from $\mathcal{C}(t_1)$, $0 < t_1 < t_2$, via $\Omega_{\mathcal{K}(t_2-t_1)}$.

An important example of a kernel which satisfies the required scale-space conditions is the Gaussian kernel $\mathcal{G}(\cdot, \sigma)$, where σ, the Gaussian-variance, controls the scale. In this case, the scale-space is linear, and $x(t), y(t)$ are obtained just by convolution of x_0, y_0 with a Gaussian. The Gaussian kernel has a large literature devoted to it in the theory of scale-spaces [4, 16, 23, 28, 58]. It has a number of interesting properties, one of them being that the family of curves $\mathcal{C}(t)$ obtained from it, is the solution of the heat equation (with $\mathcal{C}_0$ as initial condition) given by

$$\frac{\partial \mathcal{C}}{\partial t} = \Delta \mathcal{C}.$$

From the Gaussian example we see that the scale-space can be obtained as the solution of a partial differential equation called an *evolution equation.* This idea was developed in a number of different papers [1, 2, 27, 42, 45, 48] for evolution equations different from the classical heat flow. In this paper, we describe a number of scale-spaces for planar curves which are obtained as solutions of nonlinear evolution equations.

The Gaussian kernel also has several undesirable properties, principally when applied to planar curves. One of these is that the filter is not intrinsic to the curve (see Section 2). This can be remedied by replacing the linear heat equation by *geometric heat flows* [18, 19, 46, 47, 48, 50]. In particular, if the *Euclidean* geometric heat flow [18, 19] is used, a scale-space invariant to rotations and translations is obtained. If the *affine* one is used [46, 47, 50], an affine invariant multi-scale representation is obtained [48]. This and other geometric heat flows are presented in forthcoming sections.

Another problem with the Gaussian kernel is that the smoothed curve shrinks when σ increases. Several approaches, discussed in Section 3, have been proposed in order to partially solve this problem for Gaussian-type kernels (or linear filters). These approaches violate basic scale-space properties. In this paper, we show that this problem can be completely solved using a variation of the geometric heat flow methodology, which keeps the area enclosed by the curve constant. The flows which we obtain, precisely satisfy all the basic scale-space requirements. In the Euclidean case, the flow is local as well. The same approach can be used for deriving length preserving heat flows. In this case, the similarity flow exhibits locality. In short, *we can get geometric smoothing without shrinkage.*

The remainder of this paper is organized as follows. Section 2 presents basic concepts on the theory of curve evolution and describes the Euclidean geometric heat flow. Section 3 deals with an area preserving variation of this flow. Basic concepts on affine differential geometry are given in Section 4. The affine invariant heat flow is described in Section 5 and its area preserving analogue is presented in Section 6. Section 7 deals with the similarity heat flow, and gives some of it properties. In Section 8, the general approach for obtaining length preserving flows is described, and examples are given. In Section 9, an efficient numerical algorithm for the computer implementation of the proposed smoothing processes is briefly described and examples are given. Concluding remarks are given in Section 10.

2 Curve evolution and the Euclidean geometric heat flow

We consider now planar curves deforming in time, where "time" represents "scale." Let $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ denote a family of closed embedded curves, where t parametrizes the family, and p parametrizes each curve. For the case of the classical heat equation, the curves deform according to the following flow:

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \frac{\partial^2 \mathcal{C}}{\partial p^2}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p). \end{cases} \tag{2}$$

As pointed out in the Introduction, $\mathcal{C}(p,t) = [x(p,t), y(p,t)]^T$, satisfying (2), can be obtained from the convolution of $x(p,0), y(p,0)$ with the Gaussian $\mathcal{G}(p,t)$ defined by

$$\mathcal{G}(p,t) := \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-p^2}{4t}\right\}. \tag{3}$$

In order to separate the geometric concept of a planar curve from its formal algebraic description, it is useful to refer to the planar curve described by $\mathcal{C}(p,t)$ as the image (trace) of $\mathcal{C}(p,t)$, denoted by Img$[\mathcal{C}(p,t)]$ [48]. Therefore, if the curve $\mathcal{C}(p,t)$ is parametrized by a new parameter w such that $w = w(p,t)$, $\frac{\partial w}{\partial p} > 0$, we obtain

$$\mathrm{Img}[\mathcal{C}(p,t)] = \mathrm{Img}[\mathcal{C}(w,t)].$$

We see that different parametrizations of the curve, will give different results in (2), i.e, different Gaussian multi-scale representations. This is an undesirable property, since parametrizations are in general arbitrary, and may not be connected with the geometry of the curve. We can attempt to solve this problem choosing a parametrization which is intrinsic to the curve, i.e., that can be computed when only Img$[\mathcal{C}]$ is given. A natural parametrization is the *Euclidean arc-length* defined by

$$v(p) := \int_0^p \parallel \frac{\partial \mathcal{C}(\xi)}{\partial \xi} \parallel d\xi,$$

and the re-parametrization is obtained via $\mathcal{C} \circ v$. This parametrization means that the curve is traveled with constant velocity, $\parallel \mathcal{C}_v \parallel \equiv 1$. The initial curve $\mathcal{C}_0(p)$ can be re-parametrized as $\mathcal{C}_0(v)$, and the Gaussian filter $\mathcal{G}(v,t)$, or the corresponding heat flow, is applied using this parameter. The problem is that the arc-length is a time-dependent parametrization, i.e., $v(p)$ depends on time. Also, with this kind of re-parametrization, some of the basic properties of scale-spaces are violated. For example, the order is not preserved, i.e., if $\mathcal{C}_0$ and $\hat{\mathcal{C}}_0$ are two initial curves, boundaries of planar shapes, such that $\mathcal{C}_0 \subset \hat{\mathcal{C}}_0$, it is not guaranteed that this order is preserved in time. Also, the semi-group property mentioned in the Introduction can be violated with this kind of re-parametrization. The theory described below solves these problems.

Assume now that the family $\mathcal{C}(p,t)$ evolves (changes) according to the following general evolution equation

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \alpha \vec{\mathcal{T}} + \beta \vec{\mathcal{N}}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p), \end{cases} \tag{4}$$

where $\vec{\mathcal{N}}$ is the inward Euclidean unit normal, $\vec{\mathcal{T}}$ is the unit tangent [55], and α and β are the tangential and normal components of the evolution velocity $\vec{v}$, respectively.

The following lemma shows that under certain conditions, the tangential velocity does not affect Img$[\cdot]$.

Lemma 1 ([14]) *Let β be a geometric quantity for a curve, i.e., a function whose definition is independent of a particular parametrization. Then a family of curves which evolves according to*

$$\mathcal{C}_t = \alpha\vec{\mathcal{T}} + \beta\vec{\mathcal{N}}$$

can be converted into the solution of

$$\mathcal{C}_t = \bar{\alpha}\vec{\mathcal{T}} + \bar{\beta}\vec{\mathcal{N}}$$

for any continuous function $\bar{\alpha}$, by changing the space parametrization of the original solution. Since β is a geometric function, $\beta = \bar{\beta}$ when the same point in the (geometric) curve is considered.

In particular, this result shows that $\mathrm{Img}[\mathcal{C}(p,t)] = \mathrm{Img}[\hat{\mathcal{C}}(w,t)]$, where $\mathcal{C}(p,t)$ and $\hat{\mathcal{C}}(w,t)$ are the solutions of

$$\mathcal{C}_t = \alpha\vec{\mathcal{T}} + \beta\vec{\mathcal{N}}$$

and

$$\hat{\mathcal{C}}_t = \bar{\beta}\vec{\mathcal{N}},$$

respectively. For proofs of the lemma, see [14] and [48].

In other words, Lemma 1 means that if the normal component of the velocity is a geometric function of the curve, then $\mathrm{Img}[\cdot]$ (which represents the "geometry" of the curve) is only affected by this normal component. The tangential component affects only the parametrization, and not $\mathrm{Img}[\cdot]$ (which is independent of the parametrization by definition). Therefore, assuming that the normal component β of $\vec{\nu}$ (the curve evolution velocity) in (4) does not depend on the curve parametrization, we can consider the evolution equation

$$\frac{\partial C}{\partial t} = \beta\vec{\mathcal{N}}, \tag{5}$$

where $\beta = \vec{\nu}\cdot\vec{\mathcal{N}}$, i.e., the projection of the velocity vector on the normal direction.

The evolution (5) was studied by different researchers for different functions β. A key evolution equation is the one obtained for $\beta = \kappa$, where κ is the Euclidean curvature defined by [55]

$$\kappa := \left\| \frac{\partial^2 C}{\partial v^2} \right\|.$$

In this case, the flow is given by

$$\frac{\partial C}{\partial t} = \kappa\vec{\mathcal{N}}. \tag{6}$$

Equation (6) has its origins in physical phenomena [3, 17, 20]. It is called the *Euclidean shortening flow*, since the Euclidean perimeter shrinks as fast as possible when the curve evolves according to (6) [20]. Gage and Hamilton [18] proved that a planar embedded convex curve converges to a round point when evolving according to (6). (The term "round point" has the following meaning: Let $\mathcal{C}(t)$ be the curve at time t. Dilate $\mathcal{C}(t)$ to get a new curve $\tilde{\mathcal{C}}(t)$ centered at the origin and enclosing area π. Then we say that $\mathcal{C}(t)$ converges to a round point provided the dilated curves converge to the unit circle.) Grayson [19] proved that a planar embedded smooth non-convex curve, remains smooth and simple, and converges to a convex one, and from there to a round point from the Gage and Hamilton result. Note that in spite of the local character of the evolution, global properties are obtained, which is a very interesting feature of this flow. For other results related to the Euclidean shortening flow, see [3, 14, 18, 19, 20].

Next note that if v denotes the Euclidean arc-length, then [55]

$$\kappa\vec{\mathcal{N}} = \frac{\partial^2 \mathcal{C}}{\partial v^2}.$$

Therefore, equation (6) can be written as

$$\mathcal{C}_t = \mathcal{C}_{vv}. \tag{7}$$

Note that equation (7) is not linear, since v is a function of time (the arc-length gives a time-dependent parametrization). Equation (7) is also called the *(Euclidean) geometric heat flow* (compare it with the classical heat equation (2)).

Equation (7) (or (6)) has been proposed by different researchers for defining a multi-scale representation of closed curves [25, 37, 57] (see [37] for extended analysis). Note that in contrast with the classical heat flow, the Euclidean geometric one defines an intrinsic, geometric, multi-scale representation. Of course, in order to complete the theory, we must prove that all the basic properties required for a scale-space hold for the flow (7). The following lemmas do precisely this. This lemmas are obtained directly from [18, 19] on the Euclidean geometric heat flow, and [3] on more general curvature dependent flows.

Lemma 2 *Let*

$$\bar{\kappa}(t) := \oint |\kappa(p,t)| dv\,,$$

be the total absolute (Euclidean) curvature. Assume that $\mathcal{C}$ *evolves according to (6). Then,*

$$\bar{\kappa}(t) \leq \bar{\kappa}(0).$$

This result shows that the curve is getting more and more smooth when deforming via the Euclidean shortening flow, until it becomes convex according to [19].

Lemma 3 *The number of vertices, i.e., the extrema of Euclidean curvature, is a non-increasing function of time.*

Lemma 4 *The number of self-intersections is a non-increasing function of time.*

Lemma 5 *For a given local Cartesian system of coordinates, local maxima of y and κ decreases, and local minima increases. (The evolutions of y and κ are governed by parabolic equations for which the Maximum Principle [43] hold.)*

Lemma 6 *The number of inflection points, zero-crossings of the curvature, decreases with time (scale parameter).*

Lemma 7 *Let $\mathcal{C}(\cdot, t)$ satisfy (6). Then $\mathcal{C}(\cdot, t) \to \mathcal{C}(\cdot, 0)$ as $t \to 0$.*

Assume now that $\mathcal{C}_0$ and $\hat{\mathcal{C}}_0$ are initial curves such that $\mathcal{C}_0 \subseteq \hat{\mathcal{C}}_0$. It is usually also required from scale-spaces, to preserve the order, i.e., it is required that $\mathcal{C}(t) \subseteq \hat{\mathcal{C}}(t)$ for all $t \geq 0$. Of course, for the Euclidean geometric heat flow this relation holds as well:

Lemma 8 *Given a local Cartesian system such that $y_0 \leq \hat{y}_0$, then y_0 and $\hat{y}_0$ separate immediately, and the functions never cross each other (when the curve evolves according to (6)). Also, the order, as defined above, is preserved in the flow.*

It is important to note that the above properties follow principally from the fact that the studied evolution equations are parabolic. Other related properties can be proved based on this as well (see for example [1]). Further, we note that based on the theory of viscosity solutions [12], the Euclidean geometric heat flow is well defined for non-smooth initial curves [1, 10, 15].

Now the flow (7) is Euclidean invariant, i.e., admits solutions invariant to rotations and translations. This flow was extended to the affine group in [46, 47, 48, 50]. The basic results concerning the affine invariant flow will be described below.

3 Euclidean geometric heat flow without shrinkage

In the previous section, we described the Euclidean geometric heat flow, which can be used to replace the classical heat flow or Gaussian filtering in order to obtain an intrinsic scale-space for planar curves. In this section, we will show how to modify this flow in order to keep the area enclosed by the evolving curve constant.

A curve deforming according to the classical heat flow shrinks. This is due to the fact that the Gaussian filter also affects low frequencies of the curve coordinate functions [39]. Oliensis [39] proposed to change the Gaussian kernel by a filter which is closer to the ideal low pass filter. This way, low frequencies are less affected, and less shrinkage is obtained. With this approach, which is also non-intrinsic, the semi-group property holds just approximately. Note that in [4, 58] (see also [1]), it was proved that filtering with a Gaussian kernel is the unique linear operation for which the causality criterion holds, i.e., zero-crossings (or maxima) are not created at non-zero scales. Therefore, the approach presented in [39], which is closed related to wavelet approaches [34, 35], violates this important principle.

Lowe [33] proposes to estimate the amount of shrinkage and to compensate for it. The estimate is based on the amount of smoothing (σ) and the curvature. This approach, which only reduces the shrinkage problem, is again non-intrinsic, since it is based on Gaussian filtering, and works only for small rates of change. The semi-group property is violated as well.

Horn and Weldon [22] also investigated the shrinkage problem, but only for convex curves. In their approach, the curve is represented by its extended circular image, which is the radius of curvature of the given curve as a function of the curve orientation. The scale-space is obtained by filtering this representation.

We now show how to solve the shrinkage problem with the Euclidean geometric heat flow. In forthcoming sections, we will present the solution for other geometric scale-spaces as well. It is important to know that in the approach proposed below, the enclosed area is conserved exactly. (In Section 8, we will show how to keep the length constant.)

When a closed curve evolves according to (5), it is easy to prove [17] that the enclosed area $\mathbf{A}$ evolves according to

$$\frac{\partial \mathbf{A}}{\partial t} = -\oint \beta dv. \tag{8}$$

Therefore, in the case of the Euclidean geometric heat flow we obtain ($\beta = \kappa$)

$$\frac{\partial \mathbf{A}}{\partial t} = -2\pi, \tag{9}$$

and the area decreases. Moreover

$$\mathbf{A}(t) = \mathbf{A}_0 - 2\pi t,$$

where $\mathbf{A}_0$ is the area enclosed by the initial curve $\mathcal{C}_0$. As pointed out in [17, 18, 19], curves evolving according to (6) can be normalized in order to keep constant area. The normalization process is given by a change of the time scale, from t to τ, such that a new curve is obtained via

$$\tilde{\mathcal{C}}(\tau) := \psi(t)\mathcal{C}(t), \tag{10}$$

where $\psi(t)$ represents the normalization factor (time scaling). (The equation can be normalized so that the point $\mathcal{P}$ to which $\mathcal{C}(t)$ shrinks is taken as the origin.) In the Euclidean case, $\psi(t)$ is selected such that

$$\psi^2(t) = \frac{\partial \tau}{\partial t}. \tag{11}$$

The new time scale τ must be chosen to obtain $\tilde{\mathbf{A}}_\tau \equiv 0$. Define the collapse time T, such that $\lim_{t\to T} \mathbf{A}(t) \equiv 0$. Then,

$$T = \frac{\mathbf{A}_0}{2\pi}.$$

Let

$$\tau(t) = -T \ln(T - t). \tag{12}$$

Then, since the area of $\tilde{\mathcal{C}}$ and $\mathcal{C}$ are related by the square of the normalization factor $\psi(t) = \left(\frac{\partial \tau}{\partial t}\right)^{1/2}$, $\tilde{\mathbf{A}}_\tau \equiv 0$ for the time scaling given by (12). The evolution of $\tilde{\mathcal{C}}$ is obtained from the evolution of $\mathcal{C}$ and the time scaling given by (12). Taking partial derivatives in (10) we have

$$\begin{aligned} \frac{\partial \tilde{\mathcal{C}}}{\partial \tau} &= \frac{\partial t}{\partial \tau}\frac{\partial \tilde{\mathcal{C}}}{\partial t} \\ &= \psi^{-2}(\psi_t \mathcal{C} + \psi \mathcal{C}_t) \\ &= \psi^{-2}\psi_t \mathcal{C} + \psi^{-1}\kappa\vec{\mathcal{N}} \\ &= \psi^{-2}\psi_t \mathcal{C} + \tilde{\kappa}\vec{\mathcal{N}} \\ &= \psi^{-3}\psi_t \tilde{\mathcal{C}} + \tilde{\kappa}\vec{\mathcal{N}}. \end{aligned}$$

From Lemma 1, the flow above is geometric equivalent to

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau} = \psi^{-3}\psi_t < \tilde{\mathcal{C}}, \vec{\mathcal{N}} > \vec{\mathcal{N}} + \tilde{\kappa}\vec{\mathcal{N}}. \tag{13}$$

Define the *support function* ρ as

$$\rho := - < \mathcal{C}, \vec{\mathcal{N}} > .$$

Then, it is easy to show that

$$\mathbf{A} = \frac{1}{2}\oint \rho dv.$$

Therefore, applying the general area evolution equation (8) to the flow (13), together with the constraint $\tilde{\mathbf{A}}_\tau \equiv 0$, we obtain

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \tilde{\kappa}\vec{\mathcal{N}} - \frac{\pi\tilde{\rho}}{\tilde{\mathbf{A}}(\tau)}\vec{\mathcal{N}}. \tag{14}$$

Note that the flow (14) exists for all $0 \leq \tau < \infty$. Since $\tilde{\mathbf{A}}_\tau \equiv 0$ when $\tilde{\mathcal{C}}$ evolves according to (14), the enclosed area $\tilde{\mathbf{A}}(\tau)$ in (14) can be replaced by $\mathbf{A}_0$, obtaining

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(\tilde{\kappa} - \frac{\pi\tilde{\rho}}{\mathbf{A}_0}\right)\vec{\mathcal{N}}, \tag{15}$$

which gives a local, area preserving, flow.

Since $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are related by dilations, the flows (6) and (15) have the same geometric properties [17, 18, 19]. The properties of this flow can also be obtained directly from the general results on characterization of evolution equations given in [1]. In particular, since a curve evolving according to the Euclidean heat flow satisfies all the required properties of a multi-scale representation, so does the normalized flow.

4 Basic concepts in affine differential geometry

A general *affine* transformation in the plane ($\mathbf{R}^2$) is defined as

$$\tilde{\mathcal{X}} = A\mathcal{X} + B, \tag{16}$$

where $\mathcal{X} \in \mathbf{R}^2$ is a vector, $A \in \mathrm{GL}_2^+(\mathbf{R})$ (the group of invertible real 2×2 matrices with positive determinant) is the affine matrix, and $B \in \mathbf{R}^2$ is a translation vector. It is easy to show that transformations of the type (16) form a real algebraic group $\mathcal{A}$, called the *group of proper affine motions.* We will also consider the case of when we restrict $A \in \mathrm{SL}_2(\mathbf{R})$ (i.e., the determinant of A is 1), in which case (16) gives us the *group of special affine motions*, $\mathcal{A}_{sp}$.

In the case of Euclidean motions (A in (16) being a rotation matrix), it is well known that the Euclidean arc-length v is an Euclidean invariant parametrization, and the Euclidean curvature κ, is a differential invariant of the transformation. We present now the extension of this concepts for the affine group. For details see [6, 7, 8, 46, 55]. See also [38] for an extended literature on invariant theory and computer vision.

Let $\mathcal{C} : S^1 \to \mathbf{R}^2$ be an embedded, sufficiently smooth, curve with parameter p. A re-parametrization of $\mathcal{C}(p)$ to a new parameter s can be performed such that

$$[\mathcal{C}_s, \mathcal{C}_{ss}] = 1, \tag{17}$$

where $[X, Y]$ stands for the determinant of the 2×2 matrix whose columns are given by the vectors $X,\ Y \in \mathbf{R}^2$. This relation is invariant under proper affine transformations, and the parameter s is called, in analogy with Euclidean geometry, the *affine arc-length.* Setting

$$g(p) := [\mathcal{C}_p, \mathcal{C}_{pp}]^{1/3}, \tag{18}$$

the parameter s is explicitly given by

$$s(p) = \int_0^p g(\xi) d\xi. \tag{19}$$

The *affine tangent* and *affine normal* are defined as

$$\vec{\mathbf{T}} := \mathcal{C}_s, \tag{20}$$

$$\vec{\mathbf{N}} := \mathcal{C}_{ss}. \tag{21}$$

Note that in the above standard definitions, we have assumed (of course) that g (the *affine metric*) is different from zero at each point of the curve, i.e., the curve has no inflection points. In general, affine differential geometry is defined just for convex curves [8]. We will show how to overcome this problem for the evolution of non-convex curves.

By differentiating (17) we obtain

$$[\mathcal{C}_s, \mathcal{C}_{sss}] = 0. \tag{22}$$

Hence, the two vectors $\mathcal{C}_s$ and $\mathcal{C}_{sss}$ are linearly dependent and so there exists μ such that

$$\mathcal{C}_{sss} + \mu \mathcal{C}_s = 0. \tag{23}$$

The last equation implies

$$\mu = [\mathcal{C}_{ss}, \mathcal{C}_{sss}], \tag{24}$$

and μ is called the *affine curvature*. The affine curvature is the simplest non-trivial differential affine invariant of the curve $\mathcal{C}$ [8]. The conics have constant affine curvature. Therefore, the unique closed curve with constant affine curvature is the ellipse.

It is straightforward to show that ds, $\vec{\mathbf{T}}$, $\vec{\mathbf{N}}$, μ, and the area enclosed by a closed curve, are absolute invariants of the group $\mathcal{A}_{sp}$ of special affine motions and relative invariants of the group $\mathcal{A}$ of proper affine motions [46].

5 Affine geometric heat flow

In this section, we present the affine invariant evolution analogue of (6) (or (7)). We first present the flow for convex initial curves, and after that extend it for non-convex ones. For details see [46, 47, 48, 50].

As before, let $\mathcal{C}(p,t) : S^1 \times [0, \tau) \to \mathbf{R}^2$ be a family of curves, where p parametrizes the curve, and t parametrizes the family. Following [46], we consider the affine analogue of the Euclidean curve shortening flow for convex initial curves. In this case, $\mathcal{C}(p,t)$ satisfies the following evolution equation (compare with equation (7)):

$$\begin{aligned} \frac{\partial \mathcal{C}(p,t)}{\partial t} &= \mathcal{C}_{ss}(p,t), \\ \mathcal{C}(\cdot, 0) &= \mathcal{C}_0(\cdot). \end{aligned} \tag{25}$$

Since the affine normal $\mathcal{C}_{ss}$ is affine invariant, so is the flow (25). This flow was first presented and analyzed in [46]. We proved that, in analogy with the Euclidean heat flow, any convex curve converges to an elliptical point when evolving according to it (the curve remains convex as well). For other properties of the flow, see [46].

In order to complete the analogy between the Euclidean geometric heat flow (7), and the affine one given by (25), the theory must be extended to non-convex curves. In order to perform the extension, we have to overcome the problem of the "non-existence" of affine differential geometry for non-convex curves. We carry this out now. See [47, 50] for details.

Assume now that the family of curves $\mathcal{C}(p,t)$ evolves according to the flow

$$\frac{\partial \mathcal{C}(p,t)}{\partial t} = \begin{cases} 0 & p \text{ inflection point}, \\ \mathcal{C}_{ss} & p \text{ non-inflection point}, \end{cases} \tag{26}$$

with the corresponding initial condition

$$\mathcal{C}(p,0) = \mathcal{C}_0(p).$$

Since $\mathcal{C}_{ss}$ exists for all non-inflection points [8], (26) is well defined also for non-convex curves. Also, due to the affine invariance property of the inflection points, (26) is affine invariant.

From Lemma 1, we know that if we are interested only on the geometry of the curve, i.e., $\mathrm{Img}[\mathcal{C}]$, we can consider just the Euclidean normal component of the velocity in (25). In [46], it was proved that the Euclidean normal component of $\mathcal{C}_{ss}$ is equal to $\kappa^{1/3}$. Then, for a convex initial curve, $\mathrm{Img}[\mathcal{C}(p,t)] = \mathrm{Img}[\hat{\mathcal{C}}(w,t)]$, where $\mathcal{C}(p,t)$ is the solution of (25), and $\hat{\mathcal{C}}(w,t)$ is the solution of

$$\hat{\mathcal{C}}_t = \kappa^{1/3}\vec{\mathcal{N}}. \tag{27}$$

Since for an inflection point $q \in \mathcal{C}$, we have

$$\kappa^{1/3}(q) = 0,$$

the evolution given by (26) is the natural extension of the affine curve shortening of convex curves given by equation (25). Then, equation (26) is transformed into

$$\mathcal{C}_t = \kappa^{1/3}\vec{\mathcal{N}}. \tag{28}$$

If $\mathcal{C}$ is the solution of (26) and $\hat{\mathcal{C}}$ is the one of (28),

$$\mathrm{Img}[\mathcal{C}] = \mathrm{Img}[\hat{\mathcal{C}}],$$

and $\mathrm{Img}[\hat{\mathcal{C}}]$ is an affine invariant solution of the evolution (28). Note that the image of the curve is affine invariant, not the curve itself.

In [47] (see also [50]), we have proved that any smooth and simple non-convex curve evolving according to (28) (or (26)), remains smooth and simple, and becomes convex. From there, it converges into an ellipse from the results described above.

In [1], the authors gave an extensive characterization of the properties of multi-scale representations obtained via evolution equations. They also showed that under certain assumptions, equation (28) is unique in its affine invariance property. More precisely, for a two dimensional evolving function $\Phi : \mathbf{R}^2 \times [0,T) \to \mathbf{R}$, they investigated the class of second order evolution equations (the importance of this class follows from the scale-space characterization), and proved that the unique affine invariant flow is the one for which the level sets of Φ evolve according to (28) (see also Section 9, equation (54)). For results on the evolution of level sets, see also [10, 12, 15, 40] and references therein.

We have showed that the flow given by (26) (or (28)) is the (unique) affine analogue of the Euclidean geometric heat flow given by (7). Therefore this evolution is called the *affine geometric heat flow*. This flow defines an intrinsic, geometric, affine invariant multi-scale representation for planar curves. In [48], we analyzed this flow and showed that the multi-scale representation which we obtained, satisfies all the required scale-space properties (such as those given in Section 2 for the Euclidean flow). A number of affine invariant smoothing examples can be found in [48] as well. See also [49] for applications of this flow to image processing.

6 Affine geometric heat flow without shrinkage

From the general evolution equation for areas (8), and the flow (28), we have that when a curve evolves according to the affine geometric heat flow, the enclosed area evolves according to

$$\mathbf{A}_t = -\oint \kappa^{1/3} dv. \tag{29}$$

Following [6], we define the *affine perimeter* $\mathbf{L}$ as

$$\mathbf{L} := \oint [\mathcal{C}_p, \mathcal{C}_{pp}]^{1/3} dp.$$

Then it is easy to show that [46]

$$\mathbf{L} = \oint \kappa^{1/3} dv.$$

Therefore,

$$\mathbf{A}_t = -\mathbf{L}. \tag{30}$$

As in the Euclidean case, we define a normalized curve

$$\tilde{\mathcal{C}}(\tau) := \psi(t)\mathcal{C}(t), \tag{31}$$

such that when $\mathcal{C}$ evolves according to (28), $\tilde{\mathcal{C}}$ encloses a constant area. In this case, the time scaling is chosen such that

$$\frac{\partial \tau}{\partial t} = \psi^{4/3}. \tag{32}$$

(We see from the Euclidean and affine examples that in general, the exponent λ in $\frac{\partial\tau}{\partial t} = \psi^\lambda$ is chosen such that $\tilde{\beta} = \psi^{1-\lambda}\beta$.) Taking partial derivatives in (31), using the relations (8), (30), and (32), and constraining $\tilde{\mathbf{A}}_\tau \equiv 0$, we obtain the following geometric affine invariant, area preserving, flow:

$$\frac{\partial\tilde{\mathcal{C}}}{\partial\tau} = \left(\tilde{\kappa}^{1/3} - \frac{\tilde{\rho}\,\tilde{\mathbf{L}}}{2\tilde{\mathbf{A}}(\tau)}\right)\vec{\mathcal{N}}. \tag{33}$$

Since $\tilde{\mathbf{A}}_\tau \equiv 0$, $\tilde{\mathbf{A}}(\tau)$ in (33) can be replaced by $\mathbf{A}_0$ to obtain

$$\frac{\partial\tilde{\mathcal{C}}}{\partial\tau} = \left(\tilde{\kappa}^{1/3} - \frac{\tilde{\rho}\,\tilde{\mathbf{L}}}{2\mathbf{A}_0}\right)\vec{\mathcal{N}}. \tag{34}$$

Note that in contrast with the Euclidean area preserving flow given by equation (15), the affine one is not local. This is due to the fact that the rate of area change in the Euclidean case is constant, but in the affine case it depends on the affine perimeter.

As in the Euclidean case, the flow (34) satisfies the same geometric properties as the affine geometric heat flow (28). Therefore, it defines a geometric, affine invariant, area preserving multi-scale representation.

Again, based on the theory of viscosity solutions, the flow (28), as well as its normalized version (34), are well defined also for non-smooth curves.

7 Similarity geometric heat flow

Motivated by the Euclidean and affine geometric heat flows, a general theory for the formulation of invariant flows was described in [50]. Hence, given a certain transformation (Lie) group $\mathcal{L}$, it is shown how to obtain the corresponding invariant geometric heat flow. We briefly review the related theory now, and describe the corresponding similarity invariant flow.

We saw in previous sections that in order to obtain the Euclidean (affine) heat flow, derivatives in relation with the Euclidean (affine) arc-length are taken (instead of an arbitrary parameter as in the classical heat flow). Assume now that we want to formulate a *geometric heat-type flow* for plane curves which is invariant under a certain transformation group $\mathcal{L}$. Let r denote the group arc-length, i.e., the simplest invariant parametrization [21]. Then, the *invariant geometric heat-type flow* is given by

$$\begin{cases} \frac{\partial \mathcal{C}(p,t)}{\partial t} = \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p). \end{cases} \tag{35}$$

For linear groups, it is easy to prove that since r is an invariant of the group, so are $\mathcal{C}_{rr}$ and the flow (35). The flow is invariant for non-linear groups as well (see [50]). If $r \equiv v$ or $r \equiv s$, the Euclidean and affine heat flows are obtained, respectively.

The flow given by (35) is non-linear, since the group arc-length r is a function of time. This flow gives the invariant geometric flow of the group, and provides the invariant direction of the deformation. More general invariant flows are obtained if the group curvature χ is incorporated into the flow:

$$\begin{cases} \frac{\partial \mathcal{C}(p,t)}{\partial t} = \Psi(\chi)\frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p), \end{cases} \tag{36}$$

where $\Psi(\cdot)$ is a given function. Since the group arc-length and group curvature are the basic invariants of the group transformations, it is natural to formulate (36) as the most general geometric invariant flow.

The group normal $\mathcal{C}_{rr}$ is in general not perpendicular to the curve, i.e., it is not parallel to the Euclidean unit normal $\vec{\mathcal{N}}$. Based on Lemma 1, we know that the effective velocity is obtained by the projection of the group normal onto the Euclidean normal. Using this result, the flow (36) can be expressed in Euclidean terms by projecting the group normal onto the Euclidean normal (see [50] for a simple general procedure for computing this projection), and computing χ as a function of the Euclidean curvature κ and its derivatives [50].

We now derive the corresponding heat flow for the similarity group (rotations, translations, and isotropic scalings). We assume for the remainder of this section that our curves are strictly convex ($\kappa > 0$). Accordingly, let $\mathcal{C}$ be a smooth strictly convex plane curve, p the curve parameter, and as above let $\vec{\mathcal{N}}, \vec{\mathcal{T}}$ denote the Euclidean unit normal and tangent, respectively. Let

$$\sigma := \frac{\partial v}{\partial p}$$

be the speed of parametrization, so that

$$\mathcal{C}_p = \sigma\vec{\mathcal{T}}, \;\; \mathcal{C}_{pp} = \dot{\sigma}\vec{\mathcal{T}} + \sigma^2\kappa\vec{\mathcal{N}}.$$

Then clearly,

$$\mathcal{C}_p \cdot \mathcal{C}_p = \sigma^2,$$

$$[\mathcal{C}_p, \mathcal{C}_{pp}] = \sigma^3\kappa.$$

For the similarity group (in order to make the Euclidean evolution scale-invariant), we take a parametrization p such that

$$\mathcal{C}_p \cdot \mathcal{C}_p = [\mathcal{C}_p, \mathcal{C}_{pp}],$$

which implies that

$$\sigma = 1/\kappa.$$

Therefore the similarity group invariant arc-length is the standard angle parameter θ, since

$$\frac{d\theta}{dv} = \kappa,$$

where v is the Euclidean arc-length. (Note that $\vec{\mathcal{T}} = [\cos\theta, \sin\theta]^T$.) Thus the *similarity normal* is $\mathcal{C}_{\theta\theta}$, and the *similarity invariant flow* is

$$\mathcal{C}_t = \mathcal{C}_{\theta\theta}. \tag{37}$$

Projecting the similarity normal into the Euclidean normal direction, the following flow is obtained

$$\mathcal{C}_t = \frac{1}{\kappa}\vec{\mathcal{N}}, \tag{38}$$

and both (37) and (38) are geometric equivalent flows.

Instead of looking at the flow given by (38) (which may develop singularities), we reverse the direction of the flow, and look at the expanding flow given by

$$\frac{\partial \mathcal{C}}{\partial t} = -\frac{1}{\kappa}\vec{\mathcal{N}}. \tag{39}$$

For this flow, we have the following results:

Lemma 9 *The following evolution equations are obtained for a curve evolving according to (39) (see [18, 19] for the corresponding equations for the Euclidean heat flow and [46, 48, 50] for the affine one):*

1. *Evolution of Euclidean metric m ($dp = m dv$, where v is the Euclidean arc-length):*

$$m_t = m.$$

2. *Evolution of Euclidean tangent $\vec{\mathcal{T}}$:*

$$\vec{\mathcal{T}}_t = -\frac{\kappa_v}{\kappa^2}\vec{\mathcal{N}}.$$

3. *Evolution of Euclidean normal $\vec{\mathcal{N}}$:*

$$\vec{\mathcal{N}}_t = \frac{\kappa_v}{\kappa^2}\vec{\mathcal{T}}.$$

4. *Evolution of Euclidean perimeter $\mathbf{P}$:*

$$\mathbf{P}_t = \mathbf{P}.$$

5. *Evolution of area* $\mathbf{A}$*:*

$$\mathbf{A}_t = \oint \frac{1}{\kappa} dv.$$

6. *Evolution of Euclidean curvature* κ*:*

$$\kappa_t = -\left(\frac{1}{\kappa}\right)_{vv} - \kappa.$$

7. *Evolution of tangential angle* θ*:*

$$\theta_t = \frac{\kappa_v}{\kappa^2}.$$

Proof. The equations are directly obtained from general results as those described in [26]. $\square$

Theorem 1

1. *A simple convex curve remains simple and convex when evolving according to the similarity invariant flow (39).*

2. *The solution to (39) exists (and is smooth) for all* $0 \leq t < \infty$.

Proof. The proof may be obtained from the general results in [3] (see for example theorems 1.4 and 8.1). $\square$

Lemma 10 *Changing the curve parameter from* p *to* θ*, we obtain that the radius of curvature* r*,* $r := 1/\kappa$*, evolves according to*

$$r_t = r_{\theta\theta} + r. \tag{40}$$

Proof. The evolution (40) is obtained directly from the flow of κ by changing parameters and computing partial derivatives. (Note that (40) can also be used to proof Theorem 1.) $\square$

Theorem 2 *A simple (smooth) convex curve converges to a disk when evolving according to (39).*

Proof. The result is obtained analyzing the evolutions of θ and of the radius of curvature. Using separation of variables it is easy to prove that θ converges to the value in a disk, and r converges to a constant value. Since the curve remains convex and simple, we get the required conclusion. $\square$

It is important to note that in contrast with (39), (38) can deform a curve towards singularities (shocks [53]). Since (39) can be seen as a smoothing process (heat flow), the inverse equation can be seen as an enhancement process. The importance of this for image processing, as well as the extension of the theory to non-convex curves, is currently under investigation.

7.1 Similarity geometric heat flow without shrinkage

As for the Euclidean and affine flows, we can derive the area preserving flow analogue to (39). In this case, since

$$A_t = \oint \kappa^{-1} dv,$$

the non-shrinking heat flow is given by

$$\frac{\partial \tilde{C}}{\partial \tau} = \left(-\tilde{\kappa}^{-1} + \frac{\oint \tilde{\kappa}^{-1}}{2\mathbf{A}_0} \tilde{\rho} \right) \vec{\mathcal{N}}. \tag{41}$$

As in the affine case, this flow is not local. In the next section, we will derive a different non-shrinking similarity flow (length preserving) which is completely local.

8 Length preserving geometric flows

Similar techniques to those presented in previous sections, can be used in order to keep fixed other curve characteristics, e.g., the Euclidean length $\mathbf{P}$. In this case, when C evolves according to the general geometric flow

$$\frac{\partial C}{\partial t} = \beta \vec{\mathcal{N}},$$

and

$$\tilde{C}(\tau) := \psi(t) C(t), \tag{42}$$

we obtain the following length preserving geometric flow:

$$\frac{\partial \tilde{C}}{\partial \tau}(p, \tau) = \left(\tilde{\beta} - \frac{\oint \tilde{\beta} \tilde{\kappa}}{\mathbf{P}_0} \tilde{\rho} \right) \vec{\mathcal{N}}. \tag{43}$$

The computation of (43) is performed again taking partial derivatives in (42), and using the relations (see for example [17])

$$\mathbf{P}_t = -\oint \beta \kappa dv,$$

$$\mathbf{P} = \oint \kappa \rho dv,$$

together with the constraint

$$\tilde{\mathbf{P}}_\tau \equiv 0.$$

The following flows are the corresponding length preserving Euclidean, affine, and similarity heat flows respectively:

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(\tilde{\kappa} - \frac{\oint \tilde{\kappa}^2}{\mathbf{P}_0}\tilde{\rho}\right)\vec{\mathcal{N}}, \tag{44}$$

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(\tilde{\kappa}^{1/3} - \frac{\oint \tilde{\kappa}^{4/3}}{\mathbf{P}_0}\tilde{\rho}\right)\vec{\mathcal{N}}, \tag{45}$$

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(-\tilde{\kappa}^{-1} + \tilde{\rho}\right)\vec{\mathcal{N}}. \tag{46}$$

Note that in the similarity case, the flow is completely local. Another local, length preserving flow may be obtained for the Euclidean constant motion given by

$$\mathcal{C}_t = \vec{\mathcal{N}}. \tag{47}$$

This flow, obtained taking $r \equiv v$ and $\Psi(\chi) = \chi^{-1}$ in (36), models the classical Huygens principle or morphological dilation with a disk [45] (of course, it is not a geometric heat flow of the form (35)). In this case, the rate of change of length is constant and the length preserving flow is given by

$$\frac{\partial \tilde{\mathcal{C}}}{\partial \tau}(p,\tau) = \left(1 - \frac{2\pi\tilde{\rho}}{\mathbf{P}_0}\right)\vec{\mathcal{N}}. \tag{48}$$

Note that a smooth initial curve evolving according to the Euclidean constant motion (47), as well as to the flow given by (38), can develop singularities [1, 24, 45, 50]. In this case, the physically correct weak solution of the flow is the viscosity (or *entropy* [53]) one [1, 45].

9 Numerical implementation and examples

We present now an outline of the numerical algorithm for curve evolution developed by Osher and Sethian. For more details see [40, 51, 52]. Theoretical justification of this method can be found in [10, 15]. (In [48], an extended description of the algorithm is presented as well.)

Let $\mathcal{C}(p,t)$ be the family of curves satisfying the general evolution equation:

$$\frac{\partial \mathcal{C}}{\partial t} = \beta(\kappa)\vec{\mathcal{N}}. \tag{49}$$

A number of problems must be solved when developing numerical algorithms for curve evolution equations such as (49). First of all, the numerical algorithm must approximate the evolution equation, and it must be robust. Sethian [51] proved that a simple, Lagrangian, difference approximation, requires an impractically small time step in order to achieve stability. The basic problem with Lagrangian formulations is that the marker particles on the evolving curve come very close during the evolution. These can be solved by a re-distribution of the marker particles, altering the equations of motion in a non-obvious way.

If the evolving curve developes singularities, as for example in the Euclidean constant motion given by $\beta \equiv 1$ in (49), the numerical algorithm must choose the physically correct solution from the ensemble of correct weak solutions. The natural way in this case is to choose the solution which agrees with Huygens-type principles [51]. In general, the algorithm must pick the corresponding viscosity solution [12]. The numerical algorithm must deal also with possible topological changes in the evolving curve. In the Euclidean, affine, and (reverse) similarity heat flows discussed in this paper, the evolving curve does not develops singularities and no topological changes occur. This makes the implementation of the algorithm to be discussed below much easier.

Osher and Sethian [40, 51] propose an algorithm for curve (and surface) evolution for the reliable numerical solution of the above problems (see also [10, 12, 15]). It is based on Hamilton-Jacobi theory, and thus on ideas from optimal control. The algorithm can be broken down into two steps. First, the curve is embedded in a two dimensional surface. Then the equations of motion are solved using a combination of straightforward discretization, and numerical techniques derived from hyperbolic conservation laws [41, 54].

The embedding step automatically deals with possible topological changes and adds accuracy to the implementation (avoiding the use of marker particles). This step is as follows: The curve $\mathcal{C}(p,t)$ is represented by the zero level set of a smooth and Lipschitz continuous function $\Phi : \mathbf{R}^2 \times [0,\tau) \to \mathbf{R}$. In the following, we assume that Φ is negative in the interior and positive in the exterior of the zero level set. Consider the zero level set, defined by

$$\{\vec{\mathcal{X}}(t) \in \mathbf{R}^2 \,:\, \Phi(\vec{\mathcal{X}},t) = 0\}\,. \tag{50}$$

We have to find an evolution equation of Φ, such that the evolving curve $\mathcal{C}(t)$ is given by the evolving zero level $\vec{\mathcal{X}}(t)$, i.e.,

$$\mathcal{C}(t) \equiv \vec{\mathcal{X}}(t)\,. \tag{51}$$

By differentiation (50) with respect to t we obtain:

$$\nabla\Phi(\vec{\mathcal{X}},t)\cdot\vec{\mathcal{X}}_t + \Phi_t(\vec{\mathcal{X}},t) = 0\,. \tag{52}$$

Using the relation (on the level sets)

$$\frac{\nabla\Phi}{\parallel \nabla\Phi \parallel} = -\vec{\mathcal{N}}\,, \tag{53}$$

and combining equations (49) to (53), we obtain

$$\Phi_t = \beta \parallel \nabla\Phi \parallel\,. \tag{54}$$

The curve $\mathcal{C}$, evolving according to (49), is obtained by the zero level set of the function Φ, which evolves according to (54). Sethian [51] called this scheme an *Eulerian formulation* for front propagation, because it is written in terms of a fixed coordinate system.

The second step of the algorithm consists of the discretization of the equation (54). If singularities cannot develop during the evolution, as in the Euclidean or affine heat flows, a straightforward discretization can be performed [40]. If singularities can develop, as in the case of $\beta \equiv 1$, a special discretization must be implemented. In this case, the implementation of the evolution of Φ is based principally on numerical algorithms derived from the theory of hyperbolic conservation laws and viscosity solutions [12, 40, 41, 54], in order to obtain the "physically correct" weak solution. For velocity functions such as $\beta = 1 + \epsilon\kappa$, a combination of both methods is used.

It is important to note that the discretization of the evolution equations is performed on a fixed *rectangular grid* [51] (see below). This rectangular grid can be associated with the *pixel* grid of digital images, making this discretization method natural for image processing [45].

Since the evolving curve is given by the level set of the function Φ, we have to find this level set (Φ is discrete now). This is done using a very simple contour finding algorithm described in [52].

We describe now the numerical algorithm for the equation (15). Since we assume that the initial curve $\mathcal{C}_0$ is smooth (and it remains smooth from the results described in this paper), we perform, as proposed in the algorithm described above [40], a straightforward discretization of the corresponding evolution (54). (Of course, more accurate numerical approximations can be implemented [12, 40].)

If the curve $\mathcal{C}$ is a level set of a function Φ, then its curvature κ can be computed as [51]:

$$\kappa = \frac{\Phi_{yy}\Phi_x^2 - 2\Phi_x\Phi_y\Phi_{xy} + \Phi_{xx}\Phi_y^2}{(\Phi_x^2 + \Phi_y^2)^{3/2}}.$$

Since

$$\| \nabla\Phi \| = (\Phi_x^2 + \Phi_y^2)^{1/2}, \quad \vec{\mathcal{N}} = -\frac{\nabla\Phi}{\| \nabla\Phi \|},$$

and in our case

$$\beta = \kappa - \frac{\pi\rho}{\mathbf{A}_0},$$

we obtain from equations (54) and (15)

$$\Phi_t = \frac{\Phi_{yy}\Phi_x^2 - 2\Phi_x\Phi_y\Phi_{xy} + \Phi_{xx}\Phi_y^2}{(\Phi_x^2 + \Phi_y^2)} + \frac{\pi}{\mathbf{A}_0}(x\Phi_x + y\Phi_y). \tag{55}$$

The area $\mathbf{A}_0$ is computed using simple methods for the computation of the area of curves defined by implicit functions. Note that this computation is performed only once.

It is important to remember that even if the level sets (or the function Φ itself) are non-smooth, equation (55) is in general well defined based on the theory of viscosity solutions [1, 5, 10, 12, 15].

Define

$$\Phi_{ij}^n := \Phi(i\Delta x, j\Delta y, n\Delta t).$$

The time derivative is implemented using forward approximation

$$\frac{\partial \Phi}{\partial t} \approx \frac{\Phi_{ij}^{n+1} - \Phi_{ij}^n}{\Delta t},$$

and the space derivatives are implemented using central derivatives

$$\frac{\partial \Phi}{\partial x} \approx \frac{\Phi_{i+1,j}^n - \Phi_{i-1,j}^n}{2\Delta x},$$

$$\frac{\partial^2 \Phi}{\partial x^2} \approx \frac{\Phi_{i+1,j}^n - 2\Phi_{i,j}^n + \Phi_{i-1,j}^n}{\Delta x^2},$$

$$\frac{\partial^2 \Phi}{\partial x \partial y} \approx \frac{\Phi_{i+1,j+1}^n + \Phi_{i-1,j-1}^n - \Phi_{i+1,j-1}^n - \Phi_{i-1,j+1}^n}{4\Delta x^2}, \ \Delta x = \Delta y.$$

Then for implementing the normalized Euclidean scale-space (15), we have to find an embedding function Φ according to the conditions given above, and to solve equation (55) using the numerical approximations just given. The evolving curve is obtained from the zero level set of Φ. Examples are presented in Figures 1 and 2.

For the other normalized geometric flows, similar numerical implementations are performed. For example, for the affine scale-space given by equation (34), the affine perimeter $\mathbf{L}$ can be computed as

$$\sum_i \frac{\kappa_i^{1/3} + \kappa_{i+1}^{1/3}}{2} \Delta v_{i,i+1},$$

where κ_i is the curvature at the discrete contour point i, and $\Delta v_{i,i+1}$ is the distance between two consecutive contour points.

Figure 3 presents a simple example of the Euclidean constant motion (morphological dilation) and its length preserving corresponding flow.

10 Concluding remarks

In this paper, area preserving multi-scale representations for planar shapes were described. The representations are obtained by deforming the curve via the geometric heat flows while simultaneously magnifying the plane by a homethety which keeps the enclosed area constant. When the Euclidean flow is used, the scale-space is invariant to rotations. The use of the affine flow adds invariance to affine transformations. The flow is geometrically intrinsic to the curve, and exactly satisfies all the required properties of scale-spaces. For the Euclidean case, the flow is completely local as well. Based on an efficient numerical algorithm for curve evolution implementation, examples showing the smoothing property of the proposed flows were presented.

The same approach was used to derive length preserving geometric (heat) flows. In this case, locality is obtained for the similarity heat flow, which was presented and analyzed as well. The Euclidean constant motion also exhibits locality in its length preserving form.

Similar techniques can be used in order to keep other curve characteristics constant, and to transform other geometric scale-spaces [50], into analogous area (and length) preserving ones.

Different area or length preserving flows can be proposed. For example in [17, 44], non-local preserving flows are presented motivated by physical phenomena models. The advantage of the approach here described is that the non-shrinking curve is obtained by a homothety. Therefore, the resulting normalized flow keeps all the geometric properties of the original one. The flow is also local in a number of cases.

In [2], the importance of the Euclidean geometric heat flow for image enhancement was demonstrated. This was extended for the affine geometric heat flow in [11] and [49]. We are currently investigating the use of the corresponding area (or length) preserving flows for this application as well.

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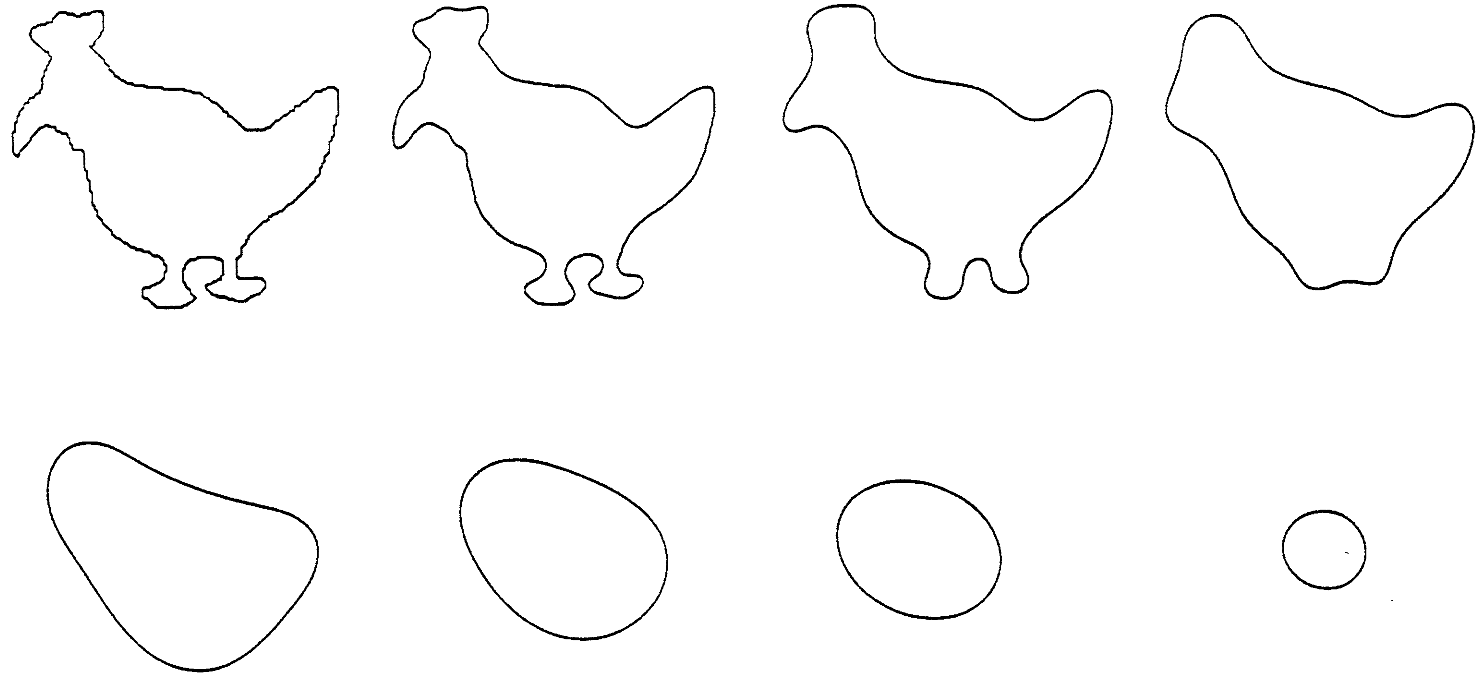

Fig. 1.a. Euclidean geometric heat flow: An evolving chicken.

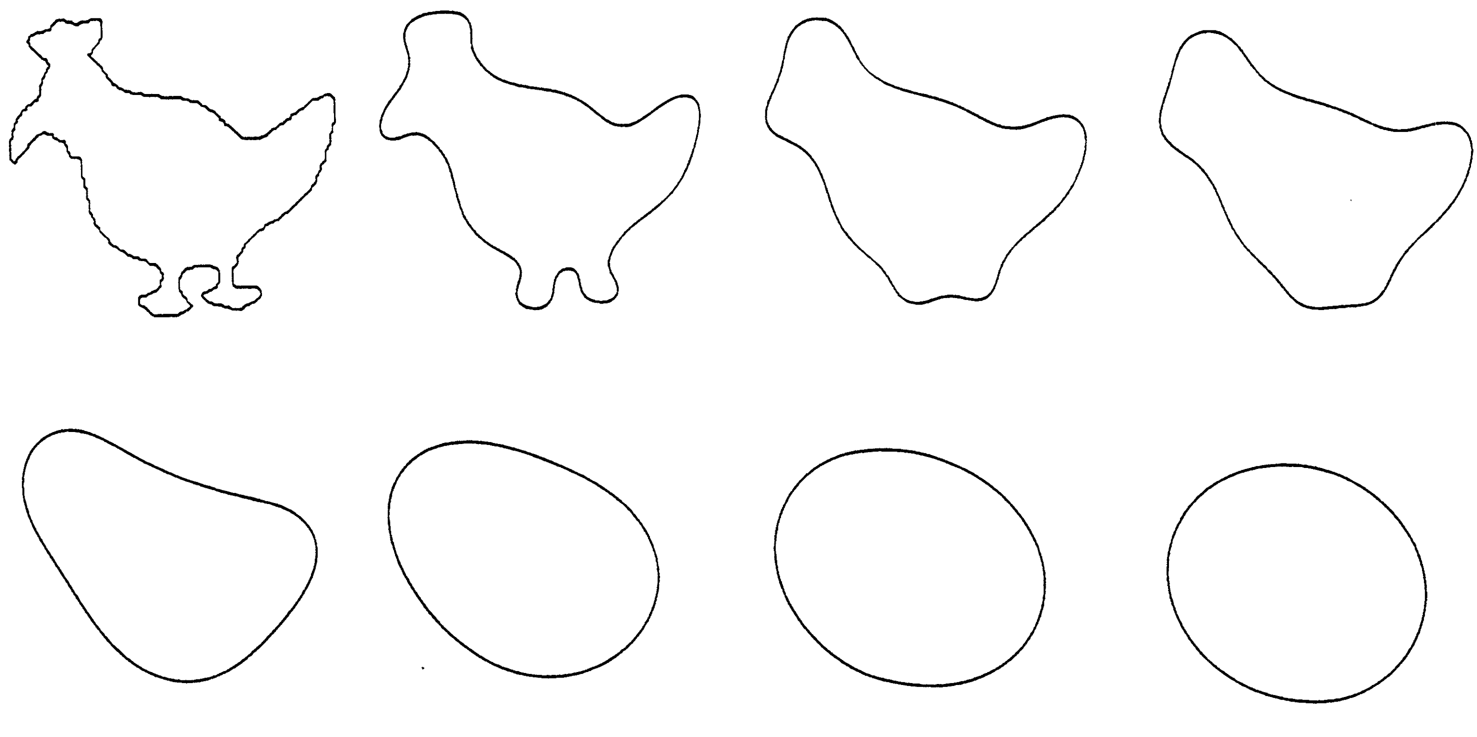

Fig. 1.b. Area preserving Euclidean geometric heat flow.

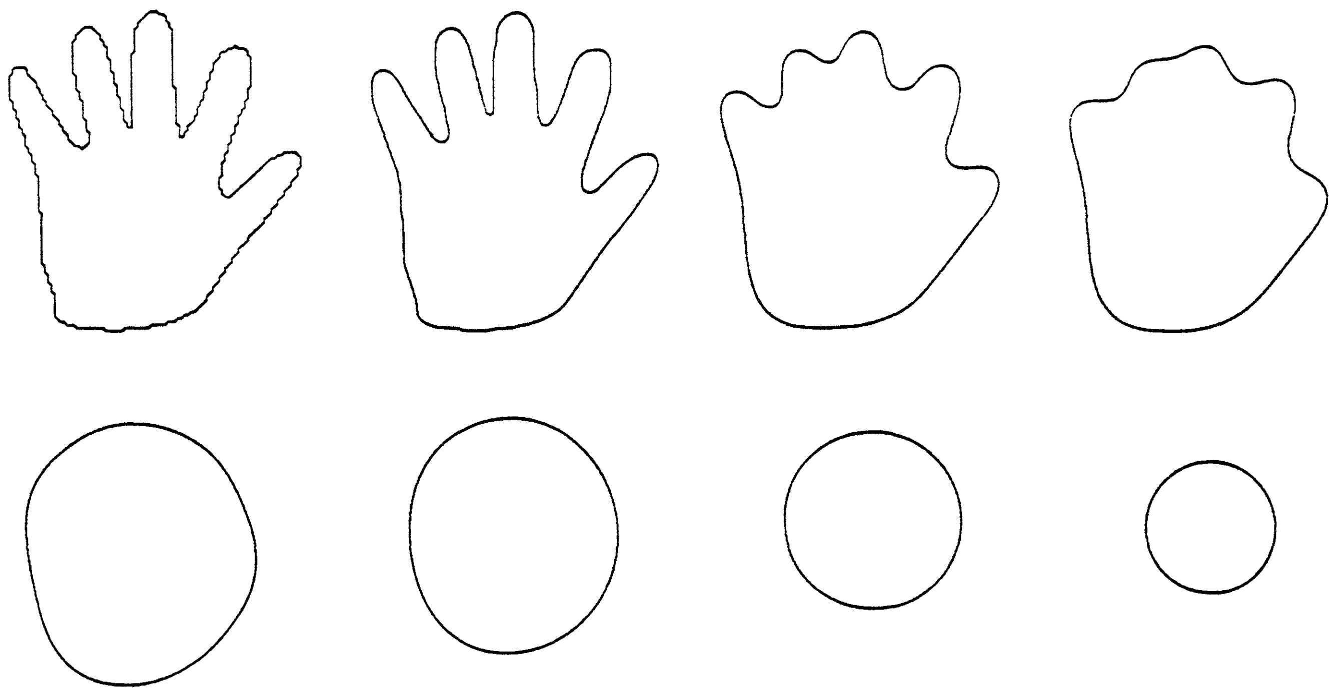

Fig. 2.a. Euclidean geometric heat flow: An evolving hand.

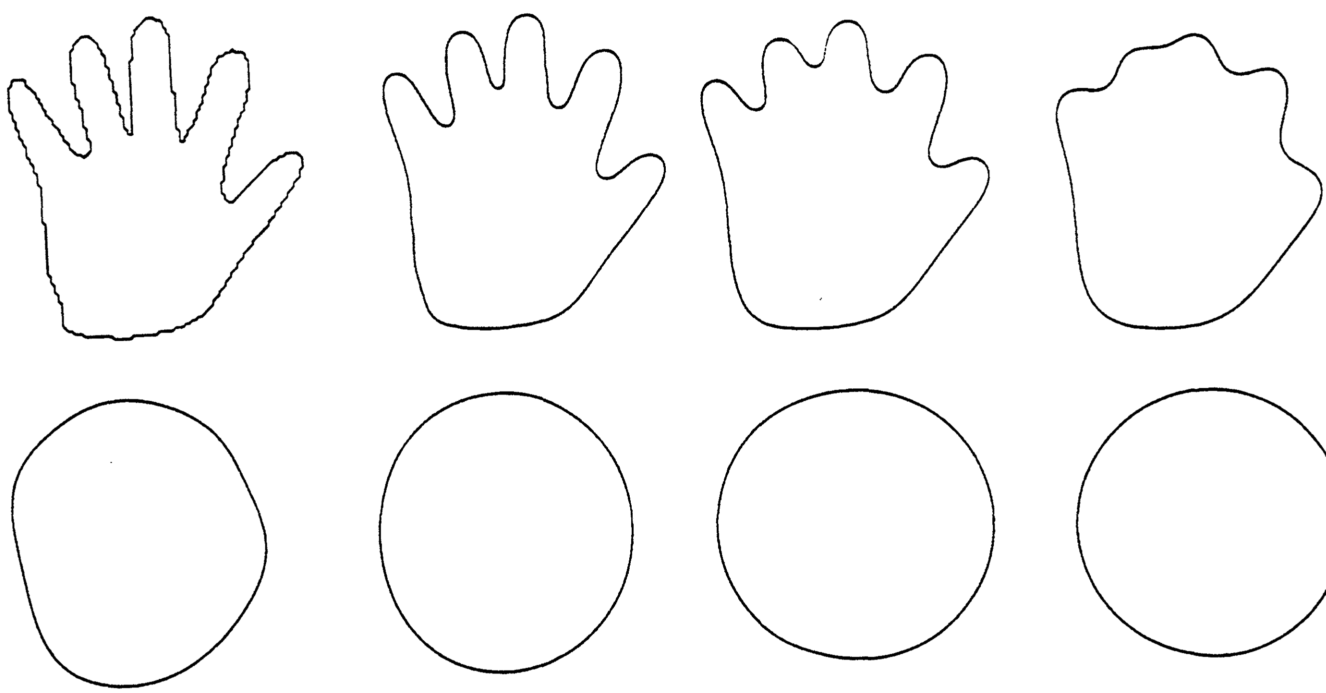

Fig. 2.b. Area preserving Euclidean geometric heat flow.

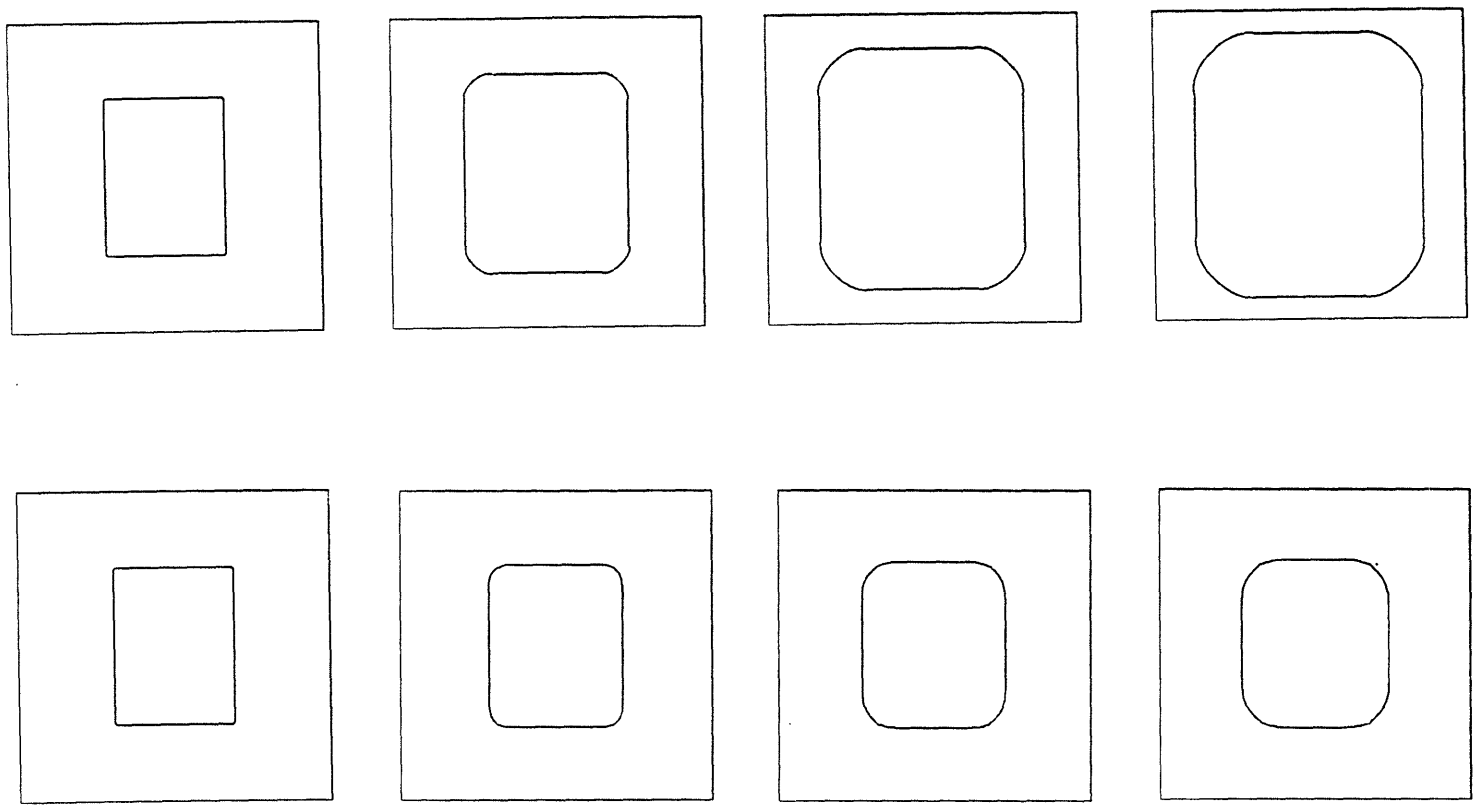

Fig. 3. Euclidean constant motion and its length preserving analogue.