July 28, 1989 LIDS-P-1896

Basis Descent Methods for Convex Essentially Smooth Optimization with Applications to Quadratic/Entropy Optimization and Resource Allocation*

by

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Abstract

Consider the problem of minimizing a convex essentially smooth function over a polyhedral set. For the special case where the cost function is strictly convex, we propose a feasible descent method for this problem that chooses the descent directions from a finite set of vectors. When the polyhedral set is the non-negative orthant or the entire space, this method reduces to a coordinate descent method which, when applied to certain dual of linearly constrained convex programs with strictly convex essentially smooth costs, contains as special cases a number of well-known methods for quadratic and entropy (either "−logx" or "xlogx") optimization. Moreover, convergence of these known methods, which (owing to the unboundedness of the dual cost) were often difficult to establish, can be inferred from a convergence result for the feasible descent method. We also give two new applications of the coordinate descent method: to "−logx" entropy optimization with linear inequality constraints and to solving certain resource allocation problems. When the cost function is not strictly convex, we propose an extension of the feasible descent method that makes descents along the elementary vectors of a certain subspace associated with the polyhedral set. The elementary vectors, instead of being stored, are generated using the dual rectification algorithm of Rockafellar. By introducing an ε-complementary slackness mechanism, we show that this extended method terminates finitely with a solution that is within ε of optimality. Because it uses the dual rectification algorithm, this method can exploit the combinatorial structure of the polyhedral set and is well suited for problems with special (e.g. network) structures.

KEY WORDS: Basis descent, coordinate descent, entropy, quadratic program, monotropic program.

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* This research is partially supported by the U.S. Army Research Office, contract DAAL03-86-K-0171 (Center for Intelligent Control Systems), and by the National Science Foundation under grant NSF-ECS-8519058.

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1. Introduction

Amongst the most important problems in optimization is that of minimizing a strictly convex cost subject to either linear equality or linear inequality constraints. A classical example is when the cost is a quadratic function. Such a problem has applications in areas including linear programming [LiP87], [Man84], [MaD88], image reconstruction [HeL78], [LeC80], and boundary value problems [CGS78], [DeT84]. Another classical example is when the cost is the entropy function (either "$-\log(x)$" or "$x\log(x)$"). This problem has applications in information theory [Ari72], [Bla72], matrix balancing [Bre67], [EvK74], [Gra71], [Kru37], [LaS81], [MaG79], [Osb60], [Sin64], [ScZ87], image reconstruction [Cen88], [Len77], speech processing [Fri75], [Jay82], [JoS84], statistical inference [DaR72], linear/nonlinear programming [Fre88], [Hua67], [Kar84], [Son88], and many other areas of science and engineering.

A popular approach to solving the above problems is to dualize the linear constraints to obtain a dual problem of the form

$$\begin{array}{ll} \text{Minimize} & h(Ep) + \langle c, p\rangle \\ \text{subject to} & p \in P, \end{array}$$

where h is a strictly convex function that is differentiable wherever it is finite, E is a matrix, c is a vector, and P is a box; and then apply a coordinate descent method to solve this dual problem. The resulting method is simple, uses little storage, has fast convergence, and, in certain cases, is highly parallelizable. Methods that take this approach include the methods of Hildreth [Hil57], Herman and Lent [HeL78] for quadratic programs, the methods of Kruithof [Kru37], Bachem and Korte [BaK79] and many others (see [LaS81]) for "$x\log(x)$" optimization problems, and the method of Censor, Lakshminarayanan and Lent [CLL79] for "$-\log(x)$" optimization problems. Coordinate descent methods for minimizing convex functions that are differentiable wherever they are finite have been well studied, but convergence of these methods typically require either that the function is strictly convex with bounded level sets [BeT89], [D'Es59], [Lue73], [SaS73] or that the function is finite everywhere [Pan84], [TsB87b], [Tse88a]. For the above dual problem, because h need not be finite everywhere and the matrix E need not have full column rank, the cost function $p \to h(Ep) + \langle c, p\rangle$ need not satisfy any of the above conditions for convergence. As a consequence, convergence for the above methods have been quite difficult to show and were shown only for special problem classes. A principal aim of this paper is to unify and extend these methods and the convergence results for them. In particular, we propose a method for solving the above problem when P is any <u>polyhedral</u> set. This method is a feasible descent method that chooses its descent directions from a <u>finite</u> set of vectors. This finite set of vector in turn is chosen to have the property that every contingent cone of P is generated by some subset of it. We show that this method is convergent, provided that the vectors in the set are chosen in an almost cyclic manner, and that the coordinate descent methods described in [BaK79], [CeL87], [HeL78], [Hil57], [Kru37], [LaS81] are all special cases of this method. We also give two new applications of this method: one to minimizing "–log(x)" entropy costs subject to linear inequality constraints and the other to solving certain resource allocation problems.

When the cost function h is only convex (not necessarily strictly convex) or when the contingent cones of P are such that the finite set of vectors cannot be stored in its entirety, we propose an extension of the above method. In this extended method, each descent direction is an elementary vector of a certain subspace related to P and, instead of being stored, is generated by the dual rectification algorithm of Rockafellar [Roc84]. This method also employs a certain ε-complementary slackness mechanism to ensure that each descent yields a "large" decrease in the cost. We show that, for any $\varepsilon > 0$, this method terminates finitely with a feasible solution that is, in some sense, within ε of optimality. A major advantage of this method, inherited from the dual rectification algorithm, is its ability to exploit the combinatorial structure of the polyhedral set P in much the same way as, say, the simplex method for linear programming. For example, when P is given by network flow constraints, this method can be implemented using graphical subroutines. To the best of our knowledge, the only other method (for minimizing convex differentiable functions over polyhedral sets) that can exploit the combinatorial structure of the constraint set in this manner is the convex simplex method of Zangwill [Zan69]. However, unlike our method, Zangwill's method requires a certain nondegeneracy assumption on the problem to guarantee convergence. Our method with its finite termination property is also reminiscent of the ε-subgradient methods [BeM73], [NgS84], [Roc84] in nonlinear optimization. However, in general the ε-subgradient methods appear to be more difficult to implement.

This paper is organized as follows: In Sec. 2 we describe our problem and in Sec. 3 we introduce the notion of a basis for the contingent cones of a polyhedral set. In Sec. 4 we describe our method for solving problems with strictly convex costs and analyze its convergence. In Sec. 5 we apply this method to solve the dual of, respectively, entropy optimization problems, quadratic programs, and certain resource allocation problems. We show in the process that a number of known methods are special cases of this method. In Sec. 6, we describe an extension of this method to problems with convex (not necessarily strictly convex) costs.

In our notation, all vectors are column vectors and superscript T denotes transpose. For any vector x, we denote by x_j the j-th coordinate of x. We denote by $\langle\cdot,\cdot\rangle$ the usual Euclidean inner product and, for any x in an Euclidean space, by $\|x\|$ and $\|x\|_\infty$, respectively, the L_2-norm and the L_∞-norm of x. For any convex function $f:\Re^m \to (-\infty,\infty]$, we denote $\mathrm{dom}(f) = \{ x \in \Re^m \mid f(x) < \infty \}$,

$$C_f \quad = \quad \text{interior of dom}(f),$$

and by $f^*:\Re^m \to (-\infty,\infty]$ the conjugate function of f [Roc70], i.e.

$$f^*(t) \quad = \quad \sup_x\{ \langle t, x\rangle - f(x) \}, \qquad \forall\, t.$$

Finally, for any open set C, we denote by bd(C) the set of boundary points of C (possibly empty).

2. Problem Description

Let E be an m×n matrix, c be an n-vector, $h:\Re^m \to (-\infty,\infty]$ be a convex function, and P be a polyhedral set in $\Re^n$. Consider the follow convex program:

$$\begin{array}{ll} \text{Minimize} & q(p) \\ \text{subject to} & p \in P, \end{array} \tag{2.1a}$$

where $q:\Re^n \to (-\infty,\infty]$ is the convex function given by

$$q(p) \quad = \quad h(Ep) + \langle c, p\rangle, \qquad \forall\, p \in \Re^n. \tag{2.1b}$$

We make the following standing assumptions about E, c, P and h:

Assumption A.

(a) h is closed and differentiable on $C_h \neq \emptyset$.

(b) For any $t \in \mathrm{bd}(C_h)$ and any sequence $\{t^k\}$ in C_h converging to t, it holds

$$\lim_{k\to\infty} \sup\{ \frac{h(t^k)-h(t)}{\|t^k-t\|} \} = -\infty.$$

(c) $P \cap C_q \neq \emptyset$, $\inf_{p\in P} q(p) > -\infty$, and the set $\{ Ep \mid p \in P,\ q(p) \le \zeta \}$ is bounded for all $\zeta \in \Re$.

Parts (a)-(b) of Assumption A imply, in the terminology of [Roc70], that h is essentially smooth (i.e., $C_h \neq \emptyset$, h is differentiable on C_h, and $\|\nabla h(t^k)\| \to \infty$ for any $t \in \mathrm{bd}(C_h)$ and any sequence $\{t^k\}$ in C_h converging to t). [To show this, it suffices to verify (see Lemma 26.2 in

[Roc70]) that $\langle \nabla h(t+\theta v), v\rangle \downarrow -\infty$ as $\theta \downarrow 0$ for any $t \in bd(C_h)$ and any v such that $t+v \in C_h$. Now, by Assumption A (b), for any $\alpha < 0$, we can find a $\hat{\theta} > 0$ such that $(h(t+\hat{\theta}v) - h(t))/\hat{\theta} \le \alpha$. This together with the fact (since h is closed) $\lim_{\theta\downarrow 0}\inf\{h(t+\theta v)\} \ge h(t)$ implies that there exists a $\bar{\theta} \in (0,\hat{\theta})$ such that $(h(t+\hat{\theta}v) - h(t+\bar{\theta}v))/(\hat{\theta} - \bar{\theta}) \le \alpha/2$. The left hand side of this inequality, by the convexity of h, is greater than or equal to $\langle \nabla h(t+\bar{\theta}v), v\rangle$, so we have $\langle \nabla h(t+\bar{\theta}v), v\rangle \le \alpha/2$. Since the choice of α was arbitrary and $\langle \nabla h(t+\theta v), v\rangle$ is monotonically decreasing with θ, this shows $\langle \nabla h(t+\theta v), v\rangle \downarrow -\infty$ as $\theta \downarrow 0$.] In fact, in the case where h is separable, i.e.

$$h(t) \quad = \sum_{j=1}^{m} h_j(t_j), \tag{2.2}$$

for some closed convex functions $h_j:\mathfrak{R}\to(-\infty,\infty]$, parts (a)-(b) of Assumption A is equivalent to the assumption that h is closed and essentially smooth. [To see this, note that if h is essentially smooth, then each h_j is essentially smooth, so that if ζ' is a left (right) boundary point of C_{h_j}, then $\lim_{\zeta\downarrow\zeta'}\{\nabla h_j(\zeta)\} = -\infty$ ($\lim_{\zeta\uparrow\zeta'}\{\nabla h_j(\zeta)\} = \infty$). This together with the fact (cf. convexity of h_j) $h_j(\zeta) - h_j(\zeta') \le \nabla h_j(\zeta)(\zeta - \zeta')$ for all $\zeta \in C_{h_j}$ implies $(h_j(\zeta) - h_j(\zeta'))/|\zeta - \zeta'| \to -\infty$ as $\zeta \to \zeta'$, $\zeta \in C_{h_j}$, so that h satisfies Assumption A (b). Since h is closed and essentially smooth, h clearly satisfies Assumption A (a).] Since h is essentially smooth and q is h composed with a linear transformation, it is easily seen that q is also essentially smooth.

As we shall see, the essential smoothness of h and q plays a crucial role in our methods and their convergence analysis. To verify part (b) seems to be difficult, but this is not exactly true. For example, part (b) trivially holds if h is real-valued (so that C_h has no boundary point) or if

$$h(t) \to \infty \text{ whenever t approaches a point in } bd(C_h) \tag{2.3}$$

(so that dom(h) equals C_h), which, as we shall see, holds in many important cases. [For example, consider the problem $\min\{\ g(t) \mid At \ge b,\ t \ge 0\ \}$, where $g:\mathfrak{R}^m\to\mathfrak{R}$ is a convex differentiable function, A is a k×m matrix, and b is a k-vector. If we use a penalty method [Ber82] to solve this problem and if we penalize the constraints $t \ge 0$ with a convex essentially smooth function $\psi:\mathfrak{R}^m\to(-\infty,\infty]$ that is finite on $(0,\infty)^m$. and tends to ∞ as t approaches a boundary point of $(0,\infty)^m$, then the penalized problem $\min\{\ g(t) + \psi(t) \mid At \ge b\ \}$ is of the form (2.1a)-(2.1b) and the cost function $h = g + \psi$ satisfies (2.3). For another example, consider the linear program $\min\{\ \langle c, p\rangle \mid Bp \ge d,\ Cp \ge f\ \}$, where B, C are respectively k×n, m×n matrices and c, d, f are vectors of suitable dimensions. If we use a penalty method to solve this problem and if we penalize the constraints $Cp \ge f$ with the above ψ, then the penalized problem $\min\{\ \langle c, p\rangle + \psi(Cp - f) \mid Bp \ge d\ \}$

is also of the form (2.1a)-(2.1b) and the cost function h given by $h(t) = \psi(t - f)$ satisfies (2.3).] On the other hand, a convex functions that is essentially smooth need not satisfy (2.3). A simple example of this is the function $h:\Re \to (-\infty,\infty]$ given by $h(t) = t \log(t)$ if $t \geq 0$ and $h(t) = \infty$ otherwise. Another example is the function $h:\Re \to (-\infty,\infty]$ given by $h(t) = -t^{\alpha}$ if $t \geq 0$ and $h(t) = \infty$ otherwise, where α is some fixed scalar in (0,1).

Part (c) of Assumption A, together with the fact that q is closed, implies that (2.1a)-(2.1b) has an optimal solution. [To see this, note that since $P \cap C_q \neq \emptyset$, then $\inf_{p \in P} q(p) < \infty$ so that $\inf_{p \in P} q(p)$ is finite. Consider any sequence $\{p^0, p^1, \ldots\}$ in P such that $\{q(p^r)\} \downarrow \inf_{p \in P} q(p)$. Since $\{ Ep \mid p \in P, q(p) \leq q(p^0) \}$ is bounded, then $\{Ep^r\}$ is bounded and, by Lemma 2 in Appendix B, $\{\langle c, p^r \rangle\}$ is bounded. For every r, let $\bar{p}^r$ denote the least L-norm solution of the feasible linear system $Ep = Ep^r$, $\langle c, p \rangle = \langle c, p^r \rangle$, $p \in P$. Then $q(\bar{p}^r) = q(p^r)$ for all $r \in R$ and, by Lemma 1 in Appendix A, $\{\bar{p}^r\}$ is bounded. Let p^{∞} be any limit point of $\{\bar{p}^r\}_{r \in R}$. Then $p^{\infty} \in P$ and (since q is closed) $q(p^{\infty}) \leq \lim_{r \to \infty} \inf\{q(p^r)\} = \inf_{p \in P} q(p)$, so that p^{∞} is an optimal solution of (2.1a)-(2.1b).] Moreover, since $P \cap C_q \neq \emptyset$ and q is essentially smooth, every optimal solution of (2.1a)-(2.1b) is in $P \cap C_q$. [To see this, note from Lemma 26.2 in [Roc70] that, for any $p \in P \cap \text{bd}(C_q)$, it holds that $\langle \nabla q(p+\theta(p'-p)), p'-p \rangle \downarrow -\infty$ as $\theta \downarrow 0$, where p' is any point in $P \cap C_q$, so that there exists a point p'' on the line segment joining p and p' for which $q(p'') < q(p)$. Since both p and p' are in P and P is convex, p'' is also in P.] This latter fact is quite useful since it implies that ∇q is continuous near an optimal solution. The assumption in part (c) that the set $\{ Ep \mid p \in P, q(p) \leq \zeta \}$ is bounded for every ζ may be difficult to verify in general (except when P is bounded), but, as we shall see, it holds naturally for many problems of practical interest.

3. Basis for the Contingent Cones of a Polyhedral Set

Central to the development of our methods is the notion of a "basis" for the contingent cones of the polyhedral set P. For each $p \in P$, let K(p) denote the contingent cone [Roc81] of P at p, i.e.

$$K(p) \;=\; \{ u \in \Re^n \mid p + \theta u \in P \text{ for some } \theta > 0 \}.$$

[Loosely speaking, K(p) is the set of directions at p that point into P.] We say that a finite set of vectors $u^1, u^2, \ldots, u^b$ in $\Re^n$ form a basis for the contingent cones of P (abbreviated as "basis for K(P)") if, for any $p \in P$, K(p) is generated by some subset of $\{\pm u^1, \ldots, \pm u^b\}$. [We say that a convex cone is generated by vectors $\tilde{u}^1, \ldots, \tilde{u}^k$ if it is precisely the set $\{ \lambda_1 \tilde{u}^1 + \ldots + \lambda_k \tilde{u}^k \mid \lambda_1 \geq 0,$

..., $\lambda_k \geq 0$ }.] For example, if P is a box in $\Re^n$ with its sides parallel to the coordinate axes, then the coordinate vectors in $\Re^n$ form a basis for K(P). If P is a subspace of $\Re^n$ spanned by some linearly independent vectors $u^1, \ldots, u^b$, then these vectors form a basis for K(P) (both in our sense and in the usual sense of linear algebra).

A property of basis that is most useful to us is the following:

Fact 1. Let $\{u^1, \ldots, u^b\}$ be any basis for K(P). For any $p \in P \cap C_q$, if there exists $u \in K(p)$ such that $\langle \nabla q(p), u \rangle < 0$, then there exists $\tilde{u} \in \{\pm u^1, \ldots, \pm u^b\} \cap K(p)$ such that $\langle \nabla q(p), \tilde{u} \rangle < 0$.

Proof. Since u is in the contingent cone of P at p, it can be expressed as a linear convex combination of the generators of this cone, i.e. there exist $\tilde{u}^1, \ldots, \tilde{u}^k$ from the set $\{\pm u^1, \ldots, \pm u^b\}$ and positive scalars $\lambda_1, \ldots, \lambda_k$ satisfying

$$u = \lambda_1 \tilde{u}^1 + \ldots + \lambda_k \tilde{u}^k, \qquad (3.1a)$$
$$\tilde{u}^i \in K(p), \qquad \forall\ i. \qquad (3.1b)$$

Since $\langle \nabla q(p), u \rangle < 0$, (3.1a) implies that $\langle \nabla q(p), \tilde{u}^i \rangle < 0$ for some i, which together with (3.1b) establishes our claim. **Q.E.D.**

Fact 1 states that if we are at a non-optimal feasible solution of (2.1a)-(2.1b), then it is always possible to find a feasible descent direction from amongst the basis vectors. This motivates a method for solving (2.1a)-(2.1b) whereby at each iteration one of the basis vectors is chosen and q is minimized along this direction. The two key issues related to this method are: (i) which of the basis vectors should we choose at each iteration and (ii) how should we generate the chosen basis vector? We will address these two issues in the following sections where we describe the details of our methods and analyze their convergence.

4. A Basis Descent Method for Problems with Strictly Convex Costs

In this section we consider a special case of the problem (2.1a)-(2.1b) where the cost function h is furthermore <u>strictly convex</u>. We develop a method for solving this problem based on descent along basis vectors whereby the basis vectors are chosen in an almost <u>cyclical</u> manner. Because this method cycles through all of the vectors in a basis, it is only suited for problems for which the constraint set P has a simple structure (such as a box). Nonetheless, many important problems, including the Lagrangian dual of linearly constrained problems with strictly convex essentially smooth costs, do possess such structures and are well-suited for solution by this method.

Throughout this section, we make (in addition to Assumption A) the following assumption about (2.1a)-(2.1b):

Assumption B. h is strictly convex on C_h.

Assumption B, together with the essentially smooth property of h (cf. Assumption A (a)-(b)), implies that the pair (C_h,h) is, in the terminology of Rockafellar [Roc70], a convex function of the Legendre type. A nice property of such a function, which we will use in Sec. 5, is that its conjugate function is also a convex function of the Legendre type. Assumption B also implies that the m-vector Ep is the same for all optimal solutions p of (2.1a)-(2.1b). [If both p and p′ solve (2.1a)-(2.1b), then $p\in C_q$, $p'\in C_q$ and, by convexity of q, $q(p) = q(p') = q((p+p')/2)$. This implies that $Ep\in C_h$, $Ep'\in C_h$ and $h((Ep+Ep')/2) = \frac{h(Ep)+h(Ep')}{2}$ (cf. (2.1b)), so that the strict convexity of h on C_h implies $Ep = Ep'$.] We will denote this m-vector by t^*.

Under Assumption B, part (c) of Assumption A can be simplified considerably as the following result shows:

Fact 2. Under Assumption B and parts (a)-(b) of Assumption A, part (c) of Assumption A holds if and only if $P\cap C_q \neq \emptyset$ and (2.1a)-(2.1b) has an optimal solution.

The only if part was already argued in Sec. 2. To argue the if part, it suffices to show that if $P\cap C_q \neq \emptyset$ and (2.1a)-(2.1b) has an optimal solution, then the set $\{ Ep \mid p\in P, q(p) \leq \zeta \}$ is bounded for every $\zeta\in\Re$. If this were not true, then the convex set $\{ (t,p,\zeta) \mid t = Ep, p\in P, q(p) \leq \zeta \}$ in $\Re^{m+n+1}$ has a direction of recession (v,u,0) satisfying $v \neq 0$. Then $v = Eu$ and, for any $p\in P$, it holds that $p+\theta u\in P$ and $q(p+\theta u) \leq q(p)$ for all $\theta \geq 0$. Let p be an optimal solution of (2.1a)-(2.1b). Then it furthermore holds that $q(p+\theta u) = q(p)$ for all $\theta \geq 0$, so that $h(Ep+\theta v) + \langle c,p+\theta u\rangle = h(Ep) + \langle c,p\rangle$ or, equivalently, $h(Ep+\theta v) = h(Ep) - \theta\langle c,u\rangle$ for all $\theta \geq 0$. Also, since p is an optimal solution of (2.1a)-(2.1b), p is in C_q (cf. discussion in Sec. 2), so that Ep is in C_h and, by Theorem 8.3 in [Roc70], $Ep+\theta v$ is in C_h for all $\theta \geq 0$. Since $v \neq 0$, this contradicts the assumption that h is strictly convex on C_h. We will use Fact 2 in the dual applications of Sec. 5.

We formally state our method for solving (2.1a)-(2.1b) (under Assumptions A and B) below. This method, at each iteration, chooses a basis vector and performs a line minimization of q over P along this direction:

Basis Descent Method

Iter. 0. Choose a basis $\{u^1, \ldots, u^b\}$ for K(P) and a $p^0 \in P \cap C_q$.

Iter. r. Choose a $w^r \in \{u^1,\ldots,u^b\}$. If $\langle \nabla q(p^r), w^r \rangle = 0$, set $\theta^r = 0$; otherwise compute

$$\theta^r = \operatorname{argmin}\{ q(p^r + \theta w^r) \mid p^r + \theta w^r \in P \}.$$

Set $p^{r+1} = p^r + \theta^r w^r$.

The line minimization is attained at every iteration r, for otherwise since the set $\{ Ep \mid p \in P, q(p) \le q(p^r) \}$ is bounded (cf. Assumption A (c)), either (i) w^r is a direction of recession for P and satisfies $Ew^r = 0$, $\langle c,w^r \rangle < 0$ or (ii) $-w^r$ is a direction of recession for P and satisfies $Ew^r = 0$, $\langle c,w^r \rangle > 0$. In case (i), we have $p^r + \theta w^r \in P$ for all $\theta \ge 0$ and $q(p^r + \theta w^r) = h(Ep^r) + \theta \langle c,w^r \rangle \to -\infty$ as $\theta \to \infty$, so the optimal value of (2.1a)-(2.1b) is $-\infty$. In case (ii), we also have, by a symmetric argument, that the optimal value of (2.1a)-(2.1b) is $-\infty$. Hence Assumption A (c) is contradicted in either case. Note also that p^r is in $P \cap C_q$ for all r. [It is easily seen that each p^r is in P. That p^r is also in C_q follows from an argument in Sec. 2 that showed that each optimal solution of (2.1a)-(2.1b) is in C_q.]

The above method is not convergent in general. To ensure convergence, we will consider the following restriction on the directions $w^1, w^2, \ldots$ (cf. [SaS73], [HeL78]):

Almost Cyclic Rule. There exists $\Delta \ge b$ such that $\{u^1,\ldots,u^b\} \subseteq \{w^{r+1},\ldots,w^{r+\Delta}\}$ for all r.

[Loosely speaking, the above rule requires that each basis vector be used for descent at least once every Δ successive iterations. An important special case is when the basis vectors are chosen in a cyclical manner, i.e. $\{w^0,w^1,\ldots\} = \{u^1, \ldots,u^b,u^1,\ldots,u^b,\ldots\}$.] In Sec. 6 we will consider an alternative strategy that examines only a subset of the basis vectors in the course of the method.

In the special case when $P = \Re^n$, the basis descent method is reminiscent of a method of Sargent and Sebastian [SaS73] based on descent along directions that are, in their terminology, uniformly linearly independent. [It can be shown that if the directions $w^0,w^1,\ldots$ satisfy the Almost Cyclic rule, then they are uniformly independent in their sense.] It is also reminiscent of a periodic basis ascent method of Pang [Pan84] (also see [LiP87, §4.2]). Moreover, if we choose the basis for $K(P) = \Re^n$ to comprise the coordinate vectors of $\Re^n$, then the basis descent method reduces to the classical coordinate descent method for minimizing strictly convex differentiable functions (e.g. [D'Es59], [Lue73], [SaS73]).

Convergence of coordinate descent methods for minimizing strictly convex essentially smooth functions has been fairly well studied. In particular, it is known that if P is a box and if q is a strictly convex, continuously differentiable function over P, and has compact level sets over P, then the coordinate descent method (using cyclic relaxation) applied to minimize q over P is convergent (e.g. [BeT89, Chap. 3.3.5]). For our problem however, P is any polyhedral set and q may be neither strictly convex (since E may be row rank deficient) nor continuosly differentiable over P (since C_q need not contain P). Nonetheless, by using a more delicate argument, we can show that the basis descent method is convergent in the sense that it always computes t^*. This result is stated below. We discuss its applications in Sec. 5.

Proposition 1. Let $\{p^r\}$ be a sequence of iterates generated by the basis descent method using the Almost Cyclic rule. Then $p^r \in P \cap C_q$ for all r and $\{Ep^r\} \to t^*$.

Proof. We clearly have $q(p^r) \geq q(p^{r+1})$ for all r and from the earlier discussion we also have $p^r \in P \cap C_q$ for all r. Since $q(p^r)$ is monotonically decreasing with r and q is bounded from below on P, $\{q(p^r)\}$ must converge to a finite limit, say q^∞. Let $t^r = Ep^r$ for all r. Then by Lemma 3 in Appendix C, $\{t^r\}$ is bounded and every limit point of $\{t^r\}$ is in C_h. We claim that

$$t^{r+1} - t^r \to 0.$$

To argue this, suppose the contrary. Then there exist $\varepsilon > 0$ and subsequence $R \subseteq \{1,2,\ldots\}$ such that $\|t^{r+1} - t^r\| \geq \varepsilon$ for all $r \in R$. Since $\{t^r\}$ is bounded, we will (passing into a subsequence if necessary) assume that $\{t^r\}_{r\in R}$ and $\{t^{r+1}\}_{r\in R}$ converge to, say, t' and t'' respectively. Then $t' \neq t''$ and both t' and t'' are in C_h, so that the continuity of h on C_h ([Roc70, Theorem 10.1]) implies $\{h(t^r)\}_{r\in R} \to h(t')$ and $\{h(t^{r+1})\}_{r\in R} \to h(t'')$ or, equivalently (since $q(p^r) = h(Ep^r) + \langle c, p^r\rangle$),

$$\{\langle c, p^r\rangle\}_{r\in R} \to q^\infty - h(t'), \qquad \{\langle c, p^{r+1}\rangle\}_{r\in R} \to q^\infty - h(t''). \tag{4.1}$$

Also, since p^{r+1} is obtained from p^r by performing a line search of q along the direction $p^{r+1} - p^r$, the convexity of q yields

$$q(p^{r+1}) \quad \leq \quad q((p^r + p^{r+1})/2) \quad \leq \quad q(p^r), \qquad \forall\, r.$$

Upon passing into the limit as $r \to \infty$, $r \in R$, and using (2.1b), (4.1) and the continuity of h on C_h, we obtain

$$q^\infty \quad \le \quad h((t' + t'')/2) + q^\infty - (h(t'') + h(t'))/2 \quad \le \quad q^\infty,$$

which contradicts the strict convexity of h on C_h, e.g. $h((t' + t'')/2) < (h(t'') + h(t'))/2$.

Let t^∞ be any limit point of $\{t^r\}$. We will show that $t^\infty = t^*$. Let P be expressed as $\{ p \in \Re^n \mid Ap \ge b \}$ for some k×n matrix A and k-vector b. Let R be any subsequence of $\{1,2,\ldots\}$ such that $\{t^r\}_{r \in R}$ converges to t^∞. Then since $p^r \in P$, we have $Ap^r \ge b$ for all $r \in R$. By further passing into a subsequence if necessary, we can assume that, for each $i \in \{1,\ldots,k\}$, either $A_i p^r < b_i + 1$ for all $r \in R$, or $A_i p^r \ge b_i + 1$ for all $r \in R$, where A_i denotes the i-th row of A. Let $I = \{ i \mid A_i p^r < b_i + 1$ for all $r \in R \}$. For each $r \in R$, let $\tilde{p}^r$ denote the least L_2-norm solution of the following linear system

$$Ep = t^r, \qquad A_i p = A_i p^r \quad \forall\, i \in I, \qquad A_i p \ge b_i + 1 \quad \forall\, i \notin I,$$

This linear system is feasible since p^r is a solution. Furthermore, since the right hand side of this system is bounded for all $r \in R$, the sequence $\{\tilde{p}^r\}_{r \in R}$ is bounded (cf. Lemma 1 in Appendix A). Let p^∞ be a limit point of $\{\tilde{p}^r\}_{r \in R}$, so that $Ep^\infty = t^\infty$ and $Ap^\infty \ge b$. By further passing into a subsequence if necessary, we will assume that $\{\tilde{p}^r\}_{r \in R} \to p^\infty$. Also, by the Almost Cyclic rule, we can (by further passing into a subsequence if necessary) assume that the ordered set $\{w^r,\ldots,w^{r+\Delta-1}\}$ is the same set, say $\{\hat{w}^0,\ldots,\hat{w}^{\Delta-1}\}$, for all $r \in R$. Since $t^{r+1} - t^r \to 0$ and $\{t^r\}_{r \in R} \to t^\infty$, we have

$$\{t^{r+j}\}_{r \in R} \to t^\infty, \qquad j = 0, 1,\ldots,\Delta. \tag{4.2}$$

Since $t^\infty \in C_h$ (cf. Lemma 3 (c) in Appendix C) and h is continuous on C_h, this implies $h(t^{r+j})\}_{r \in R} \to h(t^\infty)$ for $j = 0, 1,\ldots,\Delta$, which, together with the facts

$$q(p^{r+j+1}) - q(p^{r+j}) \;=\; h(t^{r+j+1}) - h(t^{r+j}) + \theta^{r+j}\langle c,\hat{w}^j\rangle, \qquad j = 0,\ldots,\Delta-1,\ \forall\, r \in R$$

(cf. (2.1b) and $p^{r+1} - p^r = \theta^r w^r$ for all r) and $q(p^{r+1}) - q(p^r) \to 0$, implies $\{\theta^{r+j}\langle c,\hat{w}^j\rangle\}_{r \in R} \to 0$, for $j = 0,\ldots,\Delta-1$. Since $\theta^r E w^r = t^{r+1} - t^r$ for all r and $t^{r+1} - t^r \to 0$, we also have $\{\theta^{r+j} E\hat{w}^j\rangle\}_{r \in R} \to 0$, for $j = 0,\ldots,\Delta-1$. Hence if $E\hat{w}^j \ne 0$ or if $\langle c,\hat{w}^j\rangle \ne 0$, then $\{\theta^{r+j}\}_{r \in R} \to 0$. Otherwise $E\hat{w}^j = 0$ and $\langle c,\hat{w}^j\rangle = 0$ so that, for every $r \in R$, $\langle \nabla q(p^{r+j}),\hat{w}^j\rangle = \langle E^T \nabla h(t^{r+j}) + c,\hat{w}^j\rangle = 0$, implying that $\theta^{r+j} = 0$. This shows that

$$\{p^{r+j+1} - p^{r+j}\}_{r\in R} \to 0, \qquad j = 0,\ldots,\Delta-1. \tag{4.3}$$

Let $I' = \{\, i \mid A_i p^\infty = b_i \,\}$ and $\delta = \max_{i\notin I'}\{A_i p^\infty - b_i\}$ (so $\delta > 0$). Then $A_i p^r \geq b_i + \delta/2$, for all $i\notin I'$ and all $r\in R$ sufficiently large (since for each i, either $A_i p^r \geq b_i + 1$ for all $r\in R$ or $A_i p^r = A_i \tilde{p}^r \to A_i p^\infty$ for all $r\in R$). Eq. (4.3) then implies that

$$A_i p^{r+j} \geq b_i + \delta/4, \qquad \forall\ i\notin I',\ \ \forall\ j\in\{1,\ldots,\Delta\},\ \ \forall\ r\in R \text{ sufficiently large}. \tag{4.4}$$

Now, consider any $j\in\{0,\ldots,\Delta-1\}$ such that $\hat{w}^j\in K(p^\infty)$. Then

$$A_i \hat{w}^j \geq 0 \qquad \forall\ i\in I', \tag{4.5}$$

so that, by (4.4) and (4.5), $\hat{w}^j\in K(p^{r+j+1})$, for all $r\in R$ sufficiently large. This, together with the fact that, for every $r\in R$, p^{r+j+1} is obtained by performing a line search of q over P from p^{r+j} in the direction $\hat{w}^j$, implies

$$\langle \nabla q(p^{r+j+1}), \hat{w}^j \rangle \quad \geq \ 0, \qquad \forall\ r\in R \text{ sufficiently large},$$

or, equivalently (by (2.1b)),

$$\langle E^T \nabla h(t^{r+j+1}) + c, \hat{w}^j \rangle \quad \geq \ 0, \qquad \forall\ r\in R \text{ sufficiently large}.$$

Since (cf. (4.2)) $\{t^{r+j+1}\}_{r\in R} \to t^\infty$, upon passing into the limit as $r \to \infty$, $r\in R$, and using the continuity of ∇h on C_h ([Roc70, Theorem 25.5]), we obtain

$$\langle E^T \nabla h(t^\infty) + c, \hat{w}^j \rangle \quad \geq \ 0,$$

or, equivalently (by (2.1b) and the fact $Ep^\infty = t^\infty$),

$$\langle \nabla q(p^\infty), \hat{w}^j \rangle \quad \geq \ 0.$$

By an analogous argument, we also have that, for any $j\in\{0,\ldots,\Delta-1\}$ such that $-\hat{w}^j\in K(p^\infty)$, it holds

$$\langle \nabla q(p^\infty), \hat{w}^j \rangle \quad \leq \ 0.$$

Since (by the Almost Cyclic rule) $\{u^1,\ldots,u^b\} \subseteq \{\hat{w}^0,\ldots,\hat{w}^{\Delta-1}\}$, this then implies

$$\begin{aligned} \langle \nabla q(p^\infty), u^j \rangle &\geq 0, && \forall \text{ i such that } u^j \in K(p^\infty), \\ \langle \nabla q(p^\infty), u^j \rangle &\leq 0, && \forall \text{ i such that } -u^j \in K(p^\infty), \end{aligned}$$

or equivalently, $\langle \nabla q(p^\infty), \tilde{u} \rangle \geq 0$ for all $\tilde{u} \in \{\pm u^1, \ldots, \pm u^b\} \cap K(p^\infty)$. Hence, by Fact 1, $\langle \nabla q(p^\infty), u \rangle \geq 0$ for all $u \in K(p^\infty)$. Since q is convex, this shows that p^∞ minimizes q over P and therefore $Ep^\infty = t^*$. Q.E.D.

We remark that the line search stepsizes $\theta^0, \theta^1, \ldots$ in the basis descent method can be computed inexactly in the following manner: we fix a scalar $\alpha \in [0,1)$ and, at each r-th iteration, we choose $\hat{\theta}^r$ to be any scalar θ for which $\langle \nabla q(p^r + \theta w^r), w^r \rangle$ is between 0 and $\alpha \langle \nabla q(p^r), w^r \rangle$ and choose θ^r to be the stepsize θ nearest to $\hat{\theta}^r$ that is between 0 and $\hat{\theta}^r$ and satisfies $p^r + \theta w^r \in P$. [Loosely speaking, $\hat{\theta}^r$ is a stepsize for which the directional derivative is increased to a fraction α of its starting value.] For $\alpha = 0$, we recover the exact line search procedure. It can be verified that Proposition 1 still holds for the basis descent method that uses this inexact line search procedure (in conjunction with the Almost Cyclic rule).

Notice that Proposition 1 does not assert convergence of the sequence $\{p^r\}$. In general, convergence of $\{p^r\}$ appears to be very difficult to establish without making additional assumptions on the problem (such as E has full column rank). Also notice that in the special case when c is in the row space of E, say $c = E^T\eta$ for some m-vector η, the problem (2.1a)-(2.1b) is equivalent to

$$\begin{aligned} &\text{Minimizing} && h(t) + \langle \eta, t \rangle \\ &\text{subject to} && t \in T. \end{aligned} \tag{4.6}$$

where T denotes the polyhedral set $\{ Ep \mid p \in P \}$. Now it is easily seen that if $\{u^1, \ldots, u^b\}$ is a basis for K(P), then $\{Eu^1, \ldots, Eu^b\}$ is a basis for K(T). Hence, we can apply the basis descent method to solve (4.6) (using $\{Eu^1, \ldots, Eu^b\}$ as the basis), i.e., at each iteration the objective function $h(\cdot) + \langle \eta, \cdot \rangle$ is minimized over T along a direction chosen from $\{Eu^1, \ldots, Eu^b\}$. In the special case where E has full column rank, this method is equivalent to the basis descent method applied to solve (2.1a)-(2.1b) (using $\{u^1, \ldots, u^b\}$ as the basis). [By this we mean that if Ew^0, $Ew^1, \ldots$ is the sequence of directions used in the former and $w^0, w^1, \ldots$ is the sequence of directions used in the latter, then the two methods generate identical sequences of iterates in T.] In general, however, the two methods are not equivalent.

5. Coordinate Descent Method and Dual Applications

In this section we apply the result developed in Section 4 to problems of minimizing strictly convex, essentially smooth functions subject to linear constraints. For the special cases of quadratic programming and entropy optimization, we show that a number of methods for solving these problems are applications of a coordinate descent method and that convergence of these methods follows from that for the basis descent method (cf. Proposition 1). We also give two new applications of this coordinate descent method: one to minimizing the "$-\log(x)$" entropy cost subject to linear inequality constraints and the other to solving certain resource allocation problems having separable costs and network constraints.

Let A be an $n_1 \times m$ matrix, B be an $n_2 \times m$ matrix, b be an n_1-vector, and d be an n_2-vector. Let f be a convex function on $\Re^m$. Consider the following convex program

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & Ax = b, \\ & Bx \geq d. \end{array} \tag{P}$$

We make the following standing assumptions about (P):

Assumption C.

(a) $f:\Re^m \to (-\infty,\infty]$ is closed convex and essentially smooth.

(b) For any t in $\mathrm{bd}(C_{f^*})$ and any sequence $\{t^k\}$ in C_{f^*} converging to t, it holds

$$\lim_{k\to\infty} \sup\left\{\frac{f^*(t^k)-f^*(t)}{\|t^k-t\|}\right\} = -\infty.$$

(c) f is strictly convex on C_f.

(d) The constraint set $X = \{\, x\in\Re^m \mid Ax = b,\ Bx \geq d \,\}$ has a nonempty intersection with C_f and the set $\{\, x\in X \mid f(x) \leq \zeta \,\}$ is bounded for all ζ.

Since f has nonempty compact level sets over X (cf. parts (a), (d) of Assumption C), (P) has an optimal solution which (since f is strictly convex) is unique. Let x^* denote this optimal solution. Since f is essentially smooth and $C_f \cap X \neq \emptyset$, it can be seen from Lemma 26.2 in [Roc70] that x^* is in $C_f \cap X$.

Now, by attaching Lagrange multiplier vectors λ and $\mu \geq 0$ with, respectively, the constraints $Ax = b$ and $Bx \geq d$, we obtain the following dual problem

$$\begin{array}{ll} \text{Minimize} & \phi(\lambda,\mu) \\ \text{subject to} & \mu \ge 0, \end{array} \tag{D}$$

where $\phi:\Re^n \to (-\infty,\infty]$ $(n = n_1 + n_2)$ is the dual functional

$$\begin{aligned} \phi(\lambda,\mu) &= \max_x\{ \langle Ax - b,\lambda\rangle + \langle Bx - d,\mu\rangle - f(x) \} \\ &= f^*(A^T\lambda + B^T\mu) - \langle b,\lambda\rangle - \langle d,\mu\rangle. \end{aligned} \tag{5.1}$$

The problem (D) is clearly a special case of the problem (2.1a)-(2.1b) with $h = f^*$ and

$$E = [\, A^T \;\; B^T \,], \qquad c = \begin{bmatrix} -b \\ -d \end{bmatrix}, \qquad P = \Re^{n1}\times[0,\infty)^{n2}. \tag{5.2}$$

Since f is essentially smooth and strictly convex on C_f, by Theorem 26.3 in [Roc70], its conjugate function f^* is essentially smooth and strictly convex on C_{f^*}. By Theorem 12.2 in [Roc70], f^* is also closed. Hence f^* satisfies Assumption B and parts (a), (b) of Assumption A. Also, since (P) has at least one feasible solution in C_f (cf. Assumption C (d)), Corollary 28.3.1 in [Roc70] shows that there exists an n-vector (λ^*,μ^*) satisfying with x^* the Kuhn-Tucker conditions for (P). Since $x^*\in C_f$, so that f is differentiable at x^*, the Kuhn-Tucker conditions for (P) imply $\mu^* \ge 0$ and $A^T\lambda^* + B^T\mu^* = \nabla f(x^*)$. This together with the fact ∇f is a one-to-one mapping from C_f onto C_{f^*} (see [Roc70, Theorem 26.5]) implies that $A^T\lambda^* + B^T\mu^*$ is in C_{f^*}. Therefore (λ^*,μ^*) is in $P\cap C_\phi$, so that $P\cap C_\phi \ne \varnothing$. Also, by Corollary 28.4.1 in [Roc70], (λ^*,μ^*) is an optimal solution of (D), so that, by Fact 2, Assumption A (c) holds.

Since Assumptions A and B hold, we can apply the basis descent method to solve (D), and since the constraint set is a box, we can choose as basis the set of coordinate vectors in $\Re^n$. This leads to the following coordinate descent method for solving (D) (let e^i denote the i-th coordinate vector in $\Re^n$):

Dual Coordinate Descent Method

Iter. 0. Choose a $p^0 \in C_\phi \cap \Re^{n1}\times[0,\infty)^{n2}$.

Iter. r. Choose an $i^r \in \{1,\ldots,n\}$ and set

$$p^{r+1} = p^r + \theta^r e^{i^r},$$

where $\theta^r = \operatorname{argmin}\{ \phi(p^r + \theta e^{i^r}) \mid p^r + \theta e^{i^r} \in \Re^{n1}\times[0,\infty)^{n2} \}$.

The order of relaxation $\{i^0,i^1,\ldots\}$ is assumed to be chosen according to the Almost Cyclic rule, i.e., there exists an integer Δ such that $\{1,\ldots,n\} \subseteq \{i^{r+1},\ldots,i^{r+\Delta}\}$ for all r. [We have given the coordinate descent method special attention because, as we shall see, it is a very frequently used method in optimization.]

Since Assumptions A and B hold, it follows from Proposition 1 that the dual coordinate descent method is convergent in the sense that it generates a sequence of iterates $\{p^r\}$ satisfying $Ep^r \to Ep^*$, where p^* is any optimal solution of (D). By Theorem 26.5 in [Roc70], $\nabla f^*(Ep^*) = x^*$, so that this convergence result can be restated in the primal space as follows:

Propostion 2. Let $\{p^r\}$ be a sequence of iterates generated by the dual coordinate descent method. Then $Ep^r \in C_{f^*}$ for all r and $\{\nabla f^*(Ep^r)\}$ converges to the optimal solution of (P), where E is given by (5.2).

5.1. "Log(x)" Entropy Optimization

Consider the following entropy program [CeL87], [Fre88], [JoS84], [Hua67], [Son88]

$$\begin{array}{lll} \text{Minimize} & -\sum_{j=1}^{m} \log(x_j) & (EP_1) \\ \text{subject to} & Ax = b, \; x > 0, & \end{array}$$

where A is an n×m matrix, b is an n-vector, and "log" denotes the natural logarithm. We make the following standing assumption about (EP_1):

Assumption D. The set $X = \{ x \in \Re^m \mid Ax = b, x > 0 \}$ is nonempty and bounded.

Assumption D is both necessary and sufficient for (EP_1) to have a finite optimal value. [To see this, note that if X is unbounded, then there exists nonzero $z \geq 0$ such that $Az = 0$, so that the cost of (EP_1) tends to $-\infty$ in the direction z. Conversely, if the optimal value of (EP_1) equals $-\infty$, then there must exist a sequence of vectors $\{x^k\}$ satisfying $Ax^k = b$, $x^k > 0$, for all k, and $\sum_{j=1}^{m} \log(x_j^k) \to \infty$. Since $\log(x_j) \to \infty$ only if $x_j \to \infty$, the latter implies $\|x^k\| \to \infty$. Then any limit point y of the bounded sequence $\{x^k/\|x^k\|\}$ is nonzero and satisfies $Ay = 0$, $y \geq 0$, so that X is unbounded.]

The problem (EP_1) is clearly a special case of (P) with $n_1 = n$, $n_2 = 0$ and with $f:\Re^m \to (-\infty,\infty]$ given by

$$f(x) = \begin{cases} -\sum_{j=1}^{m} \log(x_j) & \text{if } x > 0, \\ \infty & \text{else.} \end{cases} \tag{5.3}$$

It is easily seen that f is closed strictly convex and essentially smooth, so that parts (a), (c) of Assumption C hold. Since $C_f = (0,\infty)^m$, Assumption D implies that $C_f \cap \{ x \in \Re^m \mid Ax = b \} \neq \emptyset$ and f has bounded level sets on X, so that Assumption C (d) holds. Moreover, direct calculation finds that

$$\begin{aligned} f^*(t) &= \max_{x>0}\{ \langle x,t \rangle + \sum_{j=1}^{m} \log(x_j) \} \\ &= f(-t) - m, \qquad \forall\, t, \end{aligned}$$

so that $f^*(t) \to \infty$ as t approaches any boundary point of C_{f^*}. Hence Assumption C (b) also holds.

Since Assumption C holds, we immediately obtain from Proposition 2 that if $\{p^r\}$ is a sequence of iterates generated by applying the dual coordinate descent method to solve (EP_1), then $\{\nabla f^*(A^T p^r)\}$ converges to the optimal solution of (EP_1). By using the fact (cf. (5.3) and $f^*(t) = f(-t) - m$) that $\partial f^*(t)/\partial t_j = -1/t_j$ for all j and $t > 0$, we immediately obtain the following convergence result proved by Censor and Lent [CeL87]:

Proposition 3. Let $\{p^r\}$ be a sequence of iterates generated by applying the dual coordinate descent method to solve (EP_1). Then the m-vector whose j-th coordinate is the inverse of the j-th coordinate of $-A^T p^r$ converges to the optimal solution of (EP_1) as $r \to \infty$.

If, instead of the equality constraints $Ax = b$, we have inequality constraints of the form $Bx \geq d$ (or even with both equality and inequality constraints), we can still apply the dual coordinate descent method to solve this problem. By Proposition 2, the resulting method computes the (unique) optimal solution to this problem, provided that $\{ x \in \Re^m \mid Bx \geq d,\ x > 0 \}$ is nonempty and bounded. [To the best of our knowledge, this is the first result showing that the dual coordinate descent method is convergent for the inequality constrained version of (EP_1).]

5.2. "xLog(x)" Entropy Optimization

Consider the following entropy program [DaR72], [JoS84], [LaS81]

$$\begin{array}{lll} \text{Minimize} & \sum_{j=1}^{m} x_j \log(x_j) & \qquad (EP_2) \\ \text{subject to} & Ax = b, \; x > 0, & \end{array}$$

where A is an n×m matrix and b is an n-vector. We make the following standing assumption about (EP_2):

Assumption E. The set $\{ x \in \Re^m \mid Ax = b, x > 0 \}$ is nonempty.

The problem (EP_2) is clearly a special case of (P) with $n_1 = n$, $n_2 = 0$ and with $f:\Re^m \rightarrow (-\infty,\infty]$ given by

$$f(x) = \begin{cases} \sum_{j=1}^{m} x_j \log(x_j) & \text{if } x \geq 0, \\ \infty & \text{else.} \end{cases} \qquad (5.4)$$

It is easily seen that f is closed strictly convex and essentially smooth, so that parts (a) and (c) of Assumption C hold. Since $f(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, f has bounded level sets on $\{ x \in \Re^m \mid Ax = b \}$. Since $C_f = (0,\infty)^m$, we also have from Assumption E that $C_f \cap \{ x \in \Re^m \mid Ax = b \} \neq \varnothing$ so that part (d) of Assumption C holds. Moreover, direct calculation finds that

$$\begin{aligned} f^*(t) &= \max_{x \geq 0} \{ \langle x,t \rangle - \sum_{j=1}^{m} x_j \log(x_j) \} \\ &= \sum_{j=1}^{m} \exp(t_j - 1), \qquad \forall\, t, \end{aligned}$$

so that C_{f^*} has no boundary point and part (c) of Assumption C trivially holds.

Since (EP_2) is a special case of (P) and it satisfies Assumption C, we can apply the dual coordinate descent method to solve this problem. In the special case where A is the node-arc incidence matrix for a directed bipartite graph and the coordinates are relaxed in a cyclic order, this gives a balancing method of Kruithof (sometimes called the RAS method) in transportation planning. [This method was rediscovered by many others (see [LaS81]) and recently was rejuvenated for its suitability for parallel computation (e.g. [BeT89], [ZeI88]).] A modification of Kruithof's method that uses non-cyclic order of relaxation (but still using the Almost Cyclic rule) was considered by Bachem and Korte [BaK79]. Other methods that are special cases of the dual coordinate descent method applied to solve (EP_2) include the methods of Osborne [Osb60] and of Grad [Gra71] for matrix pre-conditioning and a method of Evans and Kirby [EvK74] for three dimensional balancing (see [LaS81] for a summary of these methods).

Since Assumption C holds, we immediately obtain from Proposition 2 that if $\{p^r\}$ is the sequence of iterates generated by applying the dual coordinate descent method to solve (EP_2), then $\{\nabla f^*(A^T p^r)\}$ converges to the optimal solution of (EP_2). By using the fact (cf. (5.4)) that $\partial f^*(t)/\partial t_j = \exp(t_j - 1)$ for all j and t, we immediately obtain the following convergence result (cf. [BaK79], [Bre67], [EvK74], [Osb60]):

Proposition 4. Let $\{p^r\}$ be a sequence of iterates generated by applying the dual coordinate descent method to solve (EP_2). Then the m-vector whose j-th coordinate is $\exp(t_j^r - 1)$, where t_j^r denotes the j-th coordinate of $A^T p^r$, converges to the optimal solution of (EP_2) as $r \to \infty$.

If instead of the equality constraints $Ax = b$, we have inequality constraints of the form $Bx \geq d$ (or with both equality and inequality constraints), we can still apply the dual coordinate descent method to solve this problem. By Proposition 2, the resulting method computes the (unique) optimal solution to this problem, provided that $\{ x \in \Re^m \mid Bx \geq b, x > 0 \}$ is nonempty. A special case of this method is a balancing type algorithm of Jefferson and Scott [JeS79].

Note. Proposition 4 and the above result for inequality constrained problems can alternatively be obtained as a special case of Proposition 1 in [Tse88a].

5.3. Quadratic Programming

Consider the convex quadratic problem [CoP82], [Cry71], [LiP87], [Man84], [Tse88a]

$$\begin{array}{ll} \text{Minimize} & \langle x - \bar{x}, Q(x - \bar{x}) \rangle / 2 \\ \text{subject to} & Bx \geq d, \end{array} \qquad \text{(QP)}$$

where Q is an m×m symmetric positive definite matrix, B is an n×m matrix, $\bar{x}$ is an m-vector, and d is an n-vector. We make the following standing assumption about (QP):

Assumption F. The set $\{ x \in \Re^m \mid Bx \geq d \}$ is nonempty.

The problem (QP) is clearly a special case of (P) with $n_1 = 0$, $n_2 = n$, and with $f:\Re^m \to \Re$ given by

$$f(x) = \langle x-\bar{x}, Q(x-\bar{x})\rangle/2, \qquad \forall\ x.$$

Since f is quadratic, f is closed and essentially smooth. Since Q is positive definite, f is also strictly convex, so that parts (a) and (c) of Assumption C hold. Since $C_f = \Re^m$, Assumption F and the fact $f(x) \to \infty$ as $\|x\| \to \infty$ imply that part (d) of Assumption C also holds. By direct calculation we find that

$$\begin{aligned} f^*(t) &= \max_x \{ \langle x, t\rangle - \langle x-\bar{x}, Q(x-\bar{x})\rangle/2 \} \\ &= \langle t, Q^{-1}t\rangle/2 + \langle \bar{x}, t\rangle, \end{aligned} \tag{5.5}$$

so that $C_{f^*} = \Re^m$ has no boundary point and part (c) of Assumption C trivially holds.

For this special case of (P), the corresponding dual problem (D) (cf. (5.1), (5.5)) is exactly

$$\begin{aligned} &\text{Minimizing} \quad \langle p, BQ^{-1}B^Tp\rangle/2 + \langle B\bar{x} - d, p\rangle \\ &\text{subject to} \quad p \geq 0, \end{aligned} \tag{5.6}$$

which, in view of the positive semi-definite property of $BQ^{-1}B^T$, is a well-known optimization problem called the <u>symmetric linear complementarity</u> problem [CGS78], [LiP87], [LuT89], [Man77], [MaD88], [Pan86]. The general form of this latter problem involves the minimization of a convex quadratic function $\langle p, Mp\rangle/2 + \langle w, p\rangle$ subject to $p \geq 0$, where M is an $n\times n$ symmetric <u>positive semi-definite</u> matrix and w is an n-vector. However, by expressing M as $M = BB^T$ for some $n\times m$ matrix B and by expressing w as $w = -d$ for some m-vector d (with Q taken to be the identity matrix and with $\bar{x} = 0$), we can always reduce this problem to the form (5.6).

Since (QP) is a special case of (P) and Assumption C holds, we can apply the dual coordinate descent method to solve (QP). This gives an algorithm proposed by Herman and Lent [HeL78], which is itself an extension of a method of Hildreth [Hil59] (see [CGS78], [Cry71], [LiP87], [LuT89], [Man77] for further extensions of these methods to SOR and matrix splitting methods). By Proposition 2, if $\{p^r\}$ is the sequence of iterates generated by applying the dual coordinate descent method to solve (QP), then $\{\nabla f^*(B^T p^r)\}$ converges to the optimal solution of (QP). By using the fact (cf. (5.5)) that $\nabla f^*(t) = Q^{-1}t + \xi$ for all t, we immediately obtain the following result proved by Herman and Lent [HeL78]:

Proposition 5. Let $\{p^r\}$ be the sequence of iterates generated by applying the dual coordinate descent method to solve (QP). Then $\{Q^{-1}B^T p^r + \bar{x}\}$ converges to the optimal solution of (QP).

[In the special case where the coordinates are relaxed in a cyclic manner, it was recently shown that the sequence $\{p^r\}$ in fact converges [LuT89]. The proof of this however is more difficult since the optimal solution set for the problem (5.6) may be unbounded.]

5.4. Resource Allocation Problem

In certain areas such as VLSI design, it frequently arise problems of allocating a precious resource (e.g. budget, labour) to a given set of tasks in order to meet a deadline. Moreover, these tasks typically need to satisfy some precedence relationship (e.g. one task cannot begin until another is completed). Below we propose a model, similar to one by Monma, Schrijver, Todd and Wei [MSTW88], that fits within the framework of (P). Moreover, owing to the network structure of the precedence constraints, the dual coordinate descent method applied to solve this problem can be implemented using network data structures.

Consider a directed acyclic graph with node set $\mathcal{N} = \{1,\ldots,n\}$ and arc set $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$. [The nodes correspond to the tasks and the arcs specify the precedence constraints on the tasks.] With each node i we associate a variable τ_i denoting the completion time of task i. If task i is completed at time τ_i and it requires exactly η_i time units to complete, then we incur a cost of $g_i(\tau_i) + w_i(\eta_i)$. The problem is then to minimize the total cost while meeting the task precedence constraints:

$$\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{n} g_i(\tau_i) + w_i(\eta_i) \\ \text{subject to} & \tau_j \le \tau_i + \eta_i, \quad \forall\ (i,j) \in \mathcal{A}. \end{array} \qquad \text{(RAP)}$$

[The ordered pair (i,j) represents an arc from node i to node j.] We make the following standing assumptions about the problem (RAP):

Assumption G.

(a) Each $g_i:\Re \to (-\infty,\infty]$ and $w_i:\Re \to (-\infty,\infty]$ is closed convex and essentially smooth.

(b) Each g_i and w_i is strictly convex on, respectively, C_{g_i} and C_{w_i}.

(c) The constraint set $\{ (\tau,\eta) \mid \tau_j \le \tau_i + \eta_i$, for all $(i,j)\in\mathcal{A} \}$ intersects $C_g \times C_w$ and $g(\tau) + w(\eta)$ has bounded level sets on this set, where $g:\Re^n \to (-\infty,\infty]$ and $w:\Re^n \to (-\infty,\infty]$ are the functions

$$g(\tau) = \sum_{i=1}^{n} g_i(\tau_i), \qquad\qquad w(\eta) = \sum_{i=1}^{n} w_i(\eta_i).$$

[The essential smoothness and strict convexity assumption on the g_i's and the w_i's (cf. Assumption G (a)-(b)) is not as restrictive as it may seem. For example, the function

$$w_i(\eta_i) = \begin{cases} 1/(\eta_i)^\alpha & \text{if } \eta_i \ge 0, \\ \infty & \text{else} \end{cases} \tag{5.7}$$

(α is any positive scalar), which is used in the model of [MSTW88], satisfies Assumption G (a)-(b). Constraints of the form $\tau_i \le d_i$, where d_i is a given <u>deadline</u> for task i, can be modeled by letting g_i be the function

$$g_i(\tau_i) = \begin{cases} 1/(d_i - \tau_i)^\beta & \text{if } \tau_i < d_i, \\ \infty & \text{else} \end{cases} \tag{5.8}$$

(β is any positive scalar), which also satisfies Assumption G (a)-(b). Constraints of the form $\tau_i \le s_i$, where s_i is a given <u>earliest starting time</u> for task i, can be modeled similarly.]

The problem (RAP) is clearly a special case of (P) with $n_1 = 0$, n_2 = cardinality of $\mathcal{A}$, $m = 2n$, and with $f:\Re^m \to \Re$ given by

$$f(\tau,\eta) = g(\tau) + w(\eta), \qquad \forall\, \tau,\ \forall\, \eta.$$

Since each g_i and w_i is closed convex and essentially smooth, so is f. Since each g_i and w_i is strictly convex on, respectively, C_{g_i} and C_{w_i}, f is strictly convex on C_f. Therefore parts (a) and (c) of Assumption C hold. By Assumption G (c), part (d) of Assumption C also hold. Finally, it is easily seen that f^*, the conjugate function of f, is a separable convex function of the form

$$f^*(\pi,\xi) = \sum_{i=1}^{n} g_i^*(\pi_i) + w_i^*(\xi_i), \qquad \forall\, \pi,\ \forall\, \xi, \tag{5.9}$$

and each g_i^* and w_i^* is essentially smooth on, respectively, $C_{g_i^*}$ and $C_{w_i^*}$ (see [Roc70, Theorem 26.5]); therefore, by the discussion immediately following Assumption A (cf. (2.2)), f^* satisfies Assumption C (c).

Since (RAP) is a special case of (P) and Assumption C holds, we can apply the dual coordinate descent method to solve (RAP) and, by Proposition 2, the resulting method is convergent (in the sense that it generates a sequence of vectors converging to the (unique) optimal solution of (RAP)).

We show below that, owing to the separable nature of the cost and the network structure of the constraints, the dual coordinate descent method for solving (RAP) can be implemented using network data structure and, in certain cases, is highly parallelizable. To see this, let $p_{ij} \geq 0$ denote the Lagrange multiplier associated with the constraint $\tau_j \leq \tau_i + \eta_i$. Straightforward calculation then finds the dual functional $\phi:\Re^n \to (-\infty,\infty]$ given by (5.1), (5.9) to be

$$\phi(p) = \sum_{i=1}^{n} g_i^*(\Sigma_{(i,j)\in\mathcal{A}}\, p_{ij} - \Sigma_{(j,i)\in\mathcal{A}}\, p_{ji}) + w_i^*(\Sigma_{(i,j)\in\mathcal{A}}\, p_{ij}),$$

where $p = (\ldots,p_{ij},\ldots)_{(i,j)\in\mathcal{A}}$, so that the dual problem (D) of minimizing $\phi(p)$ subject to $p \geq 0$ can be written as

$$\begin{array}{lll} \text{Minimize} & \sum_{i=1}^{n} g_i^*(\pi_i) + w_i^*(\Sigma_{(i,j)\in\mathcal{A}}\, p_{ij}) & \\ \text{subject to} & \Sigma_{(i,j)\in\mathcal{A}}\, p_{ij} - \Sigma_{(j,i)\in\mathcal{A}}\, p_{ji} = \pi_i, & i = 1,\ldots,n, \\ & p_{ij} \geq 0, & \forall\ (i,j)\in\mathcal{A}, \end{array}$$

where π_i, $i = 1,\ldots,n$, are dummy variables. The above problem we recognize to be a convex cost network flow problem [BeT89] with p_{ij} as the <u>flow</u> on arc (i,j).

It is easily verified that, for any $(i,j)\in\mathcal{A}$,

$$\partial\phi(p)/\partial p_{ij} = \nabla g_i^*(\alpha_i - \beta_i) - \nabla g_j^*(\alpha_j - \beta_j) + \nabla w_i^*(\alpha_i),$$

where $\alpha_i = \Sigma_{(i,k)\in\mathcal{A}}\, p_{ik}$, $\beta_i = \Sigma_{(k,i)\in\mathcal{A}}\, p_{ki}$, and α_j, β_j are analogously defined. Therefore in the dual coordinate descent method, the minimization of $\phi(p)$ with respect to each coordinate p_{ij} can be performed by working with the functions g_i^*, g_j^* and w_i^* only. More precisely, if we let $\rho:\Re\to(-\infty,\infty]$ be the convex function

$$\rho(\theta) \;=\; g_i^*(\alpha_i - \beta_i + \theta) + g_j^*(\alpha_j - \beta_j - \theta) + w_i^*(\alpha_i + \theta), \qquad \forall\ \theta,$$

with α_i and β_i defined as before, then the minimization of $\phi(p)$ with respect to p_{ij} reduces to the following iteration: check whether $\nabla\rho(0) < 0$; if yes, increase θ from 0 until $\nabla\rho(\theta) = 0$, otherwise decrease θ from 0 until either $\nabla\rho(\theta) = 0$ or $p_{ij} + \theta = 0$ (which ever occurs first); replace p_{ij} by $p_{ij} + \theta$. This iteration is easy to implement (consider the case where g_i, g_j are given by (5.8) and w_i is given by (5.7)) and, moreover, if two arcs (i,j) and (k,l) do not share any node in common, then p_{ij} and p_{kl} are uncoupled and can be iterated upon simultaneously.

6. An ε-Basis Descent Method Based on Elementary Vectors

Consider the base problem (2.1a)-(2.1b). As we noted earlier, the basis descent method for solving this problem has the drawbacks that it is applicable only when the cost function h is strictly convex and that it must generate the basis vectors in an almost cyclical manner. The second drawback is particularly detrimental if the polyhedral constraint set P has a complicated structure so that the size of a basis for K(P) is large. In this section, we propose an extension of the basis descent method that avoids the above drawbacks. In particular, we show that the elementary vectors [Roc69] of a certain linear subspace associated with P form a basis for K(P). By using the dual rectification algorithm of Rockafellar [Roc84] to generate these elementary vectors and by introducing an ε-complementary slackness mechanism to ensure that each descent yields a "large" decrease in the cost, we obtain a basis descent method that terminates finitely with a feasible solution that is, in some sense, within ε of optimality. [We note that, although the notion of elementary vectors was originally conceived in the context of monotropic programming, i.e. minimization of convex separable costs subject to linear constraints, it turns out to apply equally well to our problem, for which the cost is not necessarily separable but essentially smooth.]

We will assume that P is given by

$$P \;=\; \{\, p \in \Re^n \mid Ap = b,\ p \geq 0 \,\}, \tag{6.1}$$

for some k×n matrix A and some k-vector b. This representation of P allows a more direct application of the dual rectification algorithm. The case where upper bound constraints on p are also present can be handled by making minor modifications to the analysis. Let S denote the subspace

$$S \quad = \quad \{ u \in \Re^n \mid Au = 0 \}. \qquad (6.2)$$

We say that a nonzero n-vector u is an elementary vector of S [Roc69], [Roc84] if $u \in S$ and there does not exist a nonzero vector u′ in S satisfying (i) the signed support of u′ is contained in the signed support of u, i.e.

$$u_i' < 0 \Rightarrow u_i < 0, \qquad u_i' > 0 \Rightarrow u_i > 0,$$

and (ii) for at least one i it holds $u_i' = 0$ and $u_i \neq 0$. [Loosely speaking, elementary vectors of S are vectors in S that in some sense have the smallest number of nonzero components.] It is not difficult to show (using the above definition) that the elementary vectors of S, except for scalar multiples, are finite in number [Roc84, §10C]. We claim that they in fact form a basis for K(P). To show this, we will use the following property of elementary vectors [Roc84, §10C]: Every nonzero vector $\hat{u} \in S$ can be expressed in the conformal sense by the elementary vectors of S, i.e.

$$\hat{u} = \lambda_1 \tilde{u}^1 + \ldots + \lambda_k \tilde{u}^k, \qquad \text{for some } \lambda_1 \geq 0, \ldots, \lambda_k \geq 0, \qquad (6.3)$$

where each $\tilde{u}^j$ is an elementary vector of S whose signed support is contained in the signed support of $\hat{u}$. Since it is easily seen from (6.1)-(6.2) that, for any $p \in P$, an n-vector u is in K(p) if and only if $u \in S$ and

$$u_i \geq 0 \quad \text{if} \quad p_i = 0,$$

it follows that, for each $\hat{u}$ in K(p), there exist elementary vectors $\tilde{u}^1, \ldots, \tilde{u}^k$ of S satisfying (6.3) and $\tilde{u}^j \in K(p)$ for all j. Then the collection of all such elementary vectors of S (taken over all $\hat{u} \in K(p)$ and all $p \in P$), with scalar multiples excluded, form a basis for K(P).

We now describe a procedure, the dual rectification algorithm of Rockafellar [Roc84, §10J], for generating the elementary vectors of S. We will not go into the details of this algorithm, which is extensively treated in [Roc84]. For our purpose, it suffices to know that this is an algorithm that, given S and any set of closed intervals $I_1, \ldots, I_n$, finds in a finite number of steps either (i) a $\hat{v} \in S^\perp$ ($S^\perp$ denotes the orthogonal complement of S) satisfying

$$\hat{v} \in I_1 \times \ldots \times I_n,$$

or (ii) an elementary vector $\hat{u}$ of S satisfying

$\max\{ \langle \xi,\hat{u}\rangle \mid \xi \in I_1\times\ldots\times I_n \} < 0.$

Moreover, each elementary vector of S that this algorithm generates is normalized, so that the number of distinct elementary vectors of S that it can generate is finite. [In fact, instead of the dual rectification algorithm, we can use any other algorithm that resolves between the above two alternatives in the same manner.]

Below we describe a basis descent method for solving (2.1a)-(2.1b) that uses the dual rectification algorith, together with a fixed scalar $\varepsilon > 0$, to generate the descent directions:

ε-Basis Descent Method

Iter. 0 Choose a $p^0 \in P\cap C_q$.

Iter. r Given a $p^r \in P\cap C_q$, we apply the dual rectification algorithm with

$$I_i = \partial q(p^r)/\partial p_i + \begin{cases} [-\varepsilon,\varepsilon] & \text{if } p_i^r > \varepsilon, \\ [-\infty,\varepsilon] & \text{if } 0 \le p_i^r \le \varepsilon, \end{cases} \tag{6.4a}$$

for $i = 1,\ldots,n$. The algorithm returns either (i) a $\bar{v} \in S^\perp$ satisfying

$$\begin{array}{ll} -\varepsilon \le \partial q(p^r)/\partial p_i - \bar{v}_i \le \varepsilon & \text{if } p_i^r > \varepsilon, \\ -\varepsilon \le \partial q(p^r)/\partial p_i - \bar{v}_i & \text{if } 0 \le p_i^r \le \varepsilon, \end{array}$$

or (ii) an elementary vector w^r of S satisfying

$$\langle \nabla q(p^r), w^r\rangle + \textstyle\sum_{p_i^r > \varepsilon} \max_{-\varepsilon\le\zeta_i\le\varepsilon} \zeta_i w_i^r + \textstyle\sum_{0\le p_i^r\le\varepsilon} \max_{\zeta_i\le\varepsilon} \zeta_i w_i^r < 0.$$

In case (i), it can be seen that $\bar{v} \in S^\perp$ equivalently satisfies $\|p^r - [p^r - \nabla q(p^r) + \bar{v}]^+\|_\infty \le \varepsilon$, where $[\cdot]^+$ denotes the orthogonal projection onto $[0,\infty)^n$; terminate. In case (ii), it can be seen that w^r satisfies $w^r \in S$ and

$$w_i^r \ge 0 \text{ if } 0 \le p_i^r \le \varepsilon, \qquad \langle \nabla q(p^r), w^r\rangle < -\varepsilon \|w^r\|_\infty, \tag{6.4b}$$

so that, by (6.1)-(6.2), w^r is a feasible direction of descent at p^r; perform a line search of q over P from p^r along the direction w^r.

It can be seen that the ε-basis descent method (minus the termination criterion) is a special case of the basis descent method of Sec. 4 that uses, as the basis for K(P), the elementary vectors of S. However, the ε-basis descent method is not guaranteed to generate the elementary vectors of S according to the Almost Cyclic rule, so that Proposition 1 cannot be used to establish its convergence. This is the main motivation for introducing the parameter $\varepsilon > 0$ in the method, which is designed to ensure that a sufficiently large decrease in the cost occurs at each iteration. This in turn guarantees that the ε-basis descent method terminates in a finite number of iterations. We state this result below:

Proposition 6. For any $\varepsilon > 0$, the ε-basis descent method for solving (2.1a)-(2.1b) (with P given by (6.1)) terminates in a finite number of iterations with a $\bar{p} \in P \cap C_q$ and a $\bar{v} \in S^{\perp}$ satisfying $\|\bar{p} - [\bar{p} - \nabla q(\bar{p}) + \bar{v}]^+\|_\infty \le \varepsilon$.

Proof. We will argue by contradiction. Suppose that for some $\varepsilon > 0$ the ε-basis descent method does not terminate. Then at each r-th iteration the dual rectification algorithm must generate an elementary vector w^r of S satisfying (6.4b). Let $t^r = Ep^r$. Since $q(p^r) \ge q(p^{r+1})$ and $p^r \in P \cap C_q$ for all r, we have from Lemma 3 in Appendix C that $\{t^r\}$ is bounded and every limit point of $\{t^r\}$ is in C_h. Let t^∞ be any limit point of $\{t^r\}$, so that $t^\infty \in C_h$. Let $\{t^r\}_{r\in R}$ be a subsequence of $\{t^r\}$ converging to t^∞. By further passing into a subsequence if necessary, we will assume that, for each i, either (i) $0 \le p_i^r \le \varepsilon$ for all $r \in R$ or (ii) $p_i^r > \varepsilon$ for all $r \in R$. Moreover, we can assume that, for each i, either $\{p_i^r\}_{r\in R}$ is bounded or $\{p_i^r\}_{r\in R} \to \infty$.

Let $I = \{ i \mid \{p_i^r\}_{r\in R} \text{ is bounded} \}$. Since $\{p_i^r\}_{r\in R} \to \infty$ for all $i \notin I$, we will assume that $p_i^r \ge \varepsilon$ for all $r \in R$, $i \notin I$. For each $r \in R$, let $\hat{p}^r$ be the least L_2-norm solution of the following linear system

$$Ep = t^r, \qquad Ap = b, \qquad p_i = p_i^r \quad \forall\, i \in I, \qquad p_i \ge \varepsilon \quad \forall\, i \notin I. \tag{6.5}$$

This linear system is feasible since p^r is a solution. Furthermore, since the right hand side of this system is bounded as $r \to \infty$, $r \in R$, the sequence $\{\hat{p}^r\}_{r\in R}$ is bounded (cf. Lemma 1 in Appendix A). Let p^∞ be a limit point of this sequence. Then p^∞ belongs to P, satisfies $t^\infty = Ep^\infty$ and, for each i, it holds that $0 \le p_i^\infty < \varepsilon$ only if $0 \le p_i^r \le \varepsilon$ for all $r \in R$.

Since the number of distinct w^r's is finite, by further passing into a subsequence if necessary, we can assume that $w^r = w$ for all $r \in R$, for some w. Then we have from (6.4b) that

$$w_i \ge 0 \qquad \text{if } 0 \le p_i^r \le \varepsilon \ \ \forall\, r \in R, \qquad w \in S, \tag{6.6a}$$

$$\langle \nabla q(p^r), w \rangle < -\varepsilon \|w\|_\infty, \qquad \forall\, r \in R. \tag{6.6b}$$

Since $\nabla q(p^r) = E^T \nabla h(t^r) + c$ for all r (cf. (2.1b)), $\{t^r\}_{r\in R} \to t^\infty \in C_h$, and ∇h is continuous on C_h (cf. [Roc70, Theorem 25.5]), we have $\{\nabla q(p^r)\}_{r\in R} \to E^T\nabla h(t^\infty) + c$. This together with the facts $p^\infty \in C_q$ and $\nabla q(p^\infty) = E^T\nabla h(t^\infty) + c$ (cf. $Ep^\infty = t^\infty \in C_h$) implies $\{\nabla q(p^r)\}_{r\in R} \to \nabla q(p^\infty)$, so that (6.6b) yields

$$\langle \nabla q(p^\infty), w \rangle \le -\varepsilon \|w\|_\infty.$$

Hence the rate of descent of q at p^∞ in the direction w is at least $\varepsilon\|w\|_\infty$. Since ∇q is continuous around p^∞, this implies that there exist a $\bar\theta \in (0, \varepsilon/\|w\|_\infty]$ and a $\delta > 0$ such that $p^\infty + \bar\theta w \in C_q$ and

$$q(p^\infty + \bar\theta w) \le q(p^\infty) - \delta,$$

or, equivalently (by (2.1b)), $t^\infty + \bar\theta E w \in C_h$ and

$$h(t^\infty + \bar\theta E w) - h(t^\infty) + \langle c, \bar\theta w \rangle \le -\delta. \tag{6.7}$$

Since, for every $r \in R$, p^{r+1} is obtained by minimizing q over P along the direction w from p^r and (by (6.6a) and $0 < \bar\theta \le \varepsilon/\|w\|_\infty$) $p^r + \bar\theta w$ is in P, we obtain

$$\begin{aligned} q(p^{r+1}) - q(p^r) &\le q(p^r + \bar\theta w) - q(p^r) \\ &= h(t^r + \bar\theta E w) - h(t^r) + \langle c, \bar\theta w \rangle, \qquad \forall\, r \in R. \end{aligned}$$

Since $\{t^r\}_{r\in R} \to t^\infty \in C_h$ and $t^\infty + \bar\theta E w \in C_h$, the continuity of h in C_h ([Roc70, Theorem 10.1]) together with (6.7) implies that

$$q(p^{r+1}) - q(p^r) \le -\delta/2, \qquad \forall\, r \in R \text{ sufficiently large.}$$

Since $q(p^r)$ is monotonically decreasing with r, this shows that $q(p^r) \to -\infty$ as $r \to \infty$, a contradiction of Assumption A (c). Q.E.D.

By Proposition 6, the ε-basis descent method terminates finitely with a $\bar{p}\in P\cap C_q$ and a $\bar{v}\in S^{\perp}$ satisfying

$$\|\bar{p} - [\bar{p} - \nabla q(\bar{p}) + \bar{v}]^{+}\|_{\infty} \le \varepsilon. \qquad (6.8)$$

Moreover, $q(\bar{p}) \le q(p^0)$. If $\varepsilon = 0$, then it can be seen from (6.1) and the Kuhn-Tucker conditions for (2.1a)-(2.1b) that $\bar{p}$ is an optimal solution of (2.1a)-(2.1b). Hence, for $\varepsilon \ne 0$ but small, we would expect $\bar{p}$ to be nearly optimal. By making a mild assumption on h, we can quantify this notion of near optimality more precisely:

Proposition 7. Suppose that (in addition to Assumption A) P is given by (6.1) and $h(t) \to \infty$ as t approaches a boundary point of C_h. Then, for any $\zeta \ge \zeta^*$, where ζ^* denotes the optimal value of (2.1a)-(2.1b), there exists a constant γ (depending on ζ, P and q only) such that

$$\zeta^* \le q(\bar{p}) \le \zeta^* + \gamma\varepsilon,$$

for all $\varepsilon \ge 0$ and all $\bar{p}\in P\cap C_q$ satisfying $q(\bar{p}) \le \zeta$ and, together with some $\bar{v}\in S^{\perp}$, (6.8).

Proof. Fix any $\zeta \ge \zeta^*$ and let L denote the nonempty closed set $L = \{ p\in P \mid q(p) \le \zeta \}$ (L is closed since P is closed and q is a closed function). Firstly we claim that ∇q is bounded over L. To see this, note that the set $\{ Ep \mid p\in L \}$ is nonempty closed (since L is closed) and bounded (cf. Assumption A (c)). Moreover, by Lemma 2 in Appendix B, h(Ep) is bounded for all $p\in L$. Since $h(t) \to \infty$ as t approaches a boundary point of C_h, this implies that $\{ Ep \mid p\in L\}$ is a compact subset of C_h. Hence, by the continuity of ∇h on C_h ([Roc70, Theorem 25.5]), $\nabla q(p) = E\nabla h(Ep) + c$ is bounded for all $p\in L$. Let ρ_1 be a bound on $\|\nabla q(p)\|$ for all $p\in L$. It can be seen that ρ_1 depends on q, P and ζ only.

Next, for any $\varepsilon \ge 0$ and any $p\in L$, consider the following linear system

$$-\varepsilon \le \partial q(p)/\partial p_i - v_i \le \varepsilon, \qquad \forall\, i\notin I, \qquad (6.9a)$$
$$-\varepsilon \le \partial q(p)/\partial p_i - v_i, \qquad \forall\, i\in I, \qquad (6.9b)$$
$$v\in S^{\perp}, \qquad (6.9c)$$

in v, where $I = \{ i \mid 0 \le p_i \le \varepsilon \}$. Whenever this system has a solution, let $v^{\varepsilon}(p)$ denote its least L_2-norm solution. Also, for each $p\in L$, denote $v(p) = \operatorname{argmin}_{v\in S^{\perp}} \|\nabla q(p)-v\|$. Since ∇q is bounded on L, $\|\nabla q(p)-v(p)\|$ is bounded as p ranges over L. Let $\mu = \sup_{p\in L}\|\nabla q(p)-v(p)\|$ (so that $\mu < \infty$). Then, for each $p\in L$, $v = v(p)$ satisfies

$$
\begin{aligned}
&-\mu \le \partial q(p)/\partial p_i - v_i \le \mu, && \forall\ i\notin J,\\
&-\mu \le \partial q(p)/\partial p_i - v_i, && \forall\ i\in J,\\
&v\in S^{\perp},
\end{aligned}
$$

for all nonempty subsets J of $\{1,\ldots,n\}$. Comparing the above linear system with the linear system (6.9a)-(6.9c), we see that, for $J = I$, one system is just the other with each right hand side perturbed by at most μ. By a well-known Lipschitz continuity property of the solutions of linear systems ([MaS87], [Rob73]), there exists, for each $p\in L$ and $\varepsilon > 0$ such that (6.9a)-(6.9c) has a solution, a solution v of (6.9a)-(6.9c) satisfying

$$\|v-v(p)\| \le \theta\mu,$$

where θ is some constant that depends on the matrix A only. Then $\|v^{\varepsilon}(p)\| \le \|v\| \le \|v-v(p)\| + \|v(p)\| \le \theta\mu + \|v(p)\|$. Since $\|\nabla q(p)\| \le \rho_1$ for all $p\in L$, we also have $\|v(p)\| \le \|\nabla q(p)\| + \|\nabla q(p)-v(p)\| \le \rho_1 + \mu$ for all $p\in L$. Hence $\|v^{\varepsilon}(p)\| \le \theta\mu + \rho_1 + \mu$ for all $\varepsilon \ge 0$ and all $p\in L$ such that $v^{\varepsilon}(p)$ is well-defined. Let $\rho_2 = \theta\mu + \rho_1 + \mu$. It can be seen that ρ_2 depends on q, P and ζ only.

Finally, we claim that $\rho(p)$ is bounded for all $p\in L$, where $\rho(p) = \min_{p^*\in P^*} \|p-p^*\|$ and P^* denotes the set of optimal solutions of (2.1a)-(2.1b). To see this, suppose the contrary. Then there exists a sequence of n-vectors $\{p^0, p^1,\ldots\}$ in L such that $\rho(p^r) \to \infty$. Since $p^r \ge 0$ for all r, this implies that there exists a subsequence R of $\{0,1,\ldots\}$ such that, for each $i\in\{1,\ldots,n\}$, either $\{p_i^r\}_{r\in R} \to \infty$ or $\{p_i^r\}_{r\in R}$ is bounded. Let $I = \{\ i \mid \{p_i^r\}_{r\in R}$ is bounded $\}$. For each $r\in R$, let $\hat{p}^r$ denote the least L_2-norm solution of the following linear system

$$\langle c, p\rangle = \langle c, p^r\rangle, \qquad Ep = Ep^r, \qquad Ap = b, \qquad p_i = p_i^r \quad \forall\ i\in I. \tag{6.10}$$

This linear system is feasible since p^r is a solution. Furthermore, since the right hand side of this system is bounded for all $r\in R$ (cf. Assumption A (c) and Lemma 2 in Appendix B), Lemma 1 in Appendix A shows that $\{\hat{p}^r\}_{r\in R}$ is also bounded. Then since $\{p_i^r\}_{r\in R} \to \infty$ for all $i\notin I$, and p^r is a solution of (6.10) for all $r\in R$, we obtain that, for any $r\in R$ sufficiently large, the difference $u^r = p^r - \hat{p}^r$ satisfies

$$\langle c, u^r\rangle = 0, \qquad Eu^r = 0, \qquad Au^r = 0, \qquad u^r \ge 0.$$

Fix any $p^* \in P^*$. From the above set of equations we see that, for all $r \in R$ sufficiently large, it holds $p^* + u^r \in P^*$, so that $\rho(p^r) \le \|p^r - (p^* + u^r)\| = \|\hat{p}^r - p^*\|$. Since $\{\hat{p}^r\}_{r\in R}$ is bounded, this contradicts the hypothesis that $\rho(p^r) \to \infty$. Let ρ_3 be a bound on $\rho(p)$ for all $p \in L$. It can be seen that ρ_3 depends on q, P and ζ only.

Let ε be any positive scalar. Consider any $\bar{p} \in P \cap C_q$ that satisfies $q(\bar{p}) \le \zeta$ and, together with some $\bar{v} \in S^\perp$, (6.8). Then $\bar{p} \in L$, so that $\|\nabla q(\bar{p})\| \le \rho_1$ and there exists an optimal solution p^* of (2.1a)-(2.1b) satisfying $\|\bar{p} - p^*\| \le \rho_3$. Moreover, it is easily seen that $v = \bar{v}$ satisfies (6.9a)-(6.9c) with $p = \bar{p}$, so that $v^\varepsilon(\bar{p})$ is well-defined and $\|v^\varepsilon(\bar{p})\| \le \rho_2$. To simplify the notation, let $\tilde{v} = v^\varepsilon(\bar{p})$ and $J = \{\, i \mid -\varepsilon \le \partial q(\bar{p})/\partial p_i - \tilde{v}_i \le \varepsilon \,\}$. Then $\hat{v}$ satisfies $\|\tilde{v}\| \le \rho_2$ and

$$
\begin{aligned}
&-\varepsilon \le \partial q(\bar{p})/\partial p_i - \tilde{v}_i \le \varepsilon, && && \forall\, i \in J,\\
&\varepsilon < \partial q(\bar{p})/\partial p_i - \tilde{v}_i, && 0 \le \bar{p}_i \le \varepsilon, && \forall\, i \notin J,\\
&\tilde{v} \in S^\perp.
\end{aligned}
$$

[This is reminiscent of the ε-complementary slackness condition discussed in [TsB87c].] Hence

$$
\begin{aligned}
\langle \nabla q(\bar{p}), p^* - \bar{p} \rangle &= \langle \nabla q(\bar{p}) - \tilde{v}, p^* - \bar{p} \rangle \\
&= \sum_{i \in J} (\partial q(\bar{p})/\partial p_i - \tilde{v}_i)(p_i^* - \bar{p}_i) + \sum_{i \notin J} (\partial q(\bar{p})/\partial p_i - \tilde{v}_i)(p_i^* - \bar{p}_i) \\
&\ge -\varepsilon \sum_{i \in J} |p_i^* - \bar{p}_i| + \sum_{i \notin J, p_i^* < \bar{p}_i} (\partial q(\bar{p})/\partial p_i - \tilde{v}_i)(p_i^* - \bar{p}_i) \\
&\ge -\varepsilon \sum_{i \in J} |p_i^* - \bar{p}_i| - \varepsilon \sum_{i \notin J, p_i^* < \bar{p}_i} |\partial q(\bar{p})/\partial p_i - \tilde{v}_i| \\
&\ge -\varepsilon n \rho_3 - \varepsilon n(\rho_1 + \rho_2),
\end{aligned}
$$

where the first equality follows from the facts $p^* - \bar{p} \in S$ and $\tilde{v} \in S^\perp$. Let $\gamma = n(\rho_1 + \rho_2 + \rho_3)$. Then by the convexity of q,

$$
\begin{aligned}
q(p^*) &\ge q(\bar{p}) + \langle \nabla q(\bar{p}), p^* - \bar{p} \rangle \\
&\ge q(\bar{p}) - \gamma\varepsilon.
\end{aligned}
$$

On the other hand, since $\bar{p} \in P$ and p^* minimizes q over P, we have $q(\bar{p}) \ge q(p^*)$. Q.E.D.

Propositions 6 and 7 show that, when h satisfies the assumptions of Proposition 7, the ε-basis descent method computes, in a finite number of iterations, a feasible solution of (2.1a)-(2.1b) whose value is within ε multiplied by some constant (depending on P, q, and the initial iterate p^0) of the optimal value of (2.1a)-(2.1b). It should be noted that the assumptions of Proposition 7 (including Assumption A) are met for many problems in practice (for example, when P is a bounded polyhedral set and h is a convex differentiable function).

One interesting application of the ε-basis descent method is to problems for which h is separable, i.e. $h(t) = \sum_{j=1}^{m} h_j(t_j)$ for some closed convex essentially smooth functions $h_j:\Re\to(-\infty,\infty]$ (cf. (2.2)). In this case, we obtain a method that is quite similar to the fortified primal descent algorithm of Rockafellar [Roc84, §11I], with the essential difference being that the latter algorithm computes, in place of the I_j's given by (6.4a), intervals of the form

$$\left[\ \sup_{\xi<t_j}\frac{h_j(\xi)-h_j(t_j)+\varepsilon}{\xi-t_j},\ \inf_{\xi>t_j}\frac{h_j(\xi)-h_j(t_j)+\varepsilon}{\xi-t_j}\ \right], \qquad j = 1,\ldots,m,$$

at each iteration. In general, the above intervals seem to be more difficult to compute than those given by (6.4a). [On the other hand, the fortified primal descent algorithm has the advantage that each solution that it produces is guaranteed to come within ε in cost of the optimal cost.] Another interesting application of the ε-basis descent method is to network flow problems. In this case, the constraint matrix A is the node-arc incidence matrix for some directed graph and the dual rectification algorithm can be implemented by using graphical subroutines. In particular, each iteration of the dual rectification algorithm can be implemented by using, for example, a breadth-first search procedure on the graph (see [Roc84, §6]). In general, the dual rectification algorithm is implemented by means of tableau pivoting techniques similar to that found in the simplex method. This is a very nice feature of the dual rectification algorithm since it enables the algorithm to take advantage of, for example, sparse matrix techniques developed for the simplex method. To the best of our knowledge, the only other method for minimizing convex (possibly non-separable) essentially smooth functions over polyhedral sets that can exploit the combinatorial structure of the constraint set P in the same manner is the convex simplex method [Zan69]. However, the convex simplex method requires h to be real-valued and needs to make a nondegeneracy assumption on the problem to achieve convergence.

Notice that the ε-basis descent method puts a box of length 2ε around the gradient ∇q before generating a descent direction (cf. (4.4a)). This technique, designed to ensure that each descent direction offers a large rate of descent, is very similar to an ε-complementary slackness mechanism employed by the methods in [BHT87], [TsB87c]. The ε-basis descent method also restricts each descent direction to point away from those boundary hyperplanes of P to which the current iterate comes within ε in distance (cf. (4.4b)). This technique, designed to prevent jamming, is similar to ones used in ε-active set methods [Zan69], [NgS84].

The ε-basis descent method is quite closely related to the ε-subgradient methods in nondifferentiable optimization [BeM73], [NgS84], [Roc84]. To see this, consider the special case where $P = \Re^n$. In this case the problem (2.1a) simplifies to $\min_p q(p)$, and it can be seen that each iteration of the ε-basis descent method involves finding a direction w satisfying

$$\max_{u \in \nabla q(p)+\varepsilon B} \langle w,u \rangle < 0,$$

where $p \in C_q$ is the current iterate and B denotes the unit ball in the L_∞-norm, i.e. $B = [-1,1]^n$, and then performing a line search along w. When applied to solve this same problem $\min_p q(p)$, each iteration of the ε-subgradient method involves finding a direction w′ satisfying

$$\max_{u \in \partial_\varepsilon q(p)} \langle w',u \rangle < 0,$$

where $p \in C_q$ is the current iterate and $\partial_\varepsilon q(p)$ denotes the ε-subdifferential of q evaluated at p [Roc70, p. 220], i.e. $\partial_\varepsilon q(p) = \{ u \mid q^*(u) + q(p) - \langle p,u \rangle \le \varepsilon \}$, and then performing a line search along w′. Upon comparing these two iterations, we see that they differ essentially in the way that a "ball" is constructed around ∇q when computing a descent direction. This difference is nonetheless significant since in general it is much easier to compute the ball $\nabla q(p) + \varepsilon B$ and to minimize linear functions over it than to do the same for the "ball" $\partial_\varepsilon q(p)$ (for example, the ε-subgradient method of [NgS84] must solve, at each iteration, a quadratic program whose dimension grows linearly with the iteration count).

We have thusfar left open the issue of computing an initial point in $P \cap C_q$. If q is real-valued, then it suffices to find a point inside the polyhedral set P, which is a well-studied problem in linear programming. Otherwise, if P is given by a set of linear constraints, then some type of multiplier method or penalty method [Ber82] can be used to find such a point.

Finally, it should be noted that the parameter ε need not be fixed for all iterations of the ε-basis descent method. In certain cases it may be preferable to begin with a large ε and to decrease ε gradually until a desired tolerance is reached. [This ensures that cost decreases are large in the early stages of the method.] Also notice that one can employ different ε's for different coordinates.

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Appendix A.

The following lemma characterizes the boundedness of the least L_2-norm solution of perturbed linear systems:

Lemma 1. Let B be any k×r matrix. For every k-vector d such that the linear system $Bx \geq d$ has a solution, let $\xi(d)$ denote the least L_2-norm solution, i.e. $\xi(d) = \operatorname{argmin}_{Bx \geq d} \|x\|$. Then the function ξ is bounded on any bounded set of points over which it is defined.

Proof. Since the system $Bx \geq 0$ has the zero vector as a solution, it follows from a well-known Lipschitz continuity property of the solutions of linear systems ([MaS87], [Rob73]) that there exists, for each d such that $Bx \geq d$ has a solution, a solution y satisfying $By \geq d$ and

$$\|y\| \leq \theta \|d\|,$$

where θ is some constant that depends on B only. Since $\xi(d)$ is the least L_2-norm solution, it follows that $\|\xi(d)\| \leq \theta \|d\|$. Hence $\xi(d)$ is bounded for all d lying in a bounded set on which it is defined. Q.E.D.

Appendix B.

The following technical lemma characterizes the boundedness of the function h on each level set of q over P:

Lemma 2. Let ξ be any scalar for which the level set $L = \{ p \in P \mid q(p) \le \zeta \}$ is nonempty. Then (under Assumption A) both functions $p \to h(Ep)$ and $p \to \langle c, p \rangle$ are bounded on L.

Proof. We first show that the function $p \to h(Ep)$ is bounded on L. Since the set $\{ Ep \mid p \in L \}$ is bounded (cf. Assumption A (c)), this function is bounded from below on L. Therefore if it is unbounded on L, it must be unbounded from above, i.e. there exists a sequence of vectors $\{p^0, p^1, \ldots\}$ in L such that $h(Ep^r) \to \infty$. Since $q(p) = h(Ep) + \langle c, p \rangle$ (cf. (2.1b)) and $q(p) \le \xi$ for all $p \in L$, this implies that $\langle c, p^r \rangle \to -\infty$. Let P be expressed as $\{ p \in \Re^n \mid Ap \ge b \}$ for some k×n matrix A and k-vector b. Then we have $Ap^r \ge b$ for all r (since $p^r \in P$). Choose R to be any subsequence of $\{0,1,\ldots\}$ such that, for each $i \in \{1,\ldots,k\}$, either $\{A_i p^r\}_{r \in R} \to \infty$ or $\{A_i p^r\}_{r \in R}$ is bounded, where A_i denotes the i-th row of A. Let $I = \{ i \mid \{A_i p^r\}_{r \in R} \text{ is bounded} \}$. For each $r \in R$, let $\hat{p}^r$ denote the least L_2-norm solution of the following linear system

$$Ep = Ep^r, \qquad Ap \ge b, \qquad A_i p = A_i p^r \quad \forall\, i \in I. \tag{B.1}$$

This linear system is feasible since p^r is a solution. Furthermore, since the right hand side of this system is bounded for all $r \in R$, Lemma 1 in Appendix A shows that the sequence $\{\hat{p}^r\}_{r \in R}$ is also bounded. Then since $\{\langle c, p^r \rangle\}_{r \in R} \to -\infty$, $\{A_i p^r\}_{r \in R} \to \infty$ for all $i \notin I$, and p^r is a solution of (B.1) for all $r \in R$, we obtain that, for any $r \in R$ sufficiently large, the difference $u = p^r - \hat{p}^r$ satisfies

$$Eu = 0, \qquad Au \ge 0, \qquad \langle c, u \rangle < 0.$$

Then, for any $p \in P \cap C_q$, we have $p + \theta u \in P$ for all $\theta \ge 0$ and $q(p+\theta u) = q(p) + \theta \langle c, u \rangle \to -\infty$ as $\theta \to \infty$, a contradiction of the assumption that $\inf_{p \in P} q(p) > -\infty$ (cf. Assumption A (c)). This shows that the function $p \to h(Ep)$ is bounded on L. Since $q(p) = h(Ep) + \langle c, p \rangle$ for all p and q is bounded on L, the function $p \to \langle c, p \rangle$ is also bounded on L. Q.E.D.

Appendix C.

The following technical lemma characterizes a sequence of vectors in $P \cap C_q$ whose costs are monotonically decreasing:

Lemma 3. Let $\{p^0, p^1, \ldots\}$ be any sequence of n-vectors in $P \cap C_q$ satisfying $q(p^r) \geq q(p^{r+1})$ for all r. Then (under Assumption A) the following hold:

(a) $\{Ep^r\}$ is bounded.

(b) $\{\langle c, p^r \rangle\}$ is bounded.

(c) Every limit point of $\{Ep^r\}$ is in C_h.

Proof. We first prove parts (a) and (b). Since $q(p^r)$ is monotonically decreasing with r and $p^r \in P$ for all r, Ep^r is in the set $\{ Ep \mid p \in P, q(p) \leq q(p^0) \}$ for all r. Since by Assumption A (c) this set is bounded, $\{Ep^r\}$ is bounded. Also we have that p^r is in the set $\{ p \in P \mid q(p) \leq q(p^0) \}$ for all r, so that, by Lemma 2, $\{\langle c, p^r \rangle\}$ is bounded.

Now we prove part (c). Suppose the contrary. Then there exist $t^\infty \in \mathrm{bd}(C_h)$ and subsequence $R \subseteq \{1,2,\ldots\}$ such that $\{t^r\}_{r\in R} \to t^\infty$. By using the facts $\{h(t^r) + \langle c, p^r \rangle\} \to q^\infty$ and (since h is closed) $\lim_{r\to\infty, r\in R} \inf\{h(t^r)\} \geq h(t^\infty)$, we have from part (b) that $h(t^\infty) < \infty$. Furthermore, we can find a $p^\infty \in P$ such that $t^\infty = Ep^\infty$ and $q(p^\infty) \leq q^\infty$. [To see this, we solve, for each $r \in R$, the linear system

$$p \in P, \qquad Ep = t^r, \qquad \langle c, p \rangle = \langle c, p^r \rangle,$$

for the least L_2-norm solution, denoted by $\bar{p}^r$. Then $q(\bar{p}^r) = q(p^r)$ for all $r \in R$ and, since $\{t^r\}$ and $\{\langle c, p^r \rangle\}$ are bounded, $\{\bar{p}^r\}_{r\in R}$ is bounded (cf. Lemma 1 in Appendix A), so that if p^∞ is any limit point of $\{\bar{p}^r\}_{r\in R}$, then $p^\infty \in P$, $Ep^\infty = t^\infty$ and (since q is closed) $q(p^\infty) \leq \lim_{r\to\infty, r\in R} \inf\{q(p^r)\} = q^\infty$.] Since $q(p^r) \downarrow q^\infty$, we have $q(p^\infty) \leq q(p^r)$ for all $r \in R$, so that

$$\begin{aligned} 0 &\leq \frac{q(p^r) - q(p^\infty)}{\|p^r - p^\infty\|} \\ &= \frac{h(t^r) - h(t^\infty) + \langle c, p^r - p^\infty \rangle}{\|p^r - p^\infty\|} \\ &\leq \frac{h(t^r) - h(t^\infty)}{\|t^r - t^\infty\|}\|E\| + \frac{\langle c, p^r - p^\infty \rangle}{\|p^r - p^\infty\|}, \qquad \forall\, r \in R, \end{aligned}$$

which contradicts the fact $\lim_{r\to\infty, r\in R} \sup\{\frac{h(t^r)-h(t^\infty)}{\|t^r-t^\infty\|}\} = -\infty$ (cf. Assumption A (b)). Q.E.D.