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On the synthesis of a class of 2-D acausal lossless digital filters

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Abstract

Passive and lossless two-dimensional discrete systems of the fully recursive half-plane type are introduced by viewing them as 1-D filters over convolutional algebra. Necessary and sufficient conditions for 2-D transfer functions to be valid scattering as well as immittance domain description of such systems are obtained. An algorithm for the structurally passive (in fact, lossless) synthesis of filters having such recursive structure is then derived from these representation results as an extension of a recent 1-D Schur type algorithm for the synthesis of discrete lossless two-ports. Specific comments on various aspects of design and implementation of such 2-D filters potentially useful in practical problems are also made.

1. Introduction

Various recursive schemes have been proposed in the multidimensional (m-D) digital filter literature. Among these the most widely studied are the quarter plane, the asymmetric and the symmetric half-plane recursive scheme. More recently, motivated by needs for parallel processing of 2-D signals a scheme known as the fully recursive half-plane scheme has been proposed in [15], and a method of designing transfer functions of filters having this recursive structure has been outlined in [5]. The impulse responses of the class of filters just mentioned satisfies the characteristic property that the region of support is a half-plane and the filter is recursive in both horizontal and vertical direction. More specifically, the recursion equation describing the relation between the input x and output y of a filter of this type is given by:

$$\beta_0[\{y_n(m)\}] = - \sum_{i=1}^{L_D} \beta_i[\{y_{n-i}(m)\}] + \sum_{i=0}^{L_N} \alpha_i[\{x_{(n-i)}(m)\}] \qquad (1.1)$$

where $x_n(m)$, $y_n(m)$ denote the n-th row of the input and the output signal, and the (row) operations $\alpha_i[.]$ and $\beta_i[.]$ respectively denote 1-D linear shift invariant convolution operations with fixed 1-D sequences $a_i(m)$ and $b_i(m)$. Considering the 2-D Z-transform of (1.1), and assuming that the operations $\alpha_i[.]$ and $\beta_i[.]$ are all rational we then have $H(z_1,z_2)$ in (1.2) for the transfer function of the filter, where $A_i(z_1)$, $B_i(z_1)$ are rational transfer functions representing the convolutional row operations just mentioned.

$$H(z_1,z_2) = \sum_{i=0}^{L_N} A_i(z_1) z_2^i \Big/ \sum_{i=0}^{L_D} B_i(z_1) z_2^i \qquad (1.2)$$

On the otherhand, it is now well known that an input-output description such as the one expressed in (1.2), (1.3) is not enough for the successful operation of a digital filter but structural considerations need to be taken into account. The class of structurally passive filters variously known as the wave digital filters [16], orthogonal filters [17] or the lossless bounded real filters [18], when properly designed, are known to satisfy the properties of insensitivity to coefficient perturbation and non-linear arithmetic conditions resulting from overflow, finite precision arithmetic etc. These are, in fact, properties of specific realizations of transfer functions and can, therefore, also be studied via state space methods. We refer to the work in [27] for a discussion on sensetivity properties of specific realizations such as the balanced realization. Although much progress has been documented in the synthesis and design of 1-D structurally passive filters, methods for two and higher dimensions are still evolving. Synthesis methods for two and multi-dimensional wave digital filters, which are quarter plane type filters have been reported in [16], [7]. Quarter plane and asymmetric half-plane generalizations of 1-D lattice filters which are, in fact, structurally passive, have been discussed recently in the context of random field modeling in [11],[19].

Following 1-D, in the present paper (pseudo) passive or (pseudo) lossless[1] fully recursive half-plane 2-D digital filters are introduced and a method of their structurally passive synthesis and subsequently that of their design is discussed for the first time. The problem of synthesis of

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[1] our filters are (pseudo) passive or (pseudo) lossless in the sense that they dissipate or conserve discrete energy in signals with half-plane support. See Sections 2 and 4 for precise details.

quarter plane causal (thus, including filters causal in a convex cone [20]) structurally passive multidimensional filters of the type mentioned above can be equivalently viewed as the classical network theoretic problem of synthesizing a lossless but otherwise arbitrarily prescribed multidimensional transfer function as an interconnection of elementary building blocks such as capacitors and inductors (see [10] and references contained therein). This latter problem is unsolvable in multidimensions (m>2), whereas in 2-D synthesis is feasible only in an unconstrained topological structure [20], [26]. On the otherhand, it has been shown that if certain ladder-like constraints are imposed on the structure in which the filter is to be synthesized then the prescribed 2-D transfer function must satisfy further restrictions in addition to input-output losslessness [21], [22], [23]. Related other synthesis results [7], [16] in this context deal with important special cases when the multidimensional frequency response of the filter possesses certain symmetries. In contrast, the present work provides us with a synthesis of single input single output but otherwise arbitrary lossless fully recursive half-plane 2-D filters. Additionally, unlike the quarter plane case referred to earlier the synthesis is obtained in a fixed predetermined structure potentially useful for practical implementation.

As in most passive or lossless filter design techniques our synthesis method proceeds by viewing the prescribed passive transfer function as being embedded into the transfer function of a lossless two-port. The synthesis of this fully recursive half-plane lossless two-port takes advantage of a recent algorithm for the design of structurally passive 1-D filters advanced by Rao and Kailath [6] as an extension of the celebrated Schur algorithm [9]. Unlike all other methods known for the synthesis of 1-D continuous as well as discrete lossless two-ports including those available in the classical circuit theoretic literature, the algorithm of [6] enjoys the unique feature that given a transfer function associated with the lossless two-port the synthesis algorithm makes use of rational arithmetic operations only (i.e., nonrational arithmetic operations such as polynomial factorization is not required) [10]. The synthesis method for fully recursive half-plane filters to be presently described fully exploits this rational character of the 1-D algorithm in [6]. Although the details of the method differ nontrivially from 1-D due to considerations characteristic of multidimensional problems (e.g., those utilizing techniques from elementary algebraic curve theory [3], [12]), the synthesis to be outlined can be considered, at least at a conceptual level, to be a generalization of the result in [6] to two-port transfer functions the coefficients of numerator and denominator polynomials of which belong to a field of rational functions (instead of the field of rational numbers). From a different perspective the present work can also be viewed as a generalization of 1-D Schur algorithm to 2-D fully recursive half-plane schemes, thus making it possible to cast the present discussion in the closely related framework of modeling of stationary random fields and scattering theory [9].

A note regarding the stability of the filter is in order. The region of analyticity of the transfer function of our filter will be found to marginally differ from those previously considered in the 2-D half-plane literature [4], [5]. This is primarily due to the fact that the results such as those in [4], [5] are motivated by bounded-input-bounded-output considerations, whereas, in contrast, our results are driven by passivity considerations. The fact that this difference in consideration does indeed lead to diverging formulations of stability in multidimensions ($m>1$), but not in 1-D, is now known [1], [2]. Thus, there is no contradiction between our stability results and those existing in the half-plane literature so far.

The idea of considering filters with recursive structures such as the one considered in the present paper can, along with [15], be traced back to the work of Harris as referenced and described in [29]. However, although possible generalizations of 1-D lattice filters were investigated in this work, considerations of passivity or losslessness, let alone structural passivity, were not taken into account. In the present paper a complete characterization of passivity and losslessness in terms of transform domain description of systems having (partially) acausal recursive structures is given for the first time. Furthermore, it is known that due to the restricted nature of transmission zeros, (1-D) lattice filters can only realize AR type transfer functions. Structurally passive realizations of broader class of transfer functions require considerations of structures other than the lattice structure (the wave digital filters, Rao-Kailath structures etc. are examples). Since the specified transfer function need not be of the AR type (in fact, it is completely arbitrary within the class of transfer functions which are passive/lossless in the fully recursive half plane sense -- a notion to be made precise in Section 2), our results go much beyond that established by Harris [29].

In Section 2 the fully recursive half-plane passive one-ports are characterized in terms of their transfer function. In Section 3 we consider the immittance domain description of fully resursive half-plane passive systems. Characterization of fully recursive half-plane lossless two-port transfer functions form the context of Section 4. A representation theorem for fully recursive half-plane lossless two-ports analogous to that of the Belevitch canonical form [8] of representation for lossless 1-D continuous two-ports of classical network theory is developed here. In Section 5 the synthesis method based on this representation theorem is described, and in Section 6 a design methodology is proposed by taking into account the symmetry requirements [14] on the frequency response imposed by many practical multidimensional processing tasks. Some implementational considerations are also discussed here. Finally, the results are summarized and possibilities of further research are pointed out.

2. Fully recursive symmetric half-plane passive systems:

The major intent of this section is to develop transform domain characterization of single-input-single-output passive or lossless fully recursive symmetric half-plane systems. We note that in the classical theory of linear passive time invariant 1-D systems two apparently different definitions of passivity have been used [24]. As shown by Youla (see [24] for datails and references to original literature), however, the two definitions are mathematically equivalent under the additional assumption that the system under consideration is causal. Thus, causality may or may not be viewed as a consequence of passivity depending on the way this latter concept is introduced. In 2-D, since there are various ways of introducing causality in the recursive structure of the filter (the present context deals with only one such possibility), it is desirable to adopt the definition of passivity in such a way that causality may be introduced as an independent notion. In this vein, we associate the total (pseudo) energy $\Sigma\Sigma|x(n_1,n_2)|^2$ to the input $x(n_1, n_2)$ and the total (pseudo) energy $\Sigma\Sigma|y(n_1,n_2)|^2$ to the output $y(n_1, n_2)$ of the system, where the summations range from $-\infty$ to $+\infty$. The fully recursive half-plane filter is then said to be (pseudo) passive if (2.1) holds true for any choice of square summable input bi-sequence $x(n_1, n_2)$.

$$\Sigma\Sigma \; |y(n_1, n_2)|^2 \leq \Sigma\Sigma \; |x(n_1, n_2)|^2 \qquad (2.1)$$

To facilitate our discussion it will be assumed for the rest of the paper that the 1-D convolution operations $\alpha_i[.]$ and $\beta_i[.]$ in (1.1), can be viewed as convolutions with possibly infinite but rational sequences i.e., these sequences are impulse responses of 1-D IIR filters. This assumption has the consequence of making $A_i(z_1)$ and $B_i(z_1)$ 1-D rational transfer functions and thus, the 2-D transfer function $H(z_1,z_2)$ of the fully recursive filter as in (1.2) is a rational function in both z_1 and z_2.

We first examine the consequence of passivity reflected on the frequency response $H(\omega_1, \omega_2)$ of the filter[2]. By using the 2-D Parseval's theorem and the fact that $Y(\omega_1,\omega_2) = H(\omega_1,\omega_2)X(\omega_1,\omega_2)$, where $Y(\omega_1,\omega_2)$, $X(\omega_1,\omega_2)$ are the respective Fourier transforms of $y(n_1,n_2)$, $x(n_1,n_2)$ we have that (2.1) is equivalent to (2.2)

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |X(\omega_1,\omega_2)|^2 \, (1-|H(\omega_1,\omega_2)|^2) \, d\omega_1 \, d\omega_2 \geq 0 \qquad (2.2)$$

Since (2.2) is true for any input $X(\omega_1,\omega_2)$ with square summable $x(n_1, n_2)$, we have that $|H(\omega_1,\omega_2)|\leq 1$ for all real two-tuples (ω_1,ω_2) except possibly for finitely many of them. To justify this latter step note that in view of rationality of the transfer function $H(z_1,z_2)$ if $|H(\omega_1,\omega_2)|>1$ for some $(\omega_{10},\omega_{20})$ then there must exist a neighbourhood of $(\omega_{10},\omega_{20})$ in which $|H(\omega_1,\omega_2)|>1$ for all (ω_1, ω_2). Then (2.2) is violated by choosing $X(\omega_1,\omega_2)$ to have finite support inside the neighborhood $(\omega_{10},\omega_{20})$ just mentioned (the existence of such $X(\omega_1,\omega_2)$ with square summable $x(n_1,n_2)$ can be easily demonstrated). Also, if the system is lossless we have equality in (2.1) and (2.2), which via the same argument yields that $|H(\omega_1, \omega_2)|=1$ for all real 2-tuples (ω_1,ω_2) except possibly for finitely many of them. Note that the results of the preceding discussion can be succintly stated by saying that $|H(z_1,z_2)|\leq 1$ (or $|H(z_1,z_2)|=1$ in the lossless case) everywhere on $|z_1|=1$ except possibly at the nonessential singularities of the 2nd kind [20],[28].

Next, by choosing $x(n_1, n_2) = \delta(n_1, n_2)$ i.e., the 2-D impulse function, the impulse response $h(n_1, n_2)$ of the filter can be

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[2] with slight abuse of notation $H(\omega_1,\omega_2)$ is used for frequency response, whereas $H(z_1,z_2)$ is used for transfer function.

obtained as the corresponding output. Since the support of the impulse response $h(n_1, n_2)$ of the filter is restricted to the upper half plane $n_2>0$ [5] we then have that:

$$y(n_1, n_2) = h(n_1, n_2) = \sum_{k=0}^{\infty} h_k(n_1)\, \delta(n_2-k) \qquad (2.3)$$

where $h_k(n_1)$, k=0,1,...etc. are certain 1-D row sequences and $\delta(.)$ is the 1-D impulse sequence.

Considering the z-transform of (2.3) we obtain

$$H(z_1, z_2) = \sum_{k=0}^{\infty} [\sum_{n_1} h_k(n_1) z_1^{n_1}]\, z_2^{k} \qquad (2.4)$$

Using the Schwartz inequality it follows from (2.4) that

$$|H(z_1, z_2)|^2 \leq k(z_2) \,.\, \sum_{k=0}^{\infty} |\sum_{n_1} h_k(n_1) z_1^{n_1}|^2 \qquad (2.5)$$

where $k(z_2) = 1 + |z_2|^2 + |z_2|^4 + \ldots$ etc.

If we consider the special case $z_1 = \exp(j\omega_1)$ then we have (2.6) from (2.5).

$$|H(e^{j\omega_1}, z_2)|^2 \leq k(z_2) \,.\, \sum_{k=0}^{\infty} |H_k(\omega_1)|^2 \qquad (2.6)$$

where $H_k(\omega_1)$ is the Fourier transform of $h_k(n_1)$ for each $k = 0,1,2,\ldots$ etc.

On the otherhand, the total output (pseudo) energy corresponding to input $x(n_1,n_2)=\delta(n_1,n_2)$ can also be expressed as:

$$\Sigma\Sigma|h(n_1,n_2)|^2 = \sum_{k=0}^{\infty} \sum_{n_1} |h_k(n_1)|^2 = \sum_{k=0}^{\infty} (1/2\pi) \int_{-\pi}^{\pi} |H_k(\omega_1)|^2\, d\omega_1$$

$$= (1/2\pi) \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} |H_k(\omega_1)|^2\, d\omega_1$$

(2.7)

in which the first equality follows from the definition of $h_k(n_1)$ and the fact that the support of the impulse response $h(n_1, n_2)$ is in the upper half plane $n_2>0$; the second equality from 1-D Parsevals' formula; whereas the last equality follows from interchanging the integral with the infinite sum (this latter operation, although not always feasible, can be justified in the present context on the basis of monotone convergence theorem [30, p.243]).

The left hand side of (2.7) is the total (pseudo) energy in the signal $h(n_1, n_2)$, whereas the right hand side can be similarly interpreted as the sum of the total (pseudo) energies contained in the row outputs $h_0(n_1)$, $h_1(n_1)$,...etc. corresponding to the impulsive input $\delta(n_1,n_2)$. Furthermore, it follows from passivity that $\Sigma\Sigma\ |h(n_1,n_2)|^2 < 1$. Thus, the integral in the left hand side of (2.7) is finite, and consequently, the integrand in the right hand side is bounded a.e. (almost everywhere in the Lebesgue measure sense) in $[-\pi,\pi]$ i.e., we have:

$$\sum_{k=0}^{\infty} |H_k(\omega_1)|^2 < \infty \qquad \text{a.e.} \qquad (2.8)$$

In view of (2.6) and the fact that $k(z_2) < \infty$ for $|z_2| < 1$ we then conclude that $H(z_1,z_2)$ is bounded a.e. for all $|z_1| = 1$ and for all $|z_2| < 1$; and furthermore, if $H(e^{j\omega_{10}}, z_{20})$ is unbounded for some $|z_{20}| < 1$ and real ω_{10}, then $\sum_{k=0}^{\infty} |H_k(\omega_{10})|^2$ must be unbounded, and thus $H(e^{j\omega_{10}},z_2)$ must also be so for all z_2. The latter statement would then hold for all z_2 on $|z_2|=1$ in particular, which would in turn violate the previously established fact (cf. paragraph after (2.2)) that $|H(z_1, z_2)|<1$ everywhere on $|z_1|=|z_2|=1$ except possibly at the nonessential singularities of 2nd kind. Thus, $H(z_1, z_2)$ is bounded for all $|z_1| = 1$, $|z_2| < 1$.

Since the convolution operations $\alpha_i[.]$ and $\beta_i[.]$ in (1.1) can be taken to be rational IIR functions, we have that for each i, $A_i(z_1)$ and $B_i(z_1)$ in (1.2) are rational functions in z_1. Thus, as stated earlier, under the present assumption, $H(z_1,z_2)$ becomes a rational function of both z_1 and z_2, and can be expressed as the ratio of two relatively prime polynomials $n(z_1,z_2)$ and $d(z_1,z_2)$ as:

$$H(z_1,z_2) = \frac{n(z_1,z_2)}{d(z_1,z_2)} \tag{2.9}$$

We now claim that for passive systems presently under consideration, the polynomial $d(z_1,z_2)$ in (2.9) cannot have infinitely many zeros on the distinguished boundary $|z_1| = |z_2| = 1$ of the unit bi-disc. For, if $d(z_{10},z_{20}) = 0$ for some $|z_{10}| = |z_{20}| = 1$ then in view of (2.9), in order for $H(\omega_1,\omega_2)$ to be bounded we would need $n(z_{10},z_{20}) = 0$ i.e., $d(z_1,z_2)$ and $n(z_1,z_2)$ would have a common zero on $|z_1|=|z_2|=1$. However, the presence of infinitely many such zeros would, in view of Bezout's theorem in algebraic curve theory [3], require that $n(z_1,z_2)$ and $d(z_1,z_2)$ have a common factor, which has been hypothesized to be absent in (2.9).

Some essential features of the above discussion are summarized in the following result.

<u>Property 2.1</u>: A passive fully recursive symmetric half-plane filter transfer function, when expressed in irreducible rational form as in (2.9), satisfies the following two conditions: (i) $d(z_1,z_2) \neq 0$ for $|z_1| = 1$ and $|z_2| < 1$ i.e., $H(z_1,z_2)$ is analytic in $|z_2| < 1$ for every $|z_1| = 1$. (ii) $d(z_1,z_2)$ does not have infinitely many zeros on $|z_1|=|z_2|=1$.

To investigate further consequences of passivity on the transfer function $H(z_1,z_2)$, when expressed in terms of ratio of two relatively prime polynomials $n(z_1,z_2)$ and $d(z_1,z_2)$ as in (2.9) let us define $\tilde{d}(z_1,z_2)$ and $\hat{d}(z_1,z_2)$ as:

$$\hat{d}(z_1,z_2) = \tilde{d}(z_1,z_2)z_1^{d_1}z_2^{d_2}; \quad \tilde{d}(z_1,z_2) = d^*(z_1^{*^{-1}},z_2^{*^{-1}}) \tag{2.10a,b}$$

where d_1, d_2 are the partial degrees of d in z_1 and z_2, and * denotes complex conjugation.

For convenience of further discussion the following definitions will be introduced in the spirit of [2]. A polynomial d will be said to be <u>half-plane Schur</u> if it does not have any zero in $|z_1|=1$, $|z_2|<1$. Furthermore, if any polynomial d satisfies $\hat{d} = \gamma d$ for some necessarily unimodular constant γ (i.e., $|\gamma|=1$) then d will be called <u>self-reciprocal</u>. Similarly, a polynomial satisfying properties 2.1(i) and 2.1(ii) simultaneously will be called <u>half-plane scattering Schur</u>.

Condition (ii) in Property 2.1 can, in fact, be replaced by any one of the conditions expressed in the following.

<u>Assertion 2.1</u>: Let $d=d(z_1,z_2)$ be a half-plane Schur polynomial in z_1 and z_2 and let a be the primitive[3] part of d. Then the following conditions are all equivalent.

(a) $d(z_1,z_2)$ does not have infinitely many zeros on the distinguished boundary $|z_1| = |z_2| = 1$.

(b) $a(z_1,z_2)$ and $\hat{a}(z_1,z_2)$, are relatively prime polynomials.

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[3]for the purpose of the present paper the primitive part and content [20] are considered, without explicit reference, with respect to the variable z_1.

(c) Each irreducible factor of $a(z_1,z_2)$ has at least one zero in the domain $|z_1| = 1$, $|z_2| > 1$.

We first need the following Lemma.

<u>Lemma 2.1</u>: Let g be a self-reciprocal half-plane Schur polynomial. If g is nonconstant and primitive then g must have infinitely many zeros on $|z_1|=|z_2|=1$.

<u>Proof</u>: Since g is primitive, $g(z_{10},z_2)$ must be a nonconstant polynomial involving z_2 for almost all fixed values of z_{10} on $|z_1|=1$. Furthermore, since g is self-reciprocal $g(z_{10}, z_2)$ is also so, and thus, the zeros of $g(z_{10},z_2)$ must either form inverse conjugate pairs or lie on $|z_2|=1$. The former possibility, however, is ruled out by the half-plane Schur property of g. The result thus follows. Q.E.D.

<u>Proof of Assertion 2.1</u>: Let $g=\gcd(a,\hat{a})$. Then as shown in [2] we must have $\hat{g}(z_1,z_2) = \gamma g(z_1,z_2)$, where γ = constant, $|\gamma|=1$; and since a, thus g, is primitive, in view of the above Lemma, $g(z_1,z_2)$ is either a constant or must have infinitely many zeros on $|z_1| = |z_2| = 1$. In the latter case, $a(z_1,z_2)$ and thus $d(z_1,z_2)$ must have infinitely many zeros on $|z_1| = |z_2| = 1$, which is impossible if (a) holds. Thus, $g(z_1,z_2)$ = constant, and $a(z_1,z_2)$ is relatively prime with $\hat{a}(z_1,z_2)$. This shows that (a) implies (b).

To show that (b) implies (a) observe that since the content of $d(z_1,z_2)$ is nonzero on $|z_1| = 1$, if $d(z_1,z_2)$ has infinitely many zeros on $|z_1| = |z_2| = 1$ then so does $a(z_1,z_2)$. Also, if for some $|z_{10}| = |z_{20}| = 1$, $a(z_{10},z_{20}) = 0$ then $a(z_{10},z_{20}) = 0$. Consequently, if $d(z_1,z_2)$ and thus $a(z_1,z_2)$ has infinitely many zeros on $|z_1| = 1$ then $a(z_1,z_2)$ and $\hat{a}(z_1,z_2)$ would have infinitely many common zeros (on $|z_1| = |z_2| = 1$). Therefore, due to Bezout's theorem [3],

$a(z_1,z_2)$ and $\hat{a}(z_1,z_2)$ would not then be relatively prime polynomials. Thus, (a) and (b) are equivalent.

Next, if $a_1(z_1,z_2)$ is any irreducible factor of $a(z_1,z_2)$ then obviously $a_1(z_1,z_2) \neq 0$ for $|z_1| = 1$, $|z_2| < 1$. Furthermore, if $a_1(z_1,z_2)$ does not contain any zero in $|z_1| = 1$, $|z_2| > 1$ then for any z_1 on $|z_1| = 1$, $a_1(z_1,z_2) \neq 0$ in $|z_2| < 1$ as well as in $|z_2| > 1$, and thus, in view of primitive property of a_1 inherited from a, the values of z_2 such that $a_1(z_1,z_2) = 0$ must be on $|z_2| = 1$. Consequently, $a_1(z_1,z_2)$, and thus, $a(z_1,z_2)$ would have infinitely many zeros on $|z_1| = |z_2| = 1$. Therefore, (a) (or equivalently (b)) implies (c).

To prove that (c) implies (b) let $g = \gcd(a,\hat{a})$ i.e., $a = g.e$, $\hat{a} = gf$, where e and f are relatively prime polynomials. Then, as shown in [2] $\hat{g} = \gamma g$ where γ is a constant. Assuming g to be a nonconstant polynomial, if each irreducible factor of a contains at least one zero in $|z_1| = 1$, $|z_2| > 1$ then g and thus $\hat{g} = \gamma g$ must have a zero in $|z_1| = 1$, $|z_2| > 1$. However, this implies that the polynomial g and thus, in view of $a = ge$, the polynomial a must have a zero in $|z_1| = 1$, $|z_2| < 1$, which is a contradiction. Thus, g = constant and a and $\hat{a}$ are relatively prime. Q.E.D.

We also have the following important result.

<u>Property 2.2a</u>: If a rational function $H=H(z_1,z_2)$ as in (2.9) is such that $|H|\leq 1$ on $|z_1|=|z_2|=1$ except possibly at finite number of nonessential singularities of 2nd kind, if present, and if d in (2.9) is a half-plane Schur polynomial (thus, if $H(z_1,z_2)$ is transfer function of a passive fully recursive half-plane filter) then $|H|\leq 1$ for all $|z_1|=1$ and $|z_2|<1$. Furthermore, if $|H|=1$ for some (z_{10},z_{20}) with $|z_{10}|=1, |z_{20}|<1$, then $H(z_{10},z_2)$ is a constant independent of z_2. Assuming H to involve z_2, the latter situation can arise for at most finitely many values of z_{10} (with $|z_{10}|=1$).

Proof: Due to the half-plane Schur property of the denominator polynomial of $H(z_1,z_2)$, $d(z_1,z_2)$ cannot be zero for some fixed $|z_{10}|=1$ and arbitrary values of $|z_2|<1$. Thus, if for any z_{10} with $|z_{10}|=1$ we define $H_1=H_1(z_2)=H(z_{10},z_2)$ then due to our hypothesis, H_1 is well defined, analytic in $|z_2|<1$ and $|H_1|\leq 1$ for $|z_2|=1$. Thus, by maximum modulus theorem, $|H_1|\leq 1$ for all $|z_2|<1$. Since this is true for arbitrary z_{10} on $|z_1|=1$ the first part follows.

To show the second part assume that for some $|z_{10}|=1$, $|z_{20}|<1$, we have $|H(z_{10},z_{20})|=1$. Then as shown above the maximum modulus theorem applies to $H_1=H_1(z_2)=H_1(z_{10},z_2)$ and thus $|H_1(z_{20})|=1$ with $|z_{20}|<1$ implies that $H_1=H_1(z_2)=C=\text{constant}$. However, this latter statement obviously cannot hold for infinitely many values of z_{10} unless $H(z_1,z_2)$ is independent of z_2. Q.E.D.

A rational function satisfying the property $|H|\leq 1$ for $|z_1|=1$, $|z_2|<1$ will henceforth be called a <u>half-plane bounded function.</u> In fact, the following result in Property 2.2b can also be proved. This result shows that the polynomials of the type described in Properties 2.1(i) and 2.1(ii) i.e., the half-plane scattering Schur polynomials characterize denominator of irreducible rational functions satisfying the half-plane boundedness property.

Property 2.2b: If H is a nonconstant irreducible rational function as expressed in (2.9) and is such that $|H|\leq 1$ for $|z_1|=1, |z_2|<1$ then either d is a constant or satisfies Properties 2.1 (i) and 2.1 (ii) i.e., d is a half-plane scattering Schur polynomial.

Proof: Obviously, it is impossible to have $d=0$ and $n\neq 0$ for any $|z_1|=1, |z_2|<1$, because otherwise $|H|$ would be unbounded there. If $d=n=0$ for some $|z_{10}|=1$, $|z_{20}|<1$ and z_{10} is not a zero of the content of d then consider an arbitrary small arc Γ_1 of $|z_1|=1$ issuing from z_{10}. Let Γ_2 be the continuous [12] arc traced out by z_2 (beginning from z_{20}) such that $d(z_1,z_2)=0$ is satisfied. Note that since Γ_1 is assumed arbitrarily small, due to the continuity property of zeros of a polynomial as a function of its coefficients, Γ_2 must lie completely within $|z_2|<1$. On other hand, If d=n=0 for some $|z_{10}|=1$, $|z_{20}|<1$ and z_{10} is a zero of the content of d then d=0 for $z_1=z_{10}$ and for arbitrary z_2 and thus, d=0 for infinitely many values of (z_1,z_2) (in $|z_1|=1$, $|z_2|<1$ in particular). In each of the above two cases d would have infinitely many zeros in $|z_1|=1$, $|z_2|<1$. However, since n and d are relatively prime, due to Bezout's theorem [3], n cannot be zero at each of these infinitely many values of (z_1,z_2) just mentoned. Thus, we would then have $n\neq 0$, d=0 for some $|z_1|=1$, $|z_2|<1$, which has already been proved to be impossible. Thus, $d \neq 0$ for $|z_1| = 1$, $|z_2| < 1$.

Finally, if $d(z_{10},z_{20})=0$ for some $|z_{10}|=|z_{20}|=1$ then $n(z_{10},z_{20})=0$ because otherwise $|H|\leq 1$ would be violated in $|z_1|=1,|z_2|<1$ at the vicinity of (z_{10},z_{20}). Thus existence of infinitely many such (z_{10},z_{20}) would again violate the relative primeness of n and d. Q.E.D.

We then have the following characterization for passive half-plane transfer functions.

Fact 2.1: A rational function H is the transfer function of a passive fully recursive half-plane filter if and only if it satisfies the property that $|H|\leq 1$ for all $|z_1|=1$, $|z_2|<1$.

Proof: Necessity has already been established in Property 2.2a. Conversely, if $|H|\leq 1$ for $|z_1|=1$, $|z_2|<1$ then from Property 2.2b it follows that the denominator of H in irreducible rational form must be either a constant or a half-plane scattering Schur polynomial, and thus can have at most finitely many zeros on $|z_1|=|z_2|=1$. Consequently, $H(\omega_1,\omega_2)$ is well defined except possibly for a finite number of real 2-tuples (ω_1,ω_2).

Furthermore, due to rational character of H it follows by invoking continuity that $|H|\leq 1$ wherever H is well defined on $|z_1|=|z_2|=1$. Thus, $|H(\omega_1,\omega_2)|\leq 1$ for all real 2-tuples (ω_1,ω_2) except possibly a finite number of them. This latter conclusion, however, implies that (2.2) holds, where $X(\omega_1,\omega_2)$ is the Fourier transform of any square summable input signal $x(n_1,n_2)$. The pseudo-passivity of $H(z_1,z_2)$ then follows from equivalence of (2.2) and (2.1). Q.E.D.

We next assume the filter to be (pseudo) lossless in the sense described earlier i.e., equations (2.1) and (2.2) are satisfied with equality. Consequently, from (2.2) we then have that for all 2-tuples (ω_1,ω_2) with the possible exception of finitely many values (2.11) holds true.

$$|H(\omega_1,\omega_2)| = 1 \tag{2.11}$$

We first claim that the rational transfer function $H(z_1,z_2)$ of a (pseudo) lossless fully recursive half-plane transfer function satisfies the property that

$$H(z_1,z_2)\,\tilde{H}(z_1,z_2) = 1 \tag{2.12}$$

To substantiate this result we observe from the definition of the operation $\tilde{}$ that $\tilde{H}(z_1,z_2) = H^*(z_1,z_2)$ for $|z_1|=|z_2|=1$, where the superscript * denotes complex conjugation. Consequently, from (2.11) it follows that $\tilde{H}(z_1,z_2)=H^{-1}(z_1,z_2)$ for all 2-tuples (z_1,z_2) on $|z_1|=|z_2|=1$ with possible exception of at most finitely many values. Thus, the two variable rational function $\tilde{H}(z_1,z_2)$ and $H^{-1}(z_1,z_2)$ assume equal values at infinitely many distinct points (z_1,z_2), and consequently, due to analytic continuation are identically same, i.e., $\tilde{H}(z_1,z_2) = H^{-1}(z_1,z_2)$ for all z_1 and z_2.

For convenience of further exposition the following terminology will be introduced. Any rational function $H(z_1,z_2)$ as expressed in (2.9) will be said to be a <u>fully recursive half-plane all-pass function</u> if $H(z_1,z_2)$ satisfies the conditions stated in Property 2.1 and in equation (2.12). Thus, transfer functions of (pseudo) lossless fully recursive half-plane filters are fully recursive half-plane all-pass functions.

A function $A(z_1,z_2)$ of two variables z_1, z_2, when expressible as a polynomial in z_2 with coefficients as rational functions in z_1 will be said to be a <u>pseudopolynomial</u> (in z_2). Thus, if $A(z_1,z_2)$ is a pseudopolynomial then

$$A(z_1,z_2) = \alpha_0(z_1) + \alpha_1(z_1)z_2 + \ldots + \alpha_{N_2}(z_1)z_2^{N_2} \qquad (2.13)$$

where $\alpha_k(z_1)$'s are rational functions in z_1. With $A(z_1,z_2)$ as given in (2.13), where $\alpha_{N_2}(z_1)$ is not identically zero, the integer N_2 will also be denoted by $\deg_2 A$. Furthermore, the notation $\bar{A}(z_1,z_2)$ will be used to denote the pseudopolynomial obtained from $A(z_1,z_2)$ as:

$$\bar{A}(z_1,z_2) = \tilde{A}(z_1,z_2)z_2^{N_2} \qquad (2.14)$$

Two <u>pseudopolynomials</u> $B(z_1,z_2)$ and $C(z_1,z_2)$ are said to be <u>coprime</u> if there is no pseudopolynomial $D(z_1,z_2)$ <u>actually involving</u> z_2 such that $B(z_1,z_2) = D(z_1,z_2)\,B_1(z_1,z_2)$ and $C(z_1,z_2) = D(z_1,z_2)\,C_1(z_1,z_2)$ for some pseudopolynomials $B_1(z_1,z_2)$ and $C_1(z_1,z_2)$. The following property then holds true.

<u>Theorem</u> <u>2.1</u>: Any fully recursive half-plane all-pass function $H(z_1,z_2)$ (thus, rational transfer function of (pseudo) lossless fully recursive half-plane filter) can be

expressed as follows:

$$H(z_1,z_2) = -D(z_1)\ [A(z_1,z_2)/\bar{A}(z_1,z_2)] \qquad (2.15)$$

where i) $A(z_1,z_2)$ is a pseudopolynomial

ii) $D(z_1) = z_1^N \gamma\ [d(z_1)/\hat{d}(z_1)]$, where $d(z_1)$ is a polynomial in z_1, γ is a constant of unit modulus and N = integer

iii) the pseudopolynomials $A(z_1,z_2)$ and $\bar{A}(z_1,z_2)$ are coprime

iv) $\bar{A}(z_1,z_2) \neq 0$ for all $|z_1|=1$, $|z_2|<1$.

Conversely, any rational function expressible as in (2.15) with (i), (ii), (iii) and (iv) in force is a fully recursive half-plane all-pass function.

<u>Proof</u>: Let $H(z_1,z_2) = A(z_1,z_2)/B(z_1,z_2)$, where $A = A(z_1,z_2)$ and $B = B(z_1,z_2)$ are pseudopolynomials expressible as $A = a_N/a_D$ and $B = b_N/b_D$, where in turn $a_N = a_N(z_1,z_2)$, $b_N = b_N(z_1,z_2)$ are polynomials in both z_1 and z_2, whereas $a_D = a_D(z_1)$ and $b_D = b_D(z_1)$ are polynomials in z_1 only.

We further assume that $H = H(z_1,z_2)$ expressed as in (2.16) is in irreducible rational form i.e., the pairs of polynomials (a_N,a_D), (b_N,b_D), (a_N,b_N) and (b_D,a_D) are relatively prime.

$$H = (b_D a_N)/(a_D b_N) \qquad (2.16)$$

Then from (2.16), equations (2.17), (2.18) and (2.19) follows, where the generic notation n_{ip} for denoting the degree of the polynomial p in the i-th variable has been used.

$$H = \tilde{H}^{-1} = (\tilde{a}_D\tilde{b}_N)/(\tilde{b}_D\tilde{a}_N) = [(\hat{a}_D\hat{b}_N)/(\hat{b}_D\hat{a}_N)]z_1^{\nu_N-\nu_D} z_2^{N_1} \tag{2.17}$$

$$\nu_N = n_{1b_D} + n_{1a_N} \geq 0, \quad \nu_D = n_{1a_D} + n_{1b_N} \geq 0 \tag{2.18 a,b}$$

and $N_1 = n_{2a_N} - n_{2b_N}$ (2.19)

Since H is analytic in $|z_1|=1$, $|z_2|<1$ and neither $\hat{a}_D$ nor $\hat{b}_N$ can have a factor z_2, it clearly follows that $N_1 \geq 0$. Also, since H in (2.16) is in irreducible rational form, it follows by comparing (2.16) and (2.17) that

$$\alpha b_D a_N = \hat{a}_D\hat{b}_N z_1^{\nu_N} z_2^{N_1}, \quad \alpha a_D b_N = \hat{b}_D\hat{a}_N z_1^{\nu_D} \tag{2.20 a,b}$$

where $\alpha = \alpha(z_1,z_2)$ is a polynomial in z_1 and z_2. By inserting (2.20b) into (2.16) and subsequently making use of the relations between $\hat{a}_N$ and $\tilde{a}_N$, between $\hat{a}_D$ and $\tilde{a}_D$ and finally by using $A = a_N/a_D$, (2.18) and (2.19), we obtain the following

$$H = \alpha[(a_Db_D)/(\hat{a}_D\hat{b}_D)][A/\tilde{A}].z_1^{-(n_{1a_N} + n_{1b_N})} z_2^{-n_{2a_N}} \tag{2.21}$$

By defining $d = a_Db_D$ and noting the fact that $\bar{A} = \tilde{A}\, z_2^{n_{2a_N}}$ we then have:

$$H = \alpha(d/\hat{d})(A/\bar{A})\, z_1^{-(n_{1a_N} + n_{1b_N})} \tag{2.22}$$

Since H in (2.16) is irreducible and analytic in $|z_1|=1, |z_2|<1$ we note that a_Db_N cannot have a factor z_2. Invoking this fact and considering the ^ of (2.20a) we then have:

$$\hat{\alpha}\hat{b}_D\hat{a}_N = a_D b_N z_1^{-k} \tag{2.23}$$

where in (2.23) k is the total multiplicity of z_1 in $(a_D b_N)$. By substituting (2.23) into (2.20b) we obtain $\hat{\alpha}\alpha = z_1^{\nu_D - k}$ (note that since from (2.18b) ν_D = degree of $(a_D b_N)$ in z_1 it obviously follows from (2.23) that $\nu_D - k \geq 0$). Consequently, it must be true that α is a monomial involving z_1 only i.e., is of the form

$\alpha = \gamma z_1^{\nu_D - k}$ for some constant γ. Then $\alpha\hat{\alpha} = \gamma\gamma^* z_1^{\nu_D - k}$. Thus, $|\gamma| = 1$. Therefore, (2.22) yields (2.15) with $N = \nu_D - (n_{1a_N} + n_{1b_N} + k)$. Properties 2.3(i) and 2.3(ii) are thus established. To show that 2(iii) holds true note that

$$\bar{A} = (\hat{a}_N/\hat{a}_D)\, z_1^{n_{1a_D} - n_{1a_N}} \tag{2.24}$$

Consequently, if A and $\bar{A}$ has a pseudopolynomial common factor then it follows from $A_N = a_N/a_D$ and (2.24) that a_N and $\hat{a}_N$ must have a common factor involving z_2. In view of (2.20a,b) then $a_D b_N$ and $b_D a_N$ would not be relatively prime, thus violating the irreducibility of H in (2.16). Finally, to prove (iv) note that it follows from Property 2.1, (2.16) and (2.20b) that $a_D b_N$ and thus $\hat{a}_N$ is nonzero for $|z_1| = 1$, and $|z_2| < 1$.

The converse proposition follows trivially from the fact that any $H = H(z_1, z_2)$ satisfying (2.15) along with (i) through (iv) is necessarily analytic in $|z_1| = 1$, $|z_2| < 1$ and has the property of $H\tilde{H} = 1$ on $|z_1| = |z_2| = 1$. Q.E.D

<u>Theorem 2.2</u>: If a is a half-plane scattering Schur polynomial then there exists a half-plane bounded function, which in its irreducible rational form, has a as its denominator.

<u>Proof</u>: Let a=d.g, where g is the primitive part and d is the

content of a. Then $d \neq 0$ on $|z_1|=1$ and, due to Assertion 2.1(b), $\hat{g}$ is relatively prime with g. Thus, $H_1 = \hat{g}/g$ is an irreducible rational function, such that $|H_1|=1$ wherever g is nonzero on $|z_1|=|z_2|=1$. Since g can have at most finite number of zeros on $|z_1|=|z_2|=1$ we have $|H_1|=1$ on $|z_1|=|z_2|=1$ except possibly at finite number of points. It then follows from Property 2.2a by invoking the half-plane Schur property of g that $|H_1|\leq 1$ for $|z_1|=1$, $|z_2|<1$.

Consider any polynomial h relatively prime with g and d. Then $|h/d|$ is bounded on $|z_1|=1$. Thus, $H_2 = c(h/g)$ satisfies $|H_2|<1$ on $|z_1|=1$ for appropriate choice of a constant c.

Consequently, $H=H_1H_2$ satisfiess $|H|\leq 1$ for $|z_1|=1$, $|z_2|<1$. Also, since $\hat{g}$, being a primitive polynomial, cannot have a factor in common with d, dg is the denominator of H in irreducible rational form. Q.E.D.

Note that Theorem 2.2 along with Property 2.2b characterizes half-plane scattering Schur polynomials as the denominators of half-plane bounded functions in irreducible rational form.

3. Half-plane immittance functions and their properties:

A description of fully recursive half-plane one ports, which is essentially analogous to scattering parameter description of passive 1-D filters was developed in Section 2. It is now well known that an alternative formalism, namely the immittance formalism, also provides an equivalent but sometimes more efficient way of describing passive systems. For example, the split versions of Levinson and Schur algorithms of 1-D linear prediction theory as well as the 2-D wave digital filters having fan type frequency response [16] are most conveniently described via the immittance formalism. Motivated by such considerations, the class of 2-D transfer functions that characterize fully recursive half-plane passive as well as lossless filters are identified in the present section. The development on the one hand closely follows our analogous studies for the quarter plane case reported in [2] and makes use of the concept of 1-D pseudo-lossless functions [25] on the other.

A rational function $Z(z_1,z_2)$ in two-variables z_1, z_2 will be called <u>half-plane positive</u> if $\mathrm{Re}Z(z_1,z_2) \geq 0$ for $|z_1|=1$ and $|z_2|<1$. In addition, if a half-plane positive function satisfies the property $Z(z_1,z_2) + \tilde{Z}(z_1,z_2)=0$ then it will be called a <u>half-plane reactance</u> function.

Clearly, then there is a one-to-one correspondence between the class of half-plane bounded functions H and the half-plane positive functions Z via the bilinear transformation $H=(1-Z)/(1+Z)$; $Z=(1-H)/(1+H)$. The same comment holds true between the class of half-plane bounded lossless functions and the class of half-plane reactance functions.

In order to characterize the nature of numerator and denominator polynomials of half-plane positive (or reactance) functions, when expressed in irreducible rational form, we first claim that the following results hold true.

Lemma 3.1: (i) Any half-plane Schur polynomial can be expresssed as a product of a half-plane self-reciprocal Schur factor and a half-plane scattering Schur factor. (ii) a half-plane self-reciprocal Schur polynomial may not contain a half-plane scattering Schur factor involving z_2. (iii) a polynomial is half-plane self-reciprocal Schur if and only if the irreducible factors in its primitive part have the same property and its content is a self reciprocal polynomial non-zero on $|z_1| = 1$.

Proof: Let a be a half-plane Schur polynomial.
(i) Let $d=\gcd(a,\hat{a})$; $a=d.e$, $\hat{a}=d.f$, where e and f are coprime polynomials. Then d and e are half-plane Schur and, due to [2, Lemma A1], $\hat{d}=\gamma d$, where γ=constant, $|\gamma|=1$. Thus, d is half-plane self-reciprocal Schur. Also, $d.f=\hat{a}=\hat{d}.\hat{e}=\gamma.d.\hat{e}$, and consequently, $f=\gamma\hat{e}$. Thus, relative primeness of e and f implies the relative primeness of e and $\hat{e}$. Consequently, due to Assertion 2.1(b), e is half-plane scattering Schur.

Assume furthermore that a is self-reciprocal.
(ii) Any half-plane scattering Schur factor of a involving z_2 if present, would, due to Assertion 2.1(c), have a zero for $|z_1|=1$, $|z_2|>1$, and thus, in view of self-reciprocal character of a, would contribute a zero to a in $|z_1|=1$, $|z_2|<1$, which is impossible.

(iii) Let a = bd, where b is the content and d is the primitive part of a. Clearly, both b and d are self-reciprocal and half-plane Schur. Thus, any irreducible factor of d, due to part (i), is either half-plane scattering Schur or half-plane self-reciprocal Schur. The first of these two possibilities may not, however, occur due to (ii) above. The converse proposition follows trivially from the fact that a product of half-plane self-reciprocal Schur polynomials is also so. Q.E.D.

Corollary 3.1: Any factor of a half-plane scattering Schur polynomial d is also so.

Proof: Clearly, any such factor is half-plane Schur, and thus, due to Lemma 3.1(i), product of a half-plane self-reciprocal Schur factor e and a half-plane scattering Schur factor f. Thus, d and $\hat{d}$ must both contain e as a factor. However, e cannot contain a factor involving z_2 because, otherwise, due to Assertion 2.1(ii) d would not be scattering Schur. Thus, $e = e(z_1) \neq 0$ on $|z_1| = 1$. Consequently, e is half-plane scattering Schur. Q.E.D.

Associated with any polynomial $a=a(z_1,z_2)$ we next define [2] a polynomial $\phi_2(a)$ as follows, in which $n_2 > 0$ is the partial degree of a in z_2.

$$\phi_2(a)=n_2 a - 2z_2(\delta a/\delta z_2) \tag{3.1}$$

We then have the following result.

Lemma 3.2: If a is a half-plane self-reciprocal Schur polynomial involving z_2 then $\phi_2(a)/a$ is a half-plane reactance function. Additionally, if a is a nonfactorable (i.e., irreducible) polynomial then $\phi_2(a)/a$ is rational function in irreducible form.

Proof: Let $z_1=z_{10}$ be any fixed value of z_1 on $|z_1|=1$. Then $\alpha(z_2)=a(z_{10},z_2) \neq 0$ in $|z_2|<1$, because a is half-plane Schur. Thus, $\phi(\alpha)/\alpha$ is a discrete positive function [2, Lemma A2]. Furthermore, since it routinely follows that

$$[\phi_2(a)/a]_{z_1=z_{10}} = \phi_2(\alpha)/\alpha + (\deg_2 a - \deg_2 \alpha) \tag{3.2}$$

we have that $\mathrm{Re}[\phi_2(a)/a] \geq 0$ for $z_1=z_{10}$, $|z_2|<1$. Since z_{10} is arbitrary on $|z_1|=1$, it follows that $Z=\phi_2(a)/a$ is a half-plane positive function. Also, it follows via straightforward algebraic manipulations that $\tilde{Z} = \tilde{\phi}_2(a)/\tilde{a} = -\phi_2(a)/a = -Z$. Thus, $Z = \phi_2(a)/a$ is a half-plane reactance function. The last part follows from the fact that $\phi_2(a)$ and a cannot have a common factor (cf. Theorem A1 in [2]) if a is an irreducible polynomial. Q.E.D.

A polynomial is said to be <u>half-plane reactance Schur</u>, if it is half-plane self-reciprocal Schur, and none of its irreducible factors involving z_2 is of multiple order.

<u>Theorem 3.1</u>: If $d=d(z_1,z_2)$ is any half-plane reactance Schur polynomial then there exists a polynomial n such that $Z=n/d$ is a half-plane reactance function in irreducible rational form.

We first need the following two elementary results for the proof of the above result.

<u>Lemma 3.3</u>: If $Z_i=n_i/d_i$, i=1 to n are rational functions in irreducible rational form and d_i are mutually coprime polynomials then the rational function $\sum_i Z_i$ has $d=d_1 d_2 \ldots d_n$ as its denominator in irreducible rational form.

<u>Proof</u>: Straightforward for n=2. Rest follows by induction.
Q.E.D.

<u>Lemma 3.4</u>: Let $d=d(z)$ be any 1-D self-reciprocal polynomial. Then there exists a polynomial n such that n/d is a 1-D discrete pseudo-reactance[4] function in irreducible rational form.

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[4]A rational function $Z=Z(z_1)$ is discrete pseudo-positive if $\mathrm{Re}Z \geq 0$ on $|z_1|=1$ and is discrete pseudo-reactance if, additionally, $Z + \tilde{Z} = 0$ [25].

Proof: Note that d is necessarily of the form:

$$d = \prod_i (\alpha_i + z)^{m_i} \prod_j [(z+\beta_j)(1+\beta_j^* z)]^{n_j} \qquad (3.3)$$

where α_i and β_j are distinct and $|\alpha_i|=1, |\beta_j|<1$ for all i and j. Consider next the irreducible rational functions Z_i and Y_i as:

$$Z_i = h_i[(\alpha_i - z)/(\alpha_i + z)]^{m_i}; \; Y_j = (\sqrt{(-1)})(X_j + \tilde{X}_j) \qquad (3.4)$$

where $X_j = 1/(z+\beta_j)^{n_j}$, h_i=real if m_i=odd, and h_i=imaginary if m_i=even. Then it can be easily verified that each Z_i and Y_i are discrete pseudo-reactance functions with $(\alpha_i+z)^{m_i}$ and $[(z+\beta_j)(1+\beta_j^* z)]^{n_j}$ as their respective denominators in irreducible rational form. It thus follows from Lemma 3.3 that the rational function $\sum_i Z_i + \sum_j Y_j$ is pseudo-reactance with d as its denominator in irreducible rational form.

Q.E.D.

Proof of Theorem 3.1: Let $d=b(a_1.a_2...a_n)$, where b is the content of a, and a_i's are the irreducible non-constant polynomial factors of the primitive part of d. Clearly, due to Lemma 3.1(iii), b is a self-reciprocal polynomial and each a_i is a distinct self-reciprocal reactance Schur polynomial. Due to Lemma 3.4, there exists a polynomial b' such that b'/b is a discrete pseudo-reactance in irreducible rational form. Conider next the rational function:

$$n/d = \sum_{i=1}^{n} \phi_2(a_i)/a_i + b'/b \qquad (3.5)$$

It then follows from Lemma 3.2 above that $\phi_2(a_i)/a_i$ for each i, are half-plane reactance functions in irreducible rational form. Thus, $\mathrm{Re}(n/d) \geq 0$ for $|z_1|=1$, $|z_2|<1$, $(n/d)+(\widetilde{n/d})=0$, while the relative primeness of n and d follows from Lemma 3.3.

Q.E.D.

For further discussions the product of half-plane reactance polynomial and a half-plane scattering Schur polynomial will be called a half-plane immittance Schur polynomial.

Theorem 3.2: If $Z=n/d$ is a half-plane positive function in irreducible rational form then the primitive part of d (as well as of n) is necessarily a half-plane immittance Schur polynomial. If $Z=n/d$ is, in addition, a half-plane reactance function, then the primitive part of d is, in fact, a half-plane reactance Schur polynomial, whereas the content is a self-reciprocal polynomial.

Proof: We prove the stated property of d. Similar arguments apply for n. The proof is trivial if d does not involve z_2. Otherwise, let $d=b.a$, where b is the content and a is the nonconstant primitive part of d. Consider next the rational function H defined as:

$$H=(1-Z)/(1+Z)=(n-d)/(n+d) \qquad (3.6)$$

Clearly, $|H|\leq 1$ for $|z_1|=1$, $|z_2|<1$. Now, if for some $|z_{10}|=1$, $|z_{20}|<1$, $a=0$ then since (z_1-z_{10}) is not a factor of a, due to the continuity property of zeros of a polynomial as a function of its coefficients, it follows that there exists a continuous set of values of z_{10} on $|z_1|=1$ such that for some $|z_{20}|<1$ we may have $a(z_{10},z_{20})=0$ i.e., $|H(z_{10},z_{20})|=1$. However, since d and thus H involves z_2, the last conclusion has been shown to be impossible in Property 2.2a. Thus, a is half-plane Schur and consequently, due to Lemma 3.1(i), can be written as $a=ef$, where e is half-plane self-reciprocal Schur and f is half-plane scattering Schur. We thus have (3.7a).

$$Z=n/(bef); \quad Z_1=n_1/(b_1e_1f_1) \qquad (3.7a, b)$$

Let (3.7b) be obtained from (3.7a) by freezing z_1 in the corresponding polynomials and rational functions at $z_1=z_{10}$ on $|z_1|=1$. Consider a z_{10} such that $b_1\neq 0$ and the one-variable polynomials n_1 and e_1 are relatively prime. The existence of such z_{10} is guaranteed since due to relative primeness of n and e, n_1 and e_1 may have a nonconstant common factor only for a finite number of values of z_{10}.

Then since Z is half-plane positive we have that Z_1 is an 1-D positive function in z_2 having e_1 in its denominator in irreducible rational form. Since e_1 is clearly self-reciprocal Schur, due to known properties of 1-D discrete positive functions, it follows that e_1 may not contain multiple factors. Thus, e may not contain multiple factors either. Consequently, e is half-plane reactance Schur, which in turn imply that c=ef is half-plane immittance Schur.

Finally, if Z=n/d is a half-plane reactance function, then $(n/d) = -(\tilde{n}/\tilde{d}) = -(\hat{n}/\hat{d})z_1^m z_2^n$ in irreducible rational form, where m and n are integers. Since as shown above, neither n nor d may have a zero for arbitrary z_1 and $z_2=0$, we must have n=0. Thus, if $m\geq 0$ then $d=\gamma\hat{d}$, whereas if $m<0$ then $d=\gamma\hat{d}z_1^k$, where γ=constant and $k=-m$. In either case, the content as well as the primitive part of d are self-reciprocal. Since this latter factor has been shown to be half-plane immittance Schur, due to Lemma 3.1(i), it is in fact, a half-plane reactance Schur polynomial. The last part of the proof thus follows. Q.E.D.

<u>Theorem 3.3</u>: Any product of a 1-D polynomial d involving z_1 and a half-plane immittance Schur polynomial is the denominator, (and hence also the numerator), of a half-plane positive function in irreducible rational form.

<u>Proof</u>: let d=b.c, where $b=b(z_1)$ is such that all zeros of b

are on $|z_1|=1$ and c does not have any zero on $|z_1|=1$ independent of z_2. Furthermore, let c=e.f, where f is the primitive part and e is the content of c. Let f=g.h, where g=half-plane reactance Schur, and h=half-plane scattering Schur. Clearly, b is self-recipocal and thus there exists a polynomial b′ such that b′/b is a pseudo-reactance function in irreducible rational form. Furthermore, due to Theorem 3.1, there exists a polynomial g′ such that g′/g is a half-plane reactance function in irreducible rational form. Next note that since h is half-plane scattering Schur, $\hat{h}/h$ is half-plane bounded (cf. Theorem 2.1). Also, since $e \neq 0$ on $|z_1|=1$, e is relatively prime with $\hat{e}$ we have $|\hat{e}/e|=1$ for $|z_1|=1$. Thus, $\hat{u}/u$ is a half-plane bounded function in irreducible rational form, where u=e.h. Consequently, $\mathrm{Re}[1+(\hat{u}/u)] \geq 0$ for $|z_1|=1$, $|z_2|<1$. Consider next the rational function:

$$n/d = 1 + (\hat{u}/u) + (b'/b) + (g'/g) \qquad (3.8)$$

Clearly, $\mathrm{Re}(n/d) \geq 0$ for $|z_1|=1$, $|z_2|<1$. Finally, the relative primeness of n and d follows from Lemma 3.4 and mutual coprimeness of b, g, and u, which in turn follows by invoking Lemma 3.1. Q.E.D.

Theorems 3.1, 3.2 and 3.3 together characterize the denominators and numerators of half-plane positive and half-plane reactance functions in irreducible rational form.

4. Fully recursive symmetric half-plane lossless two-ports:

Characterizations of fully recursive symmetric half-plane passive as well as lossless one-ports have been established in the previous section in terms of the transfer function of the filter. In this section we make use of the results of Sections 2 and 3 to characterize fully recursive symmetric half-plane lossless multi-ports. In particular, a convenient representation for two-ports analogous to the Belevitch canonical representation of continuous time 1-D lossless circuits of classical network theory [8] is developed. This representation is then subsequently used in Section 5 to synthesize the filter in a specific structure.

A system consisting of n-ports (i.e., 2n terminal) having recursive structure of the type under consideration is lossless if (4.1) holds true for any finite (pseudo) energy inputs $x_1(n_1,n_2)$ and $x_2(n_1,n_2)$.

$$\sum_{i=1}^{n} [\Sigma\Sigma\ |y_i(n_1,n_2)|^2] = \sum_{i=1}^{n} [\Sigma\Sigma\ |x_i(n_1,n_2)|^2] \tag{4.1}$$

Consider next $x_i(n_1,n_2) \equiv 0$ for all $i = 1$ to n except k. We then have from (4.1) that for any finite (pseudo) energy $x_k(n_1,n_2)$:

$$\Sigma\Sigma\ |y_i(n_1,n_2)|^2 \leq \Sigma\Sigma\ |x_k(n_1,n_2)|^2 \tag{4.2}$$

On the otherhand, if the (n x n) rational matrix $S = S(z_1,z_2) = [S_{ij}(z_1,z_2)]$ is the transfer function of the n-port then for $X_i(z_1,z_2) \equiv 0$ for all $i = 1$ to n except k we have $Y_i(z_1,z_2) = S_{ik}(z_1,z_2)X_k(z_1,z_2)$. Thus, due to (4.2) the transfer functions $S_{ik} = S_{ik}(z_1,z_2)$ for each i,k are (pseudo) passive, and thus satisfy Property 2.1.

Furthermore, by considering 2-D Parseval's theorem (4.1) can be made to yield (4.3), where the column vector $X(\omega_1,\omega_2) = (X_1(\omega_1,\omega_2)\ldots X_n(\omega_1,\omega_2))^t$, and the superscript * denotes the combined operation of complex conjugation and matrix transposition denoted by t.

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X^*(\omega_1,\omega_2)(I_n - S^*(\omega_1,\omega_2)S(\omega_1,\omega_2))X(\omega_1,\omega_2) = 0 \tag{4.3}$$

Since (4.3) holds for any $X(\omega_1,\omega_2)$ it follows that for any 2-tuple (ω_1,ω_2) except possibly finitely many, we have $S^*(\omega_1,\omega_2)S(\omega_1,\omega_2) = I_n$, which due analytic continuation principle yields that for all z_1, z_2:

$$\tilde{S}(z_1,z_2)\; S(z_1,z_2) = I_n \tag{4.4}$$

Next, a rational matrix $S = S(z_1,z_2)$ is said to be <u>fully recursive</u> <u>half-plane</u> <u>lossless</u> if: (i) each entry of S in irreducible rational form has a half-plane scattering Schur denominator and S satisfies property (4.4).

Note that the transfer function of a fully recursive half-plane lossless n-port is necessarily of the above type. As a consequence of Property 2.2, we then have the following important conclusion.

<u>Proposition</u> <u>4.1</u>: Each entry of a fully recursive half-plane lossless matrix $S = [S_{ij}]$ satisfies $|S_{ij}| \leq 1$ for all $|z_1|=1$ and $|z_2|<1$.

<u>Proof</u>: With the possible exception of finitely many points on $|z_1| = |z_2|=1$, we have from (4.4) $S^*S = I_n$, and thus $\sum_{i=1}^{n} |S_{ij}|^2 = 1$ for all j, which in turn imply $|S_{ij}| \leq 1$ for

all i,j. The result then follows from Property 2.2a.

Q.E.D.

Consider next a fully recursive half-plane lossless bounded matrix S. Since each entry of S satisfies Property 2.1, the rational function (detS) also satisfies Property 2.1. Also, it follows from (4.4) that $(\det \tilde{S})(\det S) = 1$. Thus, S is a fully recursive symmetic half-plane all-pass function as defined in Section 2 and admits of the representation (2.15) described in Theorem 2.1, i.e., (4.5) holds.

$$\det S = -D.(A/\bar{A}) \qquad (4.5)$$

We next claim the following:

<u>Lemma 4.1</u>: If S is the transfer function matrix of a fully recursive half-plane lossless n-port and A is as in (4.5) then each entry of $\bar{A}S$ is a pseudo-polynomial.

<u>Proof</u>: From (4.4), (4.5) along with $D\tilde{D} = 1$ (cf. Theorem 2.1 (ii)) if follows after some manipulations that

$$\bar{A}S = -DA(\mathrm{Adj}\,\tilde{S}) \qquad (4.6)$$

If the ij-th entry of $\bar{A}S$ is not a pseudopolynomial then its denominator would have a factor, necessarily half-plane scattering Schur (by virtue of Corollary 3.1), involving z_2, which must also be the denominator of ij-th entry of AdjS. Thus, in view of Assertion 2.1(ii), $[\mathrm{Adj}\,\tilde{S}]_{ij}$ would then have a singularity in $|z_1| = 1$, $|z_2| > 1$. However, since S is analytic in $|z_2| < 1$ for all $|z_1| = 1$, Adj $\tilde{S}$ must be analytic in $|z_2| > 1$ for all $|z_1| = 1$, which is a contradiction. Thus, the ij-th entry of $\bar{A}S$ is a pseudopolynomial for all i,j.

Q.E.D.

Due to Lemma 4.1, S can be expressed as $S = \pi/\bar{A}$, where π is a matrix of pseudopolynomials. If, in addition, n = 2, i.e., for two-ports, it follows from (4.4) that $S = (S^{-1})^{\sim}$. Consequently, we have (4.7) via the use of (4.5) and $D\tilde{D} = 1$ (cf. Theorem 2.1 (ii)).

$$(1/\bar{A})\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = -(D/\tilde{A})\begin{bmatrix} \tilde{\pi}_{22} & -\tilde{\pi}_{21} \\ -\tilde{\pi}_{12} & \tilde{\pi}_{11} \end{bmatrix} \quad (4.7)$$

If we designate the pseudopolynomials π_{11} by B and π_{21} by C respectively then we have (4.8a) and (4.9a) in the following. Furthermore, by equating the (1,2) and (2,2) terms in (4.7), it respectively follows that (4.9b) and (4.8b) holds true.

$$S_{11} = B/\bar{A}, \; S_{22} = -D(\tilde{B}/\tilde{A}) \quad (4.8a,b)$$

$$S_{21} = C/\bar{A}, \; S_{12} = D(\tilde{C}/\tilde{A}) \quad (4.9a,b)$$

Inserting (4.8) and (4.9) in the expression for (det S) in (4.5) we then have

$$A\tilde{A} = B\tilde{B} + C\tilde{C} \quad (4.10)$$

Also, since S_{22} and S_{12} are analytic in $|z_1|=1$, $|z_2|<1$ we have from (4.8b) and (4.9b) that:

$$\deg_2 B \leq \deg_2 A \; ; \; \deg_2 C \leq \deg_2 A \quad (4.11)$$

The above discussion can be succinctly expressed in the following representation of a fully recursive symmetric half-plane lossless bounded matrix.

<u>Property</u> <u>4.1</u>: Any fully recursive symmetric half-plane lossless bounded (2x2) matrix (i.e., transfer function of a

fully recursive half-plane lossless two-port) can be represented in terms of three pseudopolynomials A, B and C as in (4.8) and (4.9), where $\bar{A}$ is half-plane scattering Schur, and furthermore (4.10), (4.11) hold true.

Conversely, any matrix, which admits of the above representation is fully recursive symmetric half-plane bounded.

<u>Proof</u>: Necessity has been proved in discussions preceeding Property 4.1. For sufficiency, note that $S\tilde{S} = 1$ trivially follows via routine algebraic manipulations with (4.8), (4.9). The proof is then completed by noting that $\bar{A}$ is half-plane scattering Schur. Q.E.D.

For convenience of exposition any S expressed as in (4.8) and (4.9) will be referred to as in <u>standard</u> <u>form</u>.

A fully recursive half-plane lossless two-port as in figure 4.1 can be alternatively described by means of a chain matrix $T = T(z_1,z_2)$ defined as in (4.12).

$$\begin{bmatrix} Y_1 \\ X_1 \end{bmatrix} = T \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} \qquad (4.12)$$

It can be easily shown from (4.8) through (4.12) that the following property characterizes the chain matrices of the type described above.

<u>Property</u> <u>4.1'</u>: The chain matrix $T = [T_{ij}]$ associated with a fully recursive half-plane two-port is lossless if and only if it can be expressed as

$$T_{11} = DA/C \; ; \; T_{12} = B/C \qquad (4.13a,b)$$

$$T_{21} = D\tilde{B}z_2{}^{n}A/C \; ; \; T_{22} = \bar{A}/C \qquad (4.14a,b)$$

where $n_A = \deg_2 A$, and A, B, C and D satisfies the same restrictions described in Property 4.1. Also, any T as in (4.13), (4.14) is said to be in standard form.

Proof: Follows from known relation between elements of chain matrix and transfer function matrix of a two-port and equations (4.8) through (4.10) along with $\tilde{D}D = 1$.

Q.E.D.

5. Synthesis of fully recursive half-plane lossless two-ports:

A procedure for synthesizing fully recursive half-plane lossless two-ports as an interconnection of more elementary building blocks of the same type will be developed in this section. The synthesis algorithm can be viewed as a generalization of the algorithm for synthesizing 1-D discrete lossless two-ports as described by Rao and Kailath in [6]. Our synthesis procedure exploits the unique feature of the algorithm described in [6] that (in 1-D) given (polynomials) A, B, C the arithmetic operations needed to be performed on the coefficients of A, B and C in each cycle of the repetitive algorithm requires rational operations only. To the best of our knowledge this is the only algorithm of the above mentioned type available for synthesis of 1-D discrete as well as continuous domain lossless two-ports including those in classical network theory [8] (all other algorithms known prior to [6] required nonrational operations e.g., polynomial factorization). The basic structure of the filter to be presently synthesized would thus be the same as in [6], whereas the elementary building blocks are certain 1-D two port sections to be referred to as the generalized Gray-Markel sections (GGM section) and z_2-type delays, each of which are fully recursive half-plane lossless.

A generalized Gray-Markel section is a 1-D two port as shown in Figure 5.1 where the 1-D transfer functions (assumed rational) $k_1 = k_1(z_1)$ and $k_2 = k_2(z_1)$ satisfy the relationship:

$$k_1\tilde{k}_1 + k_2\tilde{k}_2 = 1 \tag{5.1}$$

and are such that k_1 (and thus k_2 in view of (5.1)) satisfies $k_1\tilde{k}_1 = |k_1|^2 < 1$ almost everywhere on $|z_1| = 1$.

We first note that given any rational function k_1 of z_1 satisfying the above conditions it is always possible to find a rational function k_2 satisfying the same conditions as that of k_1 along with (5.1). (The role of k_1 and k_2 can obviously be interchanged in the present considerations). To show this let $k_1 = n_1/d_1$ where n_1, d_1 are polynomials in z_1. Then $(1 - \tilde{k}_1 k_1) = N_1/(\tilde{d}_1 d_1)$, where $N_1 = \tilde{d}_1 d_1 - \tilde{n}_1 n_1$. Thus, $N_1 = \tilde{N}_1$, and for all z_1 on $|z_1| = 1$, $N_1(z_1)$ is real and we have that $N_1(z_1) \geq 0$ as a consequence of $\tilde{k}_1 k_1 \leq 1$. Therefore, the (spectral) factorization $N_1 = \tilde{n}_2 n_2$, where n_2 is a polynomial, in z_1 holds. Also, by (possibly) rearranging the irreducible factors of $(d_1 \tilde{d}_1)$ to write $d_1 \tilde{d}_1 = d_2 \tilde{d}_2$, where d_2 = polynomial, we can have $k_2 = n_2/d_2$ such that (5.1) is satisfied. Note that since the factorizations $N_2 = \tilde{n}_2 n_2$ and $d_1 \tilde{d}_1 = d_2 \tilde{d}_2$ are not unique the k_2 so obtained is not unique unless further restrictions are imposed.

The transfer function matrix $S_G = S_G(z_1)$ associated with such a GGM section can be expressed as in (5.2a), whereas the corresponding chain matrix T is given in (5.2b).

$$S_G = \begin{bmatrix} k_1 & k_2 \\ \tilde{k}_2 & -\tilde{k}_1 \end{bmatrix} \qquad T_G = (1/\tilde{k}_2) \begin{bmatrix} 1 & k_1 \\ \tilde{k}_1 & 1 \end{bmatrix} \qquad (5.2a,b)$$

Since S_G in (4.2a) satisfies the representation described in property 4.1 with $A = 1$, $B = k_1$, $C = \tilde{k}_2$ and $D = 1$ the GGM section is indeed a fully recursive half-plane lossless two-port.

<u>Remark</u>: To proceed with the synthesis of a prescribed fully recursive half-plane lossless bounded matrix S or, equivalently, corresponding chain matrix T as described respectively in Property 3.1 or 3.1′, we first note that S_{11} (or S_{22}) is fully recursive half-plane all pass if and only

if $S_{21} \equiv S_{12} \equiv 0$. To show this, observe that if S_{11} (or S_{22}) is fully recursive half-plane all pass then for all $|z_1| = |z_2| = 1$, $|S_{11}| = 1$ (or corresp. $|S_{22}| = 1$) and thus, due to (4.4), $|S_{21}| = 0$ (or corresp. $|S_{12}| = 0$), which in turn imply that $C \equiv 0$ i.e., $S_{12} \equiv S_{21} \equiv 0$. Conversely, if $S_{12} = S_{21} \equiv 0$ then it is obvious from (4.4) that both S_{11} and S_{12} are fully recursive half-plane all pass. Similary, it can be shown that S_{21} (or S_{12}) is fully recursive half-plane all pass if and only if $S_{11} \equiv S_{22} \equiv 0$ i.e., $B \equiv 0$. In either case, the synthesis of S reduces to that of synthesis of fully recursive half-plane all pass one-ports as described in the appendix as an extension of 1-D Schur algorithm. Thus, it will henceforth be assumed without loss of generality that neither of S_{ij}'s in the prescribed two-port or in the two-ports resulting in subsequent stages of synthesis is identically zero.

Next, in view of Proposition 4.1 the rational function: $k_1 = k_1(z_1) = S_{11}(z_1,0)$ satisfies $|k_1| < 1$ a.e. on $|z_1|=1$. Therefore, in view of the preceeding discussion k_1 defines a GGM section i.e., a rational function k_2 can be found such that $|k_2|<1$ a.e. on $|z_1|=1$and that (5.1) is satisfied.

<u>Step 1:</u> The first step is to extract a GGM section with $k_1 = S_{11}(z_1,0)$ from prescribed S or T as shown in Figure 5.2. Since a cascade connection of two two-ports amounts to multiplication of the corresponding chain matrices, the chain matrix of the remaining two-port is then $T' = T_G^{-1}T$. From (4.13), (4.14) and (5.2b) we can write:

$$T' = (1/Ck_2) \begin{bmatrix} D(A-k_1\tilde{B}z_2^{\,n_A}) & B-k_1\bar{A} \\ D(\tilde{B}z_2^{\,n_A}-\tilde{k}_1A) & \bar{A}-\tilde{k}_1B \end{bmatrix} \quad (5.3)$$

We next define the pseudopolynomials A', B', C' and the 1-D rational function D' as in (5.4) and (5.5) below, where $p=p(z_1)$ is the self-reciprocal polynomial factor of largest degree present in the numerator of $\bar{A}-\tilde{k}_1B$, when expressed in irreducible rational form.

$$\tilde{p}A' = A(1-\tilde{k}_1\tilde{S}_{11}) = A-k_1\tilde{B}z_2^{\,n_A} \;;\; pC' = Ck_2 \quad (5.4a,b)$$

$$pB' = \bar{A}(S_{11}-k_1) = B-k_1\bar{A} \quad ; \quad D' = D(\tilde{p}/p) \quad (5.5a,b)$$

We claim that $deg_2A' = deg_2A$. To prove this, clearly $deg_2A' \leq deg_2A$ and note that (5.4a) yields $\tilde{p}A'/\tilde{A} = 1-\tilde{k}_1S_{11}$, which implies that if $deg_2A' < deg_2A$ then for arbitrary z_1 we would have $\tilde{k}_1k_1 = \tilde{k}_1(z_1)S_{11}(z_1,0) = 1$. As a consequence of this we can write T' as in (5.6) and (5.7), where $n_{A'} = deg_2A' = deg_2A$.

$$T'_{11} = D'A'/C' \;;\; T'_{12} = B'/C' \quad (5.6a,b)$$

$$T'_{21} = D'\tilde{B}'z_2^{\,n_{A'}}/C' \;;\; T'_{22} = \bar{A}'/C' \quad (5.7a,b)$$

We next claim that the pseudopolynomial $\bar{A}'$ satisfies the properties that $\bar{A}' \neq 0$ for $|z_1|=1$, $|z_2|<1$ and that $\bar{A}'$ is coprime with A'. To prove this we write $A' = A'_N/A'_D$ in irreducible rational form, and thus $\bar{A}' = (\hat{A}'_N/\hat{A}'_D).z_1^{\alpha}$, where α = integer and $\hat{A}'_N/\hat{A}'_D$ is in irreducible rational form. Thus, since $p\bar{A}'=\bar{A}-\tilde{k}_1B$ it follows from the definition of p that $\hat{A}'_N$ is devoid of self-reciprocal polynomial factors in z_1 only. If we assume for the purpose of a proof by contradiction that for some value of $z_1=z_{10},z_2=z_{20}$ with $|z_{10}|=1$, $|z_{20}|<1$ we have $\bar{A}'=0$ i.e., $\hat{A}'_N=0$ then since $\hat{A}'_N$ cannot have a factor

(z_1-z_{10}), by changing the value of z_1 from z_{10} along an arbitrarily small arc Γ_1 of the unit circle $|z_1|=1$ it would be possible to find a continuous [12] set (z_1,z_2) of zeros of $\hat{A}'_N$ i.e., also of $\bar{A}'$ with $z_1\varepsilon\Gamma_1\subset\{z_1;|z_1|=1\}$ and $|z_2|<1$. Also, since it follows from (5.4a) and $\deg_2 B\leq n_A=n_A{}'$ that $p\bar{A}'=\bar{A}(1-\tilde{k}_1S_{11})$ and $\bar{A}\neq 0$ in $|z_1|=1$, $|z_2|<1$ (cf. Property 4.1) we would then have that for all $z_1\varepsilon\Gamma_1$ some z_2 in $|z_2|<1$ such that $\tilde{k}_1S_{11}=1$. Since $|\tilde{k}_1|\leq 1, |S_{11}|\leq 1$ if $|z_1|=1, |z_2|<1$ (cf. Proposition 4.1) the last conclusion would then imply existence of z_2 in $|z_2|<1$ such that $|\tilde{k}_1|=|S_{11}(z_1,0)|=1$ and $|S_{11}(z_1,z_2)|=1$ for all $z_1\varepsilon\Gamma_1$. However, this in view of Property 2.2a, yields that S_{11} is independent of z_2 with $|S_{11}(z_1,z_2)| = |S_{11}(z_1)| = 1$ for all $|z_1| = 1$, which is ruled out. Thus, $\bar{A}'\neq 0$ i.e., $\bar{A}'_N\neq 0$ for $|z_1|=1, |z_2|<1$.

Further, since due to Property 2.2a $|\tilde{k}_1|=|S_{11}(z_1,0)|=1$ may hold for at most finite number of values of z_1, we have $|S_{11}|\leq 1$, and $\bar{A}\neq 0$ for $|z_1| = |z_2| = 1$ with at most finite number of exceptional points, we conclude from $p\bar{A}'=\bar{A}(1-\tilde{k}_1S_{11})$ that $\bar{A}'$, thus $\hat{A}'_N$, may have at most finite number of zeros on $|z_1|=|z_2|=1$. Since as shown earlier $\bar{A}'_N\neq 0$, and thus $\hat{A}'_N\neq 0$ in $|z_1|=1, |z_2|<1$ it follows from Assertion 2.1 that the primitive parts of A_N and $\hat{A}_N$ are relatively prime polynomials. Consequently, the pseudopolynomials $\bar{A}$ and $\bar{A}'$ are relatively prime.

Finally, straightforward algebraic manipulation along with (5.4), (5.5a) and (4.8a) yield $A'\tilde{A}' = B'\tilde{B}' + C'\tilde{C}'$, whereas $\deg_2 A' \geq \deg_2 B'$, $\deg_2 A' \geq \deg_2 C'$ follow from (5.4a,b), (5.5), (4.11) and $n_A=n_{A'}$. Since, clearly D' as in (5.5b) possesses the requisite properties for T' to be in standard form, in view of Property 3.1′ all the conditions necessary for $T' = [T'_{ij}]$, as given in (5.6), (5.7), to be a fully recursive half-plane lossless two-port chain matrix are satisfied.

We further note that as a consequence of the choice $k_1=S_{11}(z_1,0)$ we have from (5.5a) that $B'(z_1,0) = 0$ for arbitrary z_1 i.e., the pseudopolynomial B' contains z_2 as a factor. Also, from (5.4b) if C contains a pseudopolynomial factor z_2 then so does C'.

We thus have the following theorem as a result of the previous discussion.

Theorem 5.1: Let S be the transfer function matrix of a fully recursive half-plane lossless two-port as in Property 4.1. If S' is obtained by extracting from S a GGM section parametrized by $k_1(z_1)=S_{11}(z_1,0)$ as in Figure 5.1 then S' is also the transfer function of a fully recursive half-plane lossless two-port. Furthermore B' associated with S' has a pseudo-polynomial factor z_2. Also, if C has a pseudo polynomial factor z_2 then so does C'.

Step 2: In the next step we form a fully recursive half-plane two port $T^{(2)}$ by interchanging the two output terminals in each port of T' as shown in Figure 5.3. It can be easily shown that $T^{(2)}$ can then be written in terms of pseudopolynomials $A^{(2)}$, $B^{(2)}$, $C^{(2)}$ and the rational function $D^{(2)}$ in standard form as expressed in Property 3.1', where

$$A^{(2)} = A', \quad B^{(2)} = C', \quad C^{(2)} = B', \; D^{(2)} = -D' \qquad (5.8)$$

Step 3: A GGM section is then extracted from the two-port with chain matrix $T^{(2)}$ by iterating step 1 on $T^{(2)}$ to get a fully recursive half-plane lossless two-port chain matrix $T^{(3)}$. As a result, if the pseudopolynomials $A^{(3)}$, $B^{(3)}$, $C^{(3)}$ and the rational function $D^{(3)}$ represent $T^{(3)}$ in standard form as in property 3.1' then $B^{(3)}$ would have a factor z_2. Also, we have $C^{(3)} = C^{(2)}k_2^{(3)} = B'k_2^{(3)}$ (where $k_2^{(3}$ is associated with the GGM section extracted in Step 3), in which the first equality follows from (5.4b) in the context of Step 3, whereas the second equality follows from (5.8). Since the pseudopolynomial B' has a factor z_2 we conclude that $C^{(3)}$ has a factor z_2. From this and the fact that $A^{(3)}\tilde{A}^{(3)} = B^{(3)}\tilde{B}^{(3)} + C^{(3)}\tilde{C}^{(3)}$ (which is a consequence of losslessness of $T^{(3)}$) it follows that $A^{(3)}\bar{A}^{(3)} = 0$ for $z_2 = 0$ and for arbitrary z_1. Since $\bar{A}^{(3)} \neq 0$ for $|z_1| = 1$ and $|z_2| < 1$, due to lossless of $T^{(3)}$ we conclude $A^{(3)} = 0$ for $z_2 = 0$ and for arbitrary z_1. Consequently, $A^{(3)}$ has a factor z_2 and it is possible to write, for some pseudopolynomials $A^{(4)}$, $B^{(4)}$ and $C^{(4)}$ that

$$A^{(4)} = z_2A^{(3)},\ B^{(4)} = z_2B^{(3)},\ C^{(4)} = z_2C^{(3)} \qquad (5.9)$$

<u>Step 4:</u> The next step in the synthesis cycle is to extract a z_2 type delay from $T^{(3)}$ as in Figure 5.4 to produce a two-port with chain matrix $T^{(4)}$, which can be expressed in terms of $A^{(4)}$, $B^{(4)}$, $C^{(4)}$, as in (4.13) and (4.14). Furthermore, since from (5.9) $\bar{A}^{(4)} = z_2\bar{A}^{(3)}$ we have $\bar{A}^{(4)} \neq 0$ for $|z_1| = 1$, $|z_2| < 1$ and $\bar{A}^{(4)}$ can have at most finitely many zeros on $|z_1| = |z_2| = 1$ due to the same properties possessed by $\bar{A}^{(3)}$. Also, it follows from (5.9) and losslessness of $T^{(3)}$ (in particluar, counterpart of (4.10) associated with $T^{(3)}$) that

$$\tilde{A}^{(4)}A^{(4)} = \tilde{B}^{(4)}B^{(4)} + \tilde{C}^{(4)}C^{(4)} \qquad (5.10)$$

and $\deg_2 B^{(4)} \leq \deg_2 A^{(4)}$, $\deg_2 C^{(4)} \leq \deg_2 A^{(4)}$

Thus, the two-port associated with $T^{(4)}$ is fully recursive half-plane lossless. Furthermore, note that

$$\deg_2 A^{(4)} = \deg_2 A^{(3)}-1 = \deg_2 A^{(2)}-1 = \deg_2 A'-1 = \deg_2 A-1,$$

where the first equality follows from (5.9); the second and the fourth from the fact that in step 1 we have $n_A = n_{A'}$; and the third from (5.8).

Consequently, after iterating $\deg_2 A$ times the cyclic algoritm described in Steps 1 through 4, we obtain a lossless chain matrix T_f independent of z_2, which in standard form is described by $A_f = A_f(z_1)$, $B_f = B_f(z_1)$, $C_f = C_f(z_1)$ and $D = D_f(z_1)$.

The main contents of Steps 3 and 4 can be combined into the following theorem.

Theorem 5.2: If S is the transfer function matrix of a fully recursive half-plane lossless two-port as in Property 4.1 such that both B and C has a pseudo-polynomial factor z_2 then A must also have the same factor. Furthermore, a z_2 type delay can be extracted from S thus yielding another fully recursive half-plane two-port S′ such that the z_2-degree of the pseoudo-polynomial A′ associated with S′ is exactly one less than that associated with S.

Terminal Step: In the final step we extract another GGM section as in Step 1 to produce a fully recursive half-plane lossless two-port S_0 with A_0, B_0, C_0 and D_0 in standard form. Since A_f, B_f are functions of $\underset{\sim}{z}_1$ only it $\underset{\sim}{\text{f}}$ollows from (4.5a) that $B_0 \equiv 0$. Also, since $A_0\tilde{A}_0 = B_0\tilde{B}_0 + C_0\tilde{C}_0$ and $\bar{A}_0 = A_{0\sim} \neq \underset{\sim}{0}$ for all $|z_1| = \underset{\sim}{1}$ the 1-D transfer functions $(S_0)_{12} = D_0C_0/A_0$ and $(S_0)_{21} = C_0/A_0$ are both well defined and of unit modulus on $|z_1| = 1$ i.e., they are all-pass functions.[5] The realization for such a two-port is shown in Figure 4.5.

Remark: Before iterating the entire synthesis cycle after the completion of Step 4 it is once again possible, but not

---

[5]Note that $(S_0)_{12}$ and $(S_0)_{21}$ are not necessarily stable rational functions, i.e., may have poles in $|z_1|<1$ and $|z_1|>1$ as well.

necessary, to carry out the operation of interchanging the pseudopolynomials B and C as described in Step 2. However, the resulting structures are different depending on whether or not this step is incorporated in the synthesis cycle. The structure shown in Figure 4.6 is obtained when this latter step is incorporated in the synthesis cycle. In 1-D the same structure has been reported as being pipelineable in [6].

An example illustrating the above synthesis procedure will be given next.

<u>Example 5.1</u>: Consider the transfer function S of a fully recursive half-plane lossless two-port as given in the following in terms of the pseudo-polynomials A, B, C and D.

$$A=2[(z_1^2-1)/z_1(z_1+4) + z_2(z_1+2)/(2z_1+1)]$$

$$B=4(z_1+1)(2z_1+1)/(z_1+4)(z_1+2) - z_2(z_1-1)$$

$$C=3(z_1+1)(2z_1+1)/(4z_1+1)(z_1+2)$$

$$D=-(z_1+3)(2z_1+1)/(3z_1+1)(z_1+2)$$

It can be routinely verified that the A, B, C, and D specified above satisfy the conditions required in Property 4.1. The first step in the synthesis is to compute $k_1(z_1)=S_{11}(z_1,0)$, where $S_{11}=B/\bar{A}$. It follows that we have:

$$k_1(z_1) = 2(z_1+1)/(z_1+4)$$

We then compute $k_2(z_1)$ from (5.1) by effecting a 1-D spectral factorization. We have:

$$k_2(z_1)\tilde{k}_2(z_1)=9z_1/(z_1+4)(4z_1+1)$$

An obvious choice of $k_2(z_1)$ is as follows (other choices are also possible).

$$k_2(z_1)=3/(z_1+4)$$

We then extract the GGM completely specified by $k_1(z_1)$ and $k_2(z_1)$ above to obtain the fully recursive half-plane two-port having chain matrix $T'=T_G^{-1}T$ (cf. equation (5.2b)), where T_G is the chain matrix of the GGM just obtained. The resulting T' has pseudo-polynomials A', B', C' and D' as follows.

$$A'=2z_2(z_1+2)/(2z_1+1)$$

$$B'=-(z_1-1)z_2$$

$$C'=(z_1+1)(2z_1+1)/z_1(z_1+2)$$

$$D'=-(2z_1+1)(z_1+3)/(z_1+2)(3z_1+1)$$

The next step is to interchange the polynomials B' and C' and reverse the sign of D'. This corresponds to twisting the input and output terminals of the two-port and results in the chain matrix $T^{(2)}$ with $A^{(2)}$, $B^{(2)}$, $C^{(2)}$ and $D^{(2)}$ given by:

$$A^{(2)}= A',\ B^{(2)}= C',\ C^{(2)}= B',\ D^{(2)}= -D'$$

The extraction of a second GGM then follows. The parameter $k_1(z_1)$ describing this GGM is obtained by setting $z_2=0$ in the S_{11} element of the two-port obtained thus far i.e., in $B^{(2)}/\bar{A}^{(2)}$. We then have the following:

$$k_1(z_1)=(1+z^{-1})/2$$

The corresponding k_2 can then be obtained by factoring $1-k_1\tilde{k}_1$ (cf. equation (5.1)). A specific choice of this factor for $k_2(z_2)$ is

$k_1(z_1)=(1-z^{-1})/2$

The GGM with chain matrix T_G just described, when extracted from the two-port having chain matrix $T^{(2)}$, yields a two-port with a chain matrix $T^{(3)}=T_G^{-1}T^{(2)}$ (cf. equation (5.2b)) having pseudo-polynomials $A^{(3)}$, $B^{(3)}$, $C^{(3)}$ and $D^{(3)}$ associated with it, where

$$A^{(3)}=z_2(z_1+2)/(2z_1+1)$$

$$B^{(3)}=0$$

$$C^{(3)}=z_2$$

$$D^{(3)}=(z_1+3)(2z_1+1)/(3z_1+1)(z_1+2)$$

Notice that all of the pseudo-polynomials $A^{(3)}$, $B^{(3)}$ and $C^{(3)}$ have the factor z_2, which when extracted in the form of a z_2 type delay, will yield the two-port $T^{(4)}$ with associated $A^{(4)}$, $B^{(4)}$, $C^{(4)}$ and $D^{(4)}$ given by:

$$A^{(4)}=(z_1+2)/(2z_1+1)$$

$$B^{(4)}=0$$

$$C^{(4)}=1$$

$$D^{(4)}=(z_1+3)(2z_1+1)/(3z_1+1)(z_1+2)$$

This completes one entire synthesis cycle. Note further that the degree of A in z_2 has reduced by exactly one. In this specific instance, however, synthesis is essentially complete due to the fact that $A^{(4)}$, $B^{(4)}$, $C^{(4)}$ are each independent of z_2. Thus, $A_f=A^{(4)}$, $B_f=B^{(4)}$, $C_f=C^{(4)}$ and $D_f=D^{(4)}$. Also, the extraction of another GGM, as described in the terminal step, yields a trivial GGM with $k_1(z_1)=0$, $k_2(z_1)=1$. Thus, we also have in this case $A_f=A_0$, $B_f=B_0$, $C_f=C_0$ and $D_f=D_0$. The 1-D all-pass functions $(S_0)_{12}$ and $(S_0)_{21}$ are consequently given by:

$$(S_0)_{12} = (z_1+3)/(3z_1+1)$$

$$(S_0)_{21} = (z_1+2)/(2z_1+1)$$

Having described the synthesis procedure the basic reason why an analogous method does not work for the synthesis of quarter plane lossless two-ports may now be commented on. Recall that in the quarter plane case A, B, C are polynomials whereas D is a unimodular constant and the transfer function S is analytic in $|z_1|<1$, $|z_2|<1$ [2]. From this it can be shown that although $k_1(z_1)=S_{11}(z_1,0)$ is a bounded function (i.e., $|k_1|\leq 1$ in $|z_1|<1$) and a bounded $k_2(z_1)$ satisfying (5.1) can be found, extraction of the corresponding GGM leaves us with a two-port that is not necessarily analytic in $|z_1|<1$, $|z_2|<1$, which is thus not of the quarter plane lossless type. In other words, Theorem 5.1 is not valid in this case, and hence Step 1 of the synthesis procedure does not go through. The alternate strategy of extracting a constant Gray-Markel section of the conventional type parametrized by $k_1=S_{11}(0,0)$ does not, however, suffer from this last drawback i.e., a quarter plane two-port is indeed obtained after extraction. Thus, Steps 1 though 3 can be carried out without any difficulty. However, at the end of Step 3 we are left with a quarter plane lossless two-port such that $A^{(3)}=B^{(3)}=C^{(3)}=0$ for $z_1=z_2=0$, but this does not necessarily imply that z_1 or z_2 is a factor of $A^{(3)}$, $B^{(3)}$ and $C^{(3)}$ i.e., an analog of equation (5.9) does not hold. Consequently, a z_2 type delay cannot be extracted from the two-port and synthesis again breaks down.

6. Comments and Conclusions

Two-dimensional filters with various different symmetries in their magnitude responses, e.g., fan type symmetry and the circular symmetry are of practical interest. The loci of constant gain in the ω_1-ω_2 plane for the fan filters are required to be approximate straight lines, whereas those for the circularly symmetric filters are required to be closed circles in an approximate sense. In addition, we also require the pass (or the stop) region of the fan filter to be the region approximately lying within the straight lines $\omega_1 = \alpha\omega_2$ and $\omega_1 = -\alpha\omega_2$ for some $0<\alpha<1$.

A design methodology for filters of above type may proceed by requiring the transfer function $S_{21} = C/\bar{A}$ (cf. equation (4.9a)) of the lossless two-port S to have the desired characteristics. However, unlike the corresponding problem in 1-D, due to nonfactorability of m-D polynomials it is in general not possible to find a pseudopolynomial B satisfying (4.10) from A and C. To circumvent this problem it may be further assumed that the two-port is either symmetric i.e., $S_{11} = S_{22}$, $S_{21} = S_{12}$ or antimetric i.e., $S_{11} = -S_{22}$, $S_{21} = S_{12}$. Thus, in the symmetric and in the antimetric case we respectively have $B = -\tilde{D}Bz_2^{n_A}$, $B = \tilde{D}Bz_2^{n_A}$, whereas we also have $C = \tilde{D}Cz_2^{n_A}$ in both cases. We next define two rational functions S_1 and S_2 as in (6.1) and (6.2) respectively for symmetric or antimetric two ports.

$$S_1 = (B + C)/\bar{A}, \quad S_2 = (B - C)/\bar{A} \qquad (6.1a,b)$$

$$S_1 = (B + jC)/\bar{A}, \quad S_2 = (B - jC)/\bar{A} \qquad (6.1'a,b)$$

From (6.1) it is easily verified that $S_1\tilde{S}_1 = S_2\tilde{S}_2 = 1$. Thus for each i, $|S_i| = 1$ for all $|z_1| = |z_2| = 1$ except possibly finitely many values where it is undefined. Furthermore, $\bar{A} \neq 0$ for $|z_1| = 1$, $|z_2| < 1$. Thus via Property 2.2a it follows that S_i, for each i = 1,2 in (6.1) must be a fully recursive half-plane all-pass function. Exactly same conclusions hold for S_1 and S_2 in (6.1′). Consequently, S_1, S_2 can be expressed as in (6.3), where D_1, D_2 and A_1, A_2 satisfy properties analogous to D and A in Property 2.3.

$$S_1 = - D_1A_1/\bar{A}_1, \quad S_2 = - D_2A_2/\bar{A}_2 \qquad (6.3a,b)$$

Note that even if A, B, C are <u>real</u> rational functions, S_1 and S_2 are real in (5.1ab) but not in (5.1′a,b). Thus, a symmetric filter can be realized by making use of the relation $S_{21} = C/\bar{A} = (S_1-S_2)/2$, where the one-ports S_1 and S_2 are realized as in Appendix A. Although $S_{21} = C/\bar{A} = -j(S_1-S_2)$ holds true in the antimetric case, a realization in terms of this last mentioned equation is not feasible due to the presence of the factor j unless complex filter realizations are called for. In this case, the pseudopolynomials A, B, C which are real, can be found from (6.1′a,b) and subsequently S_{21} can be realized as being embedded in a real two-port S described by A, B, C in standard form. The design problem then boils down to appropriately choosing the real 1-D rational functions D_1, D_2, and real pseudopolynomials A_1, A_2 so that the frequency response requirements on $|S_{21}|$ are satisfied. This latter step may be carried out by using numerical optimization (e.g. Levenberg-Marquadt). For the purpose of numerical optimization, however, the following symmetry observations have the effect of reducing the number of parameters to be optimized.

Note that the above strategy of representing a symmetric or an antimmetric lossless two-port by means of two all-pass functions has been crucially exploited in the design of quarter plane filters having circularly symmetric and fan type frequency response [7], [16]. Additionally, in [7], [16] the symmetry dictates certain separability properties of the all-pass functions which further facilitates the solution to the approximation problem. Further investigation is needed to determine the nature of separability property, if any, imposed on S_1 and S_2 in (6.3) by the symmetries in frequency response and to take benefit of these properties in numerical approximation.

Next, a few comments on the implementational aspects of our filters will be made. The filter synthesis procedure described here is clearly minimal in terms of the number of delays of the z_2 type. Specifically, if the pseudopolynomial A in prescribed filter transfer function has degree n_2 then precisely n_2 of z_2 type delays are needed. The number of GGM needed is at most $2(n_2+1)$. However, the order of the 1-D filters $k_1(z_1)$, $k_2(z_1)$ etc. contained in each GGM can be quite large and grows rapidly not only with the degree n_1 of the specified transfer function in z_1, but with n_2 as well (Example 5.1 was purposely taylored to be simple, and thus, does not exhibit this phenomenon very well). Note however, $k_1(z_1)$, $k_2(z_1)$ and the GGMs are not necessarily causal 1-D filters. They can be implemented to process rows of data from left to right or from right to left or simultaneously from both directions. Since GGMs can be viewed as 1-D row processors, and in many 2-D applications complete rows of 2-D signal are naturally available as blocks of data, the set of data in an entire row can be processed simultaneously by a GGM. The parallelism so available can thus be potentially used to overcome the drawback resulting from large filter order of $k_1(z_1)$'s and $k_2(z_1)$'s. The fact that the present acausal filtering scheme allows us to process rows of data simultaneously without much difficulty can be viewed as a major benefit, as opposed to quarter plane filtering schemes where any concurrent processing, if possible at all, must be accompanied by cumbersome sampling schemes presently not used in practical situations. Furthermore, since our filters share the same modular structure as that of 1-D Rao-Kailath structure, and it has been noted in [6] that data flow in such 1-D structures are pipelineable, it follows after a closer examination that blocks of data in the form of 1-D rows of the 2-D signal can also be made to flow through our filter structure in an analogous fashion. Thus, in summary while the rows themselves are to be processed in parallel, the sequence in which they are to be processed are pipelineable.

Recursive structures of the type considered in the present paper can be easily extended to 3-D by requiring in (1.1) $x_n(\cdot)$'s and $y_n(\cdot)$'s to be 2-D signals and $\alpha_i[\cdot]$, $\beta_i[\cdot]$ to be 2-D convolutional operators. Alternately, the 3-D transfer function $H(z_1,z_2,z_3)$ of the filter would be then given as in (1.2) with $A_i(\cdot)$ and $B_i(\cdot)$ being rational functions of two-variables. In computational terms, this amounts to "frame recursion" i.e., in order to compute an output frame (which is now a 2-D signal), a set of previously computed frames as well as a set of input frames is needed. Such a recursive scheme, when endowed with the property of passivity or losslessness, yields to a development entirely analogous to that reported in the present paper. However, since in (5.1), which would now involve 2-variables, a rational k_1 would not necessarily determine a rational k_2 due to non factorability of 2-D polynomials. Thus, the implementation of corresponding GGM's may involve non-rational (i.e., infinite order) filtering. However, a rational approximation for k_2 which renders the associated GGM strictly passive, but not necessarily lossless, may be adopted. More importantly, symmetries in frequency response referred to earlier may potentially dictate the factorability of $(1-k_1\tilde{k}_1)$ into $k_2\tilde{k}_2$, where k_2 is rational. A detailed investigation of these issues are once again left out of the present paper.

Finally, the excellent behavior such as freedom from limit cycles, forced response stability etc. of 1-D internally passive digital filters on the face of rounding and overflow truncation can be attributed to the fact that (see [16] for details) their internal building blocks i.e., the Gray-Markel sections or adaptors behave as strictly passive elements for a large variety of roundoff and truncation schemes. In the present context of 2-D fully recursive half plane filters, the row outputs of 1-D convolution operators represented by $k_i(z_1)$, i = 1,2 in each GGM section have larger support than the corresponding inputs to them. Thus, in practical implementation, the supports of these 1-D rows must be truncated at the two boundaries. This is similar to the 1-D situation, in which the role of convolution is played by multiplication of two binary numbers -- an operation that can also be interpreted as a convolution at the bit level. On the basis of this analogy, it may be conjectured that a scheme for controlled truncation of lengths of 1-D row signals can be devised so that GGM's behave as strictly passive building blocks, and thus the advantages of internally passive realization is fully exploited. However, the details of this issue remains to be worked out.

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Appendix A:

In this appendix we prove that a fully recursive symmetric half-plane all-pass function $H=H(z_1,z_2)$ can be synthesized as an interconnection of GGM sections (cf. Section 5) and z_2-type delays. This can be considered to be a generalized form of Schur's algorithm [9].

Let $k_1 = k_1(z_1) = H(z_1,0)$. Since H is as in Theorem 2.1 it follows from Property 2.2a that $|k_1|<1$ for all $|z_1|=1$ with the possible exception of finite number of values of z_1, where $|k_1|=1$. Thus, a k_2 satisfying (4.2) can be found i.e., k_1 and k_2 defines a GGM section. Consider next the function $H_1 = H_1(z_1,z_2)$ defined as in (A1.1), which can be interpreted as the residual transfer function after extraction of the GGM section just mentioned from H_1.

$$H' = (H - k_1)/(1 - \tilde{k}H) \qquad \text{(A1.1)}$$

From (2.15) it then follows that $H' = -pA'/B'$, where $pA' = DA + k_1\bar{A}$, $B' = \bar{A} + \tilde{k}_1DA$, p being the self-reciprocal factor of largest degree present in the numerator of $(DA + k_1\bar{A})$ when expressed in irreducible rational form. Next, since we have $\tilde{D} = D^{-1}$ it follows that $\tilde{p}\tilde{A}'/\tilde{A} = \tilde{D}(1-\tilde{k}_1H)$. Consequently, if $\deg_2 A' < \deg A$ then we would have $\tilde{k}_1H(z_1,0) = |H(z_1,0)|^2 = 1$ for arbitrary z_1, which is impossible (cf. Property 2.2a). Thus, $\deg_2 A' = \deg_2 A$. It then clearly follows that $\tilde{p}\bar{A}' = \tilde{D}(\bar{A} + \tilde{k}_1DA) = \tilde{D}B'$, thus $H' = -D_1(A'/\bar{A}')$; $D_1 = D(p/\tilde{p})$. Also, since $pA' = -\bar{A}(H-k_1)$ it follows that $pA' = 0$ for $z_2=0$ and arbitrary z_1. Thus, the pseudopolynomial A' has a factor z_2. By defining A_1 via $zA_1 = A'$ we can write $H_1 = z_2H'$, where $H_1 = -D_1(A_1/\bar{A}_1)$. Note that H_1 can be constructed simply by extracting a z_2-type delay from H'. Clearly, D_1 satisfies condition (ii) of Theorem 2.1. Also, by following arguments similar to that used after (4.7a,b) it can be shown that $\bar{A}_1$ is half-plane scattering Schur. Thus, conditions (iii) and

(iv) of Theorem 2.1 are satisfied by H_1, which has now been proved to be fully recursive symmetric half-plane lossless. Since $\deg_2 A_1 = \deg_2 A' - 1 = \deg_2 A - 1$ the procedure just described when applied $\deg_2 A$ times yields a circuit as shown in Figure A.1, in which the terminating section is an all-pass (not necessarily stable) in z_1 only.

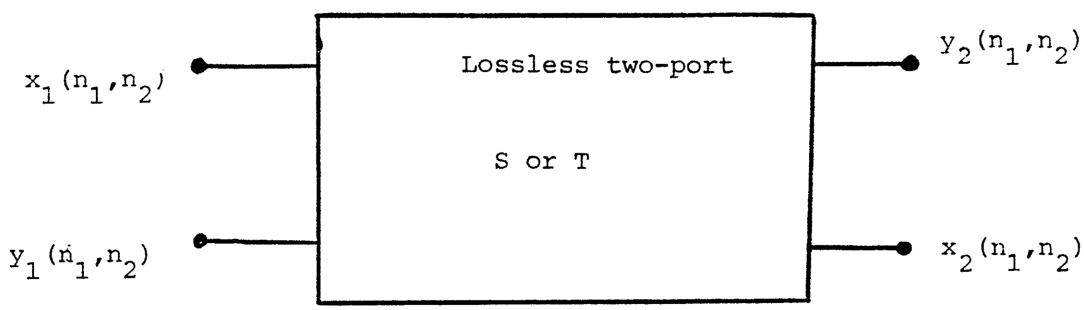

Figure 4.1: A fully recursive lossless two-port described by the transfer function matrix S or by the chain matrix T

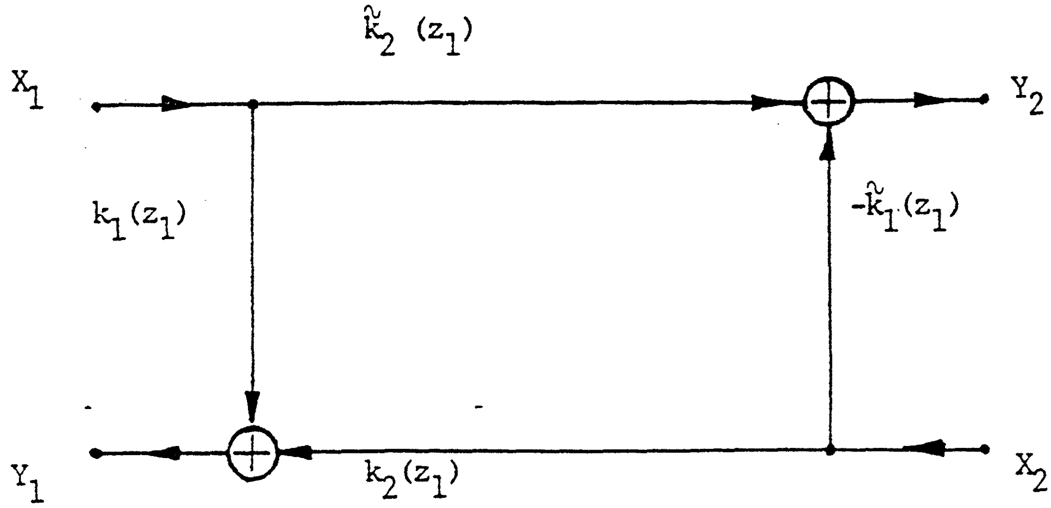

Figure 5.1: A generalized Gray Marked section (G G M)

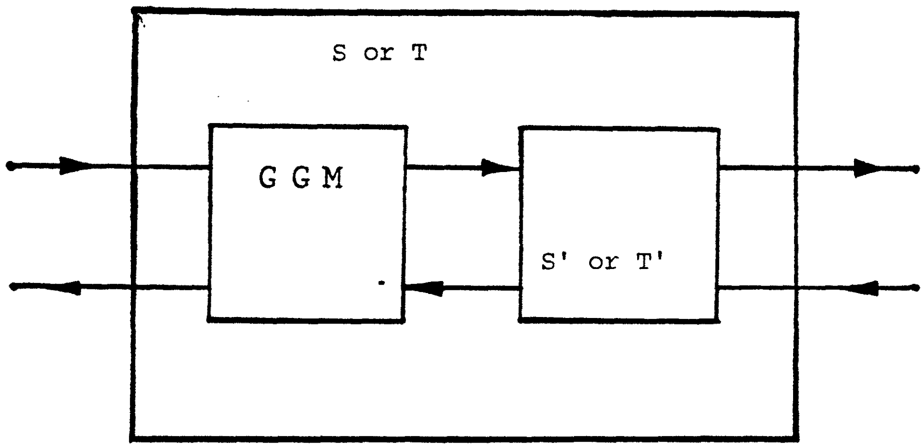

Figure 5.2: Step 1; G G M extraction

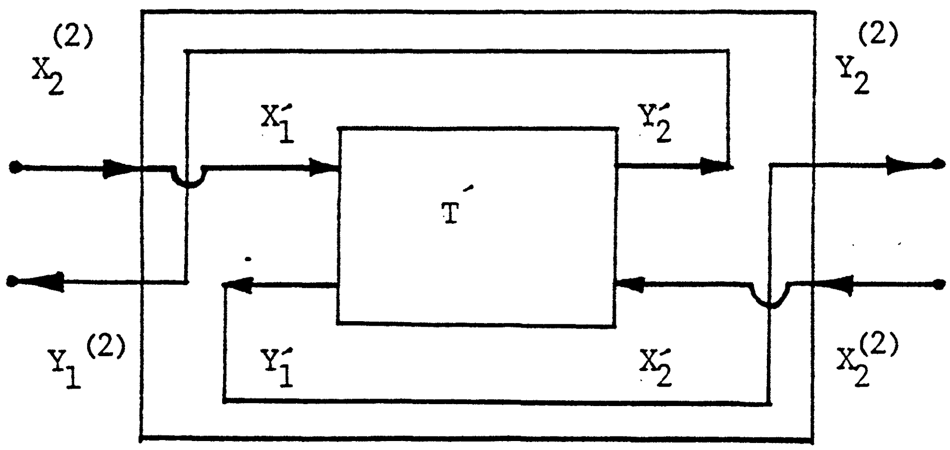

Figure 5.3: Step 2; Two-port $T^{(2)}$ obtained from T'

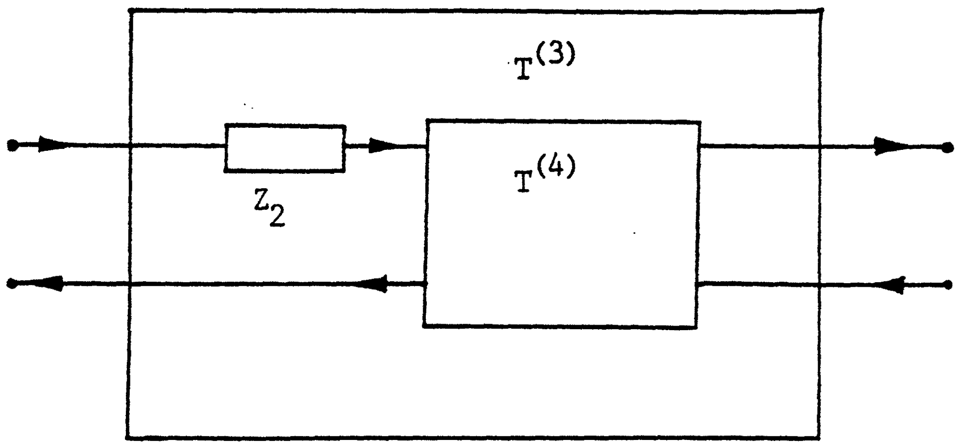

Figure 5.4: Step 4; Z_2 - delay extraction

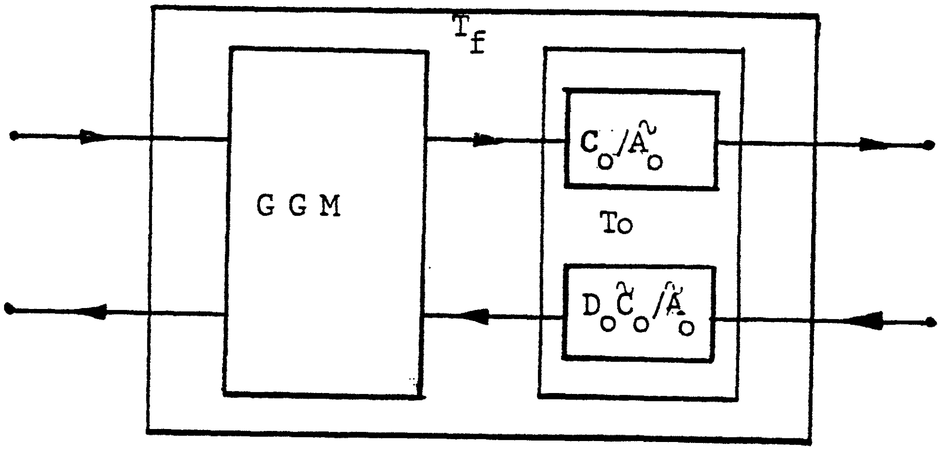

Figure 5.5: Terminal Step in Synthesis

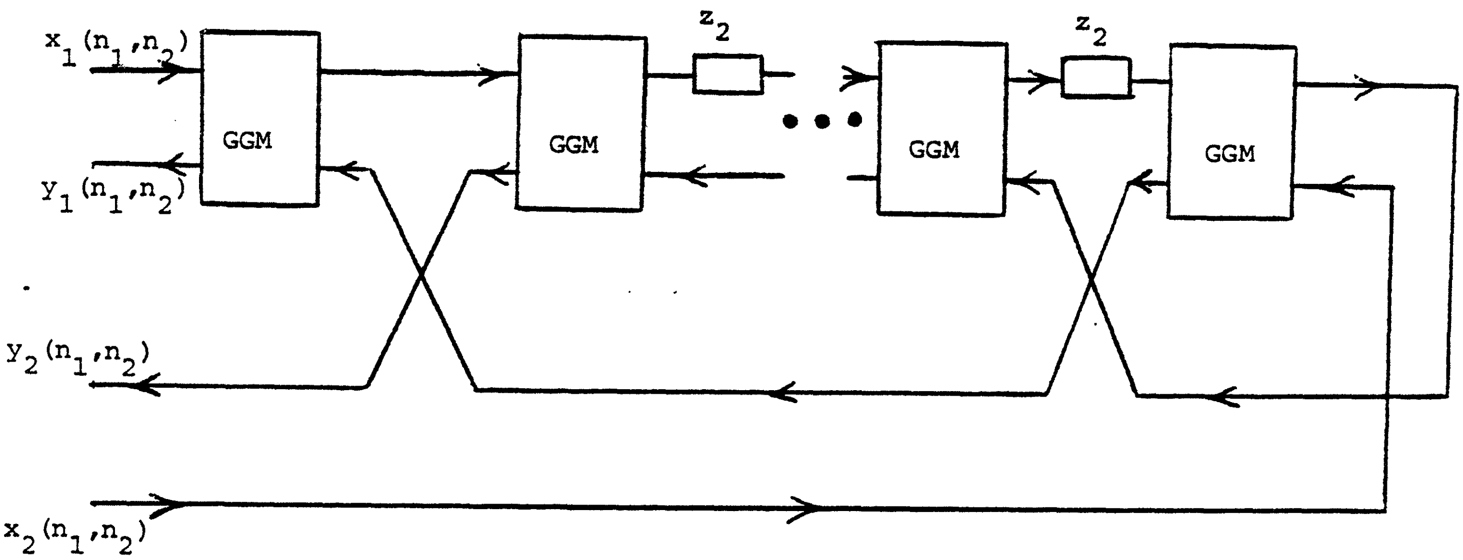

Figure 5.6: Composite filter structure obtained via $deg_2 A$ cycles of Step 1 through Step 4.

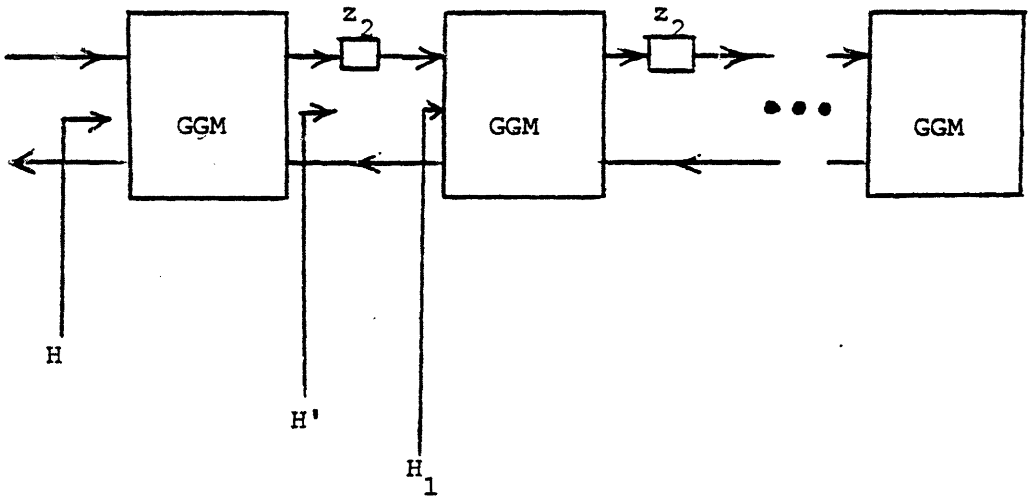

Figure A.1: Synthesis of lossless fully recursive half-plane one-port.