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Convergence of Asynchronous Matrix Iterations Subject to Diagonal Dominance*

by

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Abstract

We consider point Gauss-Seidel and Jacobi matrix iterations subject to diagonal dominance. We give conditions on the relaxation parameter under which the iteration mapping has the maximum norm nonexpansive properties given in [TBT88]. It then follows from a result in [TBT88] that, under these conditions, the associated relaxation iterations converge under asynchronous implementation. Our conditions for convergence improve upon those given by James [Jam73] for the special cases of strict diagonal dominance and irreducible diagonal dominance and synchronous implementation.

KEY WORDS: diagonal dominance, relaxation iteration, asynchronous computation

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1. Introduction

Consider stationary point relaxation methods of the Gauss-Seidel and Jacobi type for solving systems of linear equations [OrR70, Var62, You71]. For the special case where the coefficient matrix is either strictly diagonally dominant or irreducibly diagonally dominant, James [Jam73] gave general conditions on the relaxation parameters under which these methods converge. In this note we expand on the work of James. In particular, we show that these methods converge under broader conditions on the relaxation parameters and for problems where the coefficient matrix is irreducible and diagonally dominant. Moreover, our convergence results carry over to the asynchronous versions of these methods. Our arguments are based on the notion of maximum norm nonexpansive mappings and are fundamentally different from the eigenvalue analysis of James and others.

We will use the following definitions (cf. [OrR70]):

Definition 1 An n×n complex matrix $A = [a_{ij}]$ is diagonally dominant if, for all i,

$$\sum_{j\neq i} |a_{ij}| \leq |a_{ii}|, \tag{1.1}$$

and is strictly diagonally dominant if strict inequality holds in (1.1) for all i. Similarly, A is irreducibly diagonally dominant if it is irreducible, diagonally dominant, and strict inequality holds in (1.1) for at least one i.

In what follows, A will denote the coefficient matrix associated with the system of linear equations and n will denote its dimension. For simplicity we will assume that A has unity diagonal entries. The iteration matrix for the stationary point relaxation methods corresponding to A is (cf. [Jam73])

$$M(\alpha,\Omega) = (I+\alpha\Omega L)^{-1}[(I-\Omega)-(1-\alpha)\Omega L-\Omega U], \tag{1.2}$$

where Ω is an n×n diagonal matrix with positive diagonal entries, α is a scalar inside [0,1], and L (U) denotes the lower (upper) triangular part of A, i.e.

$$A = L+I+U.$$

[The familiar Jacobi (Gauss-Seidel) iteration is obtained by setting $\Omega = I$ and $\alpha = 0$ ($\alpha = 1$).] Under appropriate conditions on α and Ω, all eigenvalues of $M(\alpha,\Omega)$ lie strictly within the unit circle and the stationary relaxation method given by $M(\alpha,\Omega)$ converges [Jam73] (see also [JaR75, Var76]). However, the circle criterion is not sufficient to guarantee that the asynchronous version of this method converges. Instead, we need the following notions:

Definition 2 An n×n complex matrix $M = [m_{ij}]$ is nonexpansive if, for all i,

$$\sum_j |m_{ij}| \leq 1, \tag{1.3}$$

and is contractive if strict inequality holds in (1.3) for all i. M is block irreducibly nonexpansive if it is nonexpansive and, for any $S \subset \{1,2,\ldots,n\}$ such that $m_{ij} = 0$ for all $i \in S$ and all $j \notin S$, there exists an $s \in S$ such that $\sum_j |m_{sj}| < 1$. M is block irreducibly contractive if it is block irreducibly nonexpansive and strict inequality holds in (1.3) for at least one i.

If $M(\alpha,\Omega)$ is contractive, then the associated relaxation mapping is a maximum norm contraction and, by a result of Chazan and Miranker [ChM69] (see also [Bau78; Ber83; BeT89, §6.2]), the associated asynchronous method converges. If $M(\alpha,\Omega)$ is block irreducibly contractive or is block irreducibly nonexpansive with diagonal entries having positive real parts, then it can be shown that the associated relaxation mapping has the maximum norm nonexpansion properties described in [TBT88] and, by Proposition 2.1 in [TBT88], the associated asynchronous method converges (see also [BeT89, §7.2; LuD86; Tsi84, §4]). [The results in the above references are stated in terms of real matrices, but they can be extended to complex matrices.] Below we give conditions on α and Ω under which $M(\alpha,\Omega)$ satisfies the above convergence criteria when (i) A is strictly diagonally dominant, (ii) A is irreducibly diagonally dominant, and (iii) A is irreducible and diagonally dominant. The first two cases had been analyzed in [Jam73], but our conditions on α and Ω for the second case are more general. The third case, to the best of our knowledge, has not been analyzed previously, although it has a number of interesting applications. For example, consider the matrix EDE^T, where D is a diagonal matrix with positive diagonal entries and E is the node-arc incidence matrix for a connected directed graph. This matrix, which arises in the solution of linear network flow problems by interior point methods [Kar84], can be shown to be irreducible and diagonally dominant [TBT88, §3.2]. For another example, consider the probability transition matrix P for an irreducible Markov chain and suppose that π^* is the column vector of invariant probabilities of this

Markov chain. Then the coefficient matrix $I-P^T$ is irreducible and, upon scaling by the diagonal matrix whose ith component is the ith component of π^*, is also diagonally dominant [BeT89, §7.3.2]. [It can be seen that scaling of A by a diagonal matrix with positive diagonal entries results in the scaling of $M(\alpha,\Omega)$ by the same matrix and that the above criteria for convergence is unaffected by such scaling of $M(\alpha,\Omega)$.]

2. **Convergence Results**

Let w_i denote the ith diagonal entry of Ω and denote, for each i,

$$l_i = \sum_{j<i} |a_{ij}|, \qquad u_i = \sum_{j>i} |a_{ij}|, \tag{2.1}$$

From the definition of $M(\alpha,\Omega)$ (cf. (1.2)) we have that

$$(I + \alpha\Omega L)M(\alpha,\Omega) = I + \alpha\Omega L - \Omega A.$$

Hence, if we let m_{ij} denote the (i,j)th entry of $M(\alpha,\Omega)$, then

$$m_{ij} + \alpha w_i \left(\sum_{k<i} a_{ik}\, m_{kj}\right) = (\alpha - 1)w_i\, a_{ij}, \qquad \text{if } j < i, \tag{2.2a}$$

$$m_{ij} + \alpha w_i \left(\sum_{k<i} a_{ik}\, m_{kj}\right) = 1 - w_i, \qquad \text{if } j = i, \tag{2.2b}$$

$$m_{ij} + \alpha w_i \left(\sum_{k<i} a_{ik}\, m_{kj}\right) = -w_i\, a_{ij}, \qquad \text{if } j > i. \tag{2.2c}$$

The above identities will be used frequently in the analysis to follow. We first have the following two main results regarding $M(\alpha,\Omega)$ (cf. Theorems 2 and 3 in [Jam73]):

Proposition 1 If A is strictly diagonally dominant (i.e., $l_i + u_i < 1$ for all i), then, for any $\alpha \in [0,1]$ and any Ω such that

$$0 < w_i < 2/(1+l_i+u_i), \qquad i = 1,\ldots, n, \tag{2.3}$$

the matrix $M(\alpha,\Omega)$ is contractive.

Proof: First we note from the hypotheses of the proposition that

$$w_i(l_i + u_i) + |1 - w_i| < 1, \; i = 1,\ldots,n. \tag{2.4}$$

[To see this, note that if $0 < w_i \le 1$, then (cf. diagonal dominance) $w_i(l_i + u_i) + |1 - w_i| = w_i(l_i + u_i) + 1 - w_i < 1$; while if $1 < w_i < 2/(1+l_i+u_i)$, then (cf. (2.3)) $w_i(l_i + u_i) + |1 - w_i| = w_i(l_i + u_i) - 1 + w_i < 1$.]

For $i = 1$, we have from (2.2b)-(2.2c)

$$m_{11} = (1 - w_1),$$

$$m_{1j} = -w_1 a_{1j}, \qquad \text{if } j > 1.$$

Hence

$$\begin{aligned}\textstyle\sum_j |m_{1j}| &\le |1 - w_1| + w_1(\textstyle\sum_{j>1}|a_{1j}|)\\ &= |1 - w_1| + w_1 u_1\\ &< 1,\end{aligned}$$

where the last inequality follows from (2.4). Now suppose that

$$\textstyle\sum_j |m_{ij}| < 1, \; i = 1,\ldots, s-1, \tag{2.5}$$

for some integer $s \ge 2$. We will show that $\sum_j |m_{sj}| < 1$, thus completing our induction. Now, from (2.2a)-(2.2c) we obtain that

$$\begin{aligned}\textstyle\sum_j |m_{sj}| &\le \alpha w_s(\textstyle\sum_j \sum_{k<s}|a_{sk}|\,|m_{kj}|) + (1 - \alpha)w_s(\textstyle\sum_{j<s}|a_{sj}|) + |1-w_s| + w_s(\textstyle\sum_{j>s}|a_{sj}|)\\ &\le \alpha w_s\cdot l_s + (1 - \alpha)w_s\cdot l_s + |1-w_s| + w_s\cdot u_s\end{aligned}$$

$$= w_s(l_s + u_s) + |1 - w_s|$$
$$< 1,$$

where the second inequality follows from the inductive hypothesis (2.5) and the third inequality follows from (2.4). Q.E.D.

Proposition 2 If A is irreducibly diagonally dominant, then, for any $\alpha \in [0,1)$ and any Ω such that

$$0 < w_i \leq 2/(1+l_i+u_i), \qquad i = 1,\ldots, n, \tag{2.6}$$

where strict inequality holds for at least one row i' such that $l_{i'} + u_{i'} < 1$, the matrix $M(\alpha,\Omega)$ is block irreducibly contractive.

Proof: It is straightforward to show (by using (2.6) and repeating the proof of Proposition 1) that $\sum_j |m_{ij}| \leq 1$, for all i, and strict inequality holds for $i = i'$. Now consider any $S \subset \{1,\ldots,n\}$ such that $m_{ij} = 0$ for all $i \in S$, $j \notin S$ (if no such S exists, then $M(\alpha,\Omega)$ is irreducible and we are done). We will show that there exists an $s \in S$ such that $\sum_j |m_{sj}| < 1$. [since the choice of S is arbitray, this shows $M(\alpha,\Omega)$ to be block irreducibly contractive.]

If $i' \in S$, we can simply set s to i'. Otherwise, since A is irreducible, there must exist some $s \in S$ and some $t \notin S$ such that $a_{st} \neq 0$ and $m_{st} = 0$. We distinguish between two cases: (a) $t < s$ and (b) $t > s$. In case (a), we have from (2.2a) that

$$\alpha w_s \left(\sum_{k<s} a_{sk}\, m_{kt}\right) = (\alpha - 1) w_s\, a_{st}.$$

Hence, from (2.2a)-(2.2c) we obtain that

$$\begin{aligned}
\sum_{j\neq t} |m_{sj}| &\leq \alpha w_s\left(\sum_{j\neq t} \sum_{k<s} |a_{sk}|\, |m_{kj}|\right) + (1 - \alpha) w_s\left(\sum_{j<s, j\neq t} |a_{sj}|\right) + |1-w_s| + w_s\left(\sum_{j>s} |a_{sj}|\right) \\
&\leq \alpha w_s \cdot l_s + (1 - \alpha) w_s(l_s - |a_{st}|) + |1-w_s| + w_s \cdot u_s \\
&= w_s(l_s + u_s) + |1 - w_s| - (1 - \alpha) w_s\, |a_{st}|
\end{aligned}$$

$$\leq 1-(1-\alpha)w_s\,|a_{st}|,$$

where the second inequality follows from the fact $\sum_j |m_{kj}| \leq 1$ for all k and the last inequality follows from the observation (cf. (2.6)) that $w_s(l_s + u_s) + |1 - w_s| \leq 1$. In case (b), an analogous argument using (2.2c) instead of (2.2a) yields

$$\sum_{j\neq t} |m_{sj}| \leq 1 - w_s\,|a_{st}|.$$

Since $\alpha \in [0,1)$, we have $\sum_j |m_{sj}| < 1$ in either case. Q.E.D.

We note that the conditions on α given by Proposition 2 is <u>different</u> from that given by Theorem 3 in [Jam73] where it is assumed thate either $\alpha = 0$ or $\alpha \in (1/2,1]$. Does Proposition 2 still hold for $\alpha = 1$? One suspects that it does, but more analysis is required. [Proposition 2 can be shown to hold for $\alpha = 1$ if $m_{ii} \neq 0$ for all i, which in turn can be shown to hold if $w_i < 1/(1+l_i)$ for all i (see argument below).]

Proposition 3 If A is irreducible and diagonally dominant, then, for any $\alpha \in [0,1]$ and any Ω such that

$$0 < w_i < 1/(1+\alpha l_i), \qquad i = 1,\ldots, n, \tag{2.7}$$

the matrix $M(\alpha,\Omega)$ either is block irreducibly contractive or is block irreducibly nonexpansive with diagonal entries having positive real parts.

Proof: First note that $1/(1+\alpha l_i) \leq 2/(1+l_i+u_i)$; hence (2.7) implies that (2.6) holds. Then using (2.6) and an argument identical to that in the proof of Proposition 2, we obtain that $M(\alpha,\Omega)$ is nonexpansive and, if $M(\alpha,\Omega)$ is not irreducible, then it is block irreducibly contractive. Hence it only remains to show that the diagonal entries of $M(\alpha,\Omega)$ have positive real parts. We have from (2.2b) that

$$m_{ii} + \alpha w_i \left(\sum_{k<i} a_{ik}\, m_{ki}\right) = 1 - w_i.$$

Hence

$$\begin{aligned}
\mathrm{Re}(m_{ii}) &\geq 1 - w_i - \alpha w_i \left(\sum_{k<i} |a_{ik}||m_{ki}|\right) \\
&\geq 1 - w_i - \alpha w_i \left(\sum_{k<i} |a_{ik}|\right) \\
&= 1 - w_i(1 + \alpha l_i) > 0,
\end{aligned}$$

where the second inequality follows from the fact that $|m_{ki}| \leq 1$ for all k, and the third inequality follows from (2.7). Q.E.D.

[In practice, we can first choose any $0 < w_i \leq 2/(1+l_i+u_i)$, for all i; check if $M(\alpha,\Omega)$ is block irreducibly contractive; and if not, decrease each w_i to be below $1/(1 + \alpha l_i)$.]

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