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A Frequency-Domain Estimator For Use In Adaptive Control Systems†

Richard O. LaMaire‡, Lena Valavani§||, Michael Athans§ and Gunter Stein§◊

A robust estimation technique, developed for adaptive control systems, finds both a parameterized model and a corresponding frequency-domain error bounding function.

Key Words - Frequency-domain estimation; robust adaptive control; parameter estimation.

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‡ ALPHATECH, Inc., 111 Middlesex Turnpike, Burlington, MA 01803, U.S.A.
§ Laboratory for Information and Decision Systems, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.
|| Department of Aeronautics and Astronautics, Massachusetts Institute of Technology
◊ Honeywell Systems and Research Center, Minneapolis, MN 55418, U.S.A.

Abstract - This paper presents a frequency-domain estimator which can identify both a parameterized nominal model of a plant as well as a frequency-domain bounding function on the modeling error associated with this nominal model. This estimator, which we call a robust estimator, can be used in conjunction with a robust control-law redesign algorithm to form a robust adaptive controller.

1. INTRODUCTION AND MOTIVATION

The use of feedback control in systems having large amounts of uncertainty requires the use of algorithms that learn or adapt in an on-line situation. A control system that is designed using only a priori knowledge results in a relatively low bandwidth closed-loop system so as to guarantee stable operation in the face of large uncertainty. An adaptive control algorithm, which can identify the plant on-line, thereby decreasing the amount of uncertainty, can yield a closed-loop system that has a higher bandwidth and thus better performance than a non-adaptive algorithm. There are many problems with the adaptive control algorithms which have been developed, to date. In particular, most adaptive control algorithms are not robust to unmodeled dynamics and an unmeasurable disturbance, particularly in the absence of a persistently-exciting input signal.

In this section, we will motivate the robust estimation problem by first discussing the adaptive control problem, in general, and then presenting a perspective on the robust adaptive control problem. Further, we justify the choice of an infrequent adaptation strategy before discussing the main focus of the paper, the development of a robust estimator.

Stability of Adaptive Control Algorithms. The use of adaptive control yields systems that are nonlinear and time-varying. Thus, the stability of these systems depends on the inputs and disturbances, as well as the plant (including any unmodeled dynamics) and the compensator. However, the stability properties of a linear time-invariant (LTI) feedback system depend only on the plant and compensator, not the inputs and disturbances. Because of this fact, we take the point of view that it is desirable to make the system 'as LTI as possible'. Of course, our motivation for using adaptive control is to achieve a performance improvement (increased bandwidth) over the best non-adaptive LTI compensator. So, there is the ever present tradeoff between performance and robustness.

The preceding argument can be used to justify an infrequent control-law redesign strategy. It is envisioned that a discrete-time estimator will be used to continually update the frequency-domain estimate of the plant as long as there is useful information in the input/output data of the plant. The plant is in a closed-loop that is controlled by a discrete-time compensator that is only infrequently updated (redesigned). It can be shown that if the compensator is redesigned sufficiently infrequently, then the LTI stability of the 'frozen' system at every point in time guarantees the exponential stability of the time-varying system. In this way, the control system looks nearly LTI and consequently is more robust to disturbances than a highly nonlinear adaptive controller. It is emphasized here that a robust adaptive controller that slowly learns and produces successively better LTI compensators is the end product envisioned in this paper. The paper aims to develop only the estimation part of this robust adaptive controller. On the other end of the adaptive control spectrum are algorithms that quickly adapt to a changing system. As mentioned earlier, these types of controllers have poor robustness properties in that they are highly sensitive to unmodeled dynamics and unmeasurable disturbances, particularly in the absence of persistent excitation.

A Perspective on the Robust Adaptive Control Problem. With the solution of the adaptive control problem for the ideal case, that is, when there are no unmodeled dynamics nor unmeasurable disturbances, the problem of robustness has become a focus of current research. Recently, a new perspective on the robust adaptive control problem has appeared in the literature (Goodwin *et al.*, 1985a). Briefly, a *robust* adaptive controller is viewed as a combination of a *robust* estimator and a *robust* control law. This is an appealing point of view. For example, if the robust estimator is not getting any useful information and consequently, is not able to improve on the current knowledge of the plant, then the adaptation aspect of the algorithm can be disabled and the adaptive controller reduces to a robust control law. That is, in a situation where the adaptive algorithm is not learning, the adaptive controller becomes simply the best robust LTI control law that one could design based only on a priori information and any additional information learned since the algorithm began.

Brief Statement of the Robust Estimation Problem. The main focus of this paper is the development of a robust estimator for use in an adaptive controller. In non-adaptive robust control, the designer must first obtain a nominal model along with some measure of its goodness. A practical measure of goodness is a bounding function on the magnitude of the modeling errors in the frequency-domain. Since non-adaptive robust control requires these steps, the same steps must implicitly, or explicitly, be present in a robust adaptive control scheme, the difference being that the steps are carried out on-line rather than off-line. Thus, we assume that our robust estimator must supply:

1) a nominal plant model,

2) a frequency-domain bounding function on the magnitude of the modeling uncertainty between the true plant and this nominal model.

So, the robust estimator must provide an estimate of the parameters for the structure of the nominal model, as well as a frequency-domain uncertainty bounding function corresponding to this nominal model. Given this information, several robust control-law design methodologies could be used, including the LQG/LTR design methodology (Athans, 1986). The envisioned adaptive control system is illustrated in Fig. 1. In this paper, we will use a discrete-time model of a sampled-data control system.

The robust estimator presented in this paper is the first of its kind in that it provides guarantees concerning the current estimate of the nominal model of the plant. This requirement is essential if the estimator is to be used in a robust adaptive control situation. If the estimator cannot provide guarantees about the model it provides to the control-law redesign algorithm, then the redesign algorithm cannot guarantee stability of the closed-loop system. We will use a deterministic framework throughout the paper, since guarantees of stability are sought.

Related Literature. The work described in this paper was first presented in LaMaire (1987a) and LaMaire *et al.* (1987b). Kosut (1987, 1988) has also developed an approach to designing a robust controller using on-line measurements. The approaches of LaMaire and Kosut both use the frequency-domain estimation work of Ljung (1985, 1987) as a basis. Ljung analyzed the properties of the *empirical transfer function estimate* (ETFE), which is computed using the Fourier transforms of finite-length input/output data of the plant. Ljung (1987) developed a constant bound on the effects of using finite-length data to compute the ETFE, for strictly stable plants. This work provides the background for our development in Section 4.1 of a time-varying frequency-domain error bounding function that is computed using the DFTs of the plant input signal.

2. MATHEMATICAL PRELIMINARIES

In this section, we will present the notation and definitions that will be used in the paper, as well as some results and theorems that will be useful later on. We denote a discrete-time signal by $x[n]=x(nT)$ where $x(t)$ denotes the sampled continuous-time signal and where n is an integer and T is the sampling period. The z-transform of $x[n]$ on the unit circle is called the discrete-time Fourier transform (DTFT) and is defined as follows

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega T)n} \tag{2.1}$$

We define the N-point discrete Fourier transform (DFT) of $x[n]$ at the N frequency points, $\omega_k=(k/N)\omega_s$, for $k=0,..,N-1$, where $\omega_s=2\pi/T$ is the sampling frequency,

$$X_N(\omega_k) = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}, \quad \text{for } k=0,\ldots,N-1 \tag{2.2}$$

and where $W_N=e^{-j(2\pi/N)}$. (2.3)

Further, we define the inverse N-point discrete Fourier transform of $X_N(\omega_k)$ as follows,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_N(\omega_k)\, W_N^{-kn}, \quad \text{for } n=0,\ldots,N-1 \tag{2.4}$$

Since we will not always be working with N-point sequences that begin at 0, we define the following versions of the DFT and inverse DFT for a sequence of N points ending with time index n.

$$X_N^n(\omega_k) = \sum_{m=n-N+1}^{n} x[m]\, W_N^{km}, \quad \text{for } k=0,\ldots,N-1 \tag{2.5}$$

$$x[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_N^n(\omega_k)\, W_N^{-km}, \quad \text{for } m=n-N+1,\ldots,n \tag{2.6}$$

A useful recursive equation for computing $X_N^n(\omega_k)$ from $X_N^{n-1}(\omega_k)$ can be derived from the above definitions and is given as follows

$$X_N^n(\omega_k) = X_N^{n-1}(\omega_k) + (x[n] - x[n-N])\, W_N^{kn}, \quad \text{for } k=0,\ldots,N-1 \tag{2.7}$$

If x[n] is of finite duration, for example if $x[n]\neq 0$ only for n=0,..,N-1, then the N-point DFT of x[n] and the DTFT of x[n] are equal at ω_k,

$$X_N(\omega_k) = X(e^{j\omega T})\big|_{\omega=\omega_k}, \quad \text{for } k=0,\ldots,N-1 \tag{2.8}$$

Signal Processing Theorems. In this subsection, we will develop results that can be used to bound the effects of using finite-length data to compute frequency-domain quantities. In the later parts of this paper, the frequency-domain estimate of a stable, causal, transfer function $H(e^{j\omega T})$ will be computed based on the N-point DFTs of the transfer function's input and output signals. We will now state a theorem that bounds the error in the frequency domain between this DFT derived frequency-domain estimate and the true transfer function.

Theorem 2.1. Let y[m]=h[m]*u[m], where h[m] is an infinite-length, causal, impulse response with all its poles in the open unit disk. We denote the DTFT of h[m] by $H(e^{j\omega T})$, and the DFTs of the N-points of u[m] and y[m] ending with time index n, by $U_N^n(\omega_k)$ and $Y_N^n(\omega_k)$, respectively. Then,

$$Y_N^n(\omega_k) = H(e^{j\omega_k T})\, U_N^n(\omega_k) + E_N^n(\omega_k), \quad \text{for } k=0,\ldots,N-1, \tag{2.9}$$

where the discrete function $E_N^n(\omega_k)$ is given by

$$E_N^n(\omega_k) = \sum_{p=1}^{\infty} h[p]\, W_N^{kp} \left(U_N^{n-p}(\omega_k) - U_N^n(\omega_k) \right), \quad \text{for } k=0,\ldots,N-1, \tag{2.10}$$

where W_N is defined in equation (2.3).

Proof. See Appendix A.

Remark 2.1. The function $E_N^n(\omega_k)$ is the error in the frequency domain, at time index n, due to the use of finite-length data. That is, if the DTFTs (based on infinite-length data) of u[m] and y[m] were used in equation (2.9) instead of the DFTs (based on finite-length data), then there would be no error term $E_N^n(\omega_k)$. Note that the function $E_N^n(\omega_k) / U_N^n(\omega_k)$ is the error in the frequency domain between the DFT derived frequency-domain estimate of $H(e^{j\omega_k T})$ and the true transfer function $H(e^{j\omega_k T})$.

It will later be useful to be able to find a magnitude bounding function on $E_N^n(\omega_k)$. The following theorem provides such a bounding function by using only a finite summation of the DFT differences and therefore can be implemented in practice.

Theorem 2.2. Under the assumptions of Theorem 2.1 we find that given some finite integer M, the magnitude of $E_N^n(\omega_k)$ is bounded for each k as follows,

$$|E_N^n(\omega_k)| \leq \sum_{p=1}^{M-1} |h[p]| \, |U_N^{n-p}(\omega_k) - U_N^n(\omega_k)| + 2\, u_{max} \sum_{p=M}^{\infty} p\, |h[p]|, \quad \text{for } k=0,\ldots, N-1, \tag{2.11}$$

where

$$u_{max} = \sup_m |u[m]|.$$

Proof. See Appendix A.

3. ROBUST ESTIMATOR PROBLEM STATEMENT

In this section, we first list the assumptions required by the robust estimator and then we state the robust estimation problem. Consider the system of Fig. 1 where the discrete-time plant $G_{true}(z)$ has an input u[n] and an output y[n] that is corrupted by an additive output disturbance d[n].

A1) *Plant Assumptions.* We assume a structure for the nominal model of $G_{true}(z)$ and a magnitude bounding function on the unstructured uncertainty. That is, we assume that

$$G_{true}(z) = G(z,\theta_0)\,[1 + \delta_u(z)] \tag{3.1}$$

where $G(z,\theta_0)$ is a nominal model, $\delta_u(z)$ denotes the unstructured uncertainty of the plant, θ_0 is a vector of plant parameters and we assume,

A1.1) $G(z,\theta_0) = B(z) / A(z)$, (3.2)

where the polynomials $B(z)$ and $A(z)$ are,

$$B(z) = b_0\, z^{(m_1-n_1)} + b_1\, z^{(m_1-n_1-1)} + \dots + b_{m_1}\, z^{-n_1}, \tag{3.3}$$

$$A(z) = 1 - a_1\, z^{-1} + \dots - a_{n_1}\, z^{-n_1}, \quad n_1 > m_1, \tag{3.4}$$

and where the parameter vector is,

$$\theta_0 = [\, a_1 \dots a_{n_1} \;\; b_0 \;\; b_1 \dots b_{m_1}\,]^T. \tag{3.5}$$

A1.2) $\theta_0 \in \Theta$, where Θ is a known bounded set. (3.6)

A1.3) $|\delta_u(e^{j\omega T})| \le \Delta_u(e^{j\omega T}), \;\forall\omega.$ (3.7)

A1.4) $|d\delta_u(e^{j\omega T}) / d\omega| \le \nabla_u(e^{j\omega T}), \;\forall\omega.$ (3.8)

A1.5) $G_{true}(z)$ and $G(z,\theta_0)$ have all their poles in the open unit disk, for all $\theta_0 \in \Theta$.

A1.6) A coarse bounding function on the magnitude of the impulse response of the true plant, denoted by $g_{true}[n]$, is known such that

$$|g_{true}[n]| \le \sum_{i=1}^{I_0} g_i\, n^{(r_i)}\, p_i^n, \tag{3.9}$$

where r_i is a positive integer, and $g_i > 0$, $0 < p_i < 1$ (i.e. all the poles of $g_{true}[n]$ are in the open unit disk), and r_i are known for $i=1,..,I_0$. $g_{true}[n]$ is assumed to be causal.

A1.7) zero initial conditions.

Thus, our a priori assumptions are that we know m_1 and n_1, the degrees of $B(z)$ and $A(z)$, respectively, and the bounding functions $\Delta_u(e^{j\omega T})$ and $\nabla_u(e^{j\omega T})$. Further, we assume that the parameter vector θ_0 is in some known bounded set Θ which is only a coarse, and hence large, a priori estimate of the parameter space. The parameter vector θ_0 is not required to be unique.

A2) Disturbance Assumption. We assume that the N-point DFT of the disturbance signal d[n], whose DFT is denoted by $D_N^n(\omega_k)$, satisfies

$$|D_N^n(\omega_k)| \le \bar{\bar{D}}_N(\omega_k), \text{ for } k=0, \ldots, N-1, \forall n. \tag{3.10}$$

A3) Input Signal Assumption. We assume that the input signal u[n] is bounded and that we know u_{max} where

$$|u[n]| \le u_{max}, \forall n. \tag{3.11}$$

Remark 3.1. The discrete-time system of Fig. 1 represents a sampled-data control system. While the above plant assumptions (A1) have been presented for a discrete-time system, similar assumptions can be stated for a continuous-time plant and then used to satisfy the above discrete-time assumptions. This process, including the derivation of a discrete-time unstructured uncertainty bound from a bound on the continuous-time unstructured uncertainty, is treated in LaMaire (1987a).

Remark 3.2. Based on input/output measurements alone we cannot determine a unique θ_0 for the nominal model because of the unstructured uncertainty. That is, if we assume the structure of A1.1 above and assume only that $\delta_u(z) \in S$ where

$$S = \{ \delta(z) \mid |\delta(e^{j\omega T})| \le \Delta_u(e^{j\omega T}), \forall \omega \}, \tag{3.12}$$

then we can define a smallest set

$$\Theta^* = \{ \theta \mid G_{true}(z) = G(z,\theta)[1 + \delta_u(z)] \text{ and } \delta_u(z) \in S \} \tag{3.13}$$

in which θ_0 lies. Thus, $\theta_0 \in \Theta^* \subset \Theta$ where only Θ is known a priori. Note that, in general, Θ^* will be a point only when $\Delta_u(e^{j\omega T})=0$ for all ω.

Preparation for Problem Statement. We rewrite the true discrete-time plant of equation (3.1) as

$$G_{true}(z) = G(z,\hat{\theta}) \, [1 + \delta_{su}(z,\hat{\theta})] \tag{3.14}$$

where again $G(z,\hat{\theta})$ is the nominal model using an estimate $\hat{\theta}$ of the parameter vector θ_0 in the structure of assumption A1.1, and $\delta_{su}(z,\hat{\theta})$ denotes the modeling error due to both structured and unstructured uncertainty. That is, since a priori we only know that $\hat{\theta} \in \Theta$, where $\hat{\theta}$ is not necessarily in Θ^*, there is structured uncertainty associated with this choice of $\hat{\theta}$ as well as the ever present unstructured uncertainty.

Problem Statement. The robust estimator must provide:

1) a parameter estimate $\hat{\theta}$, and hence a nominal model $G(z,\hat{\theta})$,
2) a corresponding bounding function, $\Delta^n_{su}(e^{j\omega T},\hat{\theta})$, such that

$$|\delta_{su}(e^{j\omega T},\hat{\theta})| \leq \Delta^n_{su}(e^{j\omega T},\hat{\theta}), \quad \forall \omega. \tag{3.15}$$

That is, at a given sample time n we want to generate a new nominal model $G(z,\hat{\theta})$ (where $\hat{\theta}$ is the parameter estimate at time index n) along with a corresponding bounding function $\Delta^n_{su}(e^{j\omega T},\hat{\theta})$ in the frequency domain indicating how good the current nominal model is.

Given 1 and 2 above and a compensator we can use discrete-time versions of the stability-robustness tests of Lehtomaki *et al.* (1984) to guarantee stability in the face of bounded modeling uncertainty.

The goal of the robust estimator is to find a $\hat{\theta}$ in Θ^* and to have $\Delta^n_{su}(e^{j\omega T},\hat{\theta})$ approach $\Delta_u(e^{j\omega T})$. The viewpoint taken here is that the unstructured uncertainty $\Delta_u(e^{j\omega T})$ is the best we can do given the structure of our nominal model. Thus, even though $\Delta^n_{su}(e^{j\omega T},\hat{\theta})$ can conceivably become smaller than our a priori assumed bound $\Delta_u(e^{j\omega T})$ we will not let this occur and will instead view the function $\Delta_u(e^{j\omega T})$ as the desirable lower bound of the function $\Delta^n_{su}(e^{j\omega T},\hat{\theta})$.

The problem that we have described in this subsection will be referred to as the robust estimation problem. An algorithm which satisfies this problem will be referred to as a robust estimator since it provides a nominal model of the plant as well as a guaranteed frequency-domain bounding function on the accuracy of this nominal model.

Outline of Problem Solution. In the following sections of this paper, we will develop a solution to the robust estimation problem stated above. First, in Section 4, we will develop a method for computing a frequency-domain estimate of the true plant along with a bounding function on the additive error in the frequency domain. Then, in Section 5, the frequency-domain estimate of Section 4 will be used to find parameter estimates for the nominal model. In Section 6, the frequency-domain estimate and the frequency-domain error bounding function of Section 4 will be combined with the parameter estimates of Section 5 to yield a frequency-domain bounding function on the magnitude of the uncertainty $\delta_{su}(e^{j\omega T},\hat{\theta})$. An alternative time-domain method for finding parameter estimates is briefly discussed in Section 7. In Section 8, the results of the paper are discussed in the context of the closed-loop adaptive control problem. Conclusions are presented in Section 9.

4. FREQUENCY-DOMAIN ESTIMATION AND ERROR BOUNDING

In this section, we will develop the basic methodology for finding a frequency-domain estimate of the true plant and a corresponding error bounding function on the frequency-domain modeling error.

4.1 Basic Methodology. Consider the true discrete-time plant $g_{true}[n]$ whose input is u[n], and whose disturbance-corrupted output is y[n]. Assuming zero initial conditions, we know that

$$y[n] = g_{true}[n]*u[n] + d[n]. \qquad (4.1)$$

Then, using the notation of Section 2 and Theorem 2.1, we find that for some time index n,

$$Y_N^n(\omega_k) = G_{true}(e^{j\omega_k T})\, U_N^n(\omega_k) + E_N^n(\omega_k) + D_N^n(\omega_k), \quad \text{for } k=0,\ldots,N-1 \qquad (4.2)$$

where from Theorem 2.2 we know that for some M,

$$|E_N^n(\omega_k)| \leq \bar{E}_N^n(\omega_k), \text{ for } k=0, \ldots, N-1 \tag{4.3}$$

with

$$\bar{E}_N^n(\omega_k) = \sum_{i=1}^{M-1} |g_{true}[i]|\, |U_N^{n-i}(\omega_k) - U_N^n(\omega_k)| + 2\, u_{max} \sum_{i=M}^{\infty} i\, |g_{true}[i]|, \text{ for } k=0, \ldots, N-1, \tag{4.4}$$

where we know u_{max} from assumption A3. Assume, for example, that

$$|g_{true}[n]| \leq g_1\, p_1{}^n, \text{ for } n=0, 1, \ldots \tag{4.5}$$

where from assumption A1.6, $0 < p_1 < 1$. In this case, we can find a closed-form expression for the infinite summation term (see Appendix B). So, using equation (4.4) we find

$$\bar{E}_N^n(\omega_k) \leq \sum_{i=1}^{M-1} g_1\, p_1{}^i\, |U_N^{n-i}(\omega_k) - U_N^n(\omega_k)| + \\ 2\, u_{max}\, g_1\, p_1{}^M (M - M\,p_1 + p_1) / (1 - p_1)^2, \text{ for } k=0, \ldots, N-1. \tag{4.6}$$

The bounding function of equation (4.6) can be computed on-line by using the current N-point DFT of u[n] along with M-1 old N-point DFTs of u[n]. We note that the second line of the previous equation can be made arbitrarily small by choosing M to be sufficiently large.

Now, we define the frequency-domain estimate $G_{f,N}^n(\omega_k)$ and the corresponding frequency-domain error $E_{f,N}^n(\omega_k)$.

$$G_{f,N}^n(\omega_k) = Y_N^n(\omega_k) / U_N^n(\omega_k) \tag{4.7}$$

$$E_{f,N}^n(\omega_k) = G_{f,N}^n(\omega_k) - G_{true}(e^{j\omega_k T}), \text{ for } k=0, \ldots, N-1. \tag{4.8}$$

From equation (4.2),

$$E_{f,N}^n(\omega_k) = (E_N^n(\omega_k) + D_N^n(\omega_k)) / U_N^n(\omega_k) \tag{4.9}$$

and using the triangle inequality we find,

$$|E_{f,N}^n(\omega_k)| \leq \bar{E}_{f,N}^n(\omega_k), \text{ for } k=0, \ldots, N-1 \tag{4.10}$$

where

$$\bar{E}^n_{f,N}(\omega_k) = (\bar{E}^n_N(\omega_k) + \bar{\bar{D}}_N(\omega_k)) / |U^n_N(\omega_k)|, \tag{4.11}$$

and where $\bar{E}^n_N(\omega_k)$ is given by equation (4.4). We will refer to $G^n_{f,N}(\omega_k)$ as our frequency-domain estimate of the true plant at time index n. Note that $G^n_{f,N}(\omega_k)$ is the set of N complex numbers computed using the N-point DFTs of u[n] and y[n], which are computed on-line. (This estimate is equivalent to Ljung's *empirical transfer function estimate* (Ljung, 1987) evaluated at discrete frequencies ω_k, k=0,..,N-1.) We will refer to $\bar{E}^n_{f,N}(\omega_k)$ as the frequency-domain error bounding function at time index n. In equation (4.11), the bounding functions $\bar{E}^n_N(\omega_k)$ and $|U^n_N(\omega_k)|$ are computed on-line at each time index n, while the function $\bar{\bar{D}}_N(\omega_k)$ is known from assumption A2.

4.2 Cumulative Frequency-domain Estimate and Error Bounding Function. In this subsection, we will discuss a straight-forward technique for combining the frequency-domain estimates and the corresponding error bounding functions from different time intervals. That is, we show how to combine all of the past frequency-domain information into a cumulative estimate and cumulative error bounding function. The basic idea is that at a given frequency point ω_k we use the value of $G^n_{f,N}(\omega_k)$ that has the smallest corresponding error bounding function $\bar{E}^n_{f,N}(\omega_k)$, at that frequency. To formalize this we define the cumulative frequency-domain error bounding function at ω_k,

$$\bar{E}^n_{cumf,N}(\omega_k) = \min_{p \le n} \{ \bar{E}^p_{f,N}(\omega_k) \} \tag{4.12}$$

and the cumulative frequency-domain estimate at ω_k,

$$G^{n}_{cumf,N}(\omega_k) = \{G^{m}_{f,N}(\omega_k) \mid \bar{E}^{m}_{f,N}(\omega_k) = \bar{E}^{n}_{cumf,N}(\omega_k)\}. \quad (4.13)$$

Further, we define,

$$E^{n}_{cumf,N}(\omega_k) = G^{n}_{cumf,N}(\omega_k) - G_{true}(e^{j\omega_k T}), \text{ for } k=0,\ldots, N-1. \quad (4.14)$$

Then, equation (4.10) ensures that at time index n,

$$|E^{n}_{cumf,N}(\omega_k)| \leq \bar{E}^{n}_{cumf,N}(\omega_k), \text{ for } k=0,., N-1. \quad (4.15)$$

In practice, the following simple recursive algorithm will be used to compute $G^{n}_{cumf,N}(\omega_k)$ and $\bar{E}^{n}_{cumf,N}(\omega_k)$ at a given frequency ω_k.

Algorithm:

If $\bar{E}^{n}_{f,N}(\omega_k) < \bar{E}^{n-1}_{cumf,N}(\omega_k)$ then set

$$\bar{E}^{n}_{cumf,N}(\omega_k) = \bar{E}^{n}_{f,N}(\omega_k), \text{ and}$$

$$G^{n}_{cumf,N}(\omega_k) = G^{n}_{f,N}(\omega_k), \quad (4.16)$$

else set

$$\bar{E}^{n}_{cumf,N}(\omega_k) = \bar{E}^{n-1}_{cumf,N}(\omega_k), \text{ and}$$

$$G^{n}_{cumf,N}(\omega_k) = G^{n-1}_{cumf,N}(\omega_k).$$

Thus, our algorithm only updates the cumulative frequency-domain estimate and the corresponding cumulative error bounding function when useful information is learned, at a given frequency.

As a final note, we observe that since we are working with real-valued time-domain signals, the properties of the DFTs of real-valued signals can be used to show that,

$$G^{n}_{cumf,N}(\omega_k) = G^{n*}_{cumf,N}(\omega_{N-k}), \quad \text{for } k=1, \ldots, (N/2)-1, \tag{4.17}$$

$$\bar{E}^{n}_{cumf,N}(\omega_k) = \bar{E}^{n}_{cumf,N}(\omega_{N-k}), \quad \text{for } k=1, \ldots, (N/2)-1, \tag{4.18}$$

where '*' denotes complex conjugate and where we have assumed that N is even. This means that the information for the frequency points k=0,..,N-1 is contained in the information for the frequency points k=0,..,N/2.

5. A FREQUENCY-DOMAIN PARAMETER ESTIMATOR

In this section, we will show how the cumulative frequency-domain estimate of the previous section can be used to find parameter estimates for the nominal model. We use the structure of the nominal model, which was assumed in A1.1, and a type of weighted least-squares fit to the frequency-domain estimate $G^{n}_{cumf,N}(\omega_k)$. The procedure is best illustrated by an example. Consider the nominal model,

$$G(z,\theta_0) = b_0 / (z - a_1), \text{ where} \tag{5.1}$$

$$\theta_0 = [\, a_1 \;\; b_0 \,]^T. \tag{5.2}$$

Using this nominal model structure we can write

$$(z - a_1)\, G(z,\theta_0) = b_0, \tag{5.3}$$

or

$$z\, G(z,\theta_0) = [\, G(z,\theta_0) \;\; 1 \,]\, \theta_0. \tag{5.4}$$

Since the parameters are assumed to be real-valued, we find

$$\mathrm{Re}\{z\, G(z,\theta_0)\} = [\, \mathrm{Re}\{G(z,\theta_0)\} \;\; 1 \,]\, \theta_0, \tag{5.5}$$

$$\mathrm{Im}\{z\, G(z,\theta_0)\} = [\, \mathrm{Im}\{G(z,\theta_0)\} \;\; 0 \,]\, \theta_0. \tag{5.6}$$

Thus, if we know the complex value of $G(z,\theta_0)$ for some known z, we can find two linear equations in the parameters. Our frequency-domain estimation method yields an estimate of the plant at frequencies ω_k for k=0,..,N/2. So, letting $z=e^{j\omega_k T}$ for k=0,..,N/2 we define the following (N+2)×2 matrix (in the general case, the matrix is (N+2)×m where m is the dimension of the parameter vector θ_0),

$$A(\,G(e^{j\omega_k T},\theta_0)\,) = \begin{bmatrix} \mathrm{Re}\{G(e^{j\omega_0 T},\theta_0)\} & 1 \\ \vdots & \\ \mathrm{Re}\{G(e^{j\omega_{(N/2)} T},\theta_0)\} & 1 \\ \mathrm{Im}\{G(e^{j\omega_0 T},\theta_0)\} & 0 \\ \vdots & \\ \mathrm{Im}\{G(e^{j\omega_{(N/2)} T},\theta_0)\} & 0 \end{bmatrix} \tag{5.7}$$

and the (N+2) vector,

$$B(\,G(e^{j\omega_k T},\theta_0)\,) = \begin{bmatrix} \mathrm{Re}\{e^{j\omega_0 T}\,G(e^{j\omega_0 T},\theta_0)\} \\ \vdots \\ \mathrm{Re}\{e^{j\omega_{(N/2)} T}\,G(e^{j\omega_{(N/2)} T},\theta_0)\} \\ \mathrm{Im}\{e^{j\omega_0 T}\,G(e^{j\omega_0 T},\theta_0)\} \\ \vdots \\ \mathrm{Im}\{e^{j\omega_{(N/2)} T}\,G(e^{j\omega_{(N/2)} T},\theta_0)\} \end{bmatrix} \tag{5.8}$$

Using equations (5.2) and (5.7-8) we can write,

$$A(\,G(e^{j\omega_k T},\theta_0)\,)\,\theta_0 = B(\,G(e^{j\omega_k T},\theta_0)\,). \tag{5.9}$$

In summary, we have shown how knowledge of the complex values of $G(e^{j\omega_k T},\theta_0)$ at the (N/2)+1 frequencies $\omega_0,..,\omega_{(N/2)}$ can be used to write N+2 linear equations in the parameters. In the ideal situation where one could exactly find $G(e^{j\omega_k T},\theta_0)$ for k=0,..,(N/2), the matrix equation (5.9) will have a solution. That is, given the matrices A and B, we could solve for the true parameter vector using any *m* of the linear equations, where again *m* is the dimension of the parameter vector θ_0. However, in practice we will only have our cumulative frequency-domain estimate $\overset{n}{G}_{cumf,N}(\omega_k)$ with which to estimate the parameters.

If we use $\overset{n}{G}_{cumf,N}(\omega_k)$ instead of $G(e^{j\omega_k T},\theta_0)$ in equations (5.7-8), then the equation,

$$A(\overset{n}{G}_{cumf,N}(\omega_k))\,\hat{\theta} = B(\overset{n}{G}_{cumf,N}(\omega_k)) \tag{5.10}$$

will not, in general, have a solution. Equation (5.10) is in the form of the standard least-squares problem, which is discussed in Strang (1980).

We will choose the parameter estimate $\hat{\theta}$ as the vector that minimizes the frequency weighted norm of the error vector,

$$A(\overset{n}{G}_{cumf,N}(\omega_k))\,\hat{\theta} - B(\overset{n}{G}_{cumf,N}(\omega_k)). \tag{5.11}$$

We define, with reference to equations (5.7-8), the diagonal frequency weighting matrix,

$$W = \mathrm{diag}[\, f(\omega_0)\,..,\, f(\omega_{(N/2)})\;\; f(\omega_0)\,..,\, f(\omega_{(N/2)})\,]. \tag{5.12}$$

where f(•) is the frequency weighting function. The parameter estimate that minimizes the Euclidean norm of the error vector

$$W\,(A(\overset{n}{G}_{cumf,N}(\omega_k))\,\hat{\theta} - B(\overset{n}{G}_{cumf,N}(\omega_k))) \tag{5.13}$$

is given by the well-known result,

$$\hat{\theta} = (A^T W^T W A)^{-1} A^T W^T W B \tag{5.14}$$

where the A and B matrices in this equation depend on the values of the estimate $\overset{n}{G}_{cumf,N}(\omega_k)$.

To gain insight as to what weighting function to choose, we examine equations (5.3-4). Consider the use of the above methodology using the estimate $\hat{G}(z)$. Then, we find that the error

$$z\,\hat{G}(z) - [\,\hat{G}(z)\;\; 1\,]\,\theta_0 = (z - a_1)\,\hat{G}(z) - b_0 \tag{5.15}$$

$$= (z - a_1)\,(\hat{G}(z) - G(z,\theta_0)) \tag{5.16}$$

So, $$|\hat{G}(z) - G(z,\theta_0)| = |\, z\,\hat{G}(z) - [\,\hat{G}(z)\;\; 1\,]\,\theta_0\,| \,/\, |z - a_1|. \tag{5.17}$$

From equation (5.17) we see that if we want our parameter estimation method to be a least-squares fit in the frequency-domain, then we want to choose a weighting function that is one over the magnitude of the denominator of the nominal model. Of course, we do not know what the parameter a_1 really is, so one can only approximately choose this frequency weighting function.

6. COMPUTING A FREQUENCY-DOMAIN UNCERTAINTY BOUNDING FUNCTION

In this section, we discuss the computation of a frequency-domain uncertainty bounding function for the nominal model $G(e^{j\omega_k T},\hat{\theta})$. Specifically, we will compute a magnitude bounding function, $\Delta^n_{su}(e^{j\omega_k T},\hat{\theta})$, on $\delta_{su}(e^{j\omega_k T},\hat{\theta})$ at the frequency points ω_k for $k=0,..,N-1$.

6.1 Basic Methodology. The nominal model at time index n is obtained by using the nominal model structure and the current parameter vector estimate $\hat{\theta}$ yielded by the parameter estimator described in Section 5. Thus, we can compute the value of the nominal model $G(e^{j\omega_k T},\hat{\theta})$ for $k=0,..,N-1$. Now, using the triangle inequality, we find that at time index n, and for frequency ω_k,

$$|G(e^{j\omega_k T},\hat{\theta}) - G_{true}(e^{j\omega_k T})| \le |G(e^{j\omega_k T},\hat{\theta}) - G^n_{cumf,N}(\omega_k)| + |G^n_{cumf,N}(\omega_k) - G_{true}(e^{j\omega_k T})|. \tag{6.1}$$

and using equations (4.14-15),

$$|G(e^{j\omega_k T},\hat{\theta}) - G_{true}(e^{j\omega_k T})| \le |G(e^{j\omega_k T},\hat{\theta}) - G^n_{cumf,N}(\omega_k)| + \bar{E}^n_{cumf,N}(\omega_k). \tag{6.2}$$

We now can find a bound on $\delta_{su}(e^{j\omega_k T},\hat{\theta})$. Rewriting equation (3.14),

$$G_{true}(e^{j\omega_k T}) = G(e^{j\omega_k T},\hat{\theta})\,[1 + \delta_{su}(e^{j\omega_k T},\hat{\theta})], \quad \text{for } k=0,\ldots,N\text{-}1. \tag{6.3}$$

So, rearranging yields,

$$\delta_{su}(e^{j\omega_k T},\hat{\theta}) = [\,G_{true}(e^{j\omega_k T}) - G(e^{j\omega_k T},\hat{\theta})\,] / G(e^{j\omega_k T},\hat{\theta}). \tag{6.4}$$

Thus, using equation (6.2), we find the bounding function,

$$|\delta_{su}(e^{j\omega_k T},\hat{\theta})| \le \Delta^n_{su}(e^{j\omega_k T},\hat{\theta}), \tag{6.5}$$

where

$$\Delta^n_{su}(e^{j\omega_k T},\hat{\theta}) = \{\, |G(e^{j\omega_k T},\hat{\theta}) - G^n_{cumf,N}(\omega_k)| + \bar{E}^n_{cumf,N}(\omega_k)\,\} / |G(e^{j\omega_k T},\hat{\theta})|,$$

$$\text{for } k=0,\ldots,N\text{-}1. \tag{6.6}$$

and where we have included a superscript 'n' after the Δ_{su} to denote the fact that this bound on $|\delta_{su}(e^{j\omega_k T},\hat{\theta})|$ depends on the time index n, since $G^n_{cumf,N}(\omega_k)$, $\bar{E}^n_{cumf,N}(\omega_k)$ and also $\hat{\theta}$ depend on n.

In summary, we have shown how to compute a discrete function $\Delta^n_{su}(e^{j\omega_k T},\hat{\theta})$ that bounds the net effect of structured and unstructured uncertainty of the current nominal model $G(e^{j\omega_k T},\hat{\theta})$ relative to the true plant, at the frequencies, $\omega_0,..,\omega_{N-1}$. We used the nominal model structure of A1.1, the current parameter estimate $\hat{\theta}$; and the cumulative frequency-domain estimate $G^n_{cumf,N}(\omega_k)$ and corresponding cumulative frequency-domain error bounding function $\bar{E}^n_{cumf,N}(\omega_k)$, which were developed in Section 5.

6.2 A Smoothed Uncertainty Bounding Function. In this subsection, we discuss the computation of a smoothed, magnitude bounding function on $|\delta_{su}|$. This development is motivated by the observation that, depending upon the spectrum of the input signal, one may have a very jagged bounding function on the modeling uncertainty $|\delta_{su}(e^{j\omega_k T},\hat{\theta})|$. That is, at the frequency point ω_k the bound $\Delta^n_{su}(e^{j\omega_k T},\hat{\theta})$ may be very tight, however, at an adjacent frequency point ω_{k+1} the bound $\Delta^n_{su}(e^{j\omega_{k+1}T},\hat{\theta})$ may be very poor. In LaMaire (1987a), it is shown how the assumptions of Section 3 can be used to find a derivative bounding function $\nabla_u(e^{j\omega T})$ satisfying

$$|d\delta_{su}(e^{j\omega T},\hat{\theta}) / d\omega| \leq \nabla^n_{su}(e^{j\omega T}), \ \forall\omega. \tag{6.7}$$

If δ_{su} is analytic, then it is shown in LaMaire (1987a) that

$$|\delta_{su}(e^{j\omega T},\hat{\theta})| \leq |\delta_{su}(e^{j\omega_k T},\hat{\theta})| + (\omega-\omega_k)\, \nabla^n_{su,i}(\omega_k,\omega_{k+1}) \tag{6.8}$$

and

$$|\delta_{su}(e^{j\omega T},\hat{\theta})| \leq |\delta_{su}(e^{j\omega_{k+1} T},\hat{\theta})| + (\omega_{k+1}-\omega)\, \nabla^n_{su,i}(\omega_k,\omega_{k+1}) \tag{6.9}$$

for $\omega \in [\omega_k,\omega_{k+1}]$ where

$$\nabla^n_{su,i}(\omega_k,\omega_{k+1}) = \sup_{\omega\in[\omega_k,\omega_{k+1}]} \{\nabla^n_{su}(e^{j\omega T})\} \tag{6.10}$$

From these equations we see that it may be possible to obtain a tighter bound on $|\delta_{su}(e^{j\omega_k T},\hat{\theta})|$ than $\Delta^n_{su}(e^{j\omega_k T},\hat{\theta})$, by using the bound at an adjacent frequency point, $\Delta^n_{su}(e^{j\omega_{k-1}T},\hat{\theta})$ or $\Delta^n_{su}(e^{j\omega_{k+1}T},\hat{\theta})$, along with the smoothness information of $\nabla_{su,i}$.

6.3 Bounding Inter-sample Variations. In this brief subsection, we discuss the computation of a safety factor that must be added to the discrete bounding function $\Delta^n_{su}(e^{j\omega_k T},\hat{\theta})$ to account for inter-sample variations. Ultimately, the uncertainty bounding function at discrete frequency points will be used in stability-robustness tests to design a new robust compensator. These stability-robustness tests are meant to be used with continuous functions of frequency. Since the actual computations will be performed with an uncertainty bounding function that is a discrete function of frequency, we must add the aforementioned safety factor to the discrete function to account for the worst possible peaks that may occur between frequency samples ω_k. In LaMaire (1987a), it is shown how equations (6.8-9) can be used to choose this additive safety factor in such a way that the largest inter-sample variations lie below a line drawn between the values of the final uncertainty bounding function (including the safety factor) at two adjacent frequency samples.

7. TIME-DOMAIN PARAMETER ESTIMATION: AN ALTERNATIVE

In this section, we briefly describe an alternative method to that of Section 5 for generating the parameter estimate $\hat{\theta}$ defining the nominal model $G(z,\hat{\theta})$. This method did not perform well in our simulations, thus motivating the development of the frequency-domain parameter estimator of Section 5.

7.1 Motivation. Most current time-domain parameter estimation techniques provide unreliable estimates in the presence of unmodeled dynamics and an unmeasurable disturbance. For example, assume that a large persistently-exciting sufficiently-rich signal was present for a long time resulting in accurate parameter estimates. Then, assume that the input signal suddenly became zero but the disturbance continued to excite the system. In this case, the previously good parameter estimates could become very inaccurate. As another example, consider what happens when the plant input signal excites the high-frequency unmodeled dynamics, that is, the dynamics we constrain with the unstructured uncertainty bound. In this case, the plant output signal is greatly affected by the high-frequency unmodeled dynamics so the parameter estimates of the low-frequency nominal model may become degraded. To prevent these types of behavior, an algorithm was sought to adjust the parameter estimates selectively depending upon the usefulness of the input/output data. That is, the desired algorithm would adjust the parameter estimates when useful information is contained in the input/output data but would stop updating the estimates when no useful information is available.

In this section, we outline an algorithm that can be used with confidence in the presence of unmodeled dynamics and an unmeasurable disturbance. The resulting time-domain parameter estimator is actually a combination of a bounding mechanism that is developed in

Appendix A and a modified least-squares algorithm that was developed by Goodwin *et al.* (1985b, 1986). This modified least-squares algorithm is made robust through the use of a time-varying dead-zone. The new contribution of this paper is the development of the time-domain bounding mechanism that uses the assumptions of the robust estimator (e.g., the assumption of a frequency-domain bound on the unstructured uncertainty). Goodwin *et al.* (1985b, 1986) use a different bounding mechanism which requires somewhat different types of assumptions than those of the robust estimator.

7.2 Development of Equation Error Form. We again consider the plant of Fig. 1 where

$$y[n] = g_{true}[n]*u[n] + d[n]. \tag{7.1}$$

We can use the forward shift operator 'q' in the polynomials of assumption A1 of Section 3, to write

$$y[n] = [G_{true}(q)]\, u[n] + d[n] \tag{7.2}$$

$$= [\, (B(q) / A(q))\, [1 + \delta_u(q)]\,]\, u[n] + d[n]. \tag{7.3}$$

So,

$$y[n] = [B(q) / A(q)]\, u[n] + [B(q)\, \delta_u(q) / A(q)]\, u[n] + d[n]. \tag{7.4}$$

Multiplying both sides by the operator [A(q)] yields

$$[A(q)]\, y[n] = [B(q)]\, u[n] + [B(q)\, \delta_u(q)]\, u[n] + [A(q)]\, d[n]. \tag{7.5}$$

Rewriting yields,

$$y[n] = [1-A(q)]\, y[n] + [B(q)]\, u[n] + [B(q)\, \delta_u(q)]\, u[n] + [A(q)]\, d[n]. \tag{7.6}$$

With reference to assumption A1.1 of Section 3, we define the signal regression vector,

$$\phi[n-1] = [\; y[n-1] \quad y[n-2] \;\ldots\; y[n-n_1] \quad u[n-n_1+m_1] \quad u[n-n_1+m_1-1] \;\ldots\; u[n-n_1]\;]^T. \tag{7.7}$$

Now, equation (7.5) can be rewritten as,

$$y[n] = \phi[n-1]^T\theta_0 + e_0[n], \tag{7.8}$$

where

$$e_0[n] = [B(q)\, \delta_u(q)]\, u[n] + [A(q)]\, d[n], \tag{7.9}$$

and where θ_0 is the true parameter vector of the nominal model, as defined in A1. Goodwin *et al.* (1985b) observe that equation (7.8) will, in general, be unsuitable for parameter estimation since the error $e_0[n]$ involves "near differentiation" of the input and the disturbance. As suggested in Goodwin *et al.* (1985b), we will prefilter both the input and the output signals, u[n] and y[n], to avoid this problem. We define the filter in the forward shift operator,

$$F(q) = q^{(n_1)} / W(q) \tag{7.10}$$

where the polynomial W(q) has order n_1 or greater and has all its zeros in the open unit disk. Now, we define the filtered versions of the input and output signals,

$$u_f[n] = [F(q)]\, u[n], \tag{7.11}$$

$$y_f[n] = [F(q)]\, y[n]. \tag{7.12}$$

Multiplying both sides of equation (7.5) by the operator [F(q)] yields

$$[A(q)\, F(q)]\, y[n] = [B(q)\, F(q)]\, u[n] + [B(q)\, F(q)\, \delta_u(q)]\, u[n] + [A(q)\, F(q)]\, d[n] \tag{7.13}$$

or

$$[A(q)]\, y_f[n] = [B(q)]\, u_f[n] + [B(q)\, F(q)\, \delta_u(q)]\, u[n] + [A(q)\, F(q)]\, d[n]. \tag{7.14}$$

Rearranging yields,

$$y_f[n] = [1-A(q)]\, y_f[n] + [B(q)]\, u_f[n] + [B(q)\, F(q)\, \delta_u(q)]\, u[n] + [A(q)\, F(q)]\, d[n]. \tag{7.15}$$

We define the signal regression vector containing the filtered signals,

$$\phi_f[n-1] = [\ y_f[n-1] \quad y_f[n-2] \ \ldots\ y_f[n-n_1] \quad u_f[n-n_1+m_1] \quad u_f[n-n_1+m_1-1] \ \ldots\ u_f[n-n_1]\ ]^T. \tag{7.16}$$

Now, we see that equation (7.15) can be written as,

$$y_f[n] = \phi_f[n-1]^T \theta_0 + e_1[n], \tag{7.17}$$

where

$$e_1[n] = [B(q)\, F(q)\, \delta_u(q)]\, u[n] + [A(q)\, F(q)]\, d[n]. \tag{7.18}$$

7.3 Decomposition of the Error Signal. In this subsection, we will introduce several definitions so that we can decompose the error signal $e_1[n]$ defined in equation (7.18). First, we define the transfer functions

$$H_u(z) = B(z)\, F(z)\, \delta_u(z), \tag{7.19}$$

$$H_d(z) = A(z)\, F(z). \tag{7.20}$$

$H_u(z)$ is the transfer function from the plant input to the equation error $e_1[n]$. This transfer function describes the effect of the additive plant error, which is due to the unmodeled dynamics, on the equation error. $H_d(z)$ is the transfer function from the disturbance to the equation error $e_1[n]$. We can rewrite equation (7.18) as

$$e_1[n] = h_u[n]*u[n] + h_d[n]*d[n]. \tag{7.21}$$

where the impulse responses of $H_u(z)$ and $H_d(z)$ are denoted by $h_u[n]$ and $h_d[n]$, respectively. We decompose equation (7.21) by defining

$$e_1[n] = e_u[n] + e_d[n], \tag{7.22}$$

where

$$e_u[n] = h_u[n]*u[n] \tag{7.23}$$

$$e_d[n] = h_d[n]*d[n] \tag{7.24}$$

To bound $e_1[n]$ at each time index n, we find a magnitude bound on $e_u[n]$ and $e_d[n]$ individually. That is,

$$|e_1[n]| \leq |e_u[n]| + |e_d[n]|. \tag{7.25}$$

7.4 Outline of the Time-domain Error Bounding Technique. The assumptions of Section 3 (i.e. A1) can be used to find magnitude bounds on the frequency-responses $H_u(z)$ and $H_d(z)$ as well as magnitude bounds on the impulse responses $h_u[n]$ and $h_d[n]$. The results of Theorem A.2 of Appendix A can then be used along with the on-line computed DFT of the input signal u[n] to compute a time-varying magnitude bound on the error $e_u[n]$ due to the unmodeled dynamics. Similarly, assumption A3, which provides a magnitude bound on the DFT of the disturbance d[n], can be used to find a constant magnitude bound on the error $e_d[n]$ due to the disturbance. Given a time-varying magnitude bound on the total equation error $e_1[n]$ in equation (7.17), the robust modified least-squares algorithm of Goodwin *et al.* (1985b, 1986) can be applied. The details of this procedure are described in LaMaire (1987a).

7.5 Summary . In this section we have outlined a mechanism to bound, in the time-domain, the effects of both unmodeled dynamics and an unmeasurable disturbance. This bounding mechanism can be used together with a time-varying dead-zone (see Goodwin *et al.*, 1985b, 1986) to make a least-squares parameter estimator robust. Unfortunately, our simulations revealed that the robust time-domain parameter estimator described in this section did not perform well due to the conservatism of the magnitude bound on the equation error $e_1[n]$. The parameter estimator was "turned-off" by the time-varying dead-zone in many situations where a standard least-squares algorithm was able to continue yielding accurate parameter estimates. Consequently, we chose to use the frequency-domain method of Section 5 to find parameter estimates for the nominal model of the robust estimator.

8. APPLICATION OF THE ROBUST ESTIMATOR TO ADAPTIVE CONTROL

In this section, we will describe how the robust estimator can be used in a closed-loop adaptive control system. In Fig. 1, we showed how the robust estimator provides both a nominal plant model and a corresponding frequency-domain uncertainty bounding function to a robust control algorithm that infrequently redesigns and updates the compensator. Thus, when the plant input signal is rich enough, the robust estimator can yield a nominal model with a frequency-domain uncertainty bounding function that is smaller than the a priori uncertainty bounding function. However, when the plant input signal is not rich, the robust estimator doesn't improve its estimates and consequently, the control-law is not updated. If we want to enhance identification, that is, enable our robust estimator to reduce the frequency-domain uncertainty, we can add a probing signal in the closed-loop (see Fig. 2). This probing signal will degrade the the command-following performance of the closed-loop system; however, the increased knowledge of the plant will result in better command-following in a later period, provided that the plant remains time-invariant. This trade-off between identification goals and closed-loop performance goals has been studied in the stochastic adaptive control literature as the *dual control* problem (Bar-Shalom and Tse, 1974).

In order to evaluate the performance of the robust estimator in a closed-loop adaptive control system, a complete albeit simple adaptive controller was implemented. This adaptive control system made use of a plant inverting compensator. For the limited case of a nominal plant model with a relative degree of one or less, a robust control-law algorithm was developed (LaMaire, 1987a) that permitted the on-line redesign of a compensator based on the nominal model $G(z,\hat{\theta})$ and the frequency-domain uncertainty bounding function Δ_{su}^{n} yielded by the robust estimator. Further, a probing signal algorithm was developed (LaMaire, 1987a) that made an on-line determination of the frequency and amplitude of sinusoids that should be introduced (see probing signal v[n] in Fig. 2) to enhance the identification of the plant and thus result in an increased closed-loop bandwidth. Note that since the robust estimator produces a frequency-domain bound on the modeling uncertainty, this information can be used to synthesize a probing signal that excites the frequencies of the plant that are least well known. Additional factors that are used in the computation of the probing signal include knowledge of the target closed-loop bandwidth of the system, which determines how well the plant must be identified, and knowledge of the current compensator in the loop. As is shown in Fig. 2, the compensator affects the transfer function from v[n] to u[n] and thus determines how well the probing signal v[n] is rejected by the closed-loop.

Several simulations were performed in which a second-order nominal plant was corrupted with high-frequency, second-order unmodeled dynamics and an additive output disturbance. Different sets of *true* nominal plant parameters were used to fully understand the performance of the robust estimator. A low-frequency pseudo-random disturbance signal was used that had most of its energy below the bandwidth of the true open-loop plant. The detailed results of these simulations are presented in LaMaire (1987a). In this paper we only summarize our conclusions concerning these many simulations.

The primary conclusion drawn from our simulations was that a robust adaptive control system that uses the robust estimator can increase the closed-loop bandwidth and hence, improve the performance of a system, under the right excitation conditions. Since a range of cases were considered, several different types of behavior were observed. In some situations, the reference signal (a rich chirp-like signal) provided sufficient excitation for the robust estimator to identify the plant well, resulting in the achievement of the target closed-loop bandwidth. However, in other hard identification cases (i.e. in cases where the initial compensator was chosen such that frequencies in the range of the target closed-loop bandwidth were greatly attenuated resulting in little excitation of the plant at these frequencies), the reference signal itself had to be supplemented by the probing signal v[n] in order for the robust estimator to be able to identify the plant well enough to increase the closed-loop bandwidth. Thus, in a closed-loop context, it was our experience that given excitation at the proper frequencies, the robust estimator was able to yield an improved nominal plant model and uncertainty bound so that the robust control-law redesign algorithm could increase the bandwidth of the closed-loop system.

9. CONCLUDING REMARKS

In this paper, we presented a new estimation (identification) methodology that can be used in a robust adaptive controller to provide stability-robustness guarantees. The key feature of the robust estimator is the frequency-domain bounding function on the modeling uncertainty. In LaMaire (1987a), our simulation results revealed that the use of the robust estimator can yield improved closed-loop performance, that is, increased bandwidth as compared with the best LTI compensator that could have been designed using only a priori knowledge of the plant. In some situations, the plant input signal is not rich enough to allow identification. In these situations, one can chose to either use the best a priori control-law or introduce an external probing signal to enhance identification. As a final remark, we note that while the robust estimator provides guarantees that no other methodology can, the price of these guarantees is the large computational load of the frequency-domain calculations described in Section 4.

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APPENDIX A: DERIVATION OF SIGNAL PROCESSING THEOREMS

In this appendix, we provide proofs for Theorems 2.1 and 2.2. Further, we state and prove two additional theorems, the last of which can be used to compute a time-domain bound on the output of a linear system by using the DFT of the input of the system. This result is referred to in Section 7 of the paper.

Proof of Theorem 2.1. We know that

$$Y(e^{j\omega_k T}) = H(e^{j\omega_k T})\, U(e^{j\omega_k T}), \qquad \text{for } k = 0, \ldots, N-1, \tag{A.1}$$

where $U(e^{j\omega_k T})$ and $Y(e^{j\omega_k T})$ are the DTFTs of u[n] and y[n], respectively. Since

$$Y(e^{j\omega_k T}) = \sum_{m=-\infty}^{n-N} y[m]\, W_N^{km} + Y_N^n(\omega_k) + \sum_{m=n+1}^{\infty} y[m]\, W_N^{km}, \quad \text{for } k=0, \ldots, N-1, \tag{A.2}$$

and a similar expression holds for $U(e^{j\omega_k T})$, we can write

$$Y_N^n(\omega_k) = H(e^{j\omega_k T}) \left\{ \sum_{m=-\infty}^{n-N} u[m]\, W_N^{km} + U_N^n(\omega_k) + \sum_{m=n+1}^{\infty} u[m]\, W_N^{km} \right\}$$
$$- \left\{ \sum_{m=-\infty}^{n-N} y[m]\, W_N^{km} + \sum_{m=n+1}^{\infty} y[m]\, W_N^{km} \right\}, \quad \text{for } k=0, \ldots, N-1. \tag{A.3}$$

It can be shown that

$$\sum_{m=-\infty}^{n-N} y[m]\, W_N^{km} = h[0]\, \{ \sum_{m=-\infty}^{n-N} u[m]\, W_N^{km} \}$$

$$+ \sum_{p=1}^{\infty} h[p]\, W_N^{kp}\, \{ \sum_{m=-\infty}^{n-N} u[m]\, W_N^{km} - \sum_{m=n-N-p+1}^{n-N} u[m]\, W_N^{km} \}, \text{ for } k=0,\ldots, N-1. \tag{A.4}$$

So,

$$H(e^{j\omega_k T}) \sum_{m=-\infty}^{n-N} u[m]\, W_N^{km} - \sum_{m=-\infty}^{n-N} y[m]\, W_N^{km} =$$

$$+ \sum_{p=1}^{\infty} h[p]\, W_N^{kp}\, \{ \sum_{m=n-N-p+1}^{n-N} u[m]\, W_N^{km} \}, \text{ for } k=0,\ldots, N-1. \tag{A.5}$$

Similarly,

$$H(e^{j\omega_k T}) \sum_{m=n+1}^{\infty} u[m]\, W_N^{km} - \sum_{m=n+1}^{\infty} y[m]\, W_N^{km} =$$

$$- \sum_{p=1}^{\infty} h[p]\, W_N^{kp}\, \{ \sum_{m=n-p+1}^{n} u[m]\, W_N^{km} \}, \text{ for } k=0,\ldots, N-1. \tag{A.6}$$

Using equations (2.9), (A.3) and (A.5-6) we find that

$$E_N^n(\omega_k) = \sum_{p=1}^{\infty} h[p]\, W_N^{kp}\, \{ \sum_{m=n-N-p+1}^{n-N} u[m]\, W_N^{km} - \sum_{m=n-p+1}^{n} u[m]\, W_N^{km} \}, \text{ for } k=0,\ldots, N-1. \tag{A.7}$$

Equation (2.10) now follows using the definition of equation (2.5).

Q. E. D.

Proof of Theorem 2.2. Using the triangle inequality and equations (2.10) and (A.7) we find,

$$|E_N^n(\omega_k)| \le \sum_{p=1}^{M-1} |h[p]|\,|U_N^{n-p}(\omega_k) - U_N^n(\omega_k)| +$$

$$+ \sum_{p=M}^{\infty} |h[p]|\,\left| \sum_{m=n-N-p+1}^{n-N} u[m]\, W_N^{km} - \sum_{m=n-p+1}^{n} u[m]\, W_N^{km} \right|, \text{ for } k=0,\ldots, N-1. \tag{A.8}$$

Since,

$$\left| \sum_{m=n-N-p+1}^{n-N} u[m]\, W_N^{km} - \sum_{m=n-p+1}^{n} u[m]\, W_N^{km} \right| \le$$

$$\sum_{m=n-N-p+1}^{n-N} |u[m]| + \sum_{m=n-p+1}^{n} |u[m]| \le 2\, u_{max}\, p \tag{A.9}$$

we conclude that equation (2.11) is true.

Q. E. D.

Corollary 2.1. Under the assumptions of Theorem 2.2,

$$|E_N^n(\omega_k)| \le 2\, u_{max} \sum_{p=1}^{\infty} p\, |h[p]|, \text{ for } k = 0, \ldots, N-1. \tag{A.10}$$

Proof. Choose M=1 in Theorem 2.2. This corollary is closely related to Theorem 2.1 in Ljung (1987).

The following theorems are useful for computing the maximum output signal of a transfer function for which we have a magnitude bounding function in the frequency domain.

Theorem A.1. Let $y[m] = h[m]*u[m]$, where h[m] is an infinite-length, causal, impulse response with all its poles in the open unit disk. We denote the DTFT of h[m] by $H(e^{j\omega_k T})$, and the DFT of the N-points of u[m] ending with time index n, by $U_N^n(\omega_k)$. Then,

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} H(e^{j\omega_k T}) \, U_N^n(\omega_k) \, W_N^{-kn} + e[n], \tag{A.11}$$

where

$$e[n] = \sum_{p=N}^{\infty} h[p] \, (u[n-p] - u[n-(p \text{ modulo } N)] \,). \tag{A.12}$$

Remark A.1. The signal e[n] is the error due to the fact that the impulse response h[n] is of infinite length. We note from equation (A.12) that if $h[p]=0$ for $p \geq N$, then $e[n]=0$, $\forall n$.

Proof. From the definition of equation (2.6) we find that

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y_N^n(\omega_k) \, W_N^{-kn}. \tag{A.13}$$

Using equation (2.9) from Theorem 2.1, we find that

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} H(e^{j\omega_k T}) \, U_N^n(\omega_k) \, W_N^{-kn} + \frac{1}{N} \sum_{k=0}^{N-1} E_N^n(\omega_k) \, W_N^{-kn}. \tag{A.14}$$

Thus, the second term of the above equation is equal to e[n]. This will allow us to use equation (A.7) from the proof of Theorem 2.1 to find e[n]. However, first we will find an alternate form of equation (A.7). We observe that

$$\sum_{m=n-N-p+1}^{n-N} u[m] \, W_N^{km} - \sum_{m=n-p+1}^{n} u[m] \, W_N^{km} = \sum_{m=n-p+1}^{n} (u[m-N] - u[m] \,) \, W_N^{km}$$

$$\text{for } k = 0, \ldots, N-1, \tag{A.15}$$

since $W_N^{-kN} = 1$ for integer k. Then, using equations (A.7) and (A.15) and the inverse DFT of equation (2.6), we can express e[n] as follows.

$$e[n] = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{p=1}^{\infty} h[p] \, W_N^{kp} \sum_{m=n-p+1}^{n} (u[m-N] - u[m] \,) \, W_N^{km} \, W_N^{-kn} \tag{A.16}$$

Rearranging the summations yields

$$e[n] = \sum_{p=1}^{\infty} h[p] \sum_{m=n-p+1}^{n} (u[m-N] - u[m]) \frac{1}{N} \sum_{k=0}^{N-1} W_N^{k(m-n+p)} \tag{A.17}$$

Noting that

$$\frac{1}{N} \sum_{k=0}^{N-1} W_N^{k(m-n+p)} = \begin{cases} 1, & \text{for } m = n - p + i\,N \\ 0, & \text{otherwise} \end{cases} \tag{A.18}$$

where 'i' is an integer, we find

$$\sum_{m=n-p+1}^{n} (u[m-N] - u[m]) \frac{1}{N} \sum_{k=0}^{N-1} W_N^{k(m-n+p)} = \begin{cases} 0, & \text{for } p = 1, \ldots, N-1 \\ u[n-p] - u[n-(p \text{ modulo } N)], & \text{for } p \geq N. \end{cases} \tag{A.19}$$

Equation (A.12) follows from equations (A.17) and (A.19).

Q. E. D.

We want to be able to find a magnitude bounding function on y[n]. The following theorem provides such a bounding function by using the results of Theorem A.1.

Theorem A.2. Under the assumptions of Theorem A.1 we find that, for a real-valued impulse response h[n] and a real-valued signal u[n], the magnitude of y[n] is bounded at each n as follows,

$$|y[n]| \leq \frac{1}{N} \{ |H(e^{j\omega_0 T})|\, |U_N^n(\omega_0)| + 2 \sum_{k=1}^{(N/2)-1} |H(e^{j\omega_k T})|\, |U_N^n(\omega_k)| + |H(e^{j\omega_{(N/2)} T})|\, |U_N^n(\omega_{(N/2)})| \} + 2\, u_{max} \sum_{p=N}^{\infty} |h[p]|, \tag{A.20}$$

where

$$u_{max} = \sup_m |u[m]|, \tag{A.21}$$

and where we have assumed that N is even. An alternate form of the theorem can easily be proven for the case of an odd value of N.

Proof. By applying the triangle inequality to equation (A.11) and noting that $|W_N^{-kn}|=1$ we find,

$$|y[n]| \leq \frac{1}{N}\sum_{k=0}^{N-1} |H(e^{j\omega_k T})|\, |U_N^n(\omega_k)| + |e[n]|. \tag{A.22}$$

From equation (A.12) we obtain a bound on $|e[n]|$,

$$|e[n]| \leq \sum_{p=N}^{\infty} |h[p]|\, |(\, u[n-p] - u[n-(p \text{ modulo } N)]\,)| \leq 2\, u_{max} \sum_{p=N}^{\infty} |h[p]|. \tag{A.23}$$

To complete the proof, we observe that since h[n] and u[n] are real-valued sequences, then

$$|H(e^{j\omega_k T})| = |H(e^{j\omega_{(N-k)} T})|, \tag{A.24}$$

$$|U_N{}^n(\omega_k)| = |U_N{}^n(\omega_{(N-k)})|, \tag{A.25}$$

respectively, for k=1, . . . , (N/2)-1. Equation (A.20) follows from equations (A.22-5).

Q. E. D.

APPENDIX B: CLOSED-FORM EXPRESSIONS FOR SOME INFINITE SUMMATIONS

In this appendix, we summarize several useful results concerning the evaluation of infinite series of the geometric type. These closed-form expressions can be used to compute a bound on the infinite summation term that appears in equation (4.4). While a specific example is used in Subsection 4.1 (see equation (4.5)), the results of this appendix show that the infinite summation term of equation (4.4) can always be bounded under the assumptions of A1.6 in Section 3.

Case 1. We define

$$S_1 = \sum_{i=p}^{q} x^i, \tag{B.1}$$

where p and q are positive integers, $p < q$, and $|x| < 1$ if $q \to \infty$. Under these conditions, the following series are convergent via the ratio test. We find that

$$S_1 x = \sum_{i=p}^{q} x^{i+1} = \sum_{j=p+1}^{q+1} x^j = S_1 - x^p + x^{q+1}. \tag{B.2}$$

So,

$$S_1 = (x^p - x^{q+1}) / (1 - x). \tag{B.3}$$

Case 2.

$$S_2 = \sum_{i=p}^{q} i\, x^i. \tag{B.4}$$

We find that

$$\frac{dS_1}{dx} = \sum_{i=p}^{q} i\, x^{i-1}. \tag{B.5}$$

So,

$$S_2 = x \frac{dS_1}{dx}, \tag{B.6}$$

and it can be shown that

$$S_2 = [x^p (p - p x + x) + x^{q+1} (-q + q x - 1)] / (1 - x)^2. \tag{B.7}$$

Special Case 2a. If $q \to \infty$, then

$$S_2 = x^p (p - p x + x) / (1 - x)^2. \tag{B.8}$$

This is the result that is used in equation (4.6).

General Case. For some integer $n \geq 1$, a closed-form expression for the sum

$$S_n = \sum_{i=p}^{q} i^{n-1} x^i. \tag{B.9}$$

can be found by induction, since

$$S_n = x\frac{dS_{(n-1)}}{dx}, \tag{B.10}$$

and S_1 is given by equation (B.3).

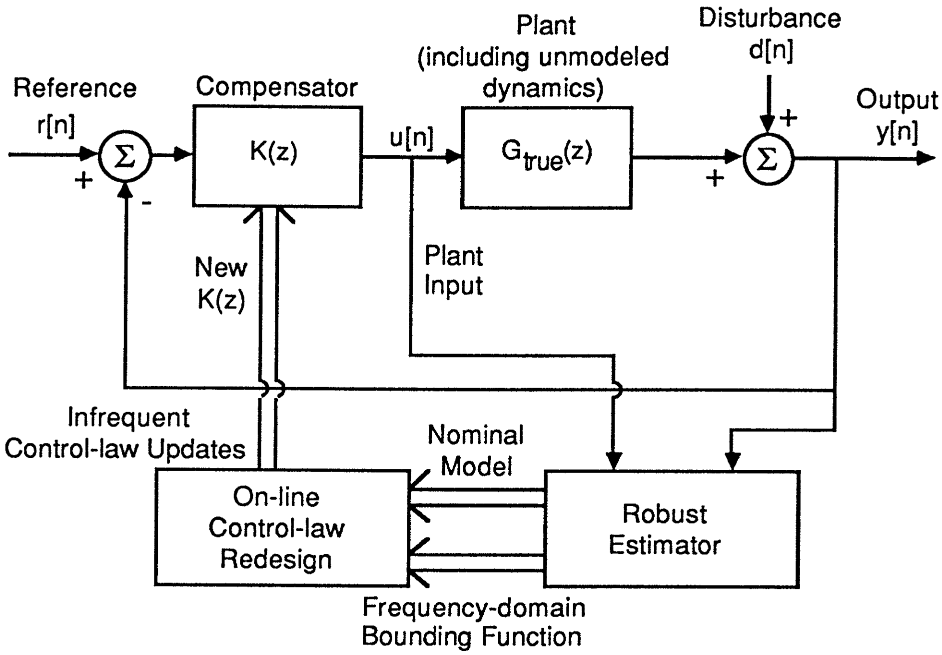

FIG. 1. A Robust Adaptive Control System

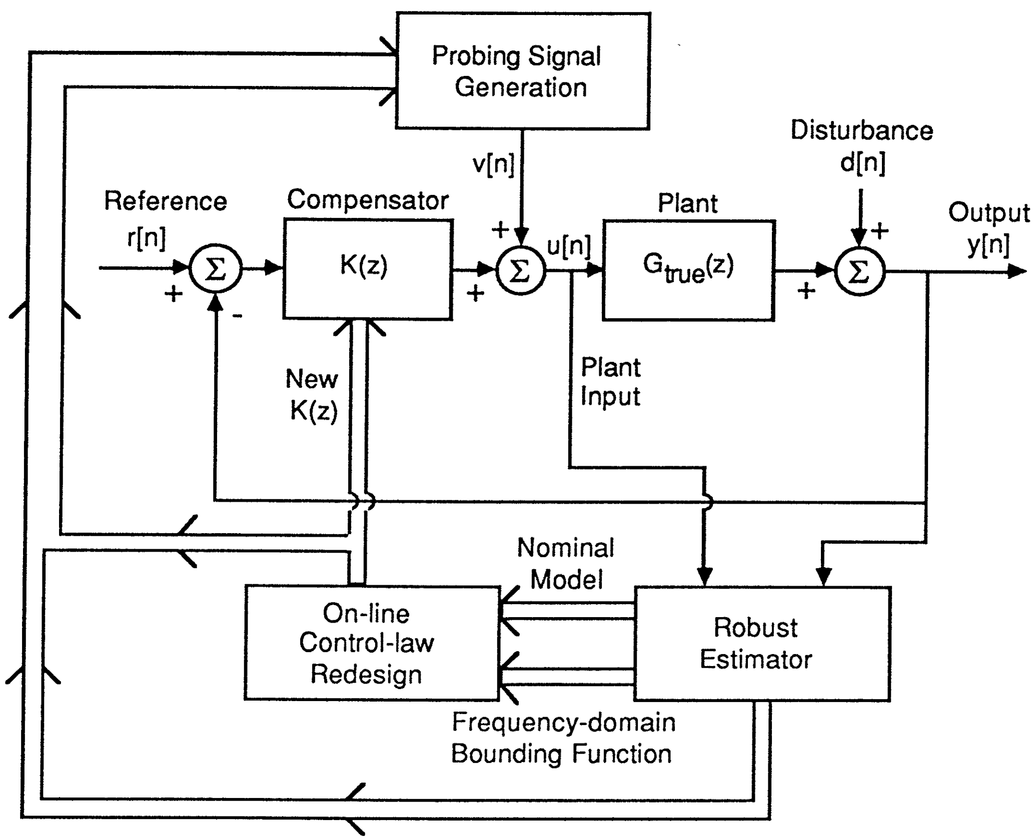

FIG. 2. A Robust Adaptive Control System with Probing Signal