STABILITY AND ROBUSTNESS OF SLOWLY TIME-VARYING LINEAR SYSTEMS†

Jeff S. Shamma and Michael Athans

Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, MA 02139

Abstract

A well known result for finite-dimensional time-varying linear systems is that if each 'frozen time' system is stable, then the time-varying system is stable for sufficiently slow time-variations. These results are reviewed and extended to a class of Volterra integrodifferential equations, specifically, differential equations with a convolution operator in the right-hand-side. The results are interpreted in the context of robustness of time-varying linear systems with a special emphasis on analysis of gain-scheduled control systems.

1. Introduction

Many applications of automatic control systems involve plants whose dynamics depend on time-varying external parameters, e.g. jet engines [18], submarines [20], or aircraft [25]. Controllers for such plants are typically designed such that for all *frozen* values of the parameters, the feedback system has certain necessary properties, such as nominal stability and robustness to unmodeled dynamics. Since the parameters are actually time-varying, it is reasonable to ask under what conditions are these properties maintained. For example, it is possible that parameter time-variations can be destabilizing [1, 27]. However, if the time-variations are sufficiently slow, then stability is maintained [8, 9, 26]. In this paper, it is shown that similar results hold for robustness to unmodeled dynamics. That is, for sufficiently slow time-variations, the feedback system maintains its stability in the presence of possibly infinite-dimensional plant perturbations, such as time-delays, actuator dynamics, or sensor dynamics.

The remainder of this paper is organized as follows. In Section 2, existing results regarding the stability of slowly time-varying finite-dimensional linear systems are reviewed. In Section 3, these results are extended to a class of Volterra integrodifferential equations. The new results are used to study a feedback system consisting of a finite-dimensional time-varying linear system in the forward loop, and a time-invariant infinite-dimensional linear system in the feedback loop. Finally, Section 4 contains a discussion of these results and their implications regarding robustness of time-varying linear systems and analysis of gain-scheduled control systems.

†This research was supported by the NASA Ames and Langley Research Centers under grant NASA/NAG 2-297.

*Proc. IEEE Decision and Control Conference
Los Angeles, CA., December 1987

Notation

R_+ denotes the set $\{t \in R \mid t \geq 0\}$. $|\cdot|$ denotes both the vector norm on R^n and its induced matrix norm. L_p and L_{pe}, $p \in [1,\infty]$, denote the standard Lebesgue and extended Lebesgue function spaces. Similarly, l_p, $p \in [1,\infty]$, denote the appropriately summable sequence spaces. The corresponding domains and ranges will be apparent from context. $A(\gamma)$ denotes the set

$$f(t) = f_a(t) + \sum_{i=1}^{\infty} f_i \,\delta(t - t_i)$$

where $f_a: R_+ \to R$, $t \longmapsto f_a(t)e^{-\gamma t} \in L_1$, $t_i > 0$, $f_i \in R$, and $i \longmapsto f_i e^{-\gamma t_i} \in l_1$. For $f \in A(\gamma)$,

$$\| f \|_{A(\gamma)} = \int_0^{\infty} | f_a(t)\, e^{-\gamma t} |\, dt \; + \; \sum_{i=1}^{\infty} | f_i\, e^{-\gamma t_i} |$$

$A^{m\times p}(\gamma)$ denotes the set of m×p matrices whose elements are in $A(\gamma)$. For $\Delta \in A^{m\times p}(\gamma)$, define $\|\Delta\|_{A(\gamma)}$ as $|\Delta'|$ where $\Delta'_{ij} = \|\Delta_{ij}\|_{A(\gamma)}$. $\hat{g}$ denotes the Laplace transform of g. $\hat{A}(\gamma)$ denotes the set of Laplace transforms of elements of $A(\gamma)$. For general facts regarding $A(\gamma)$ & $\hat{A}(\gamma)$, see [9, Appendix D].

2. Stability of Slowly Time-Varying Linear Systems

Consider the following system of linear differential equations

$$\dot{x}(t) = A(t)\, x(t); \quad x(0) = x_o \in R^n,\ t \geq 0. \tag{2.1}$$

The question addressed in this section is the following: Given that for each instant in time A(t) is a stable matrix (i.e. all eigenvalues of A(t) have negative real parts), under what conditions is (2.1) stable under time-variations? It is shown in [8, 9, 26] that if the time-variations are sufficiently slow, then (2.1) remains stable. A proof of this result which differs from those found in [8, 9, 26] is included in this paper since the proof given here carries over to the analysis of slowly time-varying robustness in a relatively straitforward manner.

The following assumption is made on (2.1):

Assumption 2.1: $A(\cdot)$ is bounded, continuously differentiable, and for some $k_A \geq 0$, $|\dot{A}(t)| \leq k_A$, for all $t \geq 0$.

The question of slowly time-varying stability is now addressed.

Theorem 2.1: Consider the linear system (2.1) under Assumption 2.1. Assume that at each instant in time A(t) is a stable matrix, and that there exist constants m & $\lambda > 0$ such that

$$| e^{A(p)t} | \leq m e^{-\lambda t}; \quad \forall\, t \geq 0, \forall\, p \in R_+. \tag{2.2}$$

Under these conditions, given any $\eta \in (0,\lambda)$,

$$k_A \leq \frac{(\lambda-\eta)^2}{4\,m\,\ln(m)} \tag{2.3}$$

implies

$$|\,x(t)\,| \leq m e^{-\eta t} |\,x_o\,|; \quad \forall\, t \geq 0, \forall\, x_o \in R^n. \tag{2.4}$$

Proof: The proof makes use of the following lemma.

Lemma 2.1 [2]: Consider the linear system

$$\dot{x}(t) = A_o x(t) + \delta A(t)\, x(t)\,; \tag{2.5}$$

$$x(0) = x_o \in R^n,\ \ \delta A(\cdot) : R_+ \to R^{n \times n}.$$

Suppose that for some m, λ, & $k \geq 0$

$$|\,e^{A_o t}\,| \leq m e^{-\lambda t}, \tag{2.6}$$

$$|\,\delta A(t)\,| \leq k, \quad \forall\, t \geq 0. \tag{2.7}$$

Under these conditions,

$$|\,x(t)\,| \leq m e^{-(\lambda - mk)\,t} |\,x_o\,|, \quad \forall\, t \geq 0, \forall\, x_o \in R^n \tag{2.8}$$

////

Note that Lemma 2.1 can be used to guarantee exponential stability of the *perturbed* time-varying system (2.5) given that the unperturbed time-invariant system ($\delta A(t) \equiv 0$) is exponentially stable. This is used in proving Theorem 2.1 as follows. Consider approximating A(t) in (2.1) by the piecewise constant matrix

$$A_{pc}(t) \equiv A(nT)\,; \quad nT \leq t \leq (n+1)T,\ \ n = 0,1,2,\ldots \tag{2.9}$$

where T is to be chosen. Rewriting (2.1)

$$\dot{x}(t) = A_{pc}(t)\, x(t) + [\, A(t) - A_{pc}(t)\,]\, x(t). \tag{2.10}$$

Now choose

$$T = \frac{2\ln(m)}{(\lambda-\eta)} \tag{2.11}$$

Then for all time $t \geq 0$

$$|\,A(t) - A_{pc}(t)\,| \leq k_A T \leq \frac{(\lambda-\eta)}{2m} \tag{2.12}$$

where k_A is chosen according to (2.3). It follows from Lemma 2.1 that on $nT \leq t \leq (n+1)T$,

$$\begin{aligned} |\,x(t)\,| &\leq m e^{-\frac{\lambda+\eta}{2}(t-nT)} |\,x(nT)\,| \\ &\leq m e^{-\frac{\lambda+\eta}{2}(t-nT)} \left(m e^{-\frac{\lambda+\eta}{2}T}\right)^n |\,x_o\,| \\ &= m e^{-\eta t} e^{-\frac{\lambda-\eta}{2}(t-nT)} \left(m e^{-\frac{\lambda-\eta}{2}T}\right)^n |\,x_o\,| \\ &\leq m e^{-\eta t} |\,x_o\,| \end{aligned} \tag{2.13}$$

which completes the proof.////

Note that in the special case where m = 1, one has that the time-variations may be arbitrarily fast.

3. Robustness of Slowly Time-Varying Linear Systems

A. Problem Definition

Consider the feedback system shown in fig. 1.

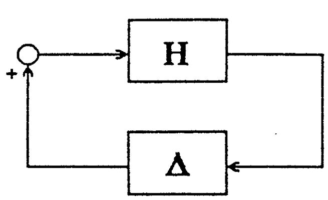

Figure 1. General Feedback System

The block in the forward loop, **H**, represents a stable finite-dimensional time-varying linear system, and the block in the feedback loop, Δ, represents a (possibly infinite-dimensional) stable time-invariant linear system. The feedback configuration of fig. 1 is quite general, and may be used to examine robust stability in the presence of a variety of possible plant perturbations [11]. Let **H** have the following state-space realization:

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, e(t); \quad x \in R^n, e \in R^m, \tag{3.1}$$

$$y(t) = C(t)\, x(t), \quad y \in R^p.$$

Furthermore, let the input/output relationship of Δ be given by

$$y'(t) = \int_0^t \Delta(t-\tau)\, y(\tau)\, d\tau. \tag{3.2}$$

Thus the feedback equations are

$$\dot{x}(t) = A(t)\, x(t) + \int_0^t B(t)\, \Delta(t-\tau)\, C(\tau)\, x(\tau)\, d\tau. \tag{3.3}$$

This equation represents a type of linear Volterra integrodifferential equation (VIDE). As in the finite dimensional case, it will be shown that if (3.3) is stable for all frozen values of time, stability is maintained for sufficiently slow time-variations.

B. Linear Volterra Integrodifferential Equations

Before time-varying robustness is discussed, some facts are presented regarding equations of the form (3.3). It was stated that these equations fall under the class of linear VIDE's. In fact, under assumptions to be stated on Δ, (3.3) actually represents a combination of VIDE's and linear delay-differential equations. Thus, both types of equations are treated under the same framework. VIDE's and their stability have been studied in, for example, [3, 13, 14, 15, 22, 23, 24], and delay-differential equations in [7, 12, 16].

In this section, assumptions on (3.3) are given, a definition of exponential stability for (3.3) is introduced, and a sufficient condition for exponential stability in the case of time-invariant **A, B,** & **C** matrices is presented. Finally, a perturbational result analogous to Lemma 2.1 is presented.

Consider the VIDE

$$\dot{x}(t) = A(t)\, x(t) + \int_0^t B(t)\, \Delta(t-\tau)\, C(\tau)\, x(\tau)\, d\tau, \quad t \geq t_o \tag{3.4}$$

with initial condition

$$x(t) = \phi(t); \quad t \in [0, t_o], \phi(t) \in L_{\infty e}. \tag{3.5}$$

Note that an initial condition for (3.4) consists of both an initial time, t_o, *and* an initial function, $\phi(t)$. Typically, the only case of interest is $t_o = 0$. However, the concept of an initial function is quite useful in analyzing stability. In addition to Assumption 2.1, the following assumptions are made on (3.4):

Assumption 3.1: $B(\cdot)$ & $C(\cdot)$ are bounded, continuously differentiable, and for some k_B & k_C, $|\,\dot{B}(t)\,| \leq k_B$ and $|\,\dot{C}(t)\,| \leq k_C$, for all $t \geq 0$.

Assumption 3.2: For some $\sigma \geq 0$, $\Delta \in A^{m \times p}(-\sigma)$.

VIDE's containing an integral operator as in Assumption 3.2 have been studied in [5, 6, 21], and references contained in [7]. Reference [5] establishes existence and uniqueness of solutions in the case of time-invariant **A, B,** & **C** matrices. Existence and uniqueness of solutions in the case of time-varying **A, B,** & **C** matrices involves standard contraction mapping techniques, and is omitted here.

In the case of time-invariant A, B, & C matrices, solutions to (3.4) can be explicitly characterized as follows:

Theorem 3.1 [5]: Consider the VIDE

$$\dot{x}(t) = A\,x(t) + \int_0^t B\,\Delta(t-\tau)\,C\,x(\tau)\,d\tau + f(t), \quad t \geq t_o \quad (3.6)$$

$$x(t) = \phi(t); \quad t \in [0, t_o],\ \phi(t) \in L_{\infty e}. \quad (3.7)$$

under Assumption 3.2. Here, $f \in L_{\infty e}$ is an exogenous input. In this case where A, B, &C are constant matrices, the unique solution to (3.6) is given by

$$x(t+t_o) = R(t)\,x(t_o) + \int_0^t R(t-\tau)\,\{\,f(\tau+t_o) + F(\tau+t_o)\}\,d\tau \quad (3.8)$$

where

$$F(t+t_o) = \int_0^{t_o} B\,\Delta(t+t_o-\tau)\,C\,\phi(\tau)\,d\tau, \quad t \geq 0, \quad (3.9)$$

and R(t) is the unique matrix satisfying

$$R(t) = I + \int_0^t \left\{ A\,R(\tau) + \int_0^\tau B\,\Delta(\tau-\beta)\,C\,R(\beta)\,d\beta \right\} d\tau \quad (3.10)$$

////

The matrix R(•) is called the 'resolvent matrix', and is analogous to the standard matrix exponential. Note that (3.10) implies that R(•) satisfies almost everywhere

$$\dot{R}(t) = A\,R(t) + \int_0^t B\,\Delta(t-\tau)\,C\,R(\tau)\,d\tau. \quad (3.11)$$

A definition of exponential stability for (3.4) is now introduced, and a sufficient condition in the case of constant A, B, & C matrices is given. First, consider the truncated exponentially weighted infinity norm defined by:

$$\| f_\beta \|_T \equiv \operatorname*{ess\,sup}_{t \in [0,T]} | e^{-\beta(T-t)} f(t) |; \quad f \in L_{\infty e}\ , \beta > 0. \quad (3.12)$$

This norm is essentially a supremum with an exponential forgetting factor backwards in time. Exponential stability for (3.4) is now defined as follows:

Definition 3.1: The VIDE (3.4) is said to be *exponentially stable* if there exist constants m, λ, & β > 0 where β ≥ λ such that for $t \geq t_o$

$$| x(t) | \leq m e^{-\lambda(t-t_o)} \| \phi_\beta \|_{t_o}; \quad \forall\, t_o \geq 0, \forall\, \phi \in L_{\infty e}.////$$

It is stressed that the constants m, λ, & β are *independent* of t_o & φ. The convention β ≥ λ follows from the reasoning that solutions to (3.4) cannot decay faster than they are forgotten.

The following theorem gives a sufficient condition for exponential stability in the case A, B, & C are constant as in (3.6).

Theorem 3.2: Consider the VIDE (3.6) with f ≡ 0. A sufficient condition for exponential stability is that there exist a constant ß > 0 such that

$$(sI - A - B\,\hat{\Delta}(s)\,C)^{-1} \in \hat{A}^{n x n}(-2ß) \quad (3.13)$$

$$\hat{\Delta}(s) \in \hat{A}^{m x p}(-2ß). \quad (3.14)$$

Proof: It is first shown that the resolvent matrix, R, is bounded by a decaying exponential. Taking the Laplace transform of (3.11) shows that

$$\hat{R}(s) = (sI - A - B\,\hat{\Delta}(s)\,C)^{-1} \quad (3.15)$$

It follows by hypothesis that $R \in A^{nxn}(-2ß)$. Since R contains no impulses, $R \in L_1$, and hence $\dot{R} \in L_1$ from (3.11). These two imply that $R \in L_\infty$. Now, write R as

$$R(t) = R'(t)\, e^{-ßt}. \quad (3.16)$$

Clearly, $R' \in A^{nxn}(-ß)$. Using the same arguments as above along with

$$\dot{R}'(t) = (A + ßI)\,R'(t) + \int_0^t B\,\Delta(t-\tau)\,C\,e^{ß(t-\tau)}\,R'(\tau)\,d\tau, \quad (3.17)$$

it follows that R' & $\dot{R}' \in L_1$ hence $R' \in L_\infty$. Thus from (3.16), there exists a constant k_1, namely $\| |R'(\bullet)| \|_{L_\infty}$, such that

$$| R(t) | \leq k_1 e^{-ßt}. \quad (3.18)$$

Now, recall that the solution to (3.6) with f ≡ 0 is given by

$$x(t+t_o) = R(t)\,x(t_o) + \int_0^t R(t-\tau)\,F(\tau+t_o)\,d\tau. \quad t \geq 0 \quad (3.19)$$

where

$$F(t+t_o) = \int_0^{t_o} B\,\Delta(t+t_o-\tau)\,C\,\phi(\tau)\,d\tau, \quad t \geq 0, \quad (3.20)$$

It is now shown that F is also bounded by a decaying exponential. Rewriting (3.20),

$$F(t+t_o) = \int_0^{t_o} B\Delta(t+t_o-\tau)e^{ß(t+t_o-\tau)}Ce^{-ß(t+t_o-\tau)}\phi(\tau)\,d\tau \quad (3.21)$$

$$F(t+t_o) = e^{-ßt}\int_0^{t_o} B\Delta(t+t_o-\tau)e^{ß(t+t_o-\tau)}Ce^{-ß(t_o-\tau)}\phi(\tau)\,d\tau \quad (3.22)$$

Since $\Delta \in A^{mxp}(-2ß)$, it follows from (3.22) that there exists a $k_2 > 0$, for example $k_2 = |B|\,\|\Delta\|_{A(-ß)}\,|C|$, such that

$$| F(t+t_o) | \leq k_2\, e^{-ßt} \| \phi_ß \|_{t_o}. \quad (3.23)$$

Substituting (3.18) & (3.23) into (3.19):

$$| x(t+t_o) | \leq k_1 e^{-ßt} | x(t_o) | + \int_0^t k_1 e^{-ß(t-\tau)} k_2 e^{-ß\tau} \|\phi_ß\|_{t_o} d\tau \quad (3.24)$$

$$\leq k_1\, e^{-ßt} \| \phi_ß \|_{t_o} \{\, 1 + k_2 t \,\}$$

$$\leq k_1 \left(1 + \frac{2\,k_2}{e}\right) e^{-\frac{ß}{2}t} \| \phi_ß \|_{t_o}. \quad (3.25)$$

Since (3.25) is true for arbitrary φ & t_o, it follows that (3.6) with f ≡ 0 is exponentially stable.*////*

Remark 3.1: Regarding the hypotheses of Theorem 3.2, it seems that condition (3.13) is necessary. This is because exponential stability implies R(•) decays exponentially. Condition (3.13) then follows from (3.15). Regarding condition (3.14), note that it is slightly stronger than the standing Assumption 3.2. Specifically, Assumption 3.2 guarantees only that $\Delta \in A^{mxp}(0)$. While this is sufficient in proving existence and uniqueness of solutions, it is definitely not necessary for exponential stability,for example the case of a finite-dimensional Δ.*////*

Remark 3.2: Note that standard results on robustness of time-invariant linear systems [e.g. 4, 10, 19] can be obtained from the previous theorem. Rewriting $\hat{R}(s)$ in (3.15):

$$\hat{R}(s) = (I - (sI - A)^{-1}\,B\,\hat{\Delta}(s)\,C)^{-1}\,(sI - A)^{-1}. \quad (3.26)$$

Now suppose that A is a stable matrix; thus $(sI - A)^{-1} \in A^{nxn}(-ß)$ for some ß > 0. Assume further that $\hat{\Delta}(s) \in \hat{A}^{mxp}(-ß)$. Thus,

$$(I - (sI - A)^{-1}\,B\,\hat{\Delta}(s)\,C) \in A^{nxn}(-ß). \quad (3.27)$$

Under these conditions, $\hat{R}(s) \in A^{nxn}(-ß)$ if and only if [9, 26]

$$\inf_{Re[s] > -ß} \det(I - (sI - A)^{-1}B\,\Delta(s)C) = \det(I - C(sI - A)^{-1}B\,\Delta(s)) > 0. \quad (3.28)$$

However, a sufficient condition for (3.28) is that

$$| C((-\beta + j\omega)I - A)^{-1}B\, \Delta((-\beta + j\omega)) | \le \gamma < 1, \ \forall\, \omega \in R \qquad (3.29)$$

As $\beta \to 0$, condition (3.30) approaches the standard robustness condition for time-invariant linear systems. Unlike previous results, however, Theorem 3.2 gives some quantitative indication of the degree of robust stability.////

The proof of slowly time-varying stability in Section 2 relied heavily on Lemma 2.1. This subsection closes with an analogous result for time-invariant VIDE's.

Theorem 3.3: Consider the VIDE

$$\dot{x}(t) = A\, x(t) + \int_0^t B\Delta(t-\tau)Cx(\tau)\, d\tau + (gx)(t), \quad t \ge t_o \qquad (3.30)$$

$$x(t) = \phi(t); \quad t \in [0, t_o], \phi \in L_\infty \qquad (3.31)$$

Here, g represents an integral operator on **x**. Assume conditions (3.13) & (3.14). Assume further that there exist constants $k > 0$ and $\alpha \ge \beta$, where β is from (3.13) & (3.14), such that

$$| (gx)(t) | \le k \,\| x_\alpha \|_t; \quad \forall x \in L_\infty\, \forall t \ge 0. \qquad (3.31)$$

Let k_1 be as in (3.25). Under these conditions,

$$k < \frac{\beta}{k_1} \qquad (3.32)$$

implies (3.30) is exponentially stable.

Proof: Define $z(t) = x(t + t_o)$. As in (3.8)

$$z(t) = R(t)z(0) + \int_0^t R(t-\tau)\{F(\tau + t_o) + (gx)(\tau + t_o)\}\, d\tau \qquad (3.33)$$

where

$$F(t + t_o) = \int_0^{t_o} B\, \Delta(t + t_o - \tau)\, C\, \phi(\tau)\, d\tau. \quad t \ge 0, \qquad (3.34)$$

As before, there exist k_1 & $k_2 > 0$ such that

$$| R(t) | \le k_1 e^{-\beta t}. \qquad (3.18)$$

$$| F(t + t_o) | \le k_2\, e^{-\beta t}\, \| \phi_\beta \|_{t_o}. \qquad (3.23)$$

Substituting (3.18), (3.23), & (3.31) into (3.33):

$$| z(t) | \le \int_0^t k_1 e^{-\beta(t-\tau)}\{k_2 e^{-\beta\tau}\|\phi_\beta\|_{t_o} + k\, \|x_\alpha\|_{\tau+t_o}\}d\tau + k_1 e^{-\beta t}|z(0)| \qquad (3.35)$$

Since $\alpha \ge \beta$,

$$| z(t) | \le \int_0^t k_1 e^{-\beta(t-\tau)}\{k_2 e^{-\beta\tau}\|\phi_\beta\|_{t_o} + k\, \|x_\beta\|_{\tau+t_o}\}d\tau + k_1 e^{-\beta t}|z(0)| \qquad (3.36)$$

Applying (3.12):

$$\begin{aligned} \| x_\beta \|_{\tau+t_o} &\equiv \sup_{\xi \in [0,\tau+t_o]} | e^{-\beta(\tau + t_o - \xi)}\, x(\xi) | \\ &\le e^{-\beta t}\{ \sup_{\xi \in [0,t_o]} |e^{-\beta(t_o - \xi)}\, \phi(\xi)| + \sup_{\xi \in [t_o,\tau+t_o]} |e^{-\beta(t_o - \xi)}\, x(\xi)| \} \end{aligned} \qquad (3.37)$$

Thus

$$e^{\beta t}|z(t)| \le k_1\{1+(k + k_2)t\}\, \|\phi_\beta\|_{t_o} + \int_0^t k_1 k \sup_{\xi \in [0,\tau]} |e^{\beta\xi}z(\xi)|d\tau. \qquad (3.38)$$

Since the right-hand-side of (3.38) is a nondecreasing function of time,

$$\sup_{\xi \in [0,t]} |e^{\beta\xi}z(\xi)| \le k_1\{1+(k+k_2)t\}\|\phi_\beta\|_{t_o} + \int_0^t k_1 k \sup_{\xi \in [0,\tau]} |e^{\beta\xi}z(\xi)|\, d\tau \qquad (3.39)$$

Rewriting (3.39) gives

$$f(t) \le \kappa_1 + \kappa_2 t + \kappa_3 \int_0^t f(\tau)\, d\tau \qquad (3.40)$$

where κ_1, κ_2, κ_3, & $f(\bullet)$ are defined in the obvious manner. Applying the Bellman-Gronwall inequality to (3.40):

$$f(t) \le \left(\kappa_1 + \frac{\kappa_2}{\kappa_3} \right) e^{\kappa_3 t} - \frac{\kappa_2}{\kappa_3} \qquad (3.41)$$

Thus,

$$|z(t)| \le \left\{ \left(k_1 + 1 + \frac{k_2}{k} \right) e^{-(\beta - k_1 k)t} - \left(1 + \frac{k_2}{k} \right) e^{-\beta t} \right\} \|\phi_\beta\|_{t_c} \qquad (3.42)$$

Exponential stability then follows from (3.32).////

Note that (3.42) implies that for some $m_k > 0$,

$$|x(t + t_o)| \le m_k e^{-\frac{1}{2}(\beta - k_1 k)t}\, \|\phi_\beta\|_{t_o} \qquad (3.43)$$

However, m_k should be chosen carefully so that as $k \to 0$, one has that $m_k \to k_1\left(1 + \frac{2k_2}{e}\right)$ as in (3.25).

C. Stability of Slowly Time-Varying Linear VIDE's

In this section, the results of Section 2 are extended to VIDE's of the form (3.4). That is, it is shown that if (3.4) satisfies (3.13) & (3.14) for all frozen values of time, then it is exponentially stable for sufficiently slow time-variations in $A(\bullet)$, $B(\bullet)$, & $C(\bullet)$.

Before proceeding with the theorem, some notations are defined. First, the following assumption is made on (3.4).

Assumption 3.3: There exists a $\beta > 0$ such that

$$(sI - A(p) - B(p)\, \hat{\Delta}(s)\, C(p))^{-1} \in \hat{A}^{n \times n}(-2\beta), \ \forall\, p \in R_+, \qquad (3.44)$$

////

This assumption guarantees uniform exponential stability for all frozen values of time. Let κ_B & κ_C be such that $|B(t)| \le \kappa_B$ & $|C(t)| \le \kappa_C$ for all $t \ge 0$. Then define

$$K \equiv k_A + k_B\, \|\Delta\|_{A(-\beta)}\kappa_C + \kappa_B\|\Delta\|_{A(-\beta)}k_C \qquad (3.45)$$

Let $R'_p(\bullet)$ denote the resolvent matrix associated with the frozen matrices $A(p)$, $B(p)$, & $C(p)$ as in equation (3.17). Then define

$$K_1 = \sup_{p \in R_+} \| \,|R'_p(\bullet)|\, \|_{L_\infty} \qquad (3.46)$$

$$K_2 = \kappa_B\|\Delta\|_{A(-\beta)}\kappa_C \qquad (3.47)$$

The question of slowly time-varying robustness in now addressed. The proof closely follows that of Theorem 2.1.

Theorem 3.3: Consider the VIDE (3.4) under Assumptions 2.1, 3.1, & 3.3. Furthermore, let $\Delta \in A^{m \times p}(-2\beta)$, with β as in Assumption 3.3. Under these conditions, given any $\eta \in (0,\beta)$, (3.4) is exponentially stable with a decay rate of $\eta/2$ for sufficiently small K, or equivalently for sufficiently slow time variations in $A(\bullet)$, $B(\bullet)$, & $C(\bullet)$.

Proof: Let t_n denote $t_o + nT$, where T is some constant to be chosen. As in the proof of Theorem 2.1, (3.4) will first be analyzed on the intervals $t_n \le t \le t_{n+1}$. Approximating $A(\bullet)$, $B(\bullet)$, & $C(\bullet)$ by piecewise constant matrices, one has that

$$\dot{x}(t) = A(t_n)\, x(t) + \int_{t_n}^t B(t_n)\, \Delta(t-\tau)\, C(t_n)\, x(\tau)\, d\tau + \int_0^{t_n} B(t)\, \Delta(t-\tau)\, C(\tau)\, x(\tau)\, d\tau + (g_n x)(t), \qquad (3.48)$$

where

$$(g_n x)(t) = [A(t)-A(t_n)]x(t) + \int_{t_n}^{t} [B(t)-B(t_n)]\Delta(t-\tau)C(t_n)x(\tau)d\tau + \int_{t_n}^{t} B(t)\Delta(t-\tau)[C(\tau)-C(t_n)]\, x(\tau)d\tau \tag{3.49}$$

Then

$$|(g_n x)(t)| \le KT\, \|x_\beta\|_{t_\cdot}, \quad t_n \le t \le t_{n+1} \tag{3.50}$$

Thus, using Theorem 3.3, it is seen that if

$$KT \le \frac{(\beta-\eta)}{2K_1} \tag{3.51}$$

then

$$|x(t)| \le \left\{ \left(K_1 + 1 + \frac{K_2}{KT}\right) e^{-\frac{1}{2}(\beta+\eta)(t-t_n)} - \left(1 + \frac{K_2}{KT}\right) e^{-\beta(t-t_n)} \right\} \|x_\beta\|_{t_n} \tag{3.52}$$

or as in (3.43)

$$|x(t)| \le m_{KT}\, e^{-\frac{1}{2}\left(\frac{\beta+\eta}{2}\right)(t-t_n)} \|x_\beta\|_{t_n} \tag{3.53}$$

for some $m_{KT} > 0$. In order to guarantee (3.51), choose

$$T = \frac{4\ln(m_{KT})}{\beta-\eta} \tag{3.54}$$

$$K \le \frac{(\beta-\eta)^2}{8K_1 \ln(m_{KT})} \tag{3.55}$$

Now, (3.53) implies that

$$\|x_\beta\|_{t_{n+1}} \le \left(m_{KT}\, e^{-\frac{1}{2}\left(\frac{\beta+\eta}{2}\right)T} \right) \|x_\beta\|_{t_n} \tag{3.56}$$

Substituting (3.54) & (3.56) into (3.53), one gets that

$$\begin{aligned} |x(t)| &\le m_{KT}\, e^{-\frac{1}{2}\left(\eta+\frac{\beta-\eta}{2}\right)(t-t_o-nT)} \left(m_{KT}\, e^{-\frac{1}{2}\left(\eta+\frac{\beta-\eta}{2}\right)T}\right)^n \|\phi_\beta\|_{t_o} \\ &\le m_{KT}\, e^{-\frac{\eta}{2}(t-t_o)} e^{-\frac{1}{2}\left(\frac{\beta-\eta}{2}\right)(t-t_n)} \left(m_{KT}\, e^{-\frac{1}{2}\left(\frac{\beta-\eta}{2}\right)T}\right)^n \|\phi_\beta\|_{t_o} \\ &\le m_{KT}\, e^{-\frac{\eta}{2}(t-t_o)} \|\phi_\beta\|_{t_o} \end{aligned} \tag{3.57}$$

which completes the proof.////

4. Concluding Remarks

The stability of a class of linear Volterra integrodifferential equations has been discussed. It was shown that if the time-varying equations frozen-time stable, then stability is maintained in the presence of slow time-variations.

As stated in the introduction, these equations can be used to study the robustness of certain gain-scheduled control systems, namely those systems whose dynamics depend on a time-varying external parameter. For example, let the dynamics of the linear parameter-varying plant, $P(\theta)$, be given by

$$\dot{x}(t) = A(\theta(t))\, x(t) + B(\theta(t))\, u(t)$$
$$y(t) = C(\theta(t))\, x(t)$$

Typically, one designs a linear parameter-varying compensator, $K(\theta)$, such that for each value of the parameter, the closed-loop system satisfies various robust stability / robust performance specifications. However, meeting such specifications is equivalent to being robust to particular uncertainties [11]. Testing this robustness then results in analyzing the stability of a feedback system of the form in fig. 1, where H is then a parameter-varying linear system. However, it is insufficient to check that the feedback system of fig. 1 is stable for each frozen value of the parameter, since the parameter is actually time-varying - hence the need for new robustness tests.

The results presented in this paper give sufficient conditions to guarantee robust stability in case of parameter time-variations; therefore, these results may prove useful in the analysis of gain-scheduled control systems. Unfortunately, the actual bounds on the parameter time-variations, as in (3.55), may be difficult -if not impossible- to compute. For example, recall that one requires the frozen-time system to be exponentially stable, which can be guaranteed by meeting the conditions of Theorem 3.2:

$$(sI - A - B\,\hat{\Delta}(s)\, C)^{-1} \in \hat{\mathcal{A}}^{n\times n}(-2\beta) \tag{3.13}$$

$$\Delta(s) \in \hat{\mathcal{A}}^{m\times p}(-2\beta) \tag{3.14}$$

Upon examining these conditions, one finds that the following information is needed. First, condition (3.14) involves knowledge about the stability margin of Δ. In other words, one must have some idea about the 'slowest pole' of the unmodeled dynamics. Given such information, one can then use the inequality (3.29) from Remark 3.2 to guarantee condition (3.13). However, this inequality involves evaluating Δ (or a bound on Δ) *off* of the $j\omega$-axis. Once these conditions are verified, one can then use (3.55) to guarantee robust stability. However, (3.55) requires numerical values for

$$K_1 = \sup_{p\in R_+} \| |R'_p(\cdot)| \|_{L_\infty} \tag{3.46}$$

$$K_2 = \kappa_B \|\Delta\|_{A(-\beta)} \kappa_C \tag{3.47}$$

which are difficult -at best- to calculate.

Note that the aforementioned difficulties may be avoided by using the small-gain theorem [e.g. 9] to guarantee the stability of fig. 1. More specifically, one can use the results from Section 2 to guarantee exponential stability of H. The L_p induced operator norm of H can then be bounded using the stated assumptions on (3.1). However, this approach completely ignores that the feedback system was designed to be robustly stable for all frozen parameter values. In this sense, this approach fails to capture the underlying philosophy behind the gain-scheduled design.

In the absence of numerical values for (3.46) & (3.47), one is limited to such qualitative statements as 'the more exponentially stable - the more tolerance to parameter time-variations.' Thus, more research is needed in 1) determining precisely what additional information on the unmodeled dynamics is needed to evaluate (3.46) & (3.47) and what properties of the unmodeled dynamics can be extrapolated from the standard assumption of a frequency domain magnitude bound & 2) finding better conditions for finite-dimensional time-varying stability, with the hopes of extending them to robustness analysis.

References

[1] J.K. Aggarwal and E.F. Infante, "Some Remarks on the Stability of Time-Varying Systems," *IEEE Transactions on Automatic Control*, Vol. AC-13, pp.722-723, December 1968.

[2] R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1970.

[3] T.A. Burton, *Volterra Integral and Differential Equations*, Acadaemic Press, New York, 1983.

[4] M.J. Chen and C.A. Desoer, "Necessary and Sufficient Condition for Robust Stability of Linear Distributed Feedback Systems," *International Journal of Control*, Vol. 35, No. 2, pp.255-267, 1982.

[5] C. Corduneanu, "Some Differential Equations with Delay," *Proceedings Equadiff 3 (Czechoslovak Conference on Differential Equations and their Applications)*, Brno, pp.105-114, 1973.

[6] C. Corduneanu and N. Luca, "The Stability of Some Feedback Systems with Delay," *Journal of Mathematical Analysis and Applicaations*, Vol. 51, pp.377-393, 1975.

[7] C. Corduneanu and V. Lakshmikantham, "Equations with Unbounded Delay: A Survey," *Nonlinear Analysis Theory, Methods, & Applications*, Vol. 4 , No 5, pp.831-877, 1980.

[8] C.A. Desoer, "Slowly Varying System $\dot{x} = A(t)\, x$," *IEEE Transactions on Automatic Control*, Vol AC-14, pp.780-781, December 1969.

[9] C.A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.

[10] J.C. Doyle and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," *IEEE Transactions on Automatic Control*, Vol. AC-26, No. 1, February 1981.

[11] J.C. Doyle, J.E. Wall, and G. Stein, "Performance and Robustness Analysis for Structured Uncertainty," *Proceedings of the 21st IEEE Conference on Decision and Control*, pp.629-636, 1982.

[12] R.D. Driver, *Ordinary and Delay Differential Equations*, Springer-Verlag, New York, 1977.

[13] R. Grimmer and G. Seifert, " Stability Properties of Volterra Integrodifferential Equations," *Journal of Differential Equations*, Vol. 19, pp.142-166, 1975.

[14] S.I. Grossman and R.K. Miller, "Perturbation Theory for Volterra Integrodifferential Systems," *Journal of Differential Equations*, Vol. 8, pp.457-474, 1970.

[15] S.I. Grossman and R.K. Miller, "Nonlinear Volterra Integrodifferential Systems with L^1-Kernels," *Journal of Differential Equations*, Vol. 13, pp.551-566, 1973.

[16] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[17] E. Hille and R. S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Coll. Publ., XXXI, New Providence, 1957.

[18] P. Kapasouris.and M. Idelchik, "Gain-Scheduled Multimode Control Design Methodology for Jet Engines Using the LQR Multivariable Control Theory," General Electric Company Report, 1986.

[19] N.A. Lehtomaki, N.R. Sandell Jr., and M. Athans, "Robustness Results in Linear-Quadratic Gaussian Based Multivariable Control Designs," *IEEE Transactions on Automatic Control*, Vol. AC-26, No. 1, February 1981.

[20] K.A. Lively, *Multivariable Control System Design for a Submarine*, Master's Thesis, Department of Ocean Engineering, MIT, 1984.

[21] N. Luca, "The Stability of the Solutions of a Class of Integrodifferential Systems with Infinite Delays," *Journal of Mathematical Analysis and Applications*, Vol. 67, pp.323-339, 1979.

[22] R.K. Miller, *Nonlinear Volterra Integral Equations*, W.A. Benjamin, Menlo Park, CA, 1971.

[23] R.K. Miller, "Asymptotic Stability Properties of Linear Volterra Integrodifferential Equations," *Journal of Differential Equations*, Vol. 10, pp.485-506, 1971.

[24] R.K. Miller, "Structure of Solutions of Unstable Linear Volterra Integrodifferential Equations," *Journal of Differential Equations*, Vol. 15, pp.129-157, 1974.

[25] G. Stein, "Adaptive Flight Control - A Pragmatic View," *Applications of Adaptive Control*, K.S. Narendra and R.V. Monopoli (eds.), Academic Press, New York, 1980.

[26] M. Vidyasagar, *Nonlinear Systems Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1978.

[27] M.Y. Wu, "A Note on Stability of Linear Time-Varying Systems," *IEEE Transactions on Automatic Control*, Vol. AC-19, p.162, April 1974.