ON RELIABLE CONTROL SYSTEM DESIGNS*

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ABSTRACT

This paper summarizes a research effort which addresses some of the current problems in interfacing systems theory and reliability. Reliability is roughly the probability that a system will perform according to specifications for a given amount of time. The reliability of a system depends on the structure of its components. Systems theory and control theory deal with the response characteristics of a system, which depend on the system dynamics. This report defines the concepts necessary to unify the structural and the dynamic properties of a system. The result is a definition of what constitutes a reliable system, from the viewpoint of systems theory, and a methodology which can be used to determine if a given design allows a reliable control system design.

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1. Introduction

A method is described which specifies whether or not a linear system which has random jump discontinuities in its dynamics can be stabilized by state feedback. The system is modeled as being a member of a set of linear systems at each time, where the current member is specified by the state of a Markov chain. The system is modeled in this paper as allowing discontinuities only in the input (actuator) matrix; this restriction is easy to remove and causes no change in the results. Only state feedback is considered; this allows exact identification of the configuraiton after a unit delay. These results do not hold under more general assumptions.

The method illustrates the unification of the concepts of reliability and stabilizability. Reliability is defined as the probability that a system will perform within specified constraints for a given period of time. Stabilizability is defined for linear time invariant systems as the existence of a state feedback control law for which the closed loop system has all its poles in the open left half of the complex plane.

Stabilizability is not as easy to define for system which can experience random discontinuities in their dynamics. The definition of stability used in this paper (cost-stability) is that, for a specified quadratic cost criterion on the state and input signals, the expected value of the cost (with respect to the statistics of the Markov chain) is finite with probability one over an infinite time horizon. A system is stabilizable if and only if this expected cost, as a function of the state feedback map, has a finite value for at least one feedback map.

We emphasize the relationship between these results and reliability. The class of systems presented here is one model of abrupt failure, reconfiguration, and repair in a linear system. Given the Markov model of jumps in the system model, the methods of robust control, in which the system is guaranteed stable in all modes of operation, are not sufficient. A system may be allowed to transit through an unstable mode of operation, and yet be cost-stable.

Cost-stability can be used to classify systems into two subclasses, those which are reliable (stabilizable with probability one), and those which are not. These classes are defined by the structural dynamics and by the continuous state dynamics. The expectation operator which will be used to define cost is with respect to the statistics of the structural model; whereas, the cost function for a given structural trajectory is with respect to the value of the state and input.

2. Previous Work

Several authors have studies the optimal control of systems with randomly varying structure. Most notable among these is Wonham [1], where the solution to the continuous time linear regulator problem with randomly jumping parameters is developed. This solution is similar to the discrete time switching gain solution presented in Section 3. Wonham also proves an existence result for the steady-state optimal solution to the control of systems with randomly varying structure; however, the conclusion is only sufficient; it is not necessary. Similar results were obtained in Beard [2] for the existence of a stabilizing gain, where the structures were of a highly specific form; these results were necessary and sufficient algebraic conditions, but cannot be readily generalized to less specific classes of systems. Additional work on the control problem for this class of systems has been done by Sworder [3], Ratner & Luenberger [4], Bar-Shalom & Sivan [5], Willner [6] and Pierce & Sworder [7]. The dual problem of state estimation with a system with random parameter variations over a finite set was studied in Chang & Athans [8].

Some of the preliminary results on which this research was based were presented in unpublished form at the 1977 Joint Automatic Control Conference in San Francisco by Birdwell, and published for the 1977 Conference on Decision and Control Theory in New Orleans by Birdwell & Athans [9]. A survey of the results was presented without proofs in [10]. This paper is based on the results in Birdwell [11].

3. Model of System Structure

Models of the structural and the system dynamics will now be presented and used in the sequel to demonstrate the concepts outlined in the introduction. Component failures, repairs, and reconfigurations are modeled by a Markov chain. Only catastrophic changes in the system structure are considered; degradations are not modeled. The hazard rate is assumed to be constant, resulting in an exponential failure distribution. In the discrete-time case, to which the sequel is confined exclusively, the hazard rate becomes the probability of failure (or repair of reconfiguration) between time t and time t + 1.

We now define the modes of operation and their dynamic transitions. The terms system configuration and system structure will be used. A system structure is a possible mode of operation for a given system, represented by the components, their interconnections, and the information flow in the system at a given time. The system configuration is the original design of the system, accounting for all modeled modes of operation, and the Markov chain governing the configuration, or structural, dynamics (transitions among the various structures). In this paper, structures are referenced by the set of non-negative integers

$$I = \{0, 1, 2, \cdot \cdot \cdot, L\} \tag{3.1}$$

Consider the system

$$\underline{x}_{t+1} = \underline{A}\,\underline{x}_t + \underline{B}_{k(t)}\,\underline{u}_t \tag{3.2}$$

where

$$\underline{x}_t \,\varepsilon\, R^n \tag{3.3}$$

$$\underline{u}_t \varepsilon R^m \tag{3.4}$$

$$\underline{A} \varepsilon R^{nxn} \tag{3.5}$$

and, for each k, an element of an indexing set I

$$k \varepsilon I = \{0, 1, 2, \cdot \; \cdot \; \cdot \; , L\} \tag{3.6}$$

$$\underline{B}_k \varepsilon R^{n \times m} \tag{3.7}$$

and

$$\underline{B}_i \neq \underline{B}_j \text{ for all } i, j \varepsilon I, i \neq j \tag{3.8}$$

The index k(t) is a random variable taking values in I which is governed by a Markov chain, and

$$\underline{\pi}_{t+1} = \underline{P}\, \underline{\pi}_t \tag{3.9}$$

$$\underline{\pi}_t \varepsilon R^{L+1} \tag{3.10}$$

where $\pi_{i,t}$ is the probability of k(t) = i, given no on-line information about k(t), and $\underline{\pi}_o$ is the initial distribution over I.

It is assumed that the following sequence of evens occurs at each time t:

1) $\underline{x}_t$ is observed exactly
2) then $\underline{B}_{k(t-1)}$ switches to $\underline{B}_{k(t)}$
3) then $\underline{u}_t$ is applied.

Consider the structure set $\{\underline{B}_k\}_{k \varepsilon I}$ indexed by I. Define the structural trajectory x_T to be a sequence of elements k(t) in I which select a specific structure $\underline{B}_{k(t)}$ at time t,

$$\bar{x}_T = (k(0), k(1), \cdot \cdot \cdot , k(T-1)) \tag{3.11}$$

The structural trajectory x_T is a random variable with probability of occurance generated from the Markov equation (3.10).

$$p(\bar{x}_T) = \prod_{t=0}^{T-1} p_{k(t);\ k(t+1)} \pi_{k(o),0} \tag{3.12}$$

where the control interval is

$$\{0, 1, 2, \cdot \cdot \cdot , T-1, T\} \tag{3.13}$$

for the finite time problem with terminal time T, and $p_{k(t);\ k(t+1)}$ is the conditional probability of the system being in the structure indexed by k(t) at time t, given that it was in the structure indexed by k(t-1) at time t-1. Then for a given state and control trajectory $(\underline{x}_t, \underline{u}_t)^{T-1}_{t=0}$ generated by (3.2) and x_T from a sequence of controls $(\underline{u}_t)^{T-1}_{t=0}$, the cost index is to be the standard quadratic cost criterion

$$J_T\left\{\bar{x}_{T'}\left(\underline{x}_t, \underline{u}_t\right)_{t=0}^{T-1}\right\} = \sum_{t=0}^{T-1}\left(\underline{x}_t^T \underline{Q}\, \underline{x}_t + \underline{u}_t^T \underline{R}\, \underline{u}_t\right) + \underline{x}_T^T \underline{Q}\, \underline{x}_T \tag{3.14}$$

4. Problem Statement

The objective is to choose a feedback control law, which may depend on any past information about $\underline{x}_t$ or $\underline{u}_t$, mapping $\underline{x}_t$ into $\underline{u}_t$

$$\underline{\Phi}_t^* : R^n \rightarrow R^m \tag{4.1}$$

$$\underline{\Phi}_t^* : \underline{x}_t \rightarrow \underline{u}_t \tag{4.2}$$

such that the expected value of the cost function $\mathcal{J}_T$ from equation (3.15)

$$J_T = E\left[\mathcal{J}_T \mid \underline{\Pi}_0\right] \tag{4.3}$$

is minimized over all possible mapping $\underline{\Phi}_t$ at $\underline{\Phi}^*_t$.

Normally, a control law of the form (4.2) must provide both a control and an estimation function in this type of problem; hence the label dual control is used. Here, the structure of the problem allows the exact determination of k(t-1) from $\underline{x}_t$, $\underline{x}_{t-1}$ for almost all values of $\underline{u}_{t-1}$. This result is stated in the following lemma:

Lemma 1: For the set $\{\underline{B}_k\}_{k \varepsilon I}$, where the $\underline{B}_k$'s are distinct, the set $\{\underline{x}_{k,t+1} = \underline{A}\,\underline{x}_t + \underline{B}_k\,\underline{u}_t\}_{k \varepsilon I}$ has distinct members for almost all values of $\underline{u}_t$.

Proof: See Appendix.

Ignoring the set of controls of measure zero for which the members of

$$\left\{\underline{x}_{k,t+1}\right\}_{k=0}^{L} \tag{4.3}$$

are not distinct, then for (almost) any control which the optimal algorithm selects, the resulting state $\underline{x}_{t+1}$ can be compared with the members of the set (4.3) for an exact match (of which there is only one with probability 1), and k(t) is identified as the generator of that matching member $\underline{x}_{k,\,t+1}$.

This approach is essentially identical to assuming that the structure of the system is perfectly observable. Assuming perfect observability does eliminate any concern about the possibility of encountering a surface of zero measure and causing the control loop to malfunction. However, in a practical application, neither the assumption of perfect state observation nor of perfect structure observation is valid, and in fact the implementer is forced to consider structure identification strategies and the dual effect of control actions on the observation process.

The optimal control law $\underline{u}^* = \underline{\Phi}_t^* (\underline{x}_t)$ can be calculated with the assumption that k(t-1) is known, since this is the case with probability one if no measurement noise is present. Thus, this solution will be labeled the <u>switching gain solution</u>, since, for each time, t, L + 1 optimal solutions are calculated <u>apriori</u>, and one solution is chosen on-line based on the past measurements $\underline{x}_t$, $\underline{x}_{t-1}$ and $\underline{u}_{t-1}$, which yield perfect knowledge of k(t-1). The solution is stated in the following theorem; the proof is contained in the Appendix.

5. The Optimal Solution

The solution is stated in the following Theorem; the proof is contained in the Appendix. Dynamic programming is used to derive the optimal solution.

Theorem 1:

At each time t, the optimal expected cost-to-go, given the system structure k(t-1), which is the minimum of the expected value of the quadratic cost over the interval {t, .., T} and is given by

$$V^*(\underline{x}_t, k(t-1), t) = \min_{\underline{u}_t = \underline{\phi}_t(\underline{x}_t)} E_{k(t)} \left\{ \underline{x}_t^T \underline{Q}\, \underline{x}_t + \underline{u}_t^T \underline{R}\, \underline{u}_t + V^*(\underline{x}_{t+1}, k(t), t+1) \mid \underline{x}_t \right\}, \tag{51}$$

is quadratic,

$$V^*(\underline{x}_t, k(t-1), t) = \underline{x}_t^T \underline{S}_{k,t} \underline{x}_t, \tag{5.2}$$

where the $\underline{S}_{k,t}$ are determined by a set of L + 1 coupled Riccati-like equations (one for each possible configuration):

$$\underline{S}_{k,t} = \underline{A}^T \left\{ \sum_{i=0}^{L} p_{ik} \underline{S}_{i,t+1} - \left[\sum_{i=0}^{L} p_{ik} \underline{S}_{i,t+1} \underline{B}_i \right] \left[\underline{R} + \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \cdot \left[\sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \right] \right\} \underline{A} + \underline{Q} \tag{5.3}$$

The optimal control, given k(t - 1) = k, is

$$\underline{u}^*_{k,t} = - \left[\underline{R} + \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \cdot \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{A} \underline{x}_t$$

Proof: See Appendix.

(5.4)From equation (5.4), the optimal linear switching gain is

$$\underline{G}_{k,t} = - \left[\underline{R} + \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \cdot \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{A} , \tag{5.5}$$

and $\underline{u}^*_t = \underline{\phi}^*_t(\underline{x}_t)$ is a switching gain linear control law which depends on k(t-1). The variable k(t-1) is determined from $\underline{x}(t)$ (see Lemma 1).

Note that the $\underline{S}_{i,t}$'s and the optimal gains $\underline{G}_{k,t}$ can be computed off-line and stored. Then at each time t, the proper gain is selected on-line from k(t-1), using Lemma 1 as in Figure 1.

This solution is quite complex relative to the structure of the usual linear quadratic solution. Each of the Riccati-like equations (5.3) involves the same complexity as the Riccati equation for the linear quadratic solution. In addition, there is the on-line complexity arising from the implementation of gain scheduling.

Conditions for the existence of a steady-state solution to equations (5.3) can be developed using the properties of the structural dynamics, as in Chizeck [12]. The development of these conditions and computational algorithms are of general theoretical importance in linear system theory. The possibility of limit cycle solutions in the switching gain computations is excluded by the following lemma:

Lemma 2: If the optimal expected cost-to-go at time t is bounded for all t, then equation (5.3) converges.

Proof: See Appendix.

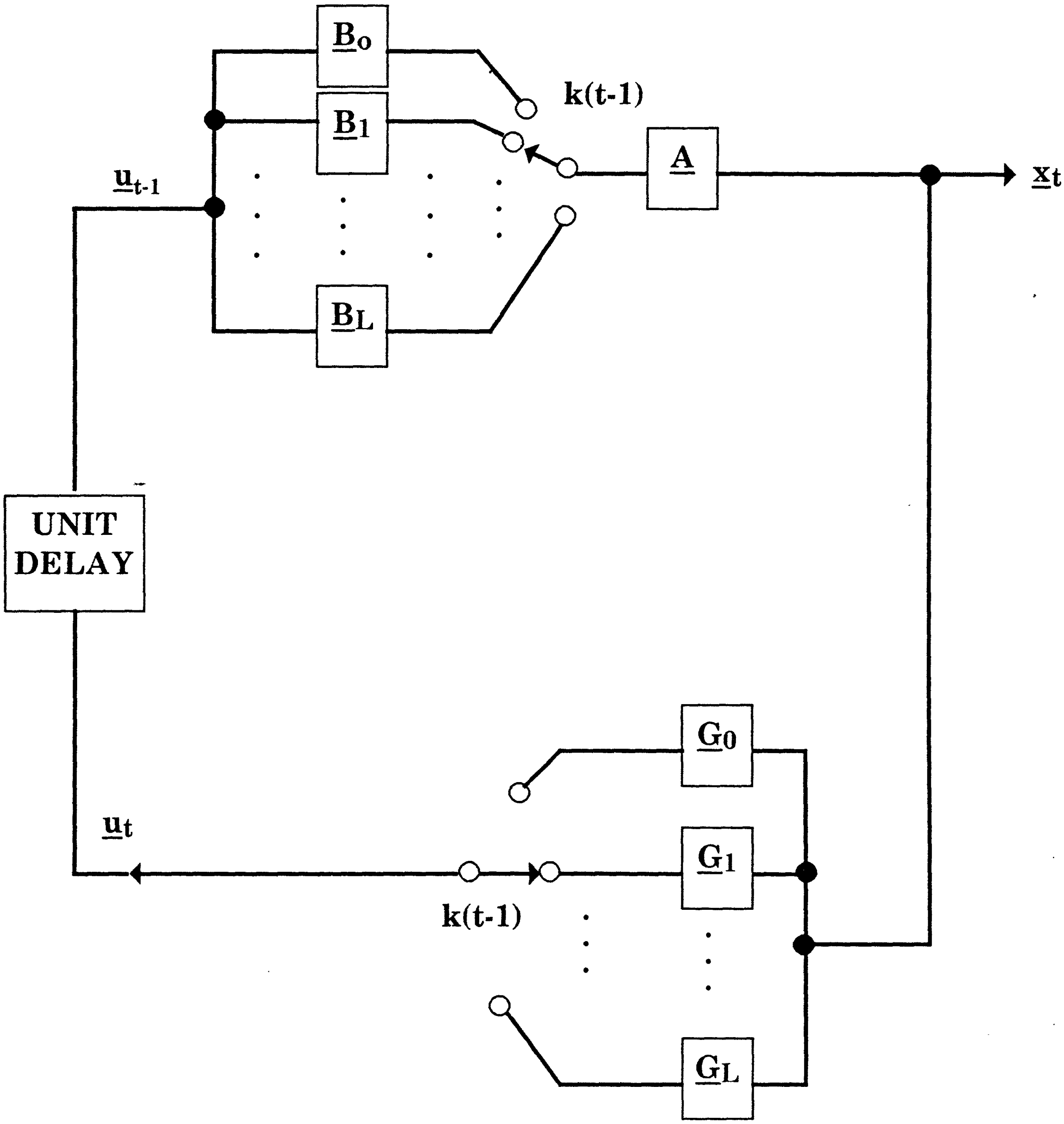

Figure 1: The switching gain control law.

Before we proceed to give necessary and sufficient conditions for the existence of a steady-state solution to equation (5.3), we must define the structure of the Markov chain. The states in the set I can be divided into closed communicating classes $C_1, \ldots, C_r$ and a group of transient states T. Let $n(C_i)$ denote the number of elements in C_i.

Lemma 3: The recursive equations (5.3) converge if and only if there exist feedback control matrices $\underline{G}_{k(t)}$ and there exist positive definite matrices $\underline{H}_i$, $i \varepsilon I$, such that

$$\underline{H}_j = \sum_{t=1}^{\infty} p_{jj}^{t-1} \left(\underline{A} + \underline{B}_j\underline{G}_j\right)^{t-1^T} \left(\underline{Q} + \underline{G}_j^T \underline{R}\,\underline{G}_j\right)\left(\underline{A} + \underline{B}_j\underline{G}_j\right)^{t-1}$$

$$+ \sum_{\substack{k \varepsilon T \\ k \neq j}} \left(\underline{A} + \underline{B}_j\underline{G}_j\right)^{t-1^T} \left(\left(\underline{A} + \underline{B}_k\underline{G}_j\right)^T H_k \left(\underline{A} + \underline{B}_k\underline{G}_j\right)\right)\left(\underline{A} + \underline{B}_j\underline{G}_j\right)^{t-1} \tag{5.6}$$

Proof: See Appendix.

This lemma is a restatement of the equivalence of Theorem 2 statements i) and iii), but with a different proof explicitly involving the Markov chain's structure. The expected cost J_T converges to a weighted sum of the matrices $\underline{H}_i$, $i \in I$ as $T \to \infty$, if it converges, and since it is the optimal cost, it must be bounded by the same weighted sum if the $\underline{H}_i$, $i \in I$ exists. Although necessary and sufficient conditions for the existence of a bounded solution in terms of the system dynamics are unknown, the following Lemma supplies sufficient conditions on the matrices $\underline{A}$, $\underline{B}$, and $\underline{P}$.

Lemma 4: The optimal expected cost is bounded for all t if there exist feedback control laws $\underline{F}_k$, k ε I, such that

1. For every state k in C_1, ..., C_r:

   If $p_{kk} < 1$, then

$$\left(1 - p_{kk}\right) \sum_{t=1}^{\infty} p_{kk}^{t-1} \| \underline{A} - \underline{B}_k \underline{F}_k \|^{t-1} \max_j \| \underline{A} - \underline{B}_j \underline{F}_k \| \leq c < 1 \tag{5.7}$$

where $\| \underline{A} \|$ is the maximum singular value of the matrix $\underline{A}$.

If $p_{kk} = 1$, then

$$\sum_{t=1}^{\infty} \| \underline{A} - \underline{B}_k \underline{F}_k \|^{t-1} < \infty \tag{5.8}$$

2. For every transient state k in T, let

$$p(k) = 1 - \sum_{j \varepsilon T} p_{kj} . \tag{5.9}$$

Let $\overline{p} = \min p(k)$. Then

$$\sum_{t=1}^{\infty} \left(1 - \overline{p}\right)^{t-1} \left\{ \max \| \underline{A} - \underline{B}_k \underline{F}_j \| \right\}^{t} < \infty \tag{5.10}$$

Proof: See Appendix.

Note that the sufficiency conditions in Lemma 4 allow the system to have structures for which no stabilizing control gain can be designed. However, the overall system can be considered reliable if the time spent in these structures is sufficiently small, as indicated by the tradeoffs between the singular values of the closed-loop matrices and the self-return probabilities p_{kk}.

6. Implications of the Solution

The existence of a steady-state solution to the switching gain problem establishes a division of system designs into those which are inherently reliable and those which are unreliable. Even though conditions to test for the existence of the steady-state solution are unavailable, software can be used with iteration for the test.

As mentioned earlier, cost stability is the appropriate definition of stability for this problem.

Definition 6: (Cost stability). The set of constant gains $\{\underline{G}_i\}_{i \varepsilon I}$ stabilizes the system (3.2) using the control law

$$\underline{u}_t = \underline{G}_k \underline{x}_t \tag{6.1}$$

where k is determined by Lemma 1 if and only if the scalar random variable

$$\sum_{t=0}^{\infty} \underline{x}_t^T \underline{Q} \underline{x}_t + \underline{u}_t^T \underline{R} \underline{u}_t < \infty \tag{6.2}$$

with probability one.

If the infinite time horizon control problem is defined as the minimization of

$$J = \lim_{T \to \infty} J_T \tag{6.3}$$

then the steady-state values of the gains calculated by equations (5.3) and (5.5) provide the minimizing control law for equation (6.3); furthermore, the $\underline{S}_{k,\,t}$ converge if and only if a solution to equation (6.3) exists.

In addition, the existence of a cost stabilizing set of gains $\{\underline{G}_i\}_{i \, \varepsilon \, I}$ is equivalent to the existence of the infinite time horizon solution. These results are summarized in the following theorem.

Theorem 1: The following statements are equivalent:

i) Equations (5.3) converge to steady-state values $\underline{S}_k$ as $T \rightarrow \infty$ (or $t \rightarrow -\infty$ for fixed T).

ii) The steady-state set of gains $\{\underline{G}^*\}_{k \, \varepsilon \, I}$ from equations (5.5) cost stabilizes the system described by equations (3.2) and (6.1).

iii) A set of gains $\{\underline{G}_k\}_{k \, \varepsilon \, I}$ exists for which J_T is bounded.

Proof: See Appendix.

7. Example

In this Section, a two-dimensional example is presented with three different switching gain solutions to illustrate the switching gain computational methodology. The computer routines which are used in the calculation of the switching gain solution are documented in [11].

The example is a two-dimensional system with four structural states corresponding to the failure modes of two actuators. In this example, failure of an actuator is modeled as an actuator gain of zero. Thus, the four structures are: I) Both actuators working ($\underline{B}_0$); ii) One actuator failed ($\underline{B}_1$ and $\underline{B}_2$), and III) Both actuators failed ($\underline{B}_3$). The system is controllable in all structures except for the structure represented by $\underline{B}_3$.

Although this example exhibits a very simple structure which models only actuator failure and self-repair, note that the Markov chain formulation does not restrict the configurations of actuators in any structural state. Therefore, this methodology can be used to model and control systems with arbitrary failure, repair, replacement, and reconfiguration structures. Neither is there any restriction that failure and repair/reconfiguration be accomplished within a single structural transition. Therefore, actuator degradation can be modeled as a sequence of discrete failures. The same technique can be applied to repair/reconfiguration modeling.

Actuator failures and repairs are assumed to be independent events with probabilities of failure and repair, per unit time, of p_f and p_r, respectively, for both actuators. Note that only exponential failure/repair distributions can be represented.

The matrices $\underline{Q}$ and $\underline{R}$ are the quadratic weighting matrices for the state $\underline{x}_t$ and the control u_t, respectively. The matrix $\underline{P}$ is the Markov transition matrix, which is calculated from knowledge of the system configuration dynamics, represented graphically in Figure 2.

There are three cases in the example. Each case assumes a different failure rate and repair rate for the actuators. Case i) has a high probability of failure and a low probability of repair, relative to Cases ii) and iii). The switching gain solution is not convergent for Case i); the gains themselves converge, but the expectecd costs do not. Only configuration state 0 is stabilized with its corresponding gain, $\underline{G}_0$.

Cases ii) and iii) both assume more reliable actuators than does Case i). Both Cases ii) and iii) have convervent switching gain solutions. Therefore, both Cases ii) and iii) represent reliable configuration designs, while Case i) is unreliable. This difference is due entirely to the different component reliabilities. Equivalently, Cases ii) and iii) are stabilized by the switching gain solution, while Case i) is not. Note that in this Example, stabilizability is not equivalent to stability in each configuration state, or robustness. For this example, no robust gain exists because the system is uncontrollable from configuration state 3.

Case ii) is interesting in that neither the cost nor the gain matrix depends on the structural state. This occurs when all the columns of the Markov transition matrix $\underline{P}$ are equal. In this case, the on-line implementation is simplified; no switching or detection of structural transitions is required.

System and Cost Matrices

$$\underline{A} = \begin{bmatrix} 2.71828 & 0.0 \\ 0.0 & .36788 \end{bmatrix}$$

$$\underline{B}_o = \begin{bmatrix} 1.71828 & 1.71828 \\ -.63212 & .63212 \end{bmatrix} \qquad \underline{B}_1 \begin{bmatrix} 0.0 & 1.71828 \\ 0.0 & .63212 \end{bmatrix}$$

$$\underline{B}_2 = \begin{bmatrix} 1.71828 & 0.0 \\ -.63212 & 0.0 \end{bmatrix} \qquad \underline{B}_3 = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

$$\underline{Q} = \begin{bmatrix} 14. & 8. \\ 8. & 6. \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

Markov Transition Matrix

$$\underline{P} = \begin{bmatrix} 1-2p_f+p_f^2 & (1-p_f)p_r & (1-p_f)p_r & p_r^2 \\ p_f(1-p_f)^2 & 1-p_f-p_r+p_fp_r & p_rp_f & p_r(1-p_r) \\ p_f(1-p_f)^2 & p_rp_f & 1-p_f-p_r+p_fp_r & p_r(1-p_r) \\ p_f^2 & (1-p_r)p_f & (1-p_r)p_f & 1-2p_r+p_r{}^2 \end{bmatrix}$$

The system dynamics are

$$\underline{x}_{t+1} = \underline{A}\,\underline{x}_t + \underline{B}_{k(t)}\,\underline{u}_t \quad ; \quad \underline{x}_t = [x_{1,t} \;\; x_{2,t}]^T$$
$$k(t)\,\varepsilon\,\{0,1,2,3\}$$

The cost, which is to be minimized, is

$$J = E\left[\sum_{t=0}^{\infty} \underline{x}_t^T\,\underline{Q}\,\underline{x}_t + \underline{u}_t^T\,\underline{R}\,\underline{u}_t \mid \underline{\pi}\right]$$

Case i)

$$p_f = .3\,;\; p_r = .7 \quad ; \quad \underline{\pi} = \begin{bmatrix} .49 \\ .21 \\ .21 \\ .09 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$

The coupled Riccati equations are non-convergent, but the gains converge:

$$\underline{G}_0 = \begin{bmatrix} -.9636 & 0 \\ -.9134 & 0 \end{bmatrix}$$

$$\underline{G}_1 = \begin{bmatrix} -.9234 & 0 \\ -.8699 & 0 \end{bmatrix}$$

$$\underline{G}_2 = \begin{bmatrix} -.8094 & 0 \\ -1.020 & 0 \end{bmatrix}$$

$$\underline{G}_3 = \begin{bmatrix} -.9636 & 0 \\ -.9134 & 0 \end{bmatrix}$$

Stability tests:

Configuration	Stable
0 ($\underline{B}_0$)	yes
1 ($\underline{B}_1$)	no
2 ($\underline{B}_2$)	no
3 ($\underline{B}_3$)	no

Case ii)

$$p_f = .1; \quad p_r = .9 \qquad \underline{\pi} = \begin{bmatrix} .81 \\ .09 \\ .09 \\ .01 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$

The coupled Riccati equations coverge:

$$\underline{G}_i = \begin{bmatrix} -.8890 & .04222 \\ -.7752 & -.9914 \end{bmatrix}$$

for i = 0,1,2,3

$$\underline{S}_i = \begin{bmatrix} 25.57 & 8.611 \\ 8.611 & 6.398 \end{bmatrix}$$

Stability tests:

Configuration	Stable
0 ($\underline{B}_0$)	yes
1 ($\underline{B}_1$)	no
2 ($\underline{B}_2$)	no
3 (B_3)	no

$p_f = .1;\quad p_r = .9\;;$

$$\underline{\pi} = \begin{bmatrix} .9799 \\ .009999 \\ .009999 \\ .0001020 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$

The coupled Riccati equations coverge:

$$\underline{G}_0 = \begin{bmatrix} -.7558 & .1270 \\ -.8073 & -.1786 \end{bmatrix}$$

$$\underline{S}_0 = \begin{bmatrix} 15.88 & 8.105 \\ 8.105 & 6.137 \end{bmatrix}$$

$$\underline{G}_1 = \begin{bmatrix} -.7060 & .1186 \\ -.8441 & -1.723 \end{bmatrix}$$

$$\underline{S}_1 = \begin{bmatrix} 16.06 & 8.074 \\ 8.074 & 8.143 \end{bmatrix}$$

$$\underline{G}_2 = \begin{bmatrix} -.8375 & .1090 \\ -.7543 & -.1669 \end{bmatrix}$$

$$\underline{S}_2 = \begin{bmatrix} 16.31 & 8.199 \\ 8.199 & 6.158 \end{bmatrix}$$

$$\underline{G}_3 = \begin{bmatrix} -.7863 & .1023 \\ -.7926 & -.1619 \end{bmatrix}$$

$$\underline{S}_3 = \begin{bmatrix} 16.54 & 8.170 \\ 8.170 & 6.162 \end{bmatrix}$$

Stability tests:

Configuration	Stable
0 ($\underline{B}_0$)	yes
1 ($\underline{B}_1$)	no
2 ($\underline{B}_2$)	no
3 ($\underline{B}_3$)	no

8. Conclusions

The concepts which allow component reliability to influence control system design in a consistent manner have been defined. When specialized to linear systems with quadratic cost functions, an optimal control problem can be defined. The resulting control law depends on the system structure, the structural dynamics, and the system dynamics. The solution to the optimal control problem defines the boundary between reliable (stabilizable) designs and unreliable designs.

In closing, we also note that the restriction that all structural changes occur in the actuator matrix can be easily removed. In this case, a structural state is completely defined by $\underline{A}_k$ and $\underline{B}_k$, rather than by $\underline{B}_k$ alone. The results in this paper are directly extendible to this case. Many of the details are available in [12].

9. Appendix

A1. Proof of Lemma 1.

Assume $\underline{x}_{k,t+1} = \underline{x}_{\ell,t+1}$ for $k \neq \ell$. Then $(\underline{B}_k - \underline{B}_\ell)\underline{u}_{t-1} = 0$, which implies $\underline{u}_{t-1}$ is in the null space of $\underline{B}_k - \underline{B}_\ell$, $N(\underline{B}_k - \underline{B}_\ell)$. Now, dimension$(N(\underline{B}_k - \underline{B}_\ell)) < m$ because the $\underline{B}_k$'s are distinct. Therefore,

$$\text{dimension}\Big(\bigcup_{\substack{k,\ell \\ k \neq \ell}} N(\underline{B}_k - \underline{B}_\ell)\Big) < m \tag{A1.1}$$

Therefore the set $\bigcup_{\substack{k,\ell \\ k \neq \ell}} N(\underline{B}_k - \underline{B}_\ell)$ has measure zero in R^m. Q.E.D.

A.2 Optimal Solution for Deterministic Problem.

For the system described in Section 3, from dynamic programming, the optimal cost-to-go at time t is given by equation (5.1). Assume the optimal cost-to-go at time t, given the structure index k(t-1) at time t-1, is quadratic:

$$V^*(\underline{x}_t, k(t-1), t) = \underline{x}_t^T \underline{S}_{k,t} \underline{x}_t \tag{A2.1}$$

This assumption will be verified by induction. Then

$$\underline{x}_t^T \underline{S}_{k,t} \underline{x}_t = \min_{\underline{u}_t = \underline{\Phi}_t(\underline{x}_t)} \Big\{ \underline{x}_t^T \underline{Q}\, \underline{x}_t + \underline{u}_t^T \underline{R}\, \underline{u}_t + \sum_{i=0}^{L} p_{ik} (\underline{A}\, \underline{x}_t + \underline{B}_i \underline{u}_t)^T \underline{S}_{i,t+1} (\underline{A}\, \underline{x}_t + \underline{B}_i \underline{u}_t) \Big\} \tag{A2.2}$$

and

$$
\text{(A2.2)} = \min_{\underline{u}_t = \underline{\phi}_t(\underline{x}_t)} \Big\{ \underline{x}_t^T \underline{Q}\, \underline{x}_t + \underline{u}_t^T \underline{R}\, \underline{u}_t + \sum_{i=0}^{L} p_{ik} \Big[\underline{x}_t^T \underline{A}^T \underline{S}_{i,t+1} \underline{A}\, \underline{x}_t + \underline{u}_t^T \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \underline{u}_t + \underline{x}_t^T \underline{A}^T \underline{S}_{i,t+1} \underline{B}_i \underline{u}_t + \underline{u}_t^T \underline{B}_i^T \underline{S}_{i,t+1} \underline{A}\, \underline{x}_t \Big] \Big\} \tag{A2.3}
$$

Differentiating the r.h.s. of equation (A2.3) w.r.t. u_t and setting it equal to zero:

$$
0 = 2 \underline{R}\, \underline{u}_t + \sum_{i=0}^{L} p_{ik} \Big[2 \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \underline{u}_t + 2 \underline{B}_i^T \underline{S}_{i,t+1} \underline{A}\, \underline{x}_t \Big] \tag{A2.4}
$$

or

$$
\underline{u}^*_{k(t-1),t} = - \Big[\underline{R} + \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \Big]^{-1} \cdot \sum_{i=0}^{L} p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{A}\, \underline{x}_t \tag{A2.5}
$$

is the optimal $\underline{u}^*$, given k(t-1).

Since no noise is present in the system, k(t-1) is obtained from $\underline{x}_t$ and $\underline{x}_{t-1}$, along with $\underline{u}_{t-1}$, as

$$
k(t-1) = i \text{ iff } \underline{x}_t = \underline{A}\, \underline{x}_{t-1} + \underline{B}_i \underline{u}_{t-1} \tag{A2.6}
$$

Substituting equation (A2.5) into equation (A2.3), and eliminating $\underline{x}_t$ because the equation must be true for all $\underline{x}_t$, and the matrix equation is symmetric, on simplification we obtain equation (5.3), which verifies assumption (A2.1) by induction, along with the initial condition

$$\underline{S}_{k,T} = \underline{Q} \tag{A2.7}$$

A3. Proof of Lemma 2.

Consider the optimization of the cost-to-go given k(t-1) at time t with final time T. This optimal cost-to-go is simply

$$V_T^*(\underline{x}_t, k(t-1), t) \tag{A3.1}$$

where T denotes the final time. For the process with final time T + 1, the optimal cost-to-go is

$$\begin{aligned} &V_{T+1}^*(\underline{x}_t, k(t-1), t) \\ &= E\left\{ \sum_{\tau=t}^{T} \underline{x}_\tau{}^T \underline{Q}\, \underline{x}_\tau + \underline{u}_\tau^T \underline{R}\, \underline{u}_\tau + \underline{x}_{T+1}^T \underline{Q}\, \underline{x}_{T+1} \,\middle|\, k(t-1) \right\} \end{aligned} \tag{A3.2}$$

Since this optimal sequence is not necessarily optimal for the problem with final time T, it must not incur less cost over {t,...,T}.

$$\begin{aligned} &V_{T+1}^*(\underline{x}_t, k(t-1), t) \\ &\geq V_T^*(\underline{x}_t, k(t-1), t) \\ &+ E\left\{ \underline{u}_T^T \underline{R}\, \underline{u}_T + \underline{x}_T^T \underline{Q}\, \underline{x}_{T+1} \,\middle|\, k(t-1) \right\} \end{aligned} \tag{A3.3}$$

Since the expectation term of equation (A3.3) is non-negative,

$$V_{T+1}^*(\underline{x}_t, k(t-1), t) \geq V_T^*(\underline{x}_t, k(t-1), t) \tag{A3.4}$$

Now, note that

$$V_T^*(\underline{x}_t, k(t-1), t) = \underline{x}_t^T \underline{S}_{i,t_T} \underline{x}_t \tag{A3.5}$$

and that equation (5.3) depends only on the number of iterations (T-t) for the calculation of $\underline{S}_{i,t_T}$, and therefore,

$$V_T^*(\underline{x}_t, k(t-1), t-1) = V_{T+1}^*(\underline{x}_t, k(t-1), t) \tag{A3.6}$$

Therefore, $\{S_{i,t}\}_{=T}$ is an increasing sequence in that

$$\underline{S}_{i,t-1} - \underline{S}_{i,t} \geq 0 \tag{A3.7}$$

Since, by hypothesis, V^* is bounded over t, the $\underline{S}_{i,t}$ converge.

A4. Proof of Lemma 3.

Equation (5.6) implies that the gains $\underline{G}_k$ result in a finite cost-to-go, expressed as an average of the matrices $\underline{H}_i$. Hence, the optimal cost is also finite, and bounded, so equation (5.3) converges. Similarly, if equation (5.3) converges, selecting $\underline{H}_k = \underline{S}_k$ and $\underline{G}_k$ according to equation (5.5) satisfies equation (5.6).

A5. Proof of Lemma 4.

Assume that the control gains $\underline{F}_k$ are used. Let τ be the time of first exit from state k. Assume τ_0 is finite with probability 1. Otherwise, $p_{kk} = 1$ and equation (5.8) applies. Then, equation (5.7) establishes that

$$\|\underline{x}_{\tau_0}\| \leq c\|\underline{x}_\tau\|$$

(A5.1)

Let

$$s = \max \| \underline{Q} + \underline{F}_j^T \underline{R} \underline{F}_j \|$$

(A5.2)

Then, the cost incured while in state k is bounded above by

$$\frac{c s \| \underline{x}_o \|^2}{1 - p_{kk}} \; .$$

(A5.3)

Consider now the new state at τ_o, and denote the time of first exit τ_1. By similar reasoning, we construct the sequence $\tau_o, \ldots, \tau_n, \ldots$.

Let C(k) be the communicating class of state k, and

$$q = \max_{j \, \varepsilon \, C(k)} p_{jj} \; .$$

(A5.4)

The overall average cost incurred can be partitioned in terms of the costs incurred between transits τ_i and τ_{i+1}, as

$$E \left\{ \sum_{t=0}^{\infty} \underline{u}_t^T \underline{R} \underline{u}_t + \underline{x}_t^T \underline{Q} \underline{x}_t \right\}$$

$$= E \left\{ \sum_{t=0}^{\tau_1 - 1} \left(\underline{u}_t^T \underline{R} \underline{u}_t + \underline{x}_t^T \underline{Q} \underline{x}_t \right) + \sum_{t=\tau_1}^{\tau_2 - 1} \left(\underline{u}_t^T \underline{R} \underline{u}_t + \underline{x}_t^T \underline{Q} \underline{x}_t \right) + \cdots \right\}$$

(A5.5)

$$\leq \frac{c s \| \underline{x}_o \|^2}{1 - q} + \frac{cs}{1 - q} E \left\{ \| \underline{x}_{\tau_1} \|^2 \right\} + \cdots$$

(A5.6)

and equation (5.7) implies

$$(A5.6) \le \frac{c\,s\|\underline{x}_o\|^2}{1-q}\,(1 + c + c^2 + \cdot\ \cdot\ \cdot\ )$$

(A5.7)

which is finite since $c < 1$.

If $p_{kk} = 1$, equation (5.9) establishes that, from structure k at time to,

$$E\left\{\sum_{t=t_o}^{\infty}\left(\underline{u}_t^T \underline{R}\,\underline{u}_t + \underline{x}_t^T \underline{Q}\,\underline{x}_t\right)\right\} \le \sum_{t=t_o}^{\infty} \|\underline{A} + \underline{B}_k \underline{F}_k\|^t < \infty\ .$$

(A5.8)

Hence, equations (5.7) and (5.8) establish that, for any initial state $k(t_o)$, $\underline{x}(t_o)$ in a closed communicating class, the cost-to-go is finite. To show the overall cost is finite, we must establish that from any initial transient state, the cost incurred until a closed communicating class is reached is finite.

Let $\tau(k)$ denote the time of first exit from T starting at $k \in T$. The expected cost incurred while in T is

$$E\left\{\sum_{t=0}^{\tau} \underline{x}_t^T\left(\underline{F}_{k(t)}^T \underline{R}\,\underline{F}_{k(t)} + \underline{Q}\right)x_t \;\middle|\; \underline{x}_o, k(o) = k\right\}$$

$$\le s E\left\{\sum_{t=o}^{\tau} \|\underline{x}_t\|^2 \;\middle|\; \underline{x}_o, k(o) = k\right\}$$

$$\le s\|\underline{x}_o\|^2 E\left\{\sum_{t=o}^{\tau}\left\{\max\|\underline{A} + \underline{B}_k \underline{F}_j\|\right\}^t \;\middle|\; k(o) = k\right\}.$$

(A5.9)

But, from the definition of p,

$$\text{Prob}\{\tau > n\} \leq (1 - p)^n .$$

(A5.10)

Hence,

$$E\left\{ \sum_{t=o}^{\tau} \underline{x}_t^T \left(\underline{F}_{k(t)}^T \underline{R}\, \underline{F}_{k(t)} + \underline{Q} \right) \underline{x}_t \;\middle|\; \underline{x}_o, k(o) = k \right\}$$

$$\leq s \| \underline{x}_o \|^2 \sum_{t=1}^{\infty} (1 - \bar{p})^t \left(\max \| \underline{A} + \underline{B}_k \underline{F}_j \| \right)^t < \infty$$

(A5.11)

by equation (5.10). Hence, the gains $\underline{F}_j$ result in finite expected cost for all initial states. The optimal expected cost-to-go will be bounded in t by this cost.

A6. Proof of Theorem 1.

i) ⇒ ii):

Suppose $\{\underline{G}^*\}_{k \varepsilon I}$ were not cost stablizing. Then for some set *M* of non-zero measure of structural trajectories (k(0), k(1), ...), J^*_T on that set is not bounded. But

$$J_T^* \geq \int_M J_T^*(m)\, dp(m) \rightarrow m \quad \text{as}\, T \rightarrow \infty$$

(A6.1)

therefore, *M* must be of measure zero.

ii) ⇒ iii): The steady-state gains $\{\underline{G}^*\}_{k \varepsilon I}$ satisfy iii).

iii) ⇒ i): By assumption, there exists a B such that

$$J_T\left(\left\{ \underline{G}_k \right\}_{k \varepsilon I} \right) < B \text{ for all } T$$

(A6.2)

Since

$$J_T^* \le J_T\left(\left\{\underline{G}_k\right\}_{k \varepsilon I}\right) < B \text{ for all } T\ , \tag{A6.3}$$

Statement i) is implied by Lemma 2.

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