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A DECOMPOSITION METHOD FOR THE APPROXIMATE EVALUATION OF CAPACITATED TRANSFER LINES WITH UNRELIABLE MACHINES AND RANDOM PROCESSING TIMES

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ABSTRACT

This paper presents a decomposition method to evaluate the performance measures of a capacitated transfer line with unreliable machines and random processing times. The decomposition is based on approximating the (k-1)-buffer system by k-1 single-buffer systems. Numerical examples indicate that the approach is viable as long as the probability that a machine is starved and blocked at the same time is small.

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1. INTRODUCTION

This paper presents a decomposition method to evaluate the performance of a capacitated transfer line with unreliable machines and random processing times. The method follows that of Gershwin (1983). It is based on a model which approximates a (k-1)-buffer system by k-1 single buffer systems. The parameters of the single-buffer systems are determined by relationships among the flows through the buffers of the original system. An iterative search algorithm is then developed to calculate the throughput rate and the average buffer levels.

Figure 1 depicts a transfer line with a series of k servers or machines $(M_1, M_2, \ldots, M_k)$ separated by queues or buffers $(B_1, B_2, \ldots, B_{k-1})$. The buffers are each of finite storage capacity $(C_1, C_2, \ldots, C_{k-1})$. Material flows from outside the sytem to M_1, then to B_1, and through all the machines and buffers in sequence until it exits via M_k.

Each machine spends a random amount of time processing each item. The randomness of M_i is characterized by three exponentially distributed random variables: the service time (with mean $1/\mu_i$), the time to fail (with mean $1/p_i$) and the time to repair (with mean $1/r_i$). When machine M_i is in a failure condition or taking a relatively long time to process an item, buffer B_{i-1} tends to accumulate material and buffer B_i tends to lose material. If this condition persists, it will lead to *blockage* of machine M_{i-1} and *starvation* of machine M_{i+1}. M_{i-1} is blocked when it has finished a piece and finds that there is no place to tranfer it. M_{i+1} is starved when there is no piece for it to process.

The great dimensionality of the state space renders the analysis of such system a formidable task. Each machine can be in one of two states: *operational* or *under repair*. Buffer B_i can be in one of C_i+3 states: $n_i = 0, 1, \ldots C_i$, C_i+1, C_i+2. C_i is the actual storage capacity of buffer B_i and n_i is the level of material in transit between M_i and M_{i+1}. It is convenient to define $N_i = C_i+2$ as an extended storage capacity between M_i and M_{i+1}. The distinction is explained in Section 2.

As a consequence, the Markov chain representation of a k-machine line with k-1 buffers has a state space of dimensionality

$$2^k \prod_{i=1}^{k-1} (C_i+3).$$

A 20-machine line with 19 buffers each of storage capacity 8, for example, has over 6.41×10^{25} states.

1.1 Decomposition

As shown in Figure 2, the transfer line L is decomposed into a set of two-machine lines L(i), i=1,...,k. Their buffers have the same capacities as those of L in Figure 1.

Pseudo-machine $M_u(i)$ models the part of the line upstream of B_i and $M_d(i)$ models the part of line downstream from B_i. We assume that the random behaviors governing flow into and out of B_i can be characterized by six exponentially distributed random variables with parameters: $\mu_u(i)$, $p_u(i)$, $r_u(i)$; and $\mu_d(i)$, $p_d(i)$ and $r_d(i)$, respectively. The key to the decomposition method is to find these parameters so that the material flow into and out of the buffers of the two-machine lines closely matches the flow into and out of the corresponding buffers of the long line L.

Six equations per buffer, or 6(k-1) conditions are required to determine the parameters.

1.2 Literature Survey

The transfer line model analyzed here is an extension of Gershwin and Berman's (1981) two stage line model. Related literature on the modeling and analysis of transfer lines is documented in that paper.

The concept of approximate decomposition of transfer lines was discussed by Hunt (1956), Hillier and Boling (1956), Takahashi *et al.* (1980), Altiok (1982, 1984), Altiok and Stidham (1983), Jafari (1984), Suri and Diehl (1983), Gershwin (1983), and others. In all papers surveyed except Gershwin's, the numerical method tends to sweep the line from one end (generally the upstream end) to the other. The symmetry of the line (as observed by Muth (1979) and Ammar (1980) has not previously been exploited. In addition, the interrupted nature of both the arrival and service processes in the decomposed line has not been considered.

A similar approach to ours was recently described by Sastry (1985), who also presented an extensive literature survey.

1.3 Contribution of this paper

Gershwin's (1983) decomposition method applies to transfer lines with constant, identical service times. This paper adapts that method to transfer lines with random service times. The accuracy of this method is evaluated via numerical simulation.

1.4 Outline

Section 2 describes the assumption and model of the production lines. Section 3 describes the derivation of the 6(k-1) equations. Section 4 evaluates the accuracy of the decomposition method. Conclusions and new research directions are discussed in Section 5.

2. MODEL DESCRIPTION AND ASSUMPTIONS

The development here is an extension of the 2-machine line model in Gershwin and Berman (1981). The storage and blockage condition are explicitly defined in response to the comment raised by Altiok and Stidham (1982).

2.1 Machine Operations

A machine can be in one of two states: *operational* or *under repair*. The machine state is denoted by the binary variable α as

$$\alpha_i = \begin{cases} 1, & \text{when } M_i \text{ is operational;} \\ 0, & \text{when } M_i \text{ is under repair.} \end{cases}$$

When a machine goes from state 1 to state 0, it is said to *fail*. A *repair* takes place when the transition from α=0 to α=1 occurs. When a machine is in state 1, it can process workpieces only when it is not starved or blocked. It processes no pieces when it is under repair. After the machine is repaired, it resumes work on the same piece it was working on when the failure occured. It is assumed here that the first machine is never starved and the last machine is never blocked.

Since blocked or starved machines are not processing, they are not vulnerable to failure. This assumption differs from Wijngaard (1979), who allows failures of idle machines.

Service, failure and repair times for M_i are assumed to be exponential random variables with parameters μ_i, p_i, and r_i; $i=1,\ldots,k$, the service rate, failure rate, and repair rate, respectively. When a machine is under repair, it remains in that state for a period of time which is exponentially distributed with mean $1/r_i$. This period is unaffected by the states of the other machine or of the storage.

When a machine is operational, it operates on a piece if it is not starved or blocked. It continues operating until either it finishes or a failure occurs, whichever happens first. Either event can happen during the time interval $(t,t+\delta t)$ with probability approximately $\mu_i\delta t$ or $p_i\delta t$ respectively, for small δt.

The repair and workpiece completion models are similar to those of Buzacott (1972). However, in Buzacott's failure model, the probability of a failure before a completion is independent of the time spent on the piece. Here, instead, the longer an operation takes, the more likely it is that a failure occurs before the work is complete. Buzacott's model would seem, therefore, to be appropriate where the predominant cause of failure is the transfer mechanism, clamping, or some other action that takes place exactly once during an operation. The model presented here would seem to better represent failures in mechanisms that are vulnerable during an entire operation.

2.2 Buffer State and Blockage Convention

The state of buffer B_i is denoted by the integer n_i. This is the number of pieces in buffer B_i plus the piece in machine M_{i+1}. When M_i is blocked, n_i also includes the finished piece in M_i. The storage capacity between M_i and M_{i+1} is denoted by C_i+1 which includes one space in M_{i+1} .

Machine M_i is blocked at the instant when it has completed a piece and there is no storage space in B_i. The convention for blockage is n_i=C_i+2. Gershwin and Berman have used this same convention in their mathematical formulation of a two-machine line. There, the blockage convention is (n_i=N_i) where N_i=C_i+2. This definition of blockage agrees with the one proposed by Altiok and Stidham.

Machine M_i is starved at the instant when it has completed a piece and there is no piece in B_{i-1}.

In the following, we use N_i to denote the storage capacity in between machines M_i and M_{i+1}.

The state s of the continuous time Markov chain model of the transfer line is denoted by

$$\{n_1,\ldots,n_{k-1},\alpha_1,\ldots,\alpha_k\}.$$

2.3 Performance Measures

The probability that machine M_i is processing a workpiece is termed the *efficiency* E_i and is given by

$$E_i = \text{prob}[\alpha_i=1,\ n_{i-1}>0,\ n_i<N_i]. \qquad (1)$$

The *production rate* (throughput rate) of machine M_i in parts per time unit, is

$$P_i = \mu_i E_i. \tag{2}$$

The mean buffer level $\bar{n}_i$ is defined as the average of materials in transit between the machines M_i and M_{i+1}. It is given by

$$\bar{n}_i = \sum_s n_i \text{ prob}(s). \tag{3}$$

Formulas for these and related quantities for two-machine lines can be found in Gershwin and Berman (1981).

2.4 Characteristics of the transfer line.

Conservation of Flow

Since there is no creation nor destruction of workpieces, the flow is conserved. That is

$$P = P_1 = P_2 = \ldots = P_k. \tag{4}$$

Flow Rate-Idle Time Relationship

Define e_i to be the *isolated efficiency* of machine M_i. It is given by (Buzacott, 1967)

$$e_i = \frac{r_i}{r_i+p_i} \tag{5}$$

and it represents the fraction of time that M_i is operational. The *isolated production rate*, ρ_i is then given by

$$\rho_i = \mu_i e_i. \tag{6}$$

and it represents what the production rate of M_i would be if it were never impeded by other machines or buffers. The actual production rate P_i is less because of blocking and starvation.

The efficiency satisfies

$$E_i = e_i \text{ prob}[\ n_{i-1}>0 \text{ and } n_i<N_i\ ]. \tag{7}$$

See Gershwin and Berman (1981), and Gershwin (1984).

The flow rate-idle time relations can be stated as

$$P_i = \rho_i \text{ prob}[\ n_{i-1}>0 \text{ and } n_i<N_i\ ]. \tag{8}$$

This follows from equation (1).

While it is possible for $n_{i-1}=0$ and $n_i=N_i$ simultaneously, it is not likely. This corresponds to an event in which M_i finishes a piece and finds that it is both blocked and starved. The probability of this event is small because such states can only be reached from states in which $n_{i-1}=1$ and $n_i=N_i-1$ by means of a transition in which

1. Machine M_{i-1} is either under repair, starved, or taking a long time to process a piece, and

2. Machine M_i completes an operation, and

3. Machine M_{i+1} is either under repair, blocked, or taking a long time to process a piece.

The production rate may therefore be approximated by

$$P_i = \rho_i \left(1 - \text{prob}[\ n_{i-1}=0\ ] - \text{prob}[\ n_i=N_i\ ]\right). \qquad (9)$$

However, if there is a great variation in efficiencies and service rates among the machines, this assumption may not hold and the method may break down. An example illustrates this in Section 4.

3. CHARACTERISTICS OF THE DECOMPOSITION

For every buffer B_i, the states of the line upstream of B_i are aggregated into two groups, represented by the *up* and *down* states of $M_u(i)$. A similar aggregation applies to the states of the line downstream of B_i. The decomposition method assumes that the transition between the up and down states of $M_u(i)$ ($M_d(i)$) can be characterized by three exponential processes with the parameters $\mu_u(i)$, $r_u(i)$, and $p_u(i)$, ($\mu_d(i)$, $r_d(i)$, and $p_d(i)$). This aggregation is not exact. It is adopted here to characterize the most important features of the behavior of the transfer line in a simple approximate way.

In this section, we derive the equations for the unknown quantities: $\mu_u(i)$, $r_u(i)$, $p_u(i)$, $\mu_d(i)$, $r_d(i)$ and $p_d(i)$, $i=1,\ldots,k$. They are based on conservation of flow (4), the flow rate-idle time relationship (9), and a set of equations (13), (14), (19), (20) and (28) developed below.

We first define the *up* and *down* states of the pseudo machines $M_u(i)$ and $M_d(i)$.

<u>Definitions : Up and Down</u>

With reference to Figure 3, $M_u(i)$ is <u>down</u> if, for some $j \le i$:

(1) M_j is down and,

(2) M_h is up, all h: $j<h\le i$; and

(3) $n_h=0$, $j\le h<i$.

Therefore the time interval when $M_u(i)$ is down is the period when the flow into B_i is interrupted due to an upstream machine failure. This is distinguished from the period during which the flow is interrupted due to a slow upstream machine while the intermediate buffers are empty. In that case, $M_u(i)$ is considered up.

An equivalent recursive definition is: $M_u(i)$ is <u>down</u> iff

(1) M_i is down, or

(2) $n_{i-1}=0$ and $M_u(i-1)$ is down.

$M_u(i)$ is up for all other states of the transfer line upstream of buffer B_i. Therefore $M_u(i)$ is <u>up</u> iff

(1) M_i is operational and $n_{i-1}>0$, or

(2) M_i is operational, $n_{i-1}=0$ but $M_u(i-1)$ is up.

Similarly, $M_d(i)$ is <u>down</u> iff

(1) M_{i+1} is under repair, or

(2) $n_{i+1}=N_{i+1}$ and $M_d(i+1)$ is down.

$M_d(i)$ is <u>up</u> iff

(1) M_{i+1} is operational and $n_{i+1}<N_{i+1}$, or

(2) M_{i+1} is operational, $n_{i+1}=N_{i+1}$ but $M_d(i+1)$ is up.

With the classification of the up and down states of the pseudo-machines and the exponential assumption of the transition between the states, the parameters $\mu_u(i)$, $r_u(i)$, $p_u(i)$, $\mu_d(i)$, $r_d(i)$ and $p_d(i)$ are the rates for the exponential distributed processes. They have the meanings:

$r_u(i)\ \delta t\ (r_d(i)\ \delta t)$: $\left\{\begin{array}{l}\text{probability that } M_u(i)\ (M_d(i)) \text{ goes from} \\ \text{down to up in } (t,t+\delta t) \text{ for small } \delta t;\end{array}\right.$

$p_u(i)\ \delta t\ (p_d(i)\ \delta t)$: $\left\{\begin{array}{l}\text{probability that } M_u(i)\ (M_d(i)) \text{ goes from} \\ \text{up to down in } (t,t+\delta t) \text{ for small } \delta t, \\ \text{given that } B_i \text{ is not full (empty) at } t;\end{array}\right.$

$\mu_u(i)\ \delta t\ (\mu_d(i)\ \delta t)$: $\left\{\begin{array}{l}\text{probability that a piece flows into} \\ \text{(out of) } B_i \text{ in } (t,t+\delta t) \text{ for small } \delta t \\ \text{when } M_u(i)\ (M_d(i)) \text{ is up,} \\ \text{given that } B_i \text{ is not full (empty) at } t.\end{array}\right.$

3.1 Derivation of Equations

Now we are ready to derive the 6(k-1) equations needed to characterize the transfer line.

Interruption of Flow

The first two sets of equations are based on the flow interruption phenomenon caused by machine failures.

By definition,

$$p_u(i)\ \delta t = \text{prob}\left[\ M_u(i) \text{ down at } t+\delta t \ \middle|\ M_u(i) \text{ up and } n_i<N_i \text{ at } t\ \right] \quad (10)$$

Substituting the definition of $M_u(i)$ down, we have

$$p_u(i)\delta t = \text{prob}\left[\begin{array}{l} \{M_i \text{ down at } t+\delta t\} \text{ or} \\ \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down at } t+\delta t\} \ \Big| \\ M_u(i) \text{ up and } n_i<N_i \text{ at } t \end{array}\right] \quad (11)$$

Since $\{M_i$ down$\}$ and $\{n_{i-1}=0$ and $M_u(i-1)$ down$\}$ are mutually exclusive events, equation (11) can be written as

$$p_u(i)\ \delta t = \text{prob}\left[\ M_i \text{ down at } t+\delta t \ \middle|\ M_u(i) \text{ up and } n_i<N_i \text{ at } t\ \right]$$

$$+ \text{ prob}\left[\begin{array}{l} n_{i-1}=0 \text{ and } M_u(i-1) \text{ down at } t+\delta t \ \Big| \\ M_u(i) \text{ up and } n_i<N_i \text{ at } t \end{array}\right] \quad (12)$$

The first term is the probability that M_i fails in $(t,\ t+\delta t)$ while processing a workpiece; that is, $p_i\delta t$.

Using Bayes' relationship, the second term can be written as

$$\frac{\text{prob}\left[\begin{array}{l} \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down at } t+\delta t\} \text{ and} \\ \{M_u(i) \text{ up and } n_i<N_i \text{ at } t\} \end{array}\right]}{\text{prob}[\ M_u(i) \text{ up and } n_i<N_i \text{ at } t\ ]}.$$

The denominator is the efficiency of $M_u(i)$; that is, $E_u(i)$.

With the definition of $M_u(i)$ up and the fact that the events $\{n_i<N_i\}$ and $\{n_i=N_i$ and $M_d(i)$ up$\}$ are mutually exclusive, the numerator can be written as

$$\text{prob}\begin{bmatrix} \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down at } t+\delta t\} \text{ and} \\ \{n_{i-1}>0 \text{ or } \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ up}\} \text{ at } t\} \text{ and} \\ \{M_i \text{ up and } \{n_i<N_i \text{ or } \{n_i=N_i \text{ and } M_d(i) \text{ up}\}\} \text{ at } t\} \end{bmatrix}$$

$$- \text{prob}\begin{bmatrix} \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down at } t+\delta t\} \text{ and} \\ \{n_{i-1}>0 \text{ or } \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ up}\} \text{ at } t\} \text{ and} \\ \{M_i \text{ up and } \{n_i=N_i \text{ and } M_d(i) \text{ up}\} \text{ at } t\} \end{bmatrix}.$$

The second term is zero because: (1) when M_i is blocked, the probability of transition from $\{n_{i-1}>0\}$ to $\{n_{i-1}=0\}$ is zero; and (2) we have assumed that the probability of the event $\{n_{i-1}=0$ and $n_i=N_i\}$ is zero.

The event

$\{ M_i$ up and $\{n_i<N_i$ or $\{n_i=N_i$ and $M_d(i)$ up$\}\} \}$

is the event $\{M_d(i-1)$ up$\}$. Hence the numerator is just the steady state transition probability of line L(i-1) into state (0,0,1). This is the same as the steady state transition probability out of state (0,0,1) and this is only possible via the "repair" of $M_u(i-1)$. Thus the numerator is the probability

$r_u(i-1)\ p(i-1;001)\ \ \delta t,$

where p(i-1;001) is the steady state probability that line L(i-1) is in state (0,0,1).

Therefore the parameter $p_u(i)$ is given by

$$p_u(i) = p_i + \frac{r_u(i-1)\ p(i-1;001)}{E_u(i)} \tag{13}$$

In a similar manner, we obtain

$$p_d(i) = p_{i+1} + \frac{r_d(i+1)\ p(i+1;N10)}{E_d(i+1)} \tag{14}$$

Here, p(i+1;N10) is the steady state probability that line L(i+1) is in state (N,1,0) and $E_d(i+1)$ is the efficiency of $M_d(i+1)$.

Resumption of Flow

This second pair of equations describes the recovery from machine failures. The derivation follows from that in Gershwin (1983) for transfer lines with constant service times.

By definition,

$$r_u(i)\ \delta t = \text{prob}\Big[\ M_u(i) \text{ up at } t+\delta t \ \Big|\ M_u(i) \text{ down and } n_i<N_i \text{ at } t\ \Big]. \tag{15}$$

Substituting the definition of $M_u(i)$ up, we get

$$r_u(i)\ \delta t = \text{prob}\left[\begin{array}{l} \left\{\begin{array}{l} M_i \text{ up and} \\ \{n_{i-1}>0 \text{ or } \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ up}\}\} \text{ at } t+\delta t \end{array}\right\} \ \Big| \\ \{M_u(i) \text{ down and } n_i<N_i \text{ at } t\} \end{array}\right]. \tag{16}$$

Using the equivalent notation

$$\begin{aligned} &\{n_{i-1}>0 \text{ or } \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ up}\}\} \\ &\qquad = \text{NOT}(n_{i-1}=0 \text{ and } M_u(i-1) \text{ down}), \end{aligned} \tag{17}$$

and decomposing the conditioning event, equation (16) can be written as

$$\begin{aligned} r_u(i)\ \delta t \ = \ &\text{prob}\left[\begin{array}{l} \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down and } n_i<N_i\} \text{ at } t \ \Big| \\ \{M_i \text{ down or } \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down}\} \\ \text{and } n_i<N_i\} \text{ at } t \end{array}\right] \\ &\cdot \text{prob}\left[\begin{array}{l} \{M_i \text{ up and NOT}(n_{i-1}=0 \text{ and } M_u(i-1) \text{ down}) \text{ at } t+\delta t\} \ \Big| \\ \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down and } n_i<N_i\} \text{ at } t \end{array}\right] \\ &+ \text{prob}\left[\begin{array}{l} \text{complement of the event of the first} \\ \text{factor in the first term} \end{array}\right] \\ &\cdot \text{prob}\left[\begin{array}{l} \{M_i \text{ up and NOT}(n_{i-1}=0 \text{ and } M_u(i-1) \text{ down}) \text{ at } t+\delta t\} \ \Big| \\ \{M_i \text{ down and } n_i<N_i \text{ at } t\} \end{array}\right] \end{aligned} \tag{18}$$

This decomposition is possible because $\{n_{i-1}=0\}$ and $\{M_i \text{ down}\}$ are disjoint events. This is because M_i cannot fail when it is starved from processing and when it fails, it has to be processing one piece of material. We now evaluate the four conditional probabilities in (18).

Using Bayes' rule, the first factor of the first term of (18) can be written as

$$\frac{\text{prob}\left[\begin{array}{l}\{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down}\} \text{ and} \\ \{M_i \text{ down or } \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down}\}\} \\ \text{and } \{n_i<N_i\} \text{ at } t\end{array}\right]}{\text{prob}\left[\ \{M_i \text{ down or } \{n_{i-1}=0 \text{ and } M_u(i-1) \text{ down}\} \text{ and } n_i<N_i\} \text{ at } t\ \right]}.$$

Noting that $\{M_i$ down$\}$ and $\{n_{i-1}=0\}$ are disjoint, the numerator is the probability that line L(i-1) is in state (0,0,1). This probability is p(i-1;001).

The denominator is the probability of the conditional event of (15), that is probability of $\{M_u(i)$ down and $n_i<N_i\}$. This probability can be calculated by using the relationship

$$r_u(i) \text{ prob}[\ M_u(i) \text{ down and } n_i<N_i\ ]$$
$$= p_u(i) \text{ prob}[\ M_u(i) \text{ up and } n_i<N_i\ ],$$

as given by Gershwin and Berman.

Thus, the denominator is

$$\text{prob}[\ M_u(i) \text{ down and } n_i<N_i\ ] = \frac{p_u(i)\ E_u(i)}{r_u(i)}.$$

Hence, the first factor of the first term of (18) is

$$\frac{p(i-1;001)r_u(i)}{p_u(i)\ E_u(i)} \equiv X_i\ .$$

Now the second factor of the first term of (18) is the probability of the transition of $M_u(i-1)$ being down to being up in $(t,t+\delta t)$; that is $r_u(i-1)\ \delta t$.

The first factor of the second term is just $1-X_i$.

The second factor of the second term of (18) is the probability of M_i been repaired in $(t,t+\delta t)$; that is $r_i\delta t$.

Therefore, equation (18) can now be written as

$$r_u(i) = r_u(i-1)\ X_i + r_i\ (1-X_i), \quad i=2,\ \ldots,\ k-1. \tag{19}$$

Equation (19) shows that $r_u(i)$ is a convex combination of r_i and $r_u(i-1)$.

A similiar analysis yields

$$r_d(j) = r_d(j+1)\, Y_j + r_{j+1}\,(1-Y_j), \quad j=1,\ \ldots,\ k-2 \qquad (20)$$

where

$$Y_j = \frac{p(j+1;N10)\ r_d(j)}{p_d(j)\ E_d(j)}.$$

Conservation of Flow

The conservation of flow states that the production rates of the decomposed two-machine lines, L(i) and the transfer line are the same. Thus

$$P = P_i = P_k = P(i) = P_u(i) = P_d(i),\ i=1,\ldots,k-1. \qquad (21)$$

Flow Rate-Idle Time Relationship

The flow rate-idle time relationship gives

$$P_i = e_i\,\mu_i\,(\ 1-\text{prob}[\ n_{i-1}=0\ ]\ -\ \text{prob}[\ n_i=N_i\ ]\ ) \qquad (22)$$

Since the buffers in the decomposed two-machine lines behave similarly to the corresponding buffers in the transfer line, we have

$$\text{prob}[n_{i-1} = 0] = p_s(i-1) \qquad (23)$$

and

$$\text{prob}[n_i = N_i] = p_b(i) \qquad (24)$$

where $p_s(i-1)$ is the probability the buffer in line L(i-1) becomes empty and $p_b(i)$ is the probability that $M_u(i)$ is blocked. These probabilities are calculated from the two-machine line formulas in Gershwin and Berman.

Gershwin and Berman show that the flow rate-idle time relationship for a two-machine line is

$$e_d(i-1)\ (\ 1-p_s(i-1)\ ) = \frac{P_d(i-1)}{\mu_d(i-1)} \tag{25}$$

and

$$e_u(i)\ (\ 1-p_b(i)\ ) = \frac{P_u(i)}{\mu_u(i)}\ . \tag{26}$$

Using these relationships and the conservation of flow, (22) can be written as

$$\frac{1}{e_i\ \mu_i} + \frac{1}{P} = \frac{1}{e_d(i-1)\ \mu_d(i-1)} + \frac{1}{e_u(i)\ \mu_u(i)}; \qquad i=2,\ldots,k-1. \tag{27}$$

Boundary Conditions

The remaining 6 equations are satisfied by the boundary conditions

$$\left.\begin{aligned} r_u(1) &= r_1 \\ p_u(1) &= p_1 \\ \mu_u(1) &= \mu_1 \\ r_d(k-1) &= r_k \\ p_d(k-1) &= p_k \\ \mu_d(k-1) &= \mu_k \end{aligned}\right\} \tag{28}$$

Sastry's (1985) method is quite similar to this one. The principal difference is that Sastry propogates the mean times to fail ($1/p_u$ and $1/p_d$) instead of the failure rates in the interruption of flow equations; and the mean times to repair ($1/r_u$ and $1/r_d$) instead of the repair rates in the resumption of flow equations.

3.2 Numerical Technique

There are a total of 6(k-1) equations among (13), (14), (19), (20), (21), (27) and (28) in 6(k-1) unknowns: $r_u(i)$, $p_u(i)$, $\mu_u(i)$, $r_d(i)$, $p_d(i)$, $\mu_d(i)$; $i=1, \ldots, k-1$.

The equations can be thought of as defining a two-point boundary value problem of the form

$$f(x_i, x_{i+1}) = 0;\ i=1,\ ..,\ k-1. \tag{29}$$

where x_i is a 6-vector of the parameters of the line L(i). The nonlinear function $f(\cdot,\cdot)$ involves the evaluation of $P(i)$, $p(i;N10)$ and $p(i;001)$ by means of the two-machine-line formulas in Gershwin and Berman.

We have developed an algorithm similar to that of Gershwin (1983). It makes use of the two-point boundary value problem structure, and works by iteratively evaluating the two-machine lines. This algorithm is used to analyze several transfer lines in the next section.

4. NUMERICAL RESULTS AND DISCUSSION

In this section, we report on numerical experience with the algorithm. We examined a variety of transfer lines, and we compared decomposition and simulation results. Simulations were all run for at least 100,000 time units.

4.1 Short Lines

We used the decomposition method to analyze a three-machine line. The only varying parameter is μ_2, which in turn causes ρ_2, the isolated production rate of M_2, to vary. The other parameters are chosen to make the line symmetric.

When ρ_2 is small, M_2 is the bottleneck; the production rate of the line is slightly less than ρ_2. As ρ_2 increases to infinity, the upper bound is given by the production rate of a two-machine line with the parameters of M_1 and M_3 and with buffer capacity equal to C_1+C_2 plus the one storage space in M_2. This upper bound is .258 parts per time unit.

The results are in Table 2. Figure 4 shows the variation of production rate with ρ_2. At the lower end, both the decomposition method and the simulation agree very well in terms of production rate and mean buffer levels. This agreement breaks down when ρ_2 exceeds 0.48 (about 1.5 times ρ_1 or ρ_3). The production rate from the simulation approaches the upper bound while that of the decomposition method exceeds this bound.

We believe the error in the decomposition method is due, in part, to the non-negligible probability of the joint event $\{n_1=0$ and $n_2=N_2\}$ when ρ_2 exceeds 0.48. The last column in Table 2 shows, from numerical simulation, that the fraction of time M_2 is simultaneously starved and blocked increases with ρ_2.

4.2 Longer Lines

Tables 3-9 list the results of the decomposition method on longer lines. The parameters of the machines are chosen to represent a typical range of variation of a transfer line. The efficiencies are between 85% and 95% and the isolated production rates vary from 0.18-1.28 pieces per time unit. The mean processing times in Tables 3 and 9 are about an order of magnitude larger that the mean repair times. The mean processing times in Tables 4-8 are on the same order as the mean repair times.

The results indicate the accuracy of the decomposition method in these cases. Typical errors in the throughput rate are less than 5%. The error does not seem to be sensitive to the length of the line.

4.3 Algorithm Behavior

When the algorithm converges, it often converges rapidly. For example, the three-machine case of Table 1 in which $\mu_2 = .5$ converged after 30 evaluations of the two-machine line, and the five machine case of Table 4 converged after 134 evaluations. The eight-machine cases of Tables 8 and 9 converged after 345 and 882 iterations, respectively. These evaluations are very rapid; they are the analytic formulas of Gershwin and Berman (1981).

There were many cases in which this algorithm failed to converge. By comparison, the very similar algorithm of Gershwin (1983) is quite stable. We conjecture that the difference is due to the Interruption of Flow equations ((13) and (14)). Further evidence comes from other experiments (not reported elsewhere) that were performed with the model of Gershwin (1983). The algorithm for that model was modified so that the Resumption of Flow equations were replaced by the Interruption of Flow equations. We were never able to make the modified algorithm converge.

4.4 Transfer Line Behavior

In Tables 10.1-10.4, a 3-machine line is used to show the relationship between machine efficiency and speed. Tables 10.1 lists the parameters of the base case; Tables 10.2-10.4 show the results as one parameter is varied at a time. The results suggest that machine efficiency contributes more to line production rate than machine speed does.

5. CONCLUSIONS AND FURTHER RESEARCH

A decomposition method has been developed to analyze capacitated transfer lines with unreliable machines and random processing times. Numerical examples show that this method is accurate as long as the probability that a machine is simultaneously starved and blocked is small. The accuracy of the algorithm does not seem to be sensitive to the length of the line. However, it does appear to be unstable and does not always converge.

Future research effort should be directed at incorporating into the decomposition method the event that a machine can be starved and blocked at the same time. An alternate effort is to quantify the error bound of the method. The convergence properties of the algorithm should be studied. This decomposition method should be extended to more general networks such as assembly/disassembly networks and Jackson-like networks with blocking.

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i	Machine Parameters					N
	r_i	p_i	e_i	μ_i	ρ_i	
1	.05	.03	.625	.5	.3125	10
2	.06	.04	.600	-	-	10
3	.05	.03	.625	.5	.3125	-

TABLE 1. Parameters of the Transfer Line

μ_2	ρ_2	Decomposition Method			Simulation			
		P	$\bar{n}_1$	$\bar{n}_2$	P	$\bar{n}_1$	$\bar{n}_2$	%M_2 S & B
0.1	0.06	.060	9.618	0.382	.060	9.618	0.396	0.00
0.2	0.12	.115	8.836	1.164	.114	8.961	1.177	0.00
0.3	0.18	.159	7.872	2.128	.159	7.792	2.181	0.10
0.4	0.24	.189	6.957	3.043	.191	7.012	2.982	0.14
0.5	0.30	.209	6.203	3.796	.210	6.174	3.728	0.24
0.6	0.36	.223	5.623	4.376	.218	5.711	4.481	0.24
0.7	0.42	.231	5.180	4.820	.230	5.250	4.742	0.47
0.8	0.48	.238	4.841	5.159	.235	4.961	5.115	0.62
0.9	0.54	.243	4.577	5.423	.239	4.811	5.411	0.61
1.0	0.60	.246	4.365	5.635	.242	4.468	5.424	0.69
1.2	0.72	.252	4.052	5.948	.245	4.294	5.860	0.75
1.5	0.90	.257	3.744	6.256	.251	4.091	6.104	0.92
1.6	0.96	.258	3.669	6.331	.249	3.902	5.932	0.90
2.5	1.50	.264	3.275	6.725	.255	3.692	6.277	1.08
3.0	1.80	.266	3.151	6.849	.253	3.613	6.276	1.29

%M_2 S & B : PERCENTAGE OF TIME M_2 IS STARVED AND BLOCKED CONCURRENTLY.

TABLE 2. RESULTS FROM THE DECOMPOSITION METHOD AND SIMULATION

i	Machine Parameters					N
	r_i	p_i	e_i	μ_i	ρ_i	
1	.3	.02	.9375	1.	.9375	6
2	.5	.05	.9091	1.3	1.1818	4
3	.1	.01	.9091	1.5	1.3636	6
4	.4	.08	.8333	1.6	1.333	-

	P	$\bar{n}_1$	$\bar{n}_2$	$\bar{n}_3$
Decomposition	.7839	2.8479	1.4697	1.6013
Simulation	.7873	2.7504	1.4291	1.5564
% Error	-.43	3.5	2.8	2.9

TABLE 3. Example of a 4-Machine Line

i	Machine Parameters					N
	r_i	p_i	e_i	μ_i	ρ_i	
1	.1	.01	.909	.25	.2273	4
2	.3	.02	.938	.20	.1875	4
3	.5	.04	.926	.30	.2778	4
4	.5	.02	.939	.20	.1875	4
5	.5	.04	.926	.30	.2778	-

	P	$\bar{n}_1$	$\bar{n}_2$	$\bar{n}_3$	$\bar{n}_4$
Decomposition	0.1377	2.6571	1.6488	1.9740	.9243
Simulation	0.1439	2.7033	1.6285	2.0016	.9816
% Error	-4.31	-1.71	1.25	-1.38	-5.84

TABLE 4. First Example of a 5-Machine Line

i	Machine Parameters					N
	r_i	p_i	e_i	μ_i	ρ_i	
1	.3	.02	.9375	.20	.1875	4
2	.4	.05	.8889	.23	.2044	4
3	.1	.01	.9091	.30	.2727	4
4	.4	.07	.8511	.26	.2213	4
5	.3	.02	.9375	.21	.1969	-

	P	$\bar{n}_1$	$\bar{n}_2$	$\bar{n}_3$	$\bar{n}_4$
Decomposition	0.1385	2.2318	1.6804	1.9433	1.6307
Simulation	0.1407	2.1883	1.6565	1.9601	1.7058
% Error	-1.56	1.99	1.44	-.86	-4.40

TABLE 5. Second Example of a 5-Machine Line

i	Machine Parameters					N
	r_i	p_i	e_i	μ_i	ρ_i	
1	.3	.02	.9375	.20	.1875	4
2	.4	.05	.8889	.23	.2044	4
3	.1	.01	.9091	.30	.2727	6
4	.4	.07	.8511	.26	.2213	4
5	.3	.03	.9091	.21	.1909	4
6	.1	.03	.7692	.27	.2077	6
7	.4	.06	.8696	.26	.2261	-

	P	$\bar{n}_1$	$\bar{n}_2$	$\bar{n}_3$	$\bar{n}_4$	$\bar{n}_5$	$\bar{n}_6$
Decomposition	0.1333	2.3460	1.9121	3.8464	2.4237	1.6054	1.4906
Simulation	0.1304	2.3596	1.9789	3.7193	2.3258	1.5908	1.3502
% Error	2.22	-.58	-3.38	3.42	4.21	0.92	10.40

TABLE 6. First Example of a 7-Machine Line

i	r_i	p_i	e_i	μ_i	ρ_i	N
	Machine Parameters					
1	.3	.02	.9375	.20	.1875	4
2	.4	.05	.8889	.23	.2044	4
3	.1	.01	.9091	.30	.2727	5
4	.4	.07	.8511	.26	.2213	4
5	.3	.02	.9375	.21	.1909	4
6	.5	.06	.8929	.23	.2054	5
7	.1	.01	.9091	.27	.2455	-

	P	$\bar{n}_1$	$\bar{n}_2$	$\bar{n}_3$	$\bar{n}_4$	$\bar{n}_5$	$\bar{n}_6$
Decomposition	0.1356	2.2984	1.8243	2.9069	2.2105	1.6129	1.1853
Simulation	0.1371	2.2761	1.8173	2.9493	2.1293	1.5856	1.1916
% Error	-1.09	0.98	0.39	-1.44	3.81	1.72	-0.53

TABLE 7. Second Example of a 7-Machine Line

i	r_i	p_i	e_i	μ_i	ρ_i	N
	Machine Parameters					
1	.3	.02	.9375	.20	.1875	4
2	.5	.05	.9091	.23	.2091	4
3	.1	.01	.9091	.30	.2727	4
4	.4	.07	.8511	.26	.2213	4
5	.3	.05	.8571	.40	.3429	4
6	.1	.01	.9091	.27	.2455	4
7	.3	.02	.9375	.20	.1875	4
8	.4	.05	.8889	.23	.2044	-

	P	$\bar{n}_1$	$\bar{n}_2$	$\bar{n}_3$	$\bar{n}_4$	$\bar{n}_5$	$\bar{n}_6$	$\bar{n}_7$
Decomposition	0.1376	2.2591	1.8183	2.1040	1.7734	2.4501	2.3831	1.4980
Simulation	0.1388	2.2392	1.8142	2.1274	1.7092	2.3325	2.3088	1.5273
% Error	-0.86	0.89	0.23	-1.10	3.76	5.04	3.22	-1.92

TABLE 8. First Example of an 8-Machine Line

i	Machine Parameters					N
	r_i	p_i	e_i	μ_i	ρ_i	
1	.3	.03	.9091	1.5	1.3636	6
2	.5	.05	.9091	1.3	1.1818	5
3	.1	.01	.9091	2.0	1.8182	6
4	.4	.06	.8696	1.6	1.3914	4
5	.3	.04	.8824	2.0	1.7648	6
6	.1	.01	.9091	1.7	1.5455	4
7	.3	.02	.9375	1.2	1.1250	7
8	.4	.05	.8889	1.6	1.4222	-

	P	$\bar{n}_1$	$\bar{n}_2$	$\bar{n}_3$	$\bar{n}_4$	$\bar{n}_5$	$\bar{n}_6$	$\bar{n}_7$
Decomposition	0.8382	4.2720	2.4007	3.6399	1.9680	3.3653	2.1976	1.6642
Simulation	0.8304	4.2908	2.4285	3.6010	1.9279	3.2885	2.1591	1.5684
% Error	0.94	-0.44	-1.14	1.08	2.08	2.33	1.78	6.11

TABLE 9. Second Example of an 8-Machine Line

i	Machine Parameters					N
	r_i	p_i	e_i	μ_i	ρ_i	
1	.05	.05	.5	.6	.3	4
2	.05	.05	.5	.6	.3	4
3	.05	.05	.5	.6	.3	-

TABLE 10.1 Fixed Parameters of a Three Machine Line

r_1	p_1	e_1	μ_1	P	$\bar{n}_1$	$\bar{n}_2$
5.00	10^{-9}	1.00	0.3	.1668	2.526	1.474
0.03	0.01	0.75	0.4	.1612	2.517	1.483
0.05	0.05	0.50	0.6	.1561	2.510	1.490
0.01	0.03	0.25	1.2	.1432	2.515	1.485

TABLE 10.2 Effects of Variations of the Parameters of the First Machine

r_2	p_2	e_2	μ_2	P	$\bar{n}_1$	$\bar{n}_2$
5.00	10^{-9}	1.00	0.3	.1687	2.459	1.548
0.03	0.01	0.75	0.4	.1616	2.494	1.523
0.05	0.05	0.50	0.6	.1561	2.510	1.490
0.01	0.03	0.25	1.2	.1378	2.508	1.361

TABLE 10.3 Effects of Variations of the Parameters of the Second Machine

r_3	p_3	e_3	μ_3	P	$\bar{n}_1$	$\bar{n}_2$
5.00	10^{-9}	1.00	0.3	.1684	2.456	1.544
0.03	0.01	0.75	0.4	.1616	2.477	1.506
0.05	0.05	0.50	0.6	.1561	2.510	1.490
0.01	0.03	0.25	1.2	.1365	2.625	1.479

TABLE 10.4 Effects of Variations of the Parameters of the Third Machine

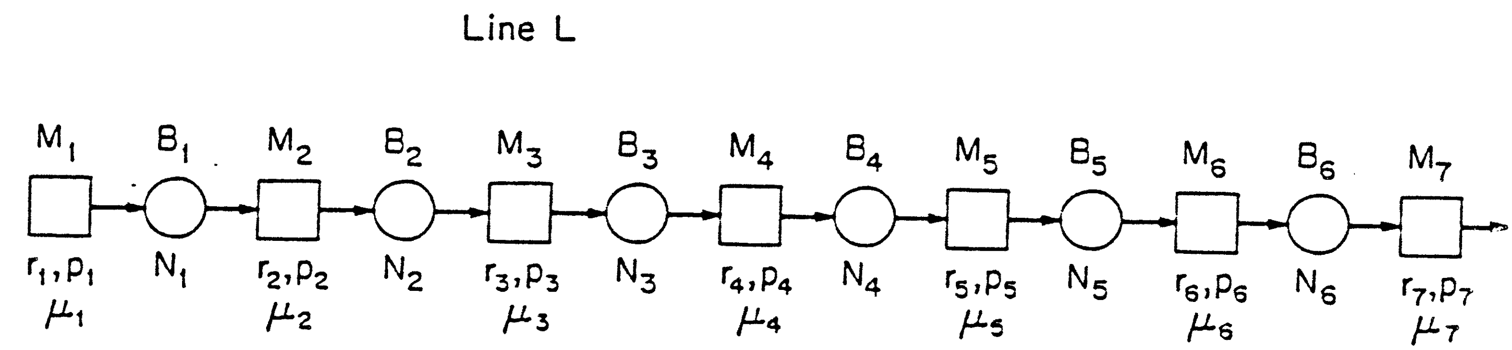

FIGURE 1: TRANSFER LINE

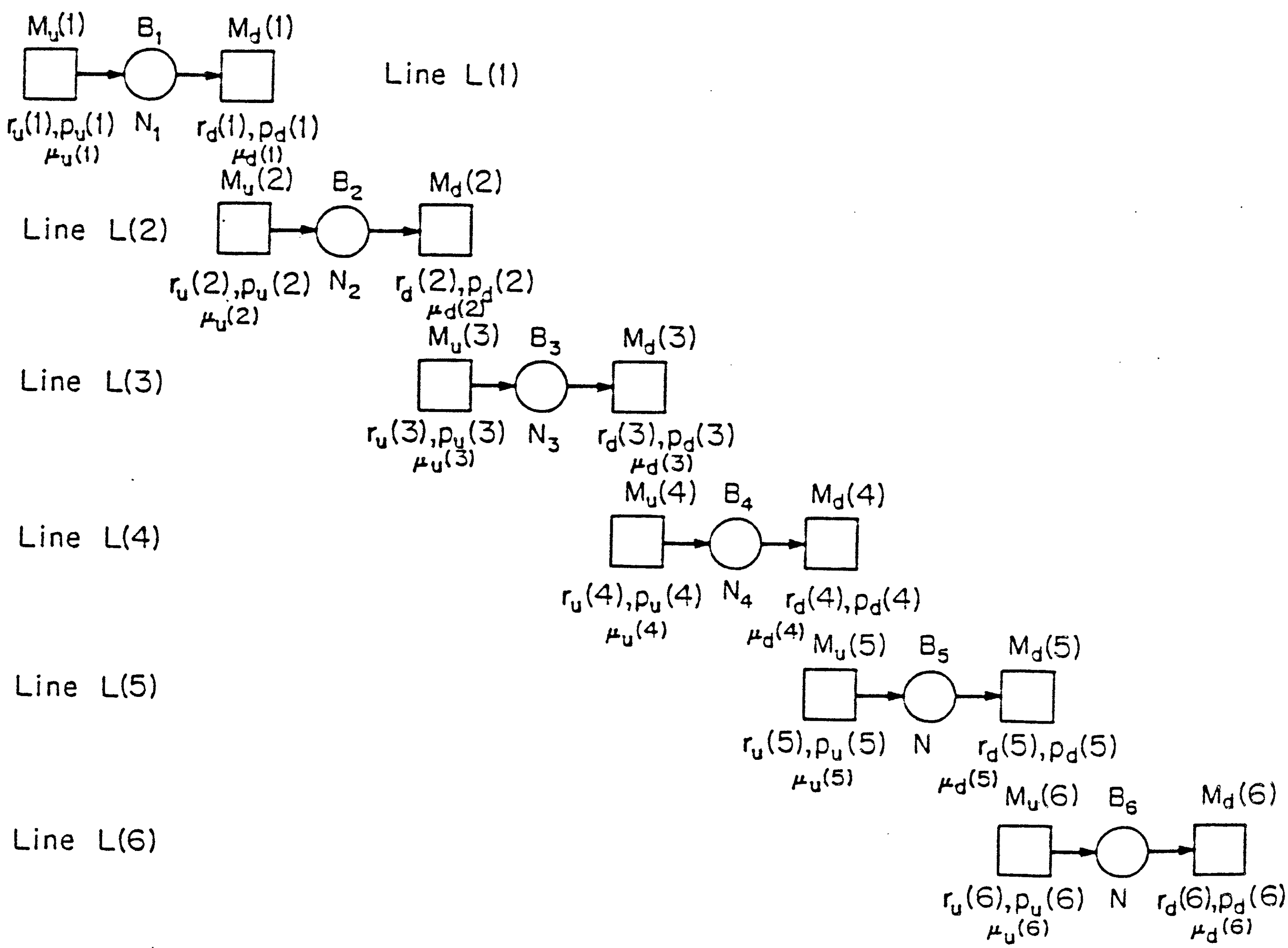

FIGURE 2: DECOMPOSED TWO-MACHINE LINES

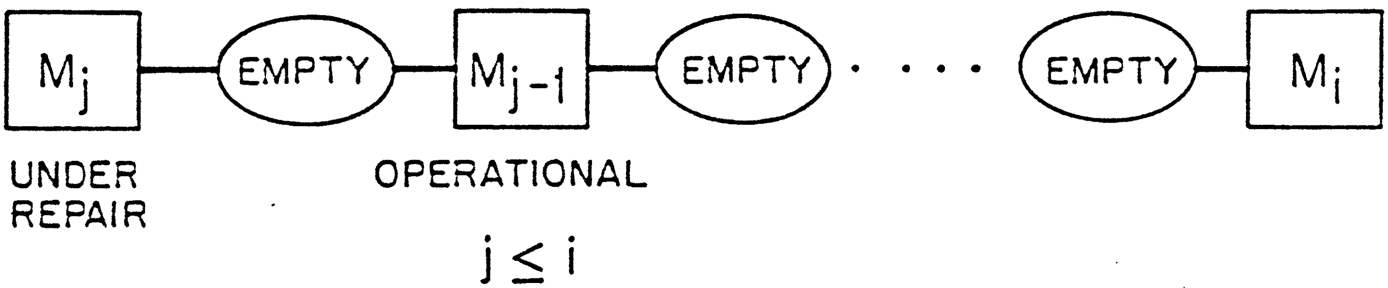

FIGURE 3: DEFINITION OF $M_u(i)$ DOWN

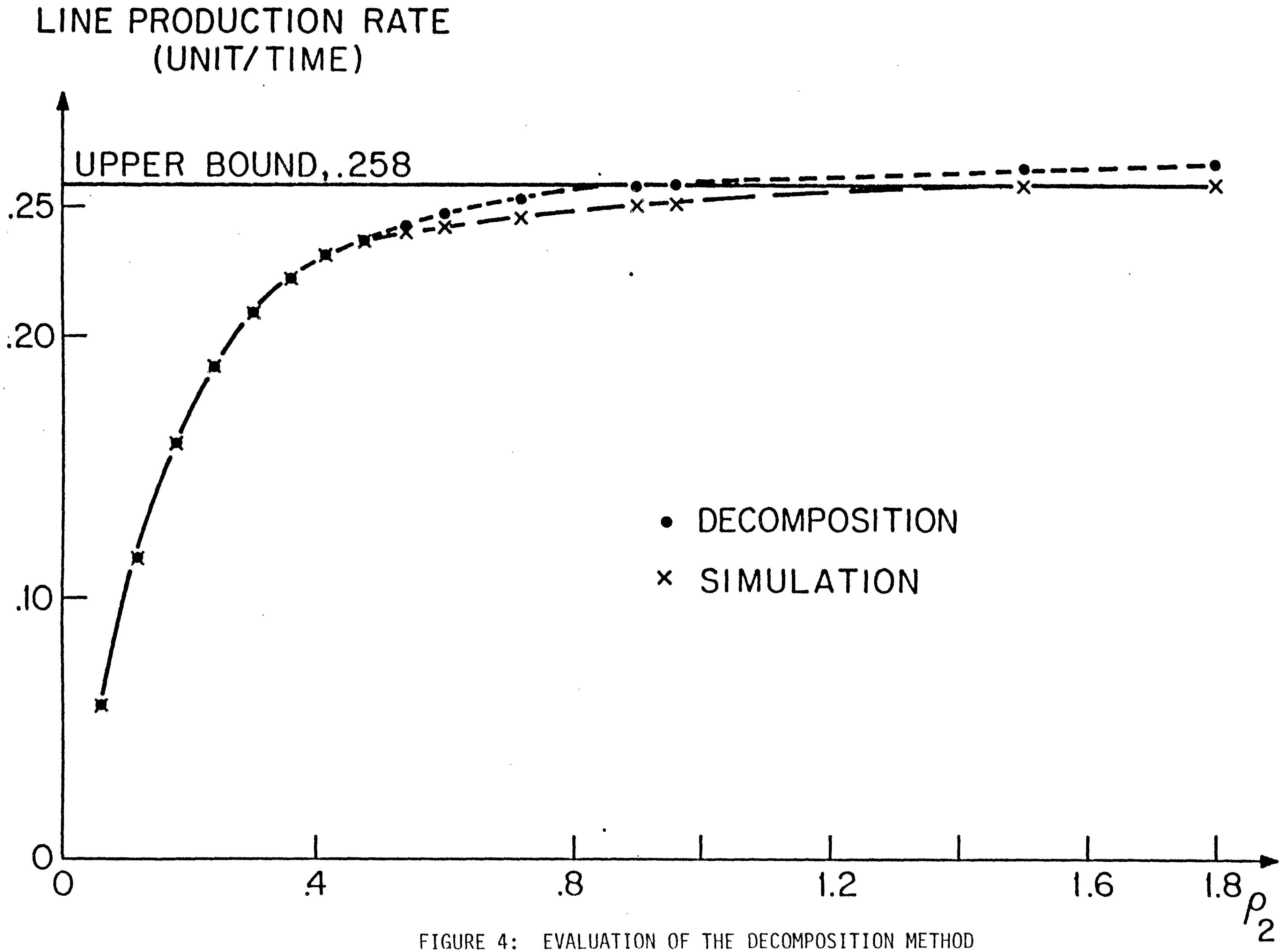

FIGURE 4: EVALUATION OF THE DECOMPOSITION METHOD