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DIFFUSION APPROXIMATIONS FOR THREE-STAGE TRANSFER LINES WITH UNRELIABLE MACHINES AND FINITE BUFFERS

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SUMMARY

In this paper, we derive an approximate model for the flow of parts in a three-stage transfer line with unreliable machines and finite storage elements. The analysis of exact models of these systems was described in Gershwin and Schick [1], where the basic notation for this paper is established. In [1], it was noted that exact analysis of three-stage or higher problems is very difficult, if not altogether impossible. In this paper we develop an approximation methodology which is used to study the three-stage transfer line model. This methodology can be extended to n-stage transfer lines without introducing additional levels of complexity. These extensions are discussed in [2].

Figure 1 describes the basic three-stage transfer line. Machine i processes parts from the preceding storage element until no parts are left in it. The output of each machine goes into the subsequent buffer. It is assumed that the first buffer is an infinite source, and the last buffer an infinite sink. In addition, an exponential failure and repair process is used to model the operation of the machines, with compensation for eliminating failures when the machine is not in use. It is also assumed that flow is conserved, and that parts are infinitesimal, and that machine i will process at maximum rate $\overline{b}_i$ whenever possible

The stochastic equations which describe the flow of parts through buffers 1 and 2 can be described as follows: Let x_i denote the level of flow into buffer i. Let a_i be a binary variable which is 0 when machine i is off, and 1 when machine i is on. Let N denote the buffer capacities. The accumulation of flow in buffer i can be described by:

$$dx_i = 0 \quad \text{if } x_i = N \text{ and } (a_{i+1}\overline{b}_{i+1} - a_i\overline{b}_i) < 0$$

or

$$x_i = 0 \text{ and } (a_{i+1}\overline{b}_{i+1} - a_i\overline{b}_i) > 0 \qquad (1.a)$$

$$dx_i = (a_i\,\overline{b}_i - a_{i+1}\overline{b}_{i+1})\,dt \quad \text{otherwise} \qquad (1.b)$$

Equation 1 is a random evolution for the level of flow in each buffer, because the on-off state of each machine is a random process. Assuming that machines can slow up their production rate $\overline{b}_i$ if the storages are either empty or full, the production rate of machine i can be described by the function

$$b_i(\underline{x}, \underline{a}) = \overline{b}_i \text{ if } x_i < N \qquad (2.a)$$

$$x_{i-1} > 0$$

$$= a_{i+1}b_{i+1}(\underline{x}, \underline{a}) \text{ if } x_i = N \text{ and} \qquad (2.b)$$

$$a_{i+1}b_{i+1} < a_ib_i$$

$$= a_{i-1}b_{i-1}(\underline{x}, \underline{a}) \text{ if } x_{i-1} = 0 \text{ and} \qquad (2.c)$$

$$a_ib_i > a_{i-1}b_{i-1}$$

Then,

$$dx_i = (a_ib_i(\underline{x}, \underline{a}) - a_{i+1}b_{i+1}(\underline{x}, \underline{a}))dt \qquad (3)$$

The evolution of the random process a_i is described by stochastic differential equation

$$da_i = (1-a_i)dR_i + a_i(1-I\{b_i(\underline{x}, \underline{a})=0\})dF_i \qquad (4)$$

where R_i, F_i are independent counting processes with exponential jump rates r_i, f_i respectively. Note that the last term on the right side of (4) implies that machines cannot fail unless they are processing some flow. The presence of this coupling term implies that the $\underline{a}$ process is not a Markov process; rather, the full $(\underline{x}, \underline{a})$ process is Markov, described by equations (3) and (4).

Consider now two scaled processes, y_i^1 and y_i^2, defined as follows:

$$y_i^1\ (t) = \frac{1}{N}\, x_i\ (Nt) \qquad (5.a)$$

$$y_i^2\ (t) = \frac{1}{N}\, x_i\ (N^2t) \qquad (5.b)$$

The main results of the paper can be stated as follows:

Let

$$d_i = \frac{r_i\ \overline{b}_i}{r_i + f_i} - \frac{r_{i+1}\ \overline{b}_{i+1}}{r_{i+1} + f_{i+1}} \qquad (6)$$

and let T denote the time of first exit of the process $(y_1^1(t), y_2^1(t))$ from the unit square D = (0, 1) x (0, 1). Define the process $\underline{z}^o(t)$ by

$$z_i^o(t) = y_i^1(0) + d_it$$

Theorem 1: Unbalanced line approximation
Assume d_i is O(1) for some i. Then, for $o \le t < T$, the process $\underline{y}^1(.)$ converges uniformly almost surely as $N \to \infty$ to the process $\underline{z}^o(t)$. That is, for any e>o,

$$\lim_{N\to\infty}\ \sup_{o \le t \le T}\ \text{Prof}\{|\underline{z}^o(t) - \underline{y}^1(t)| > e\} = 0$$

Furthermore, define the process v(t) by

$$v_i(t) = \sqrt{N}\ (y_i^1(t) - z_i^o(t))$$

The process $\underline{v}(t)$ converges weakly to a zero-mean Wiener process $\underline{w}(.)$ with covariance.

$$E\{w(t)w^T(s)\} = \Sigma \min(t, s)$$

where

$$\Sigma = \begin{pmatrix} \frac{2(\bar{b}_1^2 f_1 r_1)}{(f_1 + r_1)^3} + \frac{2\bar{b}_2^2 f_2 r_2}{(f_2 + r_2)^3} & -\frac{2\bar{b}_2^2 f_2 r_2}{(f_2 + r_2)^3} \\ -\frac{2\bar{b}_2^2 f_2 r_2}{(f_2 + r_2)^3} & \frac{2\bar{b}_2^2 f_2 r_2}{(f_2 + r_2)^3} + \frac{2\bar{b}_3^2 f_3 r_3}{(f_3 + r_3)^3} \end{pmatrix}$$

Theorem 2: Assume that f_1, f_2 are both of order $(1/N)$ as $N \to \infty$. Then, for any $K < \infty$, the process $\underline{y}^2(t)$ converges weakly on $C([o, K]; \bar{D})$ to a diffusion process on $\bar{D}$ with instantaneous reflection on the boundary ∂D. The infinitesimal generator of this process in D is given by L, where

$$Lg(\cdot) = N\left(f_1 \frac{\partial}{\partial x_1} g + f_2 \frac{\partial}{\partial x_2} g\right) + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \Sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} g$$

with dense domain

$$\mathrm{Dom}(L) = \{g \in C^2(D) \mid C(\bar{D}) \text{ satisfying a-d}\}$$

where

a) $\frac{\partial}{\partial x_1} g(0, x_2) = \frac{\partial}{\partial x_2} g(0, x_2)$

b) $\frac{\partial g}{\partial x_1}(x_1, 1) = \frac{\partial g}{\partial x_2}(x_1, 1)$

c) $\frac{\partial g}{\partial x_1}(1, x_2) = 0$

d) $\frac{\partial g}{\partial x_1}(x_1, 0) = 0$

The directions of reflection of the limit process in Theorem 2 are shown in Figure 2. Note that the limit process has a discontinuous reflection field at (0, 0), (0, 1) and (1, 1), as well as existing in a domain with non-differentiable boundaries. Hence, the classical theory of reflected Markov processes, as developed in [5], cannot be used to establish this theorem. Rather, the theorem is established by developing a strong characterization of the boundary process associated with the limiting diffusion process.

The properties of the limiting diffusion process in Theorem 2 can be used to approximate the properties of the corresponding exact process defined by (2) - (5). In particular, the ergodic distribution of the diffusion process serves as an approximation to the ergodic distribution of equations (2) - (5). Hence, long-term average lost production and other ergodic properties can be computed using this approximation, with a resulting simplification of solving only one second order elliptic PDE, rather than 8 first order hyperbolic PDE's. The results presented here are part of a larger paper [2], which contains the proofs of the main theorems.

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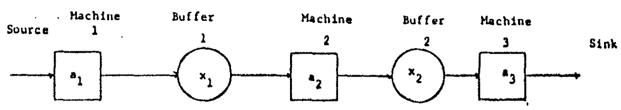

Figure 1: Three Stage Transfer Line

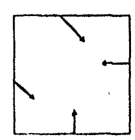

Figure 2. Directions of Reflection for Diffusion Approximation.