

A Methodology for Demand Learning  
with An Application to the Optimal  
Pricing of Seasonal Products

by  
Gabriel R. Bitran  
Hitendra K. Wadhwa

WP #3898-96

April 1996

# **A Methodology For Demand Learning With An Application To The Optimal Pricing Of Seasonal Products**

**Gabriel R. Bitran\***  
**Hitendra K. Wadhwa\*\***

**MIT Sloan School of Management**

**April, 1996**

## **Abstract**

Retailers who sell seasonal products often suffer substantial losses due to the mismatch between supply and demand caused by demand uncertainty, supply inflexibility and the short sale season. A major source of the retailer's demand uncertainty is often the lack of information about how attractive the product will be to customers, and in such cases the sales observed over the course of the season help to provide this crucial information. In this paper, we describe a modeling approach for using observed sales data to update demand information over time, and show how this can be embedded in an optimal dynamic pricing model for seasonal products. Our technique utilizes the Bayesian approach commonly employed in dynamic learning models such as inventory models that incorporate demand learning. It is distinguished from existing approaches, however, by its ability to address important sources of non-stationarity in the demand distribution, such as price changes and changes in customers' values for the product over time. We present results from some preliminary computational tests which indicate that the methodology is effective in estimating demand under a range of conditions. They also suggest that the incorporation of demand learning can lead to swift price corrections early in the season, and that this can substantially improve revenues in the seasonal product pricing context. In addition, the computational results provide insights into the nature of optimal dynamic pricing strategies in situations of over- or under-estimation of demand.

---

\* Sloan School of Management, MIT, 50 Memorial Drive, E53-355, Cambridge, MA 02139.  
Email: gbitran@MIT.EDU

\*\* Sloan School of Management, MIT, 50 Memorial Drive, E53-356A, Cambridge, MA 02139.  
Email: hitendra@MIT.EDU

# 1. INTRODUCTION

## **Pricing Of Seasonal Products**

Retailers who carry seasonal merchandise, such as summer wear or ski equipment, often have to place orders for these items much in advance of the sale season. Once the initial order is placed, they often have limited, if any, flexibility in placing reorders. In addition, such merchandise is only in demand at the stores during the season, and is therefore liquidated by the retailers at highly discounted prices at the end of the season through clearance sales, sales in outlet stores and bulk sales to discount merchandisers. Since the demand for the merchandise is not predictable at the order time, retailers may be left with an excess or shortage of merchandise as the sale season progresses. These mismatches between supply and demand can be very costly to retailers due to the forced markdowns and stockouts that result from them.

The U.S. apparel industry provides a good example for our discussion. According to one study (Frazier, 1986), the cost to retailers for forced markdowns and stockouts equal about 8% and 6% of net retail apparel sales. A retail industry journal (Stores, Dec. 1994) reported that the costs to apparel retailers in the U.S. for mismatches between supply and demand amounted to \$25 billion in 1993. A number of efforts have been made by apparel manufacturers and retailers under a movement called Quick Response to address this problem. These efforts have primarily focussed on improving the timely flow of information and goods between different nodes in the apparel supply chain, leading to shorter production and distribution leadtimes that allow production and procurement decisions to be made closer to the start of the sales season. This facilitates more informed decision making and thereby reduces the risk of demand misestimation. Hammond (1990) provides an overview of the Quick Response movement. Related to Quick Response is an approach for demand estimation and production planning called Accurate Response (Fisher and Raman, 1994, Fisher et al, 1994).

One area of retail merchandise management that complements Quick Response strategies is the dynamic pricing of the product by the retailer over the course of the season. Often a substantial amount of demand uncertainty is still present once the final ordering decision has been made for the season, and so the pricing decision provides the retailer with an important mechanism through which to ultimately minimize the supply-demand imbalance over the course of the season. Gallego and Van Ryzin (1994) and Bitran and Mondschein (1993, 1995) have recently formulated analytical models of this problem and studied the behavior of optimal pricing policies over time.

## **Demand Learning**

The demand uncertainty faced by the retailer can be separated into two components. One source of the uncertainty is often the lack of information about how attractive the product will be to customers. This lack of information often constitutes a major source of uncertainty, particularly in the case of products such as fashion goods, where styles change every season. Sales observed during the course of the season can in such cases provide valuable information about market conditions. This sales-driven demand learning allows retailers to refine their demand forecasts and alter prices accordingly over time. In some situations where Quick Response strategies have helped to reduce production lead times substantially, retailers may also be in a position to use

early sales information to place a second, and final, order for the product. For instance, two leading retailers, The Limited and Benetton, test market products in a few representative outlets in their chains early in the season and use the resulting demand information to determine the right quantities to order for the rest of the season (Hammond, 1992).

The second component of the uncertainty consists of unpredictable factors, such as the weather, and store traffic. Note that while the first component of uncertainty (related to the lack of information on product attractiveness) is resolvable through sales observations, the second one is not.

A realistic dynamic pricing model for seasonal products needs to incorporate a mechanism for demand learning in order to allow sales observations to be used to resolve the first component of demand uncertainty identified above. Existing dynamic pricing models (such as those developed by Gallego & Van Ryzin and Bitran & Mondschein) have not considered the possible correlation in demand over time, and therefore could not be used to explore the potential for demand learning. In this paper, we describe a modeling approach for using observed sales data to update demand information over time, and show how this can be embedded in an optimal dynamic pricing model. Our technique utilizes the Bayesian approach commonly employed in dynamic learning models such as inventory models that incorporate demand learning. It is distinguished from existing approaches, however, by its ability to address important sources of non-stationarity in the demand distribution, such as price changes and changes in customers' values for the product over time.

In addition to a description of the methodology, we present results from some preliminary computational tests based on representative data. These results indicate that the methodology is effective in estimating demand under a range of conditions. They also suggest that the incorporation of demand learning can lead to swift price corrections early in the season, and that this can substantially improve revenues in the seasonal product pricing context. In addition, the computational results provide insights into the nature of optimal dynamic pricing strategies in situations of over- or under-estimation of demand and in situations where prices are required to be non-increasing over time.

The remainder of this section discusses the demand model on which our demand learning methodology is based and the optimal pricing model within which we have embedded it, and comments on the other sections in this paper.

## **Demand Model**

We use the demand model developed by Bitran and Mondschein (1993, 1995). The entire sales season is divided into  $T$  discrete periods. We assume that for each period there is a reservation price distribution associated with the customer population. Customers in each period arrive at the store in the form of a Poisson process with an arrival rate that is known to the planner, and that is independent of price. (These assumptions are based on the following observations. While demand for individual products may be highly unpredictable, the volume of customer arrivals in retail outlets such as department stores follows a much more stable pattern. There are, however, random variations in the number of store arrivals in any period, and these variations are modeled

through the Poisson arrival process. This arrival process may have different rates in different time periods to reflect periods of high and low store traffic. Also, with the exception of a few advertised products, arriving customers are usually un-informed about specific product prices). Each store arrival is drawn randomly from the reservation price distribution. A store arrival is assumed to purchase the product if the price of the product is not larger than its reservation price. As observed by Bitran and Mondschein, for any given price  $p$  and any period  $t$ , the resulting distribution of the number of purchases in period  $t$  at price  $p$  is itself a Poisson distribution. This result will be stated in more technical terms later in this paper. A more substantive discussion of this demand model can be found in Bitran and Mondschein (1995).

The demand model described above incorporates two sources of uncertainty - the number of store arrivals in any period (which is a Poisson process) and the "maximum willingness to pay" of the arriving customers (which are random samples from the reservation price distribution). These sources relate to the second (unresolvable) component of demand uncertainty discussed above. In this paper, we extend the demand model by introducing a third source of uncertainty related to the retailer's lack of information about how attractive the product is to the customer population. This source relates to the first (resolvable) component of demand uncertainty discussed above. We assume that there is some parameter of the reservation price distribution that is unknown to the retailer at the beginning of the season. This parameter is revised at the end of each period based on observed sales in that period.

### **Optimal Pricing Model**

We utilize and extend the discrete-time optimal pricing model developed by Bitran and Mondschein (1993). (We will term this the *basic* optimal pricing model, to distinguish it from the extended model that we develop in this paper to incorporate demand learning.) There are  $T$  discrete periods in the season, and the retailer has a fixed stock of the product at the beginning of the season. Price revisions are allowed only at the beginning of each period. Product value at the end of the planning horizon is taken to be zero. (Our methodology is also applicable when there is a salvage value function at the end of the last period, and our results and conclusions should remain valid as long as this function is concave.) The retailer's objective is to maximize the expected revenues over the planning horizon. Product cost is assumed to be a sunk cost, and is therefore not incorporated in the model. The resulting optimal pricing problem is formulated as a stochastic dynamic program, with the stages corresponding to the different time periods and the state in each period being the inventory level of the product at the start of the period.

To incorporate demand learning in the above model, we include two additional variables in the state space that allow us to transfer demand information from one period to the next. The resulting increase in the state space of the dynamic program can in certain cases effect the computational time in a significant manner, and we have developed a heuristic solution approach to the dynamic program that runs very efficiently. This heuristic has performed very well in the computational tests performed by us, as reported later in this paper.

The rest of this paper is organized as follows. Section 2 provides a review of the literature related to our work, and Section 3 describes our demand learning methodology under various demand conditions. Section 4 defines the basic optimal pricing model and shows how the demand learning

technique can be embedded in the model. Section 5 presents computational results and draws certain conclusions and insights from them. Finally, in Section 6, we summarize the key contributions made by our research and present further research issues arising from it.

## 2. LITERATURE REVIEW

Our research relates to the literature on dynamic demand learning in the context of inventory management. We present a brief review of this research below, and then comment on two recent contributions to the topic of seasonal product pricing that were cited in Section 1.

### **Demand learning models**

The issue of dynamic demand learning has been studied by many researchers in inventory control contexts. A number of them involve the application of a Bayesian approach to incorporate demand learning in a periodic review stochastic inventory model. We describe this Bayesian learning approach - in the inventory control context - below and then briefly discuss some specific work in this area. We then cite some additional work on modeling demand learning that is not based on the Bayesian approach.

The periodic review stochastic inventory problem with demand learning is typically formulated as a dynamic program. Demand in any period is assumed to be a random variable with a distribution function that has an unknown parameter, say  $\alpha$ . The planner is expected to specify a prior distribution on  $\alpha$ . After each time period, the observed demand is used to revise the prior into a posterior distribution through the application of Bayes' rule. The prior is required to be conjugate to the demand distribution, thus allowing the posterior to be calculated in a simple manner. A parameter that summarizes the posterior distribution is included in the state space of the dynamic program so that the updated demand information is passed on from one period to the next. A second variable in the state space is the current inventory level.

Some researchers (Scarf, 1959, 1960, Azoury, 1979, 1985) have identified conditions under which the Bayesian model can be reformulated as a dynamic program with a single state variable that incorporates information both on the current inventory level as well as on the observed demand from past periods.

Murray and Silver (1966) present an inventory model for style goods where the number of potential buyers in each period is known. Each of these buyers purchases the product with probability  $\rho$ , and this probability is unknown at the start of the season. The authors assume a beta prior on  $\rho$ , which is revised over time as demand is observed. The price of the product is assumed to be fixed throughout the season. The paper contains an interesting discussion of a 'state aggregation' procedure to help decrease the computational requirements of the resulting model.

Popovic (1987) presents an inventory control model with demand learning where the demand distribution is non-stationary. He uses a Bayesian approach with demand modeled as a Poisson distribution whose arrival rate is unknown and changing over time. The arrival rate  $\lambda_t$  at time  $t$  is

modeled as  $\lambda_t = \lambda(k+1)t^k$ , for some known positive integer  $k$  and some unknown  $\lambda$ . By assuming a gamma prior distribution on  $\lambda$ , the author shows how the Bayesian approach could be applied to the problem.

An alternative Bayesian learning model has been proposed by Chang and Fyffe (1971). The authors consider demand over a season consisting of  $T$  periods. Aggregate demand for the season is represented through a random variable,  $D$ , which is assumed to be normally distributed with an unknown mean  $\mu$  and variance  $v$ . The demand in any period  $t$  is assumed to be of the form  $d_t = D s_t + X_t$ , where  $v_t$  is the estimated proportion of the aggregate season demand that falls in period  $t$  (so that  $\sum_{t=1}^T s_t = 1$ ), and  $X_t$  is a noise term that is independent of  $D$  and the other  $X_j$ 's and is distributed normally with zero mean and variance  $v_t$ . Both  $s_t$  and  $v_t$  are constants that may be computed using historical data. As demand is observed, the distribution of  $D$  is updated through an application of Bayes' rule, leading to revised forecasts for sales in each time period through the above equation. This model of demand has been used by Crowston, Hausman and Kampe (1973) in a multistage production planning context.

Iyer and Eppen (1995a, 1995b) describe a methodology in which demand in each period is assumed to be based on one of a set of 'pure demand processes'. The actual underlying demand process is unknown to the user, and a (discrete) prior distribution is defined over the set of demand processes. This prior is updated after each demand observation using Bayes rule. The demand processes are required to satisfy certain conditions that allow the demand learning process to be embedded in a dynamic optimization model in an analytically convenient fashion. Among the demand processes that meet these conditions are the normal, negative binomial and Poisson.

Hausman (1969) shows that under certain circumstances, the ratios of successive demand forecasts can, as a first approximation, be treated as independent random variables distributed according to the lognormal distribution. Based on his observations, some researchers have studied the problem of demand forecasting for style goods by making a markovian assumption on demand under which the demand  $D_t$  in period  $t$  is related to past demand only through the demand  $D_{t-1}$  in the last period. Hausman and Peterson (1972) use this approach in studying a multiproduct problem.

All the models discussed above suffer from one key limitation that prevents us from utilizing their approaches in our setting - they assume that there is no price change affecting demand over time. Incorporating this non-stationary price behavior in a Bayesian learning framework is a key contribution of this paper to the literature on demand learning.

Recently, Smith et al (1994) have presented a two-stage sales forecasting methodology that models demand as a function of price and a combination of other marketing and environmental factors. The parameters of the model are first estimated through regression analysis using historical data. In stage 2, the key parameters of this model are updated through a discounted least squares procedure.

## Dynamic pricing models

Gallego and Van Ryzin (1994) and Bitran and Mondschein (1993,1995) have recently developed and analyzed models for dynamic pricing of seasonal products. While Gallego and Van Ryzin have developed a continuous-time model, Bitran and Mondschein have used both discrete- and continuous-time frameworks. The models by Gallego and Van Ryzin and Bitran and Mondschein (1993) allow for prices to be altered freely (both upwards and downwards) over time. These models assume that the merchandise can only be ordered once before the start of the sales season, with no reordering capability during the season. Gallego and Van Ryzin model demand as a Poisson process where the arrival (more appropriately, 'purchase') rate at time  $t$  depends on the price at time  $t$ . Bitran and Mondschein model price in a two-phased manner. They consider a Poisson process based store arrival distribution, which is independent of the product price, and a reservation price distribution for the product that determines the fraction of arrivals that will purchase the product at any given price. This demand model appears to be a more behaviorally appealing representation of the store purchase process. The authors show, however, that this model is actually equivalent to one with a price-dependent Poisson purchase process.

We have chosen to base our optimal pricing model on the model that was formulated by Bitran and Mondschein (1993), and we will term this as the *basic pricing model* in this paper. This model will be presented in technical terms in Section 4. We have chosen a discrete time framework since we believe it is more representative of actual retail pricing contexts. In their 1995 paper, Bitran and Mondschein focus on a version of the basic pricing model where the price is required to be non-increasing over time. While we do not discuss this issue specifically when describing our demand learning methodology, our approach can be extended to this context in a straightforward manner, and in the computational results section, we present some test results based on this version of the model.

Gallego and Van Ryzin present two single-price heuristics for their model and show that these heuristics provide asymptotically optimal price policies (as the level of initial inventory or the length of the time horizon approaches infinity). They present results of computational tests that suggest that these heuristics provide near-optimal solutions. The test problems, however, are based on the following assumptions:

1. The purchase arrival distribution at any given price is stationary
2. The retailer knows the purchase arrival rate at any given price accurately

In actual retail environments, these assumptions are likely to be violated. The purchase arrival distribution at a given price may change over time due to changes in the customer mix and changes in customers' values for the product, and the retailer may have very limited information about how attractive the product is to customers' at the beginning of the season. Wadhwa (1996, Chapter 3) shows that if the first assumption is violated, a single-price heuristic may not be expected to perform well, and provides an alternative heuristic that can be efficiently implemented and that provides near-optimal solutions. In this paper, we show that if the second assumption is violated, a single-price heuristic may again not perform well. By incorporating our demand learning technique in Bitran and Mondschein's (1993) pricing model and applying the heuristic solution technique developed in Wadhwa (1996, Chapter 3), we develop a fast heuristic solution scheme that provides substantially improved performance.



### 3. DEMAND LEARNING

In this section, we describe our demand learning methodology in the context of the demand model discussed in Section 1. We assume that the store arrival rate in each period is known, and that the reservation price distributions are all functions of an unknown parameter  $\delta$ . For instance, such a distribution may be exponential, with cumulative distribution function given by  $F(r) = 1 - e^{-\delta r}$ , where the mean of the distribution is  $1/\delta$  and  $\delta$  is the unknown parameter.

Our goal is to utilize observed sales data to update the value of this parameter  $\delta$  from one period to the next. Our methodology is an extension of the standard Bayesian learning technique. The Bayesian approach applies to a context where we get observations from the same distribution over time and use these to update our information (represented in the form of a prior distribution) on some unknown parameter of that distribution. We have found it necessary to seek an extension to this approach since in our case the demand distribution (from which the demand data is observed) changes from one period to the next, due, primarily, to changes in price. The Bayesian approach utilizes both the planner's initial estimate of demand and the observed sales data to revise the demand forecast. This is in contrast to alternative statistical estimation approaches such as MLE (maximum likelihood estimation), where only the observed sales data is utilized. As the volume of sales data grows, however, the planner's initial estimate has a progressively smaller impact on the demand estimate calculated using the Bayesian approach, and the Bayesian estimate becomes very similar to the maximum likelihood estimate (Hines and Montgomery, 1980, pg. 579). We have chosen to utilize the Bayesian approach since we believe that in actual applications it would be important to capture both the planners' initial judgement as well as the initial sales data in arriving at an accurate and robust estimate of demand during the early part of the season.

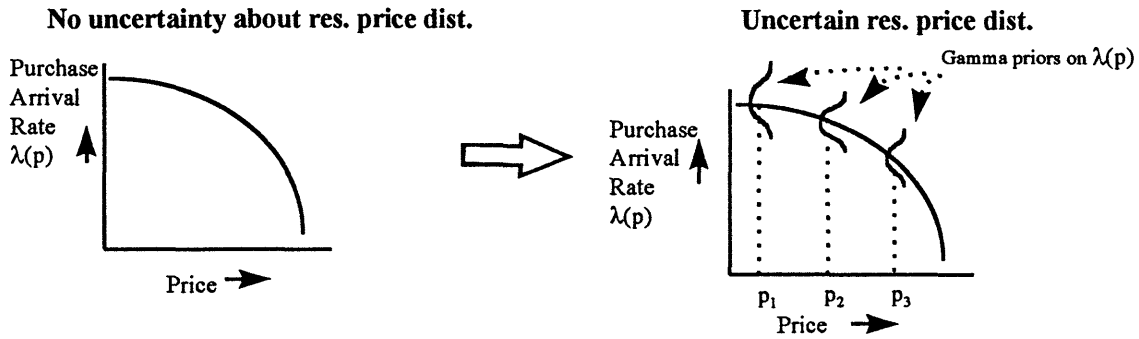
The discussion below is organized as follows. We first describe the Bayesian approach to demand learning under the assumption that the price, the store arrival rate, and the length of time periods stay constant (Case 0). The stationarity assumptions are then peeled away one by one. We describe how the methodology can be extended to allow for price changes from one time period to the next (Case 1), and then show how non-stationary store arrivals can be incorporated (Case 2). Finally, we describe a way of incorporating changes in the reservation price distribution over time (Case 3). At various points in the discussion, we illustrate our approach by using the example of the exponential reservation price distribution mentioned above. Examples of more general reservation price distributions to which our approach can be applied are provided at the end of this section.

We assume for now that:

- The Poisson store arrival rate  $A_t$  is constant ( $= A$ ) across all time periods
- The reservation price distribution  $F_t(\cdot|\delta)$  is constant ( $= F(\cdot|\delta)$ ) across all time periods.

As discussed earlier, and stated later in Lemma 4.1, the purchase distribution at price  $p$  in time period  $t$  can be shown to be a Poisson distribution with the arrival rate  $\lambda_t(p|\delta)$ . Thus the purchase arrival rate  $\lambda_t(p|\delta) = A_t(1 - F_t(p|\delta))$  may be written simply as  $\lambda(p|\delta)$  for all time periods under the

above assumptions. Note that  $\lambda(p|\delta)=A(1-F(p|\delta))$ . We have explicitly shown the dependence of  $F$ ,  $F_t$ ,  $\lambda$ ,  $\lambda_t$  on  $\delta$ . Due to its dependence on the unknown parameter  $\delta$ , the purchase arrival rate (for each price level  $p$ ) is not known to the planner a priori. Instead, this rate is a random variable, which we will denote by  $\lambda(p)$ . Utilizing the Bayesian approach, we assume that the planner initially has a family of prior distributions on  $\lambda(p)$ , one for each price  $p$ , which will be updated at the end of every period ( $t=1, 2, \dots, T$ ) through the incorporation of observed demand data. Note that the prior distribution is *not on*  $\delta$ , the unknown parameter, but on  $\lambda(p)$ , the unknown arrival rate at price  $p$  (in fact, there is one such distribution for each price  $p$ ). The priors are illustrated in Figure 3.1.



**Figure 3.1**

It would be more natural to consider a prior distribution on the unknown parameter  $\delta$  directly. However, such an approach does not allow for a computationally and analytically convenient way to perform Bayesian updates on the prior distribution over time, primarily since the observed data does not consist of samples from the reservation price distribution.

We assume that the following condition holds:

**Condition 3.1:** The function  $F(p|\delta)$  is invertible in  $\delta$  for each price  $p$ .

This condition implies that for any given value  $\lambda$  of  $\lambda(p)$ , the equation  $\lambda = A(1-F(p|\delta))$  has a unique solution in  $\delta$ . Note that the exponential distribution (with  $\delta = 1/\text{mean}$ ) satisfies Condition 3.1.

### Case 1. Constant price

If the price  $p$  were held constant over time, this Bayesian updating of the prior distribution would follow by a straightforward application of Bayes' rule, as described below:

Suppose we enter period  $t$  with a prior distribution  $f_p(\lambda)$  on  $\lambda(p)$ , and we observe a demand of  $n$  units in this period at price  $p$ . The posterior distribution on  $\lambda$ ,  $f_p(\lambda|n)$ , can then be calculated by using Bayes' rule as follows:

$$f_p(\lambda|n) = \frac{P[\text{Demand} = n|\lambda] \cdot f_p(\lambda)}{\int_0^{\infty} P[\text{Demand} = n|x] \cdot f_p(x) dx} = \frac{\frac{e^{-\lambda} \lambda^n}{n!} f_p(\lambda)}{\int_0^{\infty} \frac{e^{-x} x^n}{n!} f_p(x) dx}$$

The last equation follows from the fact that the demand distribution is Poisson, as mentioned in Section 1 and discussed further in Section 4. The revised distribution  $f_p(\cdot|n)$  will now represent the prior distribution for demand in period  $t+1$ , and the updating process will repeat itself.

For the updating process to be computationally feasible, one would need to devise a way of efficiently computing the posterior distribution. This can be achieved by starting the process with a prior that belongs to the family of *conjugate distributions* for the Poisson. One such conjugate for the Poisson is the two-parameter gamma distribution. The gamma includes a wide range of distribution forms over  $[0, \infty)$ , and its density function is given by  $g(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$ , where  $a$

$\geq 1$  and  $b > 0$  are the parameters of the distribution. When the prior  $f_p(\cdot)$  is gamma with parameters  $(a_p, b_p)$ , the posterior can be shown (DeGroot, pg. 323) to be given by

$$f_p(\lambda|n) = \frac{(b_p + 1)^{a_p+n}}{\Gamma(a_p + n)} x^{a_p+n-1} e^{-(b_p+1)x}, \text{ which is gamma with parameters } a_p + n, b_p + 1. \text{ Hence the}$$

calculation of the posterior distribution is reduced to simple parameter updates for  $a_p$  and  $b_p$ .

## Case 2. Varying prices

Our discussion above has assumed that the price remains fixed from one period to the next. We now need to include a mechanism that will allow us to use observed demand data at price  $p$  to learn about demand at any other price  $p^*$ .

As in the above discussion, we will require that, for each price  $p$ , the prior distribution on  $\lambda(p)$  is gamma with parameters  $(a_p, b_p)$ . In order to use demand observations at a certain price  $p$  to update the prior distribution for the purchase rate at another price  $p^*$ , we will connect together the parameters  $(a_p, b_p)$  of the gamma priors on  $\lambda(p)$  across different prices  $p$ . This will facilitate the updating of the parameters  $(a_{p^*}, b_{p^*})$  whenever we update the parameters  $(a_p, b_p)$ . Since there are two parameters  $(a_p, b_p)$  for each price  $p$ , we need two equations to link them together across  $p$ . We will derive these two equations by defining two desirable properties that should be satisfied by these parameters.

The first property is the expected purchase rate  $E[\lambda(p)]$  (under the prior distribution for  $\lambda(p)$ ) should have the same form across  $p$  as the true underlying value of  $\lambda(p)$ , i.e.,

$$E[\lambda(p)] = A(1 - F(p|\delta)), \text{ for some } \delta \text{ that is independent of } p$$

Since the mean of a gamma  $(a_p, b_p)$  distribution is given by  $a/b$ , this implies that

$$a_p/b_p = A(1 - F(p|\delta)), \text{ for some } \delta \text{ that is independent of } p \tag{P1}$$

The second property is that the coefficient of variation of the gamma  $(a_p, b_p)$  distributions should be independent of  $p$ , i.e.,

$$1/a_p = S \text{ for some } S \text{ that is independent of } p \tag{P2}$$

We now discuss the motivation that underlies properties (P1) and (P2). These properties limit the families of prior distributions  $\{\text{gamma}((a_p, b_p) \text{ for all prices } p\}$  on  $\lambda(p)$  that we are allowed to

consider. For instance, consider the set of means  $\{a_p/b_p$  for all prices  $p\}$  for one such family of prior distributions. The mean value  $a_p/b_p$  could in principle vary as a function of  $p$  in an almost arbitrary manner. Property (P1) requires that this value in fact should vary as a function of  $p$  in a specific way. Let the actual underlying value of the unknown parameter be  $\delta = \delta_0$ . Then we have

$$\lambda(p|\delta_0) = A(1 - F(p|\delta_0)) \text{ for all } p.$$

In the current situation, where  $\delta_0$  is unknown, it is reasonable to take the mean  $a_p/b_p$  of the purchase rate distribution as an estimate of the purchase rate at each price  $p$ . Then, motivated by the previous equation, we require that there exists a  $\delta$  value such that

$$a_p/b_p = E[\lambda(p)] = A(1 - F(p|\delta)) \text{ for all } p.$$

This is property (P1). The associated  $\delta$  value itself may be taken to represent our present estimate of  $\delta_0$ . Property (P2) is based on the following motivation: for any price level  $p$ , the coefficient of variation of the gamma  $(a_p, b_p)$  distribution reflects the level of uncertainty of the planner with respect to the arrival rate  $\lambda(p)$ . Property (P2) states that this level of uncertainty does not vary with price. Alternative versions for this property could also be used. For instance, if the planner's level of uncertainty about demand was much greater for some 'medium' range of prices, but much lower for some extreme (low or high) range of prices, property P2 could instead be defined as  $1/a_p = S/(p-m)^2$  for some constants  $S, m$ .

These equations allow us to link together the  $(a_p, b_p)$  parameters across  $p$ . If we know the values of  $a_p, b_p$  for some price  $p$ , we can calculate the values of  $a_{p^*}, b_{p^*}$  for some other price  $p^*$  using equations (P1) and (P2) as follows. First, we use (P1) and (P2) with price  $p$  to calculate the values for  $\delta$  and  $S$ . Then we use these values of  $\delta$  and  $S$ , along with price  $p^*$ , to calculate  $(a_{p^*}, b_{p^*})$ . Note that equation (P1) will yield a unique value for  $\delta$  because of Condition 3.1.

To illustrate, suppose the reservation price distribution was exponential, as described above. Properties (P1) and (P2) can be written in this case as:

$$\begin{aligned} a_p/b_p &= Ae^{-\delta p} && \text{(P1 - Exp)} \\ 1/a_p &= S && \text{(P2 - Exp)} \end{aligned}$$

Suppose, at any time, we had the parameters  $(a_p, b_p)$  and wanted to determine the parameters  $(a_{p^*}, b_{p^*})$ . First, we would calculate  $\delta$  and  $S$  from the above equations, as follows:

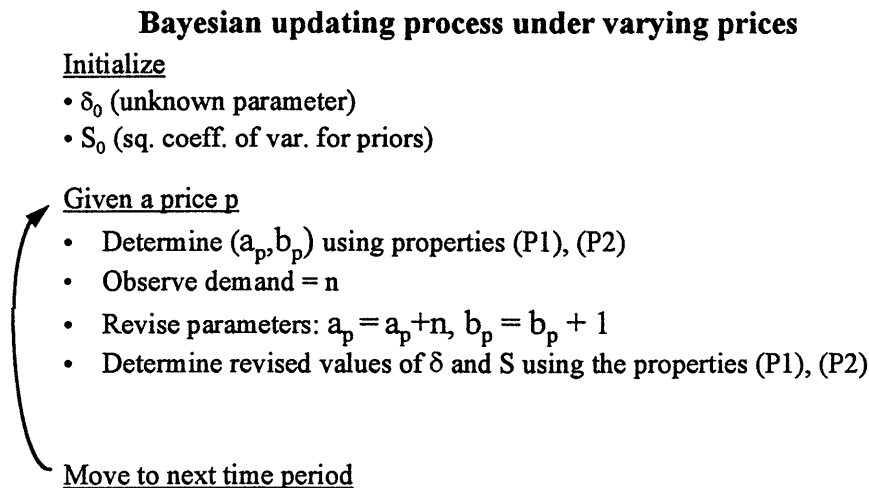
$$\delta = -(1/p)\log(a_p/Ab_p), \quad S = 1/a_p$$

We would then use equations (P1-Exp) and (P2-Exp) to calculate  $(a_{p^*}, b_{p^*})$  using these values of  $S$  and  $\delta$ , as follows:

$$a_{p^*} = 1/S, \quad b_{p^*} = 1/SAe^{-\delta p^*}$$

We observe that, given properties (P1) and (P2), information about the prior distributions for all price levels  $p$  is completely contained in the pair of parameters  $(\delta, S)$ . We therefore term these the 'linking' parameters, and it is these parameters that will get revised from one time period to the next. They would also need to be initialized by the model user, and these initial values would reflect the initial state of knowledge of the planner. Thus, in the exponential reservation price distribution case,  $\delta_0$  (the initial value of  $\delta$ ) would reflect the best estimate of the inverse of the

distribution mean, and  $S_0$  (the initial value of  $S$ ) would reflect the initial degree of uncertainty associated with this estimate. They would then get revised over time in the following manner: Suppose we are in period  $t$ , and the current values of the linking parameters are  $\delta$  and  $S$ . Suppose the price in period  $t$  is  $p$ , and we observe a demand of  $n$  units in period  $t$ . We would first calculate the parameters  $(a_p, b_p)$  for the prior on  $\lambda(p)$  based on equations (P1) and (P2) using the values  $\delta$  and  $S$  for the linking parameters. Next, this prior distribution would be updated in the standard Bayesian manner on observing the demand. This would yield a posterior distribution on  $\lambda(p)$  with parameters  $(a_p + n, b_p + 1)$ . The revised values for  $\delta$  and  $S$  would then be recalculated from equations (P1) and (P2) using these updated values for  $(a_p, b_p)$ . This process is outlined in Figure 3.2.



**Figure 3.2**

**Case 3. Non-stationary store arrivals**

We now consider the case where store arrival rates are not constant across time periods. The purchase rate now depends on the period under consideration in addition to the price, and so we denote it as  $\lambda_t(p|\delta)$ . Note that  $\lambda_t(p|\delta) = A_t(1-F(p|\delta))$ , where  $A_t$  is the store arrival rate in period  $t$ . The gamma priors also depends on the time period, and so we denote the parameters of the gamma prior on  $\lambda_t(p)$  as  $(a_{pt}, b_{pt})$  for each price  $p$  and time period  $t$ . As before, we will assume that the planner knows the value of  $A_t$  for each time period  $t$ , but that the planner still does not know the value of  $\lambda_t(p)$  since  $\delta$  is not known.

A property of the gamma distribution that is useful in this case is the following:

Lemma 3.1: If  $X$  is a random variable with a gamma  $(a,b)$  distribution, and  $Y = kX$  for some constant  $k$ , then  $Y$  has a gamma  $(a,b/k)$  distribution. □

Lemma 3.1 states a fairly standard result about the gamma distribution, and so we do not provide a proof for it here. We seek to modify the approach described in the previous case to allow for different values of  $A_t$ 's across time periods. The change we make will allow us to translate a

family of prior distributions on  $\{\lambda_t(p)\}_p$  for period  $t$  to a family of prior distributions on  $\{\lambda_{t+1}(p)\}_p$ . We explain the nature of the change required below.

Suppose we have an updated gamma  $(a_{pt}, b_{pt})$  prior on  $\lambda_t(p)$  at the end of period  $t$ , and this yields the updated values of  $\delta$  and  $S$  from equations (P1) and (P2). In the previous case, a gamma  $(a_{pt}, b_{pt})$  prior on  $\lambda_t(p)$  translated into a similar gamma  $(a_{pt}, b_{pt})$  prior on  $\lambda_{t+1}(p)$  since the two random variables  $\lambda_t(p)$  and  $\lambda_{t+1}(p)$  had the same prior distribution. In the present case, however,  $\lambda_{t+1}(p) = (A_{t+1}/A_t)\lambda_t(p)$ . Hence, Lemma 3.1 tells us that a gamma  $(a_{pt}, b_{pt})$  distribution on  $\lambda_t(p)$  converts to a gamma  $(a_{pt}, (A_t/A_{t+1})b_{pt})$  distribution on  $\lambda_{t+1}(p)$ . Thus, the parameters of the prior distribution for  $\lambda_{t+1}(p)$  satisfy the equations:

$$\frac{a_{pt+1}}{b_{pt+1}} = \frac{A_{t+1}}{A_t} \frac{a_{pt}}{b_{pt}} = \frac{A_{t+1}}{A_t} A_t (1 - F(p|\delta)) = A_{t+1} (1 - F(p|\delta)), \quad \text{and} \quad 1/a_{pt+1} = S$$

These equations yield

$$a_{pt+1} = 1/S \quad \text{and} \quad b_{pt+1} = \frac{1}{S A_{t+1} (1 - F(p|\delta))}$$

#### Case 4. Non-stationary reservation price distributions

We are now ready to address the final stationarity assumption made earlier - that the reservation price distribution is stationary (i.e., constant over time). This distribution may change over time, for instance, because customers value the product less over time, or because the high reservation price customers leave the market early in the season. The former may be true in the case of seasonal apparel such as coats, and the latter in the case of new books or fashionwear. To address this problem, we would need to know how the reservation price distribution changes over time. We assume that this change occurs in the following manner:

**Condition 3.2:** There exist constants  $R_2, \dots, R_T$  such that  $f_t(r) = f_t(R_t r)$  for all  $r$  and  $t > 1$ , where  $f_t(\cdot)$  is the density function for the reservation price distribution in period  $t$ .

Thus, under condition 3.2, the reservation price distribution in any period  $t > 1$  could be considered a 're-scaled' version of the distribution in period 1. We assume that reservation prices stay constant within any period, a reasonable assumption when the period lengths are of size, say, one day or a week. We also assume that the planner knows the trend in reservation prices as represented by the parameters  $(R_2, \dots, R_T)$ . These may have been derived, for instance, through an examination of historical sales records for similar products.

We now describe how our demand learning methodology, as described for Case 2, can be adapted to the above situation. We describe this modification for the case of the exponential reservation price distribution below. We use Case 2 only for convenience of exposition, and our approach is equally applicable to the setting in Case 3.

The variable  $\delta$  that is kept track of now represents the updated value of the parameter for the exponential reservation price distribution *corresponding only to period 1*. It is appropriately scaled in each period to derive the corresponding parameter for the reservation price distribution

for that period. Suppose we are at the beginning of period  $t$ . Suppose the current values of the linking parameters are  $\delta$  and  $S$ , and that the price in period  $t$  is  $p$ . We describe below how we calculate the revised values of  $\delta$  and  $S$  at the end of period  $t$ , given a demand observation of  $n$  in period  $t$ . First, the values  $(a_{pt}, b_{pt})$  of the gamma prior on  $\lambda_t(p)$  need to be calculated. Given the relationship in condition 3.2 between the reservation price distributions in periods 1 and  $t$ , the prior distribution on  $\lambda_t(p)$  is the same as the prior distribution (at the beginning of period  $t$ ) on  $\lambda_1(p/R_t)$ . The parameters for this distribution can be calculated using equations (P1-Exp) and (P2-Exp) with  $p/R_t$  instead of  $p$ , to yield:

$$a_p = 1/S \quad \text{and} \quad b_p = \frac{1}{SA e^{-\frac{\delta}{R_t}p}}$$

Next, these parameters are revised in the usual Bayesian fashion upon observing a demand level  $n$ , giving us the updated values  $(a_p + n, b_p + 1)$ . Now we need to calculate the revised values of  $\delta$  and  $S$  in terms of these updated values. As in the equations above, we get:

$$a_p + n = 1/S \quad \text{and} \quad b_p + 1 = \frac{1}{SA e^{-\frac{\delta}{R_t}p}}$$

which yields: 
$$\delta = -\frac{R_t}{p} \log\left(\frac{a_p + n}{A(b_p + 1)}\right), \quad \text{and} \quad S = 1/(a_p + n)$$

### Alternative reservation price distributions

We have been illustrating our methodology through an exponential reservation price distribution. Here, we show how some alternative distributions could be used in its place.

*Weibull, with unknown location parameter:* The Weibull is a three parameter family of distributions that allows us to model a fairly wide variety of unimodal reservation price distributions on  $[\delta, \infty)$  for any  $\delta \in \mathbf{R}$ . Its density and distribution functions are given by:

$$f(p) = \alpha\beta^\alpha(p-\delta)^{\alpha-1}e^{-\beta^\alpha(p-\delta)^\alpha}, \quad F(p) = 1 - e^{-\beta^\alpha(p-\delta)^\alpha}, \quad p \geq \delta$$

where  $\delta \in \mathbf{R}$  is the *location* parameter,  $\beta > 0$  is the *scale* parameter, and  $\alpha > 0$  is the *shape* parameter. By assuming that the location parameter  $\delta \leq 0$ , we can circumvent the case where we may have  $p < \delta$ . Even if  $\delta$  were allowed to be positive, it can be shown in a straightforward manner that the price will never be less than  $\delta$  in the optimal pricing model that we will discuss in the next section. Also, a comment is in order about the case where  $\delta < 0$ , since the fact that this leads to 'negative' values for the reservation price distribution may appear counter-intuitive. Actually, a Weibull distribution with density function  $f(\cdot)$  for which the location parameter  $\delta$  is negative can be replaced by a distribution with the same density function  $f(x)$  for  $x \geq 0$ , and with a probability  $\int_{\delta}^0 f(x)dx$  of being equal to zero. Since the price will never be negative, these two

distributions will be effectively equivalent in the demand behavior they model. This new distribution can be viewed as corresponding to a population of customers with a segment that has zero value for the product and the rest having values distributed as  $f(x)$  for  $x \geq 0$ .

The exponential distribution is a special case of the Weibull, derived by setting  $\alpha=1$  and  $\delta=0$ .

In this case, the location parameter may be taken as unknown while the shape and the scale of the distribution are taken as known to the planner. This means that the parameters  $\beta$  and  $\alpha$  will remain fixed at some prespecified levels, while the location parameter  $\delta$  will be revised over the course of time. This can alternatively be viewed as revising the mean of the distribution over time while maintaining the same shape and scale, since the mean of the Weibull distribution is of the form  $\delta + M(\alpha, \beta)$ , where  $M$  is a function of  $\alpha$  and  $\beta$ . This distribution can be shown to satisfy Conditions 3.1 and 3.2 as well.

In this case, we may write  $\lambda(p)$  as:  $\lambda(p) = Ae^{-\beta^\alpha(p-\delta)^\alpha}$  for  $p \geq \delta$

Property (P1) becomes:  $a_p/b_p = Ae^{-\beta^\alpha(p-\delta)^\alpha}$ ,  $p \geq \delta$ .

As before, this equation, along with (P2), provides straightforward solutions for  $\delta$  and  $S$  in terms of  $(a_p, b_p)$ , and vice versa.

We have used this case of a Weibull reservation price distribution with an unknown location parameter in our computational tests, discussed in Section 5.

*Weibull, with unknown scale parameter:* An alternative to the above would be to use just the two parameter family of Weibull distributions (with  $\delta = 0$ ) and assume that  $\beta$  was the unknown parameter. In this case, we have  $f(p) = \alpha \beta^\alpha p^{\alpha-1} e^{-\beta^\alpha p^\alpha}$ ,  $p \geq 0$   $F(p) = 1 - e^{-\beta^\alpha p^\alpha}$ ,  $p \geq 0$

In this case too Conditions 3.1 and 3.3 are satisfied. Property (P1) can now be expressed as:

$$a_p/b_p = Ae^{-\beta^\alpha p^\alpha}$$

This equation, along with (P2), provides straightforward solutions for  $\beta$  and  $S$  in terms of  $(a_p, b_p)$ , and vice versa. We have done some additional computational tests with this case.

The two cases of the Weibull distribution discussed above are illustrated in Figure 3.5.

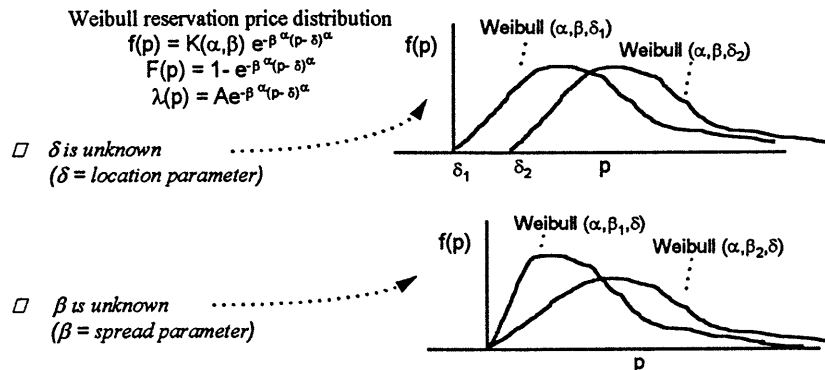


Figure 3.5

*Gamma distribution with unknown mean:* A third alternative would be to use the gamma distribution. We can express the density function  $f$  and the distribution function  $F$  of a gamma distribution with mean  $\mu$  and variance  $\sigma^2$  as:



$$f(p) = \left(\frac{\mu}{\sigma^2}\right)^{\frac{\mu^2}{\sigma^2}} p^{\frac{\mu^2}{\sigma^2}-1} e^{-\frac{\mu}{\sigma^2}p}, \quad F(p) = \int_0^p \left(\frac{\mu}{\sigma^2}\right)^{\frac{\mu^2}{\sigma^2}} x^{\frac{\mu^2}{\sigma^2}-1} e^{-\frac{\mu}{\sigma^2}x} dx$$

This distribution could be used in the following manner: It could be assumed that the planner knows how spread out the reservation price distribution is across the population of customers, though he or she may not know where this distribution is located. This means that the variance of the distribution would be fixed at some prespecified level  $\sigma^2$ , and the mean  $\mu$  would be unknown. Using the gamma instead of the exponential would entail making certain straightforward modifications in our demand learning process. Property (P1) will become:

$$a_p/b_p = A \int_p^\infty \left(\frac{\mu}{\sigma^2}\right)^{\frac{\mu^2}{\sigma^2}} r^{\frac{\mu^2}{\sigma^2}-1} e^{-\frac{\mu}{\sigma^2}r} dr \quad \text{for some } \mu > 0, \sigma^2 > 0 \quad (\text{P1-Gamma})$$

The one complicating issue that arises now is in the calculation of  $\mu$  in terms of  $p$ ,  $a_p$  and  $b_p$ . The above equation does not allow us to solve for  $\mu$  analytically. This problem may be addressed in

the following manner. Let us denote by  $I(p, \mu)$  the integral  $\int_p^\infty \left(\frac{\mu}{\sigma^2}\right)^{\frac{\mu^2}{\sigma^2}} r^{\frac{\mu^2}{\sigma^2}-1} e^{-\frac{\mu}{\sigma^2}r} dr$ . The values of

$I(p, \mu)$  could be precalculated for a range of different values of  $p$  and  $\mu$  (both  $p$  and  $\mu$  would need to be discretized), and this matrix of values could then be used to find the appropriate value of  $\mu$  for given values of  $\alpha_p$ ,  $\beta_p$ ,  $p$  and  $A$  to solve the above equation. It can be shown that  $I(p, \mu)$  is increasing in  $\mu$ , and so a simple binary search process could be employed in searching for the right value of  $\mu$ .

#### 4. OPTIMAL DYNAMIC PRICING WITH DEMAND LEARNING

In this section, we describe how the demand learning technique described in Section 3 can be embedded in the discrete time optimal dynamic pricing model developed by Bitran and Mondschein (1993). We begin with a description of the basic pricing model developed by these authors. We then show how it can be extended to incorporate the demand learning mechanism, and how the resulting model may be solved computationally. We also describe a heuristic solution approach for the model with demand learning that allows for substantial run-time savings.

##### Basic Pricing Model

We present below the discrete-time dynamic programming model for the optimal dynamic pricing problem as developed by Bitran and Mondschein (1993).

##### Basic Pricing Model

###### Notation

$N_t$	=	Number of customer arrivals in period $t$ (a Poisson random variable)
$A_t$	=	Customers' arrival rate in period $t$
$D_t(p)$	=	Number of purchases in period $t$ at price $p$ (a random variable)
$I_0$	=	Total inventory at beginning of planning horizon
$T$	=	Number of time periods in planning horizon

$f_t(\cdot)$  = probability density function for the reservation price distribution in time period  $t$   
 $F_t(\cdot)$  = cumulative distribution function for the reservation price distribution in time period  $t$   
 $V_t(I)$  = Maximum expected revenue from period  $t$  onwards when the initial inventory is  $I$ .

The indices for the time periods increase with time, i.e., the sequence of time periods is given by  $1, 2, \dots, T$ .

*Backward recursion*

$$\begin{aligned}
 V_t(I) &= \max_{p \geq 0} E[p \min\{D_t(p), I\} + V_{t+1}(I - \min\{D_t(p), I\})] \\
 &= \max_{p \geq 0} \sum_{d=0}^{I-1} P(D_t(p) = d) (pd + V_{t+1}(I - d)) + P(D_t(p) \geq I) pI
 \end{aligned}$$

*Boundary conditions*

$$\begin{aligned}
 V_T(I) &= 0 && \text{for all } I \\
 V_t(0) &= 0 && \text{for all } t
 \end{aligned}$$

The distribution for  $D_t(p)$  is given by:

$$P(D_t(p) = d) = \sum_{n=d}^{\infty} P(N_t = n) (1 - F_t(p))^d F_t(p)^{n-d} \quad (4.1)$$

The following result is from Bitran and Mondschein (1993):

**Lemma 4.1:**  $D_t(p)$  is a Poisson process with arrival rate  $\lambda_t(p) = A_t(1 - F_t(p))$ .

**Proof:** Follows by performing some straightforward algebraic manipulations on equation 4.1. □

Note that in the above model we are ignoring inventory holding costs and the time value of money - these entail straightforward modifications and do not affect the results in this paper, and so, for expositional simplicity, have not been modeled. In addition, we are assuming that there is no shortage cost aside from the lost opportunity to generate more revenues through additional sales.

## Model solution

The dynamic program can be solved backwards in time. For each stage  $t$  and state variable  $I$ , we need to solve a unidimensional non-linear optimization problem where the decision variable is the price  $p$ . This basic model is not computationally demanding.

## Incorporation of demand learning

The demand learning technique developed in Section 3 requires that the parameters  $(\delta, S)$  be transferred from one period to the next. We therefore need to include these parameters in the state space of the dynamic program. As mentioned in Section 3, these parameters are initialized by the model user, and they are then updated and transferred from one period to the next in the dynamic program. We provide below a formal description of how the demand learning process is incorporated within the dynamic programming model. To simplify the description, we assume that there are only two periods in the model, that the store arrival rate and reservation price distribution are stationary, and that the reservation price distribution is exponential with an unknown mean. The extensions to the more general cases of nonstationary store arrivals,

nonstationary reservation price distributions and alternative reservation price distribution forms, are straightforward.

## **Extension of Basic Pricing Model To Incorporate Demand Learning**

### Initialization

The user specifies the following parameters:

- $\delta_0$  = Initial value of the parameter  $\delta$
- $S_0$  = Initial value of the parameter  $S$
- $I_0$  = Initial level of inventory
- $A$  = Store arrival rate

### State space:

The state space at the beginning of period  $t$  ( $t=1,2$ ) is given by  $(I, \delta, S)$ , where  $I$  is the level of inventory, and  $(\delta, S)$  the updated values of the parameters, at the beginning of period  $t$  (or, equivalently, at the end of period  $t-1$ ).

### Solution process

The backward recursion approach is employed to solve the dynamic program. We therefore begin from period 2 (the last period), and calculate the value function for each state of the system, going then to period 1 and doing the same for the starting state  $(I_0, \delta_0, S_0)$ .

### Period 2:

In period 2, we solve the following optimization problem:

$$V_2(I, \delta, S) = \max_{p \geq 0} E[p \min\{I, D(p)\} | \delta, S]$$

for each possible state  $(I, \delta, S)$ . For a given state  $(I, \delta, S)$ , this problem is solved by performing a line search on  $p$ . We describe below how the expectation in the above equation is computed for a given state  $(I, \delta, S)$  and a given value of  $p$ .

First, the values for  $a_p$  and  $b_p$  are calculated using equations (P1 - Exp) and (P2 - Exp):

$$a_p = 1/S \quad \text{and} \quad b_p = 1/SAe^{-\delta p}$$

Now

$$\begin{aligned} E[p \min\{I, D(p)\} | \delta, S] &= \sum_{n=1}^{I-1} pnP[D(p) = n | \delta, S] + pIP[D(p) \geq I | \delta, S] \\ &= \sum_{n=1}^{I-1} pnP[D(p) = n | \delta, S] + pI \left( 1 - \sum_{n=1}^{I-1} P[D(p) = n | \delta, S] \right) \end{aligned}$$

The above term would be calculable in a straightforward manner once we knew  $P[D(p) = n | \delta, S]$  for each  $n$  from 1 to  $I-1$ . For a given  $n$ , this probability can be calculated as follows:

$$\begin{aligned}
P[\mathbf{D}(p) = n|\delta, S] &= \int_0^{\infty} P[\mathbf{D}(p) = n|\lambda] \text{gamma}(\lambda|a_p, b_p) d\lambda \\
&= \int_0^{\infty} \frac{\lambda^n}{n!} e^{-n\lambda} \lambda^{a_p-1} e^{-b_p\lambda} \frac{b_p^{a_p}}{\Gamma(a_p)} d\lambda \\
&= \frac{b_p^{a_p}}{n! \Gamma(a_p)} \int_0^{\infty} \lambda^{n+a_p-1} e^{-(b_p+1)\lambda} d\lambda \\
&= \frac{b_p^{a_p}}{n! \Gamma(a_p)} \frac{\Gamma(n+a_p)}{(b_p+1)^{n+a_p}}
\end{aligned}$$

### Period 1

In period 1, we need to solve the following optimization problem:

$$\begin{aligned}
V_1(I_0, \delta_0, S) \\
&= \max_{p \geq 0} E\left[p \min\{I_0, \mathbf{D}(p)\} + VR_2(I_0 - \min\{I_0, \mathbf{D}(p)\}, \bar{\delta}, \bar{S}) | \delta_0, S_0\right]
\end{aligned}$$

$\bar{\delta}, \bar{S}$  here are the updated values of  $\delta$  and  $S$  that are to be passed to the next period, *and they are functions of  $\delta_0, S_0, n$  and  $p$* . Here,  $n$  is the number of purchases observed in period 1. We show how these are calculated below.

The above problem is solved, again, by performing a line search on  $p$ . We describe below how the expectation in the above equation is computed for a given value of  $p$ .

As before, the values for  $a_p$  and  $b_p$  are calculated using equations (P1 - Exp) and (P2 - Exp):

$$a_p = 1/S_0 \quad \text{and} \quad b_p = 1/S_0 A e^{-\delta_0 p} \quad (4.1)$$

Now

$$\begin{aligned}
&E\left[p \min\{I_0, \mathbf{D}(p)\} + V_2(I_0 - \min\{I_0, \mathbf{D}(p)\}, \bar{\delta}, \bar{S}) | \delta_0, S_0\right] \\
&= \sum_{n=1}^{I-1} \left( pn + V_2(I_0 - n, \bar{\delta}, \bar{S}) \right) P[\mathbf{D}(p) = n | \delta_0, S_0] + p I_0 P[\mathbf{D}(p) \geq I_0 | \delta_0, S_0] \\
&= \sum_{n=1}^{I-1} \left( pn + V_2(I_0 - n, \bar{\delta}, \bar{S}) \right) P[\mathbf{D}(p) = n | \delta_0, S_0] + p I_0 \left( 1 - \sum_{n=1}^{I-1} P[\mathbf{D}(p) = n | \delta_0, S_0] \right)
\end{aligned}$$

Here,  $(\bar{\delta}, \bar{S})$  are the revised values of  $\delta$  and  $S$  got by observing demand  $n$  at price  $p$ . The above term could be computed in a straightforward manner once we have calculated  $\bar{\delta}, \bar{S}$ ,  $V_2(I_0 - n, \bar{\delta}, \bar{S})$  and  $P[\mathbf{D}(p) = n | \delta, S]$  for each  $n$  from 1 to  $I-1$ . We show below how, for a given value of  $n$ , the quantities  $\bar{\delta}, \bar{S}$  and  $P[\mathbf{D}(p) = n | \delta, S]$  are calculated. The function value  $V_2(I_0 - n, \bar{\delta}, \bar{S})$  is then known from the period 2 calculations done at the previous step of the algorithm.

$P[\mathbf{D}(p) = n | \delta, S]$  is calculated just as in the previous step, with the values of the parameters  $a_p$  and  $b_p$  now coming from equation (4.1). We are now left with the task of computing  $(\bar{\delta}, \bar{S})$ . This observed demand first results in a revision of the parameter values of the gamma prior on  $\lambda(p)$  from  $(a_p, b_p)$  to  $(a_p+n, b_p+1)$ . This updating process is based on a direct application of Bayes' rule, as described in the previous section. Let us denote these revised parameter values by  $(\bar{a}_p, \bar{b}_p)$ , so that  $\bar{a}_p = a_p + n$ , and  $\bar{b}_p = b_p + 1$ . The parameters  $(\bar{\delta}, \bar{S})$  are derived by solving equations (P1) and (P2) with  $(\bar{a}_p, \bar{b}_p)$  in place of  $(a_p, b_p)$ . This yields:

$$\bar{\delta} = -\frac{1}{p} \log\left(\frac{\bar{a}_p}{A\bar{b}_p}\right) \quad \text{and} \quad \bar{S} = \frac{1}{\bar{a}_p},$$

that is,

$$\bar{\delta} = -\frac{1}{p} \log\left(\frac{a_p + n}{A(b_p + 1)}\right) \quad \text{and} \quad \bar{S} = \frac{1}{a_p + n}.$$

For this demand learning methodology to be effective, we expect to observe the following trends:

- S should decrease over time, since it represents the amount of uncertainty that is left in the planner's knowledge about the demand function
- $\delta$  should converge to its true underlying value over time (on average).

These expectations are supported by the results of the computational tests we have performed, as discussed in the next section.

## Model Solution

The variables  $\delta$  and S need to be discretized since they are in the state space of the dynamic program. The variable  $\delta$  can be assumed to lie in some interval of reasonable length around the initial estimate  $\delta_0$ . S can be assumed to lie in some interval  $[0, W]$  where W is an upper bound on S. Once these intervals are identified, the variables can be discretized appropriately to balance solution accuracy with computational efficiency.

The increase in state space caused by the introduction of  $\delta$  and V has a significant effect on the computation time for solving the program. Depending on other model characteristics, the dynamic program may or may not have a satisfactory solution time.

A different way to reduce the computational time requirement is by adapting to the above problem the expected value heuristic described in Chapter 3 of Wadhwa (1996). We provide a brief description of this heuristic here, referring the reader to the above dissertation for a more complete discussion. The heuristic is based on a fairly common technique for solving large scale stochastic dynamic programming problems, involving the replacement of all the random variables in the model ( $\mathbf{D}_t(p)$ 's) by their expected values ( $\lambda_t(p) = A_t(1 - F_t(p))$ ). This reduces the optimal pricing model to a deterministic optimization problem. As shown in Wadhwa (1996), this problem can be formulated as a concave maximization problem and solved through an efficient line-search technique. The heuristic is based on formulating and solving such an approximation to the model

on a rolling horizon basis. Thus, at the beginning of period  $t$ , we solve the deterministic problem from period  $t$  onwards and utilize the optimal price  $p_t$  that it determines. Then we observe the demand  $D_t(p)$ , revise the state variables ( $I$ ,  $\delta$ ,  $S$ ) appropriately, and repeat the process in the next period. This heuristic has performed quite well in our computational tests, and we report on its performance in the next section.

## 5. COMPUTATIONAL TEST RESULTS

In this section, we present summary results from some of the computational tests we have performed and draw some conclusions and insights from them. The test results provide evidence about the capability of our demand learning methodology to converge to the correct underlying reservation price distribution and about the strong performance of the heuristic solution approach. These results also yield some interesting insights into pricing behavior, sales patterns, and the supply-demand imbalance under different conditions.

We begin by describing our computational testing plan.

### Test plan

Our objective in performing the computational tests was to study the following issues:

1. How well does the demand learning technique perform? Does  $\delta$  converge to the correct underlying value, and does  $S$  converge to zero?
2. What is the impact of over- or under-estimation in demand on the retailer's pricing behavior, on optimal revenues, and on the balance between supply and demand over the season?
3. Under situations of demand over- and under-estimation, what is the impact of demand learning on the retailer's pricing behavior, on optimal revenues, and on the balance between supply and demand over the season?
4. What is the impact of a non-increasing price constraint on the retailer's pricing behavior? How does the demand learning approach compare with the no demand learning approach when a non-increasing price constraint is imposed?
5. How well does the heuristic solution approach (described in Section 4) perform in relation to the more computationally demanding optimal solution approach?

To examine these issues, we developed a pair of base case problems with the following characteristics. There were four time periods, and the initial inventory was set at 50. The Poisson store arrival rate was set at 500 for each time period. The values of the parameters for the Weibull reservation price distribution were taken so as to get a mean reservation price of \$100 along with a reasonable spread around this mean value. These parameter values were 2.0, .007 and -30 for  $\alpha$ ,  $\beta$  and  $\delta$  respectively. The base case consisted of two versions - in the overestimation version,  $\delta$  was overestimated by 30 (i.e.,  $\delta_0$  was set at 0), which resulted in an overestimation of the mean reservation price by \$30. In the underestimation version, this mean was underestimated by \$30 by setting  $\delta_0 = -60$ . The initial value for the coefficient of variation,  $S$ ,

was taken to be 0.5 in both cases. The parameters  $\delta$  and  $S$  were discretized by using 7 and 5 point grids for them respectively.

In addition to the pair of base case problems themselves, several modifications of this pair - involving changes in one or more variables - were utilized to examine the impact of those changes on the solution. These problems were solved separately using the following different approaches:

1. NO DL: The basic optimal pricing model (with no demand learning)
2. DL: The optimal pricing model with demand learning
3. DL-HEUR: The optimal pricing model with demand learning, using the heuristic solution technique.
4. PERF INFO: The basic optimal pricing model under perfect information, i.e., with the correct initial value of  $\delta$ .

PERF INFO and NO DL are based on the basic pricing model of Bitran and Mondschein (1993). They differ in that PERF INFO is based on perfect demand information, and NO DL is not. DL and DL-HEUR are both based on the demand learning model we have described in this paper. They differ only in solution technique, as described above. The PERF INFO run was done in order to provide a baseline against which to compare the results from the other models - the solution under the PERF INFO model represented the ideal pricing behavior (and level of revenues) that would be evidenced if the reservation price distribution had been correctly estimated.

To analyze the solutions from these approaches for each test problem, we simulated the price policies recommended by each approach under the true demand conditions (i.e., using the true value of  $\delta$ ). The simulations allowed us to calculate the expected values under these pricing strategies of a number of quantities of interest, such as the overall revenues, the price level and sales in each time period, and the values of the parameters  $S$  and  $\delta$  at the end of each time period.

Both the basic pricing model and the model with demand learning, in the forms they were described in the previous section, assume that price can be set at any value - i.e., prices are not restricted to some discrete set of allowable levels. This may constitute a small departure from realism, for retailers would typically have a list of feasible price levels that they would select from instead of allowing price to be set at any (continuous) level. We have performed a number of runs using discrete price sets (for instance, the set  $\{\$20, \$25, \$30, \dots, \$95, \$100\}$ ). These results are similar to those described in this section under continuous prices, and we therefore have not separately discussed the discrete-price context here. See Gallego and Van Ryzin (1994) and Bitran and Mondschein (1993, 1995) for additional discussions on using continuous versus discrete price levels in the seasonal product pricing context.

We now turn to a discussion of the results of the computational tests.

### **Convergence of parameters**

Table 5.1 shows the values of the parameters  $S$  and  $\delta$  at the end of time periods  $t = 1, 2$  and  $3$  under different initial inventory levels. (Note that we have not shown the values of  $\delta$  directly but instead have shown the values of the mean reservation price, a more meaningful quantity). The

rest of the data for these problems was taken from the base case with demand overestimation. The results for the base case with demand underestimation were similar in nature.

**Average value of mean ( $\delta + 130$ )**

		Inventory			
		25	50	75	100
End of time period	1	106.5	104.5	103	102.5
	2	103	102	101.5	101
	3	102	101.5	101	101

INITIAL VALUE = \$130  
TRUE VALUE = \$100

**Table 5.1**

**Average value of Sq. Coeff. of Var. (S)**

		Inventory			
		25	50	75	100
End of time period	1	0.22	0.17	0.11	0.10
	2	0.09	0.05	0.04	0.03
	3	0.05	0.03	0.02	0.01

INITIAL VALUE = 0.5

**Table 5.2**

These results indicate that, over time, the parameter S converges to zero and the mean reservation price (and the parameter  $\delta$ ) converges to its true underlying value. The convergence is faster when there are more sales observations in each time period (which happens when the initial inventory is higher), which is intuitive. These results indicate that the demand learning procedure does an effective job of recovering the true underlying reservation price distribution. This suggestion is corroborated by a number of additional computational tests we have done using different problem data.

### **Performance of models under different conditions**

Table 5.3 shows the optimality gaps between the optimal expected revenues under PERF INFO and the expected revenues under the approaches NO DL, DL and DL- HEUR under different initial inventory levels. The optimality gap (or % suboptimality) for NO DL is defined as the following quantity:

$$100 \cdot \frac{\text{Optimal exp. revs. (under PERFINFO)} - \text{Exp. revs. (Under NO DL)}}{\text{Optimal exp. revs. (under PERFINFO)}}$$

and it is defined similarly for DL and DL-HEUR.

Table 5.4 shows the same data under different levels of demand misestimation. It is striking to observe how poorly the NO DL method performs in many cases. In contrast, the DL model



performs consistently well, even when the level of demand misestimation is very high. This suggests that demand learning can lead to a substantial improvement in revenue. We also note that DL-HEUR performs almost as well as DL in most cases. This provides further evidence of the effectiveness of the expected value heuristic described in Wadhwa (1996), and suggests that a viable heuristic solution option exists for situations where the DL model may be too computationally demanding.

**Different Levels of Initial Inventory**  
*% Suboptimality of Revenues (Compared to PERF INFO)*

$I_0 =$	Overestimation (Mean = \$130, i.e., $d = 0$ )				Underestimation (Mean = \$70, i.e., $d = -30$ )			
	25	50	75	100	25	50	75	100
No DL	19.9	20.2	19.3	18.8	6.5	6.8	7.0	7.3
DL	1.9	2.4	2.3	2.7	4.5	1.5	2.0	1.5
DL - Heur	2.9	2.3	2.1	2.1	3.6	3.2	3.2	3.1

**Table 5.3**

**Different Levels of Demand Misestimation**  
*% Suboptimality of Revenues (Compared to PERF INFO)*

Initial Mean ( $\delta_0 + 130$ ) =	Overestimation			Underestimation		
	\$160	\$130	\$115	\$40	\$70	\$85
No DL	52.5	20.2	4.9	22	6.8	2.2
DL	4.6	2.4	1.1	10.3	1.1	0.8
DL - Heur	4.5	2.3	1.3	20.7	3.2	0.9

**Table 5.4**

We note from table 5.3 that, in the demand overestimation case, the basic model under imperfect information appears to perform progressively better with increasing levels of initial inventory, and that this behavior is reversed in the demand underestimation case. At present, we do not have strong intuition or analytical support for these observations. We can, however, determine the optimality gap between the basic model with imperfect demand information (the NO DL model) and the same model with perfect information (the PERF INFO model) as the level of inventory approaches infinity, and this result is formally stated below.

**Lemma 5.1:** Let  $\delta_0$  be the initial estimate of the unknown parameter  $\delta$  and let  $\delta_a$  be its true underlying value. Let  $V^{\text{NODL}}(I)$  be the expected revenues derived from the price strategy recommended by NO DL, and  $V^{\text{PERFINFO}}(I)$  be the expected revenues from the price strategy recommended by PERF INFO, when the initial inventory level is  $I$ . For each period  $t$ , let  $p_t(\delta)$  be the price that maximizes the function  $pA_t(1 - F_t(p|\delta))$ . Then

$$\frac{V^{\text{NODL}}(I)}{V^{\text{PERFINFO}}(I)} \rightarrow \frac{\sum_{t=1}^T p_t(\delta_0) A_t (1 - F_t(p_t(\delta_0)|\delta_a))}{\sum_{t=1}^T p_t(\delta_a) A_t (1 - F_t(p_t(\delta_a)|\delta_a))} \text{ as } I \rightarrow \infty$$

**Proof:** As  $I \rightarrow \infty$ , the dynamic program for the basic pricing problem separates into  $T$  independent pricing problems, one for each period  $t$ , since the inventory constraint for the season is effectively removed. The optimal price in period  $t$  is then given by the price that maximizes the expected period  $t$  revenue,  $pA_t(1-F_t(p|\delta))$  (see the proof of Proposition 2 in Bitran and Mondschein, 1995). This price is  $p_t(\delta_0)$  in the NO DL case and  $p_t(\delta_a)$  in the PERF INFO case, and the result then follows. □

It is straightforward to show that in the case where the reservation price distributions  $F_t$  are of the exponential type, the above ratio can be arbitrarily bad (i.e., can be arbitrarily close to zero).

### Impact of initial level of demand uncertainty

Table 5.5 shows the 'optimality gaps' between the optimal expected revenues under PERF INFO and the expected revenues under the approaches NO DL, DL and DL- HEUR under different values of  $S_0$ . This data suggests that the DL and DL-HEUR models both perform fairly robustly with respect to  $S_0$  as long as  $S_0$  is not too small. This is a very desirable result since  $S_0$  is probably the most unintuitive of all the parameters that needs to be specified by the user, and would therefore require the most judgement on the user's part. The poor performance by DL and DL-HEUR for the case where  $S_0$  is very small is not surprising - a small  $S_0$  implies that the user is very certain about the initial estimate  $\delta_0$ , and the Bayesian updating approach would then continue to give a large weight to this initial estimate instead of responding more to the observed sales, thus making the demand learning model behave very similar to the basic (no-demand learning) model.

<b>Different Levels of Initial Demand Uncertainty</b>										
<i>% Suboptimality of Revenues (Compared to PERF INFO)</i>										
Overestimation (Mean = \$130, i.e., $\delta = 0$ )      Underestimation (Mean = \$70, i.e., $\delta = -30$ )										
$S_0 =$	0.01	0.3	0.5	1	2	0.01	0.3	0.5	1	2
No DL	20.2	20.2	20.2	20.2	20.2	6.8	6.8	6.8	6.8	6.8
DL	12.5	1.9	2.4	2.4	2.9	6.8	2.2	1.5	1.1	1
DL - Heur	13.9	2.6	2.3	2.3	2.2	5.2	3.4	3.2	3.4	4.5

**Table 5.5**

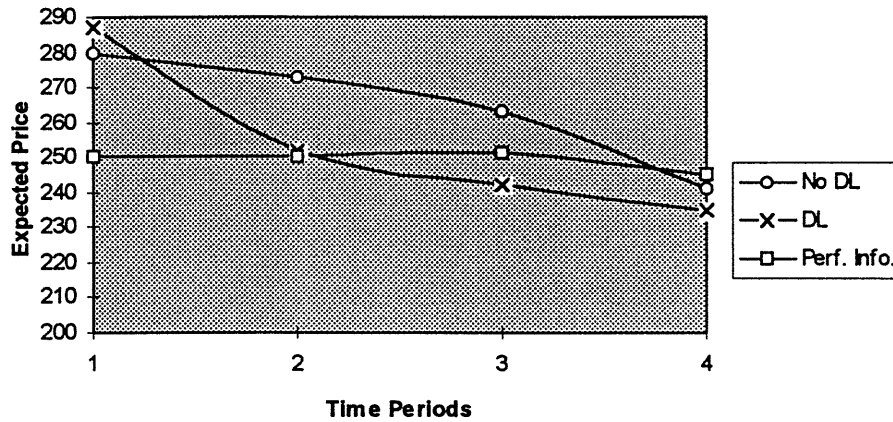
### Expected Price and Sales Trends

We now discuss the expected price and sales trends observed for the base case runs. This analysis sheds some valuable insights into dynamic pricing behavior under misestimated demand; and the effect of demand learning on this behavior.

Figure 5.1 shows the expected prices in periods 1-4 under PERF INFO, NO DL and DL for the base case, and Figure 5.2 shows the cumulative expected sales at the end of periods 1-4 for the

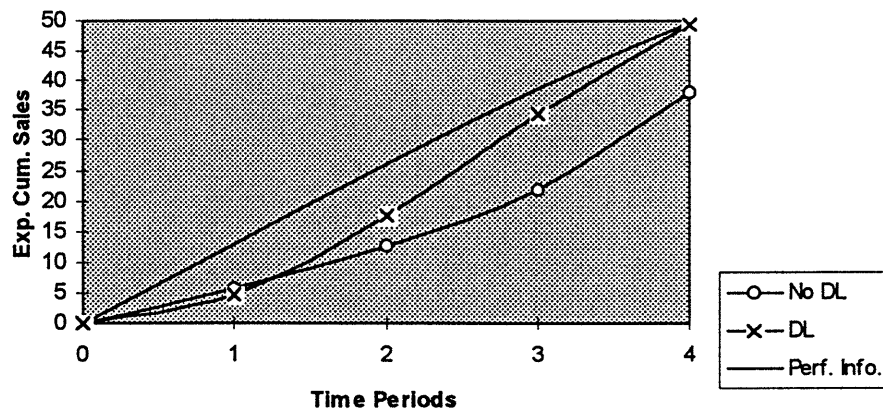
same cases. The NO DL and DL runs were performed under the demand overestimation version of the base case.

**Base Case with Overestimation  
Expected Price Trends**



**Figure 5.1**

**Base Case with Overestimation  
Expected Cumulative Sales**

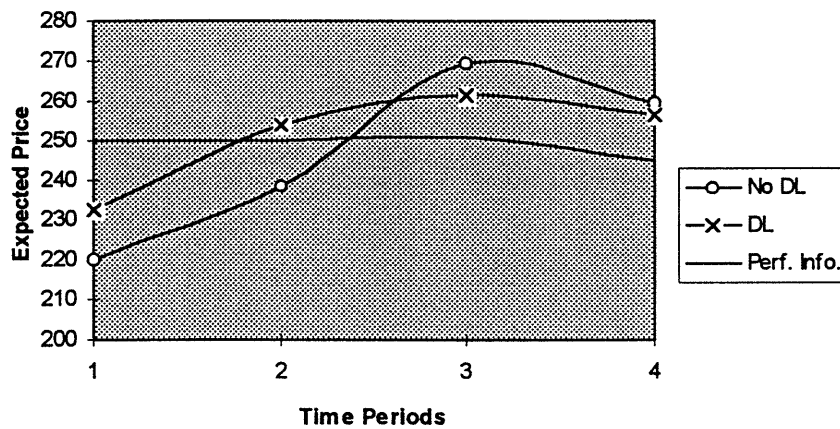


**Figure 5.2**

Under perfect information, the ideal pricing strategy is to maintain a fairly level price and sales trend throughout the season (with a slight dip in the last period). When demand is overestimated, both NO DL and DL start the season with higher-than-ideal prices, as we would expect. The NO DL approach leads to gradual markdowns over time since the model finds an unexpectedly high level of unsold inventory after each period. This occurs since it always expects demand at the price it sets to be higher than what occurs on average, since it has overestimated the reservation price distribution. As seen in Figure 5.2, the NO DL approach leads to slow initial sales, and while the successive markdowns do help to accelerate sales, they do not have an adequate enough

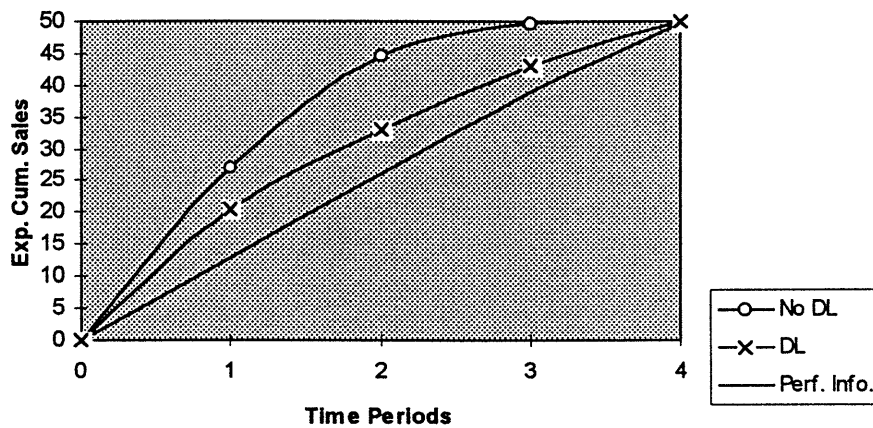
impact because the model continues to expect higher sales than what will actually occur on the average. When the season ends, there is therefore a significant amount of unsold stock left. Under the DL approach, the initial price is again higher (and sales again slower) than the ideal levels. However, the slow sales in period 1 lead this model to learn that it has overestimated demand, and it responds by dropping the price significantly in order to affect a price correction. This price is even lower than that from PERF INFO since in the DL case the period 1 sales have been very low, and so it needs to compensate for it by accelerating sales in the subsequent periods. Notice that DL does a very good job of balancing supply and demand by the end of the season by correcting its price path early in the season. Figures 5.3 and 5.4 show the same output as Figures 5.1 and 5.2 for the demand underestimation version of the base case.

**Base Case with Underestimation**  
**Expected Price Trends**



**Figure 5.3**

**Base Case with Underestimation**  
**Expected Cumulative Sales**



**Figure 5.4**

The ideal pricing policy under perfect information is exactly the same as previously seen, since this assumes perfect demand estimation (and is therefore not affected by the demand over or under estimation). When demand is underestimated, both NO DL and DL start the season with lower prices. The NO DL approach leads to gradual markups over time as the model finds an unexpectedly high level of sales in each period. This is so since it always expects demand at the price it sets to be lower than what occurs on average, since it has underestimated the reservation price distribution. As seen in Figure 5.2, the NO DL approach leads to higher-than-ideal levels of initial sales, and while the successive markups do help to slow down sales, they do not have an adequate enough impact because the model continues to expect lower sales than what will actually occur on the average. On the average, therefore, the inventory is exhausted much before the season ends, leading to a stockout. Under the DL approach, the initial price is again lower (and sales again higher) than the ideal levels. However, the high sales level in period 1 leads this model to learn that it has underestimated demand, and it responds by raising the price significantly in order to affect a price correction. This price is even higher than that from PERF INFO since in the DL case the period 1 sales have been very high, and so it is left with fewer units to sell in the subsequent periods. DL again does a very good job of balancing supply and demand by the end of the season by correcting its price path early in the season.

The increase in prices recommended by the DL and NO DL models above may be infeasible in many retail contexts, for retail practice (based, say, on consumer expectations) may require that the price of the merchandise not be marked *up* within a season. While the above analysis may still provide useful insights about optimal pricing behavior under freer pricing conditions, it is also interesting to study the performance of different approaches under such a non-increasing price requirement.

### **Impact of non-increasing price constraint**

In order to impose this requirement, the basic pricing model and the model with demand learning are modified in the following manner. The price variable is discretized (so that price can only assume one of a limited set of values). The state space for both models is augmented by a new variable that stands for the price determined for the previous period. This price variable acts as an upper bound on price in the current period. The dynamic programming formulation of this problem can be found in Bitran and Mondschein (1995). These authors also present a number of computational test results for this model. The formulation for the demand learning case can be derived in a simple manner as an extension to the one considered in this paper, and so we do not provide a technical description here. The heuristic solution approach we described in Section 4 does not transfer over to this non-increasing price context.

Figure 5.5 shows the expected price trends from a series of runs of the different models with and without the non-increasing price constraint. These runs were made using the base case data presented earlier - in particular, the initial inventory level is 50, and the initial coefficient of variation  $S$  is 0.5 for the DL runs. Under price overestimation, the impact of the non-increasing constraint on the price trends under DL and NO DL is minimal, since the price tends to be revised downwards over time. Therefore, the DL and NO DL runs shown in Figure 5.5 correspond to the

base case of demand underestimation ( $\delta_0 = -\$60$ ). All runs were done using a discrete set of prices based on \$5 increments.

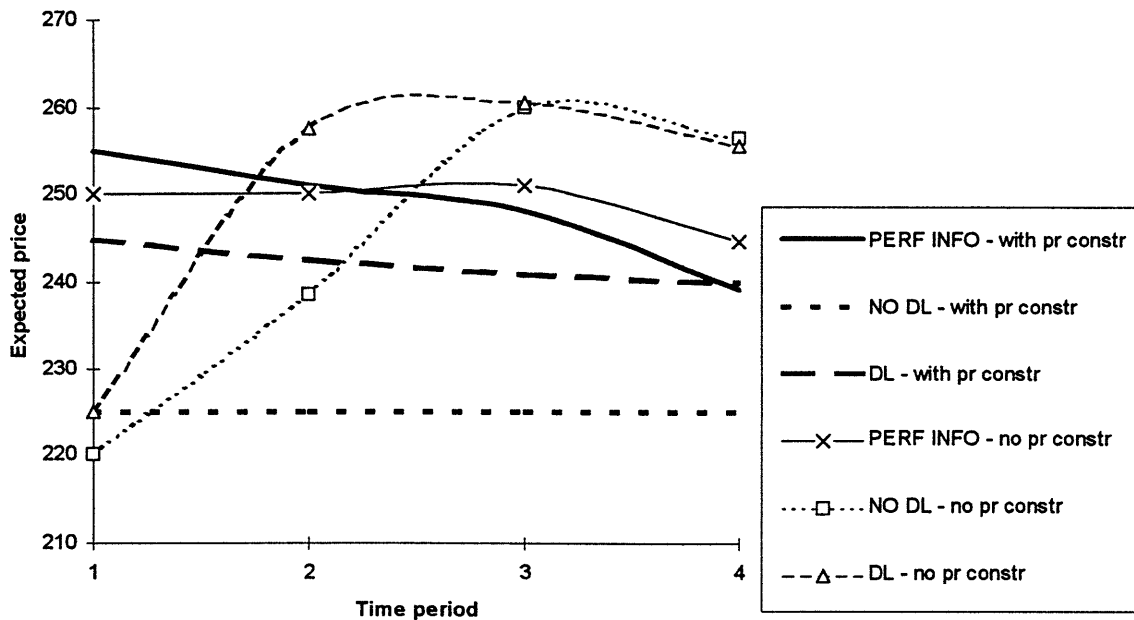


Figure 5.5

The following observations can be made about the price trends shown in the above figure:

- For each model (PERF INFO, NO DL, and DL), the imposition of the non-increasing price constraint leads to an increase in the initial (period 1) price. This is intuitive - since there will not be any recourse to revising the price upwards in the future, the retailer starts with a higher period 1 price in order to have more pricing flexibility (i.e., a greater range of feasible prices) in the future, and this flexibility is important because of the uncertainty in demand. (Note that, for the DL and NO DL models, this increase is *not* influenced by the fact that demand has been initially *underestimated*, since the first period price is determined by these models before any sales have been observed. We would therefore expect to see the same period 1 price even for the case of demand overestimation.)
- Since the DL and NO DL runs are based on demand underestimation, the initial prices under these runs are lower than the price under PERF INFO.
- For the non-increasing price case, the period 1 price under the DL model is higher than that under the NO DL model. In the DL case, the model recognizes that demand may have been significantly over or underestimated, and sets the initial price in a manner that will allow it to revise price appropriately as demand learning occurs over time. This initial price is therefore set at a higher value than for the NO DL model, since the DL model wants to give itself more flexibility in the range of prices it can adopt in future periods in response to demand learning.

## 6. CONCLUSIONS AND FUTURE RESEARCH ISSUES

In this paper, we have presented a methodology for dynamic demand learning in an environment where the demand distribution changes over time due to changes in price, customer arrival rate and customers' values for the product. We have shown how this methodology can be incorporated in a model of dynamic pricing for seasonal products. In addition, we have performed a series of computational tests to evaluate our demand learning scheme and to study the performance of dynamic pricing strategies with and without demand learning. Some key conclusions and insights that we have derived from these computational tests are the following:

- Under the demand learning methodology presented in this paper, the estimated reservation price distribution converges to the actual reservation price distribution and the variances of the priors converge to zero as more observations are gathered. The methodology is therefore successful in utilizing sales data to move from the planners' initial estimate of customers' values for the product to their true underlying values.
- When there is a significant misestimation of demand, the optimal pricing strategy under no demand learning will result in large revenue losses due to:
  - Overpricing and high levels of unsold stock (under demand overestimation), or
  - Underpricing and early stockouts (under demand underestimation)
- Under conditions of demand misestimation, dynamic demand learning can help to limit the revenue losses to relatively nominal levels.
- Under the demand learning approach, there is a swift price correction early in the season with not much learning thereafter. Thus, it is important to have good pricing flexibility early in the season.
- When a non-increasing price constraint is imposed, both the NO DL and DL approaches start with a higher initial price in order to maintain more pricing flexibility in future periods. However, the DL approach starts with a substantially higher initial price than the NO DL approach since it recognizes that, in addition to the residual uncertainty, demand may have been significantly over or underestimated initially.

While it has addressed a key limitation of existing models of seasonal product pricing, our research has also identified a number of methodological and empirical issues on which further work would be useful. These are discussed below.

- *Increased demand variability:* At any given price  $p$  and time period  $t$ , we assume that the underlying demand distribution is Poisson. Since the variance of a Poisson distribution is equal to its mean, the use of this distribution strictly limits the variance of the underlying demand distribution. Alternative distribution forms such as the negative binomial and the normal allow for higher variances, and an extension of the demand learning approach to one of these cases may therefore be useful.
- *Test marketing, reorders and channel arrangements:* In some situations, retailers may have the flexibility to place a second order for the product early in the sales season after gaining some limited demand information through observed sales at some stores or through consumer interviews. An appealing extension of the model in this paper would be one that incorporates a reordering decision after the first period. Certain aspects of manufacturer-retailer contracts, such as a limit on the reorder quantity, end-of-season returns to the manufacturer, and backup

agreements (such as those considered by Eppen and Iyer, 1995) may also be investigated through extensions of the model in this paper. An analysis of such models may provide useful insights about the retailer's market testing, pricing and ordering strategy.

- *Multiparameter (or non-parametric) estimation:* Our demand learning method assumes that the retailer's lack of information can be captured via a reservation price distribution in which a single parameter is unknown to the planner. Thus, for instance, in our computational tests, we have assumed that the planner knows the shape and scale of the (Weibull) reservation price distribution and does not know the location (or equivalently, the mean) of the distribution. A more powerful demand learning model would allow for even less of prior knowledge on the retailer's front by determining the true reservation price distribution with more limited prior information. Alternatively, one could use the "Weibull with unknown location parameter"-based context of our computational study and perform additional tests to evaluate the performance of the demand learning scheme in situations where the planner has misjudged the shape and scale of the reservation price distribution.
- *Non-stationary reservation price distribution:* Another useful extension may be in contexts where the reservation price distribution changes with time. Our approach assumes that this change follows a certain pattern which is known to the planner, and this may in certain contexts be a limiting assumption. An alternative technique might apply differential weights to the sales data from past periods to capture the trend in customers' values for the product in addition to revising the retailer's initial estimate of this distribution.
- *Empirical study:* The discussion in the last section has highlighted the importance of demand learning and of correcting prices swiftly in response to such learning. This analysis leads to a number of questions on industry practice that merit empirical investigation, such as the following: How effective are retail organizations in learning about demand early in the season and to what extent do they react through early price corrections? How does our methodology perform in comparison with the decisions made by merchandise managers at these organizations?

## REFERENCES

1. Azoury, K.S. (1979), "Bayesian policies for dynamic inventory models," Ph.D. Dissertation, Engineering, UCLA, 1979.
2. Azoury, K.S. (1985), "Bayes solution to dynamic inventory models under unknown demand distribution," *Management Science*, 31, 9, 1150-60.
3. Bitran, G.R. and S.V. Mondschein (1993), "Perishable product pricing: An application to the retail industry," *MIT working paper*, July.
4. Bitran, G.R. and S.V. Mondschein (1995), "Periodic pricing of seasonal products in retailing," to appear in *Management Science*.
5. Chang, H.S. and D.E. Fyffe (1971), "Estimation of forecast errors for seasonal-style-goods sales," *Management Science*, 18, 2, B-89 - B-96.
6. Crowston, W.B., W.H. Hausman and W.R. Kampe II (1973), "Multistage production for stochastic seasonal demand," *Management Science*, 19, 8, 924-935.
7. DeGroot, M.H., "Probability and statistics," Second Edition, Addison-Wesley Publishing Co.



8. Fisher, M. L., J. H. Hammond, W. R. Obermeyer, and A. Raman (1994), "Making supply and demand meet in an uncertain world", *Harvard Business Review*, May-June 1994, 83-93.
9. Fisher, M. L. and A. Raman (1992), "Reducing the cost of demand uncertainty through accurate response to early sales", to appear in *Operations Research*.
10. Gallego, G. and G. van Ryzin (1994), "Optimal dynamic pricing of inventories with stochastic demand over finite horizons," *Management Science*, 40, 8, 999-1020
11. Hammond, J. H. and M. G. Kelly (1990), "Quick Response in the Apparel Industry" Harvard Business School Note N9-690-038.
12. Hammond, J. H. (1992), "Quick Response in Retail/Manufacturing Channels," Chapter 8, *Globalization, Technology and Competition*, Harvard Business School Press, Boston, Massachusetts.
13. Hausman, W.H. (1969), "Sequential decision problems: A model to exploit existing forecasters," *Management Science*, 16, 2, 93-111
14. Hausman, W.H. and R. Peterson (1972), "Multiproduct production scheduling for style goods with limited capacity, forecast revisions and terminal delivery," *Management Science*, 18, 7, 370-383.
15. Hines, W. W. and D. C. Montgomery (1980), "Probability and Statistics in Engineering and Management Science," John Wiley and Sons, New York, New York.
16. Iglehart, D. L. (1964), "The dynamic inventory problem with unknown demand distribution," *Management Science*, 10, 3, 429-440.
17. Iyer, A. V. and G. D. Eppen, (1995a), "Improved fashion buying with Bayesian updates" Working Paper, Graduate School of Business, University of Chicago.
18. Iyer, A. V. and G. D. Eppen, (1995b) "Backup agreements in fashion buying - the value of upstream flexibility" Working Paper, Graduate School of Business, University of Chicago.
19. Karlin, S. (1960), "Dynamic inventory policy with varying stochastic demands," *Management Science*, 6, 231-258.
20. Murray, G.R. and E.A. Silver (1966), "A Bayesian analysis of the style goods inventory problem," *Management Science*, 12, 11, 785-796.
21. Popovic, J.B. (1987), "Decision making on stock levels in cases of uncertain demand," *European Journal of Operations Research*, 32, 276-90.
22. Scarf, H. (1959), "Bayes solutions of the statistical inventory problem," *Annals of Mathematical Statistics*, 30, 490-508.
23. Scarf, H. (1960), "Some remarks on bayes solutions to the inventory problem," *Naval Research Logistics Quarterly*, 7, 591-596.
24. Wadhwa, H. K. (1996), "Models for pricing and inventory management of seasonal products," Doctoral Dissertation, MIT Sloan School of Management, MIT, Cambridge, MA..