A DYNAMIC MODEL FOR REQUIREMENTS PLANNING WITH APPLICATION TO SUPPLY CHAIN OPTIMIZATION

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This paper develops a new model for studying requirements planning in multi-stage production-inventory systems. We first characterize how most industrial planning systems work, and then develop a mathematical model to capture some of the key dynamics in the planning process. Our approach is to use a model for a single production stage as a building block for modeling a network of stages. We show how to analyze the single-stage model to determine the production smoothness and stability for a production stage, and the inventory requirements. We also show how to optimize the tradeoff between production capacity and inventory for a single stage. We then can model the multi-stage supply chain, using the single stage as a building block. We illustrate the multi-stage model with an industrial application and conclude with some thoughts on a research agenda.

1. Introduction

Most discrete-parts manufacturing firms plan their production with MRP (materials requirements planning) systems, or at least, with logic based on the underlying assumptions of MRP. A typical planning system starts with a multi-period forecast of demand for each finished good or end item. The planning system then develops a production plan (or master schedule) for each end item to meet the demand forecast. These production plans for the end items, after offsetting for lead times, then act as the requirement forecasts for the components needed to produce the end items. The requirements forecast for each component gets translated into production plans for the component, similar to how the production plan for the end items was created. The planning system continues in this way developing requirement forecasts and production plans for each level of the bill of materials.

Implicit in this planning process are assumptions about the production and demand process. The production plan is developed assuming that the forecast is accurate and will not change. Within the production process, requirements are generated assuming that there are deterministic production lead times and deterministic yields. Needless to say, these assumptions of a benevolent world don't match reality. Inevitably, the forecast changes and uncertainties in the production process arise that result in deviations from the plan. To respond to these changes, most planning systems will completely revise their plan after some time period, say a week or a month. Again, the planning process starts with the (new) forecast and repeats the steps necessary to regenerate a plan for each level in the products' bills of materials.

The intent of this paper is to present a model that captures the basic flavor of this planning process, and does so in such a way that it can be used to look at various tradeoffs within the production and planning systems. In particular, we model the forecasts for the planning system as a stochastic process; in this way we try to represent a dynamic input to the planning system, namely how forecasts change and evolve over time. The forecast process is the key exogenous input for the model. The other key is how the forecasts get converted into production plans or master schedules. We model this process as a linear system, with which we can represent the logic for MRP systems and from which we get significant analytical tractability. Finally, the model is structured so that it can describe multi-stage production-inventory systems.

We are not aware of very much work that is directly related to the dynamic modeling of requirements planning. Baker (1993) provides a nice survey and critique of the literature relevant to the general topic of requirements planning. However, most of the work deals with specific issues like lot sizing, or determination of buffer levels. Karmarkar (1993) discusses tactical issues of lot sizing, order release and lead times in the context of dynamic planning systems. But neither of these papers reports on work that attempts to model a dynamic forecast process. One exception is Graves, Meal et al. (1986) in which we modeled a two-stage production-inventory system with a dynamic forecast process. In contrast with the present paper, Graves, Meal et al. (1986) focused on issues of how to disaggregate an aggregate plan in the two-stage context. Although the present paper does not consider the disaggregation issue, it does provide a more powerful model that is applicable to general multi-stage systems.

The model for converting the forecast into a production plan is related to earlier work by the first co-author in that it uses linear systems for a production-inventory context. See Graves 1986; 1988a, 1988b, 1988c, and Fine and Graves, 1989.

There are four more sections to this paper. In the next section we develop the model for a single production-inventory stage. As part of the development, we present our model for the forecast process, and develop the analyses to generate three performance measures for the stage: production smoothness, production stability, and inventory requirements. In the third section we examine an optimization for the tradeoff between production capacity and inventory for a single stage. Although the development is somewhat involved, the final results are surprisingly simple and we believe, of interest. We report on an application of the model to a supply-chain study in the fourth section. The application demonstrates the value of a system-wide perspective for optimizing the supply chain. In the final section we briefly summarize the paper and then lay out a research agenda for enhancing this thrust, namely modeling dynamic requirements planning.

2. Single-Stage Model

In this section we present the model for a single production stage that produces one (aggregate) product and that serves demand from a finished good inventory. The single-stage model serves as a building block for creating models of multi-stage, multi-item systems. We first describe the forecast process that generates the demand requirements for the product. We then give a model for determining the schedule for production outputs from the production stage and show how to manipulate this model to obtain three measures of interest: the production variance as a measure of production smoothing; the inventory variance as a measure of safety stock; and the stability of the production schedule as a measure for the forecast process passed on to any upstream stages.

Forecast Process

We assume that in each period t we have a *requirements forecast*, where $f_t(t+i)$ is the forecast made at time t for the requirements in period t+i, i = 1, 2, ... H, and H is the forecast horizon. At time t, we have no forecast of demand beyond time t+H; in effect, for i > H we assume that $f_t(t+i) = \mu$, where μ equals the long-run average demand rate. We denote the demand observed in period t by $f_t(t)$; that is, the forecast made in period t for requirements in period t is synonomous with the realized demand.

We assume a forecast process in which we generate a new set of forecasts $f_t(t+i)$ each period; we model the forecast process for requirements as a martingale process. The updates of the forecasts from period to period are generated by the *forecast revision*, $\Delta f_t(t+i)$, defined by

(1) $\Delta f_t(t+i) = f_t(t+i) - f_{t-1}(t+i)$ for i = 0, 1, ... H,

where $f_{t-1}(t+H) = \mu$ by assumption. We assume that for a fixed index i, $\Delta f_t(t+i)$ is an i.i.d. random variable over time t with zero mean.

Thus, in the current period we have a forecast for each time period (e. g., month or week) in our planning horizon. We assume that in the next period we will completely revise the forecast. However, we expect the revision to have zero mean and to be independent from period to period.

The vector for the revisions to the forecast process is given by $\underline{\Delta f}_t$, where $\Delta f_t(t+i)$ is the $i+1^{st}$ element, $i = 0, 1, 2, \ldots H$. By assumption, $\underline{\Delta f}_t$ is an i.i.d. random vector with $E[\underline{\Delta f}_t] = 0$ and $Var[\underline{\Delta f}_t] = \Sigma$, the covariance matrix. We note that if we can observe the forecast process, then we can assess whether or not the forecasts are unbiased (i.e., $E[\underline{\Delta f}_t] = 0$) with independent revisions, and can estimate the covariance matrix Σ.

The *i-period forecast error* is the difference between the actual demand in period t and the forecast of this demand made i periods earlier:

$$f_t(t) - f_{t-i}(t) = \Delta f_t(t) + \Delta f_{t-1}(t) + \cdots + \Delta f_{t-i+1}(t).$$

Since $\Delta f_s(t)$ for $s = t-i+1, \ldots t$, are independent random variables, each with zero mean, the i-period forecast, $f_{t-i}(t)$, is unbiased and the forecast of a particular requirement will improve (in terms of a smaller variance of the forecast error) over time. That is, the variance of the i-period forecast error, $f_t(t) - f_{t-i}(t)$, is smaller than the variance of the (i+1)-period forecast error, $f_t(t) - f_{t-i-1}(t)$. We note that the variance of the H-period forecast error is the trace of the covariance matrix Σ.

Schedule for Production Outputs

Given the forecast vector for period t, we need to convert it into a schedule or plan for production. This is often termed the master schedule. We focus on production outputs from the production stage; later we will discuss how to translate a plan for production outputs into production starts. Production starts will be of interest since they serve as the requirements forecast for the next upstream production stage.

Let $F_t(t+i)$ equal the *planned production outputs* for period t+i as of period t, where $F_t(t)$ is the actual production completed in period t. We assume that the production plan extends out only for the next H periods, and that beyond this horizon, the plan is just to produce the average demand, that is, $F_t(t+i) = \mu$ for $i > H$.

Each period, after we obtain the new forecast, we update or revise the plan for production outputs. We define $\Delta F_t(t+i)$ as the *plan revision*:

$$\Delta F_t(t+i) \;=\; F_t(t+i) - F_{t-1}(t+i).$$

From this definition and the fact that $F_t(t+i) = \mu$ for $i > H$, we see that

$$(2) \qquad F_t(t+i) = \mu + \Delta F_{t+i-H}(t+i) + \cdots + \Delta F_t(t+i) \quad \text{for } i = 0, 1, \ldots H.$$

Thus, to model the production plan, we need to model the plan revision $\Delta F_t(t+i)$. To do this, we first define the inventory process. For I_t being the inventory at time t, the inventory balance equation is

$$(3) \qquad I_t = I_{t-1} + F_t(t) - f_t(t).$$

The *planned inventory* at time t+i is the expected level of inventory in a future period given the current forecast and the current production plan as of time t:

$$(4) \qquad I_t(t+i) = I_t + F_t(t+1) + \cdots + F_t(t+i) - f_t(t+1) - \cdots - f_t(t+i).$$

We assume that for each time t, we set the production plan $F_t(t+i)$, i = 0, 1, ... H, so that the planned inventory at the end of the planning horizon, $I_t(t+H)$, is a given constant. That is, we will set the production plan and maintain it from period to period so that the end-of-horizon inventory neither grows nor decreases, but remains constant. We term the level to which the inventory is targeted as the *safety stock*. In a later section we will discuss how to set this level; for now, all we need to know is that this level remains constant.

From (3) and (4), we obtain by equating $I_{t-1}(t-1+H)$ and $I_t(t+H)$ that

(5) $$\Delta F_t(t) + \Delta F_t(t+1) + \cdots + \Delta F_t(t+H) = \Delta f_t(t) + \Delta f_t(t+1) + \cdots + \Delta f_t(t+H).$$

That is, to assure that the end-of-horizon inventory remains constant, we require that the cumulative revision to the production plan should equal the cumulative forecast revision in each period.

Each period we revise the production schedule to ensure (5). To do this, we model the schedule update as a linear system:

(6) $$\Delta F_t(t+i) = \sum_{j=0}^{H} w_{ij} \ \Delta f_t(t+j) \qquad \text{for } i = 0, 1, \ldots H.$$

where w_{ij} denotes how the forecast revision affects the schedule. In particular, w_{ij} is the proportion of the forecast revision for period t+j that is added to the schedule of production outputs for period t+i.

We expect that $0 \leq w_{ij} \leq 1$; to ensure that (5) is true, we require that for each j:

$$\sum_{i=0}^{H} w_{ij} = 1.$$

We will refer to w_{ij} as a *weight* or proportion. In general, these weights are smoothing parameters. To smooth production we will see that we set the weights w_{ij} for a fixed j to be as nearly constant as possible (e.g., $w_{ij} = 1/(H+1)$ for $i = 0, 1, \ldots H$); to minimize inventory, we set the weights so that the production plan tracks the forecast as closely as possible (e.g., for fixed j, $w_{ij} = 1$ for $i = j$ and $w_{ij} = 0$ otherwise). The specification of the weights permits one to balance the tradeoff between production smoothing and inventory requirements, as will be seen.

We can use (6) to model how most implementations of MRP systems translate forecast revisions into schedule revisions. For instance, in the simplest case at time t the schedule is frozen for periods t+j, j = 0, 1, 2 ... k for some value of k < H, and is totally free to change for periods t+j, j = k+1, ... H. Then any revision to the forecast within the frozen zone results in a schedule revision for the first period beyond the frozen zone; that is, for $0 \leq j \leq k$, $w_{ij} = 0$ for $i \neq k+1$ and $w_{ij} = 1$ for $i = k+1$. Any revision to the forecast beyond the frozen zone results in a one-for-one schedule revision in the same period: for $k+1 \leq j \leq H$, $w_{ij} = 1$ for $i = j$ and $w_{ij} = 0$ otherwise. Occasionally there is an intermediate zone, between the frozen and free zones, in which changes to the schedule are permitted but are restricted in size, e.g., no more than 10% increase or decrease in the scheduled amount. The model given by (6) cannot exactly capture this policy, but can approximate its behavior by using fractional weights.

In matrix notation, we can rewrite (6) as

$$(7) \qquad \underline{\Delta F}_t = \mathbf{W}\, \underline{\Delta f}_t$$

where $\mathbf{W} = \{ w_{ij} \}$ is an (H + 1) x (H + 1) matrix, and $\underline{\Delta F}_t$ and $\underline{\Delta f}_t$ are column vectors with elements $\Delta F_t(t+i)$ and $\Delta f_t(t+i)$, for i = 0, 1, ... H. From this, we observe that $\underline{\Delta F}_t$ is an independent random vector, has zero mean, and has a covariance matrix $\mathbf{W}\, \Sigma\, \mathbf{W'}$. (This observation is important since from $\underline{\Delta F}_t$ we obtain directly the forecast revision for the upstream production stage.)

We can express the production plan in matrix notation by

$$(8) \qquad \underline{F}_t = \mathbf{B}\, \underline{F}_{t-1} + \mu \underline{U}_{H+1} + \underline{\Delta F}_t$$

where $F_t(t+i)$ is the $i+1^{st}$ element of the vector $\underline{F}_t$ for i = 0, 1, ... H, $\underline{U}_{H+1}$ is a unit vector with $u_i = 0$ for i = 1, ... H and $u_{H+1} = 1$, and **B** is a matrix with elements $b_{ij} = 1$ for j = i+1 and $b_{ij} = 0$ else. Premultiplying a column vector by **B** replaces the i^{th} element in the vector with the $i+1^{st}$ element and replaces the last element with a zero.

From (7) and (8) and repeated substitution, we obtain

$$(9) \qquad \begin{aligned} \underline{F}_t &= \mathbf{B}\underline{F}_{t-1} + \mu \underline{U}_{H+1} + \mathbf{W}\underline{\Delta f}_t \\ &= \mathbf{B}^{H+1}\underline{F}_{t-H-1} + \underline{\mu} + \sum_{i=0}^{H} \mathbf{B}^i \mathbf{W}\underline{\Delta f}_{t-i}, \end{aligned}$$

where $\underline{\mu}$ is the vector with each element equal to μ. We can simplify (9) by noting that premultiplying an (H + 1) x 1 vector by $\mathbf{B}^{H+1}$ gives the null vector:

$$(10) \qquad \underline{F}_t = \underline{\mu} + \sum_{i=0}^{H} \mathbf{B}^i \mathbf{W}\underline{\Delta f}_{t-i}.$$

Production Starts

We have described a model for setting production outputs for a single stage based on a dynamic forecast process of the requirements for that stage. We need to relate this to the production inputs or starts for the production stage. Production starts are critical for planning in a multi-stage system in that the production starts for one stage will determine the demand or requirements for other (upstream) stages that supply the raw inputs.

Suppose the production stage has a fixed and known lead time: production started in period t completes production and is available to meet demand in period t + L, where L is the lead time. We assume for now that there is no yield variability, so that production outputs equal production starts. For a fixed lead time L, we expect $\Delta F_t(t+i)$ to equal zero for i < L, since the quantity

$F_t(t+i)$ was started at time t+i-L and we presume that it cannot be stopped or altered once it is in the production stage. This implies that, for a fixed lead time of L, $w_{ij} = 0$ for $i < L$ for all j.

We define $G_t(t+i)$ to be the *planned production starts* in period t+i, as of period t. If we assume a fixed lead time L, the output in period t+L+i is the input or production starts in period t+i, namely $G_t(t+i) = F_t(t + L + i)$. Hence, the description and characterization of $F_t(t+i)$ provides a complete description and characterization for $G_t(t+i)$, once we offset the output plan to reflect the lead time.

From the prior analysis, we see that the revision of the plan for the production starts, $\Delta G_t(t+i) = G_t(t+i) - G_{t-1}(t+i)$, is an i.i.d. random variable. This is important since $\underline{G}_t = \{G_t(t+i)\}$ is the forecast of requirements for upstream stages over some horizon (suitably scaled to reflect "goes into" relationships and expected yield factors). *The fact that* $\underline{\Delta G}_t$ *is an i.i.d. random vector implies that the previous analysis, which assumed a dynamic forecast process* $\underline{f}_t$ *with i.i.d. revisions, can be applied directly to the upstream stages which see derived demand. Thus, the model of a single stage will give a building block to analyze a system of interrelated production stages.*

Although the case of fixed lead times seems most natural, the model can also be adapted to other cases. To preserve the above highlighted property, we just need to be able to model the relationship between the plan for the production starts and the output plan as a linear system; that is, for some matrix **A**, we can write $\underline{G}_t = \mathbf{A}\ \underline{F}_t$. If we can relate the starts to the outputs in this way, then $\underline{\Delta G}_t$ remains an i.i.d. random vector and the prior analysis can be reapplied using $\underline{G}_t$ as the forecast process for the upstream stages.

For instance, we might use the matrix **A** to model yield factors within the production stage (need to start 1.2 units to get output 1.0), or to model the fact that production starts occur on a different time scale (biweekly rather than weekly) from the production outputs. Also, we might want to model other policies for setting production starts. For example, we can set **A** to model a constant work-in-process policy where production starts for the period exactly equal production outputs. Indeed, in this way, for general multi-stage systems, we can use this general approach to compare push policies, where starts equal planned output L periods from now, with pull policies, where starts "replace" the outputs produced in the current period.

<u>Measures of Interest</u>

There are three categories of measures for the single-stage model: the *smoothness* of the production outputs, the *safety stock* for the end-item inventory, and the *stability* of the production plan.

The *smoothness* of the production outputs is of interest because more variable (less smooth) production is expected to require more production resources or capacity. Furthermore, we can influence the smoothness of production via our inventory and control policies.

An output of the model is the variability of the inventory process, which will dictate how much *safety stock* is needed to ensure an acceptable service level; if the inventory process is more variable, more safety stock will be needed.

The *stability* of the production plan, particularly the production starts, is of interest since the plan for production starts acts as the forecast for upstream stages. We will, in effect, equate the stability of the production plan to the accuracy of the forecast process for the upstream stages. More stablility means a more accurate forecast process. This measure is critical as we try to understand the workings of a multi-stage system since the inventory requirements and the variability of the production outputs for a stage will depend heavily on the accuracy of the forecast process.

We first develop the measures for production smoothing and for the stability of the production plan. We will need a more extensive development to obtain the variability of the inventory process in order to set the safety stock.

Production Smoothing

A common measure of production smoothing is the variance of the production output, Var[$F_t(t)$]. From (10) we see immediately that the random vector $\underline{F}_t$ has mean $\underline{\mu}$ and has a covariance matrix given by

$$\text{(11)} \quad \text{Var}(\underline{F}_t) = \sum_{i=0}^{H} \mathbf{B}^i \mathbf{W} \Sigma \mathbf{W}' \mathbf{B}'^i.$$

We can use the covariance matrix to obtain the first measure of the production smoothing, Var[$F_t(t)$]. Indeed, one can show that

$$\text{(12)} \quad \text{Var}\,[\,F_t(t)\,] = \text{tr}\,(\mathbf{W}\,\Sigma\,\mathbf{W}')$$

where tr(**A**) is the trace of matrix **A**.

A second measure of production smoothing is given by $F_t(t) - F_{t-1}(t-1)$, the change in production outputs from one period to the next. In matrix notation we see from (9) that

$$\underline{F}_t - \underline{F}_{t-1} = \mu U_{H+1} + \mathbf{W}\,\underline{\Delta f}_t - [\,\mathbf{I} - \mathbf{B}\,]\,\underline{F}_{t-1}$$

where **I** is the identity matrix. Since $\underline{\Delta f}_t$ and $\underline{F}_{t-1}$ are independent of each other, we find that the covariance matrix for $\underline{F}_t - \underline{F}_{t-1}$ is given by

$$\text{(13)} \quad \text{Var}(\underline{F}_t - \underline{F}_{t-1}) = \mathbf{W}\Sigma\mathbf{W}' + \sum_{i=0}^{H}[\mathbf{I} - \mathbf{B}]\,\mathbf{B}^i\,\mathbf{W}\Sigma\mathbf{W}'\,\mathbf{B}'^i\,[\mathbf{I} - \mathbf{B}]'$$

$$= (\mathbf{I} - \mathbf{B}) \, \mathrm{Var}(\underline{F}_t) + \mathrm{Var}(\underline{F}_t) \, (\mathbf{I} - \mathbf{B})'$$

From this covariance matrix, we can determine the second measure of production smoothing, namely Var[$F_t(t) - F_{t-1}(t-1)$].

Production Stability

For the stability of the production plan, we use $\underline{\Delta F}_t$: the random vector for the one-period revision to the production plan, which is the basis for the revision to the forecast of requirements for upstream stages. (The production starts, as described earlier, would generate the actual forecast seen by the upstream stages; but since the starts are usually just the production plan offset by the lead time, we can use the revision to the production plan for defining stability.) From (7) we obtain its expectation and covariance matrix:

$$E(\underline{\Delta F}_t) = \underline{0}$$

$$\text{(14)} \qquad \mathrm{Var}(\underline{\Delta F}_t) = \mathbf{W} \, \Sigma \, \mathbf{W}' .$$

We propose the covariance matrix $\mathbf{W} \, \Sigma \, \mathbf{W}'$ as a measure of the stability of the production plan. A more stable production plan will have a smaller covariance matrix, and will yield more accurate forecasts for the upstream stages. When analyzing the upstream stages, the dynamics of the requirements depend upon this covariance matrix; in this sense, for the upstream stages, the covariance matrix in (14) is analogous to Σ for the downstream stage, namely it is the covariance matrix for the relevant requirements forecast process.

One measure of the size of a covariance matrix is its trace. We note that with this interpretation the tr($\mathbf{W} \, \Sigma \, \mathbf{W}'$) signifies not only the stability of the production plan but also the variance of the requirements forecast for the upstream stages over the planning horizon. Furthermore, we see that, according to the proposed measures (12) and (14), smoothing production is essentially equivalent to stabilizing the production plan and requirements forecast for the upstream stages.

Inventory

We focus on the end-item inventory for the single stage. We assume that the requirements for the single stage are to be met from the end-item inventory and that typical service expectations apply, e.g., the inventory should stock out in no more than 2% of the periods or that the inventory should provide a 97% fill rate. In (3) we have defined the end-item inventory as I_t. The measure of inventory is the expectation of I_t, which we will show to be equal to the end-of-horizon inventory level; we term this level to be the safety stock. To determine this measure, we first need to evaluate $Var(I_t)$, since this quantity will determine the safety stock required to achieve a prespecified service level. For instance, if the forecast errors are normally distributed, then we can show that I_t has a normal distribution with expectation $E(I_t)$ and variance $Var(I_t)$. To achieve a desired service level expressed as the stockout probability, we need to set the safety stock level so that

(15) $E(I_t) > k\,\sigma(I_t)$

where k is such to ensure the service level and $\sigma()$ denotes the standard deviation.

Recall that in (4), we defined $I_t(t+i)$ to be the planned inventory level in period t+i as of time t; that is, $I_t(t+i)$ is the expected inventory in period t+i where the expectation is as of period t. For notational convenience, $I_t(t)$ denotes the actual inventory in period t, i.e., $I_t(t)$ is the same as I_t. As stated in the earlier development of (5), we assume that the end-of-horizon inventory $I_t(t+H)$ is targeted to equal some constant, which we call the safety stock and denote by *ss*.

The inventory flow equation for the planned inventory is

(16) $I_t(t+i) = I_t(t) + F_t(t+1) + \cdots + F_t(t+i) - f_t(t+1) - \cdots - f_t(t+i)$.

Define $\Delta I_t(t+i) = I_t(t+i) - I_{t-1}(t+i)$. From (3) and (16), we find that

$$\Delta I_t(t+i) = \Delta F_t(t) + \cdots + \Delta F_t(t+i) - \Delta f_t(t) - \cdots - \Delta f_t(t+i)$$

for i = 0, 1, ... H-1. By assumption, since we keep the inventory constant at *ss* beyond the horizon, we have

$$\Delta I_t(t+H) = I_t(t+H) - I_{t-1}(t+H) = ss - ss = 0.$$

In matrix notation, let $\underline{I}_t$ be an (H+1) x 1 column vector with $I_t(t+i)$ as its i+1st element. Then

$$\underline{\Delta I}_t = \mathbf{T} [\underline{\Delta F}_t - \underline{\Delta f}_t]$$

where **T** is a matrix with element $t_{ij} = 1$ for i >= j and $t_{ij} = 0$ else. We can now write the inventory random vector as

(17) $\underline{I}_t = \mathbf{T} [\underline{\Delta F}_t - \underline{\Delta f}_t] + \mathbf{B}\, \underline{I}_{t-1} + ss\underline{U}_{H+1}$

where we use the fact that $I_{t-1}(t+H) = ss$. We can simplify (17) by repeated substitution, by substitution of (7) and by noting that premultiplication of an (H + 1) x 1 vector by $\mathbf{B}^{H+1}$ gives the null vector:

(18) $\underline{I}_t = \sum_{i=0}^{H} \mathbf{B}^i\, \mathbf{T}\, [\mathbf{W} - \mathbf{I}]\ \underline{\Delta f}_{t-i} + \underline{ss}$

where $\underline{ss}$ denotes the column vector with each element equal to *ss*. From (18) we see that the random vector $\underline{I}_t$ has mean equal to the $\underline{ss}$, and has a covariance matrix given by

$$\text{(19)} \quad \mathrm{Var}(\underline{I}_t) = \sum_{i=0}^{H} \mathbf{B}^i \, \mathbf{T} \, [\mathbf{W} - \mathbf{I}] \, \Sigma \, [\mathbf{W} - \mathbf{I}]' \, \mathbf{T}' \, \mathbf{B}'^i .$$

We can use (19) to find Var[$I_t(t)$], which is necessary to determine how to set the safety stock level *ss*. From (19), we can show with some effort that

$$\text{(20)} \quad \mathrm{Var}[\, I_t(t)] = \mathrm{tr}(\, \mathbf{T} \, [\, \mathbf{W} - \mathbf{I} \,] \, \Sigma \, [\, \mathbf{W} - \mathbf{I} \,]' \mathbf{T}' \,) = \sum_{k=0}^{H} \sum_{i=0}^{k} \sum_{j=0}^{k} a_{ij}$$

where

$$\mathbf{A} = \{a_{ij}\} = [\, \mathbf{W} - \mathbf{I} \,] \, \Sigma \, [\, \mathbf{W} - \mathbf{I} \,]'.$$

Now from (15) we set the safety stock by $ss = k \, \sigma[\, I_t(t)]$, where $\sigma[I_t(t)]$ is obtained from (20) and k is such to provide the desired service level from the inventory.

<u>Summary</u>

We have developed in this section a single-stage model of requirements planning. The primary input to the model is the specification of the forecast process. The model determines how to satisfy the demand requirements by production and inventory planning. The model permits some control via the setting of the weight matrix **W**, which indicates how the production plan is revised to accomodate revisions to the forecast. Three types of measures for the model have been identified and characterized. We obtain the smoothness of production from (12), the stability of the production plan from (14) and a measure of the end-item inventory requirements from (20).

The model has been developed to extend directly to a general acyclic network of multiple stages. If the stage under study is a downstream stage that passes component requirements to one or more upstream stages, then the upstream stage(s) will have a forecast process with covariance matrix given by

(14). With this observation, the analysis presented here can be replicated directly to the upstream stage(s) and so on. For this extension to multiple stages, we need to assume that an upstream stage never starves a downstream stage, i. e., there is always adequate (raw material) inventory for a stage to make its production starts. This is an approximation which is likely to be reasonable if each stage operates with an inventory policy in which stockouts are rare.

3. Optimal Weights For Single-Stage Model

For a single stage it is natural to wonder how to choose the weights in (6) that determine how a forecast revision is converted into a revision of the production plan. To gain some insight into this question, we pose and solve an optimization problem for choosing the weights. The tradeoff between production smoothing in the stage and the end-item inventory requirements should govern the choice of weights. This tradeoff is the basis for stating the optimization problem:

(21) $\text{MIN}\ \sigma\,[\,F_t(t)\,]$

subject to

$$\sigma\,[\,I_t(t)\,] \le K$$

$$\sum_{i=0}^{H} w_{ij} = 1 \qquad\qquad \forall\, j.$$

The optimization problem minimizes production smoothing, as given by the standard deviation of the production output, subject to a constraint on the standard deviation of the inventory and the requirement that the weights sum to one. We can interpret the objective as minimizing production capacity; we can view the nominal capacity required at the stage as being the expected production requirements plus some number of standard deviations (see Graves (1988a) for further discussion of this viewpoint). The constraint on the standard deviation of the inventory is effectively a constraint on the amount of safety stock required; the safety stock is likely to be some multiple of $\sigma\,[\,I_t(t)\,]$, depending on the service expectations. An alternative formulation would be to minimize the standard deviation of the inventory, equivalently minimize the safety stock, subject to a constraint on the standard deviation of the production output.

There are no restrictions in the optimization on the weights, other than the convexity constraint. We have not imposed any non-negativity constraints nor any restrictions on the weights due to a fixed production lead time. That is, in the optimization we ignore the requirement that for a fixed lead time of L, $w_{ij} = 0$ for $i < L$ for all j. Rather, we allow the weights to be totally free; in this sense, the optimization will produce a lower bound for the case with fixed lead times.

To develop some insights on the optimal weights, we will transform the original optimization problem (21) into an equivalent form by restating it in terms of the variances of the production and inventory variables:

(21a) $\text{MIN Var} [F_t(t)]$

subject to

$$\text{Var} [I_t(t)] \leq K^2$$

$$\sum_{i=0}^{H} w_{ij} = 1 \qquad \forall j .$$

To analyze this equivalent problem, we consider the Lagrangian relaxation:

(21b) $L(\lambda) = \text{MIN Var} [F_t(t)] + \lambda \text{Var} [I_t(t)] - \lambda K^2$

subject to

$$\sum_{i=0}^{H} w_{ij} = 1 \qquad \forall j .$$

By solving this problem over a range of positive values for the Lagrange multiplier λ, we can find the tradeoff surface between production smoothing and inventory requirements for a single stage. We will also obtain some intuition for the form of the optimal weighting function.

In the remainder of this section, we will focus on solving (21b). To solve (21a), and equivalently (21), we would need to search over λ until the solution to (21b) satisfies the relaxed constraint.

We consider the case when the covariance matrix for the forecast revision process is diagonal. That is, we assume that the forecast revisions are uncorrelated, and $\mathrm{Var}[\underline{\Delta f}_t] = \Sigma = \{\sigma_i^2\}$, where $\sigma_i^2 = \mathrm{Var}[\Delta f_t(t+i)]$ is the i+1st element on the diagonal, i = 0, ... H.

For this case, we can simplify (12) and (20) to be

$$\text{(12*)} \qquad \mathrm{Var}[F_t(t)] = \sum_{i=0}^{H} \sum_{j=0}^{H} \left(w_{ij}\ \sigma_j\right)^2$$

and

$$\text{(20*)} \qquad \mathrm{Var}[I_t(t)] = \sum_{i=0}^{H} \sum_{j=0}^{H} \left(b_{ij}\ \sigma_j\right)^2$$

where

$$\text{(22)} \qquad b_{ij} = w_{1j} + \cdots + w_{ij} \qquad \text{for } i < j$$

$$= w_{1j} + \cdots + w_{ij} - 1 \qquad \text{for } i \geq j.$$

By substituting (12*) and (20*) into (21b), we observe that the minimization problem separates into H+1 subproblems, one for each period j:

$$\text{(21c)} \qquad L(\lambda) = \sum_{j=0}^{H} L_j(\lambda) - \lambda K^2$$

where

$$\text{(23)} \qquad L_j(\lambda) = \mathrm{MIN} \sum_{i=0}^{H} \left(w_{ij}\ \sigma_j\right)^2 + \lambda \sum_{i=0}^{H} \left(b_{ij}\ \sigma_j\right)^2$$

subject to

$$\sum_{i=0}^{H} w_{ij} = 1.$$

We now characterize the solution to $L_j(\lambda)$ with a series of propositions.

P1 The optimal weights in (23) are independent of σ_j^2.

Each term in the objective function of $L_j(\lambda)$ in (23) is proportional to σ_j^2, which can then be factored out. *Thus, we can determine the optimal weights in the Lagrangian (21b) without knowing the covariance matrix for the forecast revision.* We only need to know that the covariance matrix is diagonal. However, to solve the original problem, (21) or (21a), does require knowledge of the covariances to ensure satisfaction of the inventory constraint. Nevertheless, the determination of the optimal weights for the Lagrangian provides general insight into the character of the optimal solution to (21).

In the subsequent development we will focus on a single subproblem $L_j(\lambda)$ for some index j.

The Kuhn-Tucker conditions for (23) consist of the convexity constraint over the weights, plus the following set of equations:

$$(24) \quad w_{ij} + \lambda \sum_{k=i}^{H} \left(w_{0j} + \cdots + w_{kj} - u_{kj} \right) = \gamma \qquad \text{for } i = 0, \ldots H$$

where $u_{kj} = 1$ if $k >= j$, $u_{kj} = 0$ if $k < j$, and γ is the (scaled) dual variable for the single convexity constraint in (23). Since (23) is a convex program, the Kuhn-Tucker conditions are both sufficient and necessary, and identify a unique solution.

To find the solution, we equate (24) for i-1 and i to obtain

$$(25) \quad w_{ij} = w_{i-1,j} + \lambda \left(w_{0j} + \cdots + w_{i-1,j} - u_{i-1,j} \right) \qquad \text{for } i = 1, \ldots H.$$

We can construct a solution to (24) by selecting a value for w_{0j} and repeatedly applying (25). To satisfy the convexity constraint, we could search over values for w_{0j}. Alternatively, we describe in the next two propositions how to find w_{0j} analytically.

P2 For a given value of λ, the solution to (25) for w_{ij} is a linear function of w_{0j} given by

(26a) $w_{ij} = P_i(\lambda)\, w_{0j}$ for $i = 0, 2 \ldots j$,

(26b) $w_{ij} = P_i(\lambda)\, w_{0j} - R_{i-j}(\lambda)$ for $i = j+1, 2 \ldots H$,

where $P_i(\lambda)$ is a polynomial in λ of degree i, and $R_{i-j}(\lambda)$ is a polynomial in λ of degree i-j. In particular, we can show by induction that for n = 0, 1, ... H

$$P_n(\lambda) = \sum_{i=0}^{n} \frac{(n+i)!}{(2i)!\,(n-i)!}\,\lambda^i,$$

and that for n = 1, 2, ... H-j

$$R_n(\lambda) = \sum_{i=1}^{n} \frac{(n+i-1)!}{(2i-1)!\,(n-i)!}\,\lambda^i.$$

P3 The optimal choice for w_{0j} that solves (23) is given by

$$(27) \qquad w_{0j} = \frac{1 + \sum\limits_{i=j+1}^{H} R_{i-j}(\lambda)}{\sum\limits_{i=0}^{H} P_i(\lambda)} = \frac{P_{H-j}(\lambda)}{\sum\limits_{i=0}^{H} P_i(\lambda)},$$

which simplifies to

$$w_{0j} = \frac{\sum\limits_{i=0}^{H-j} \frac{(H-j+i)!}{(2i)!(H-j-i)!}\,\lambda^i}{\sum\limits_{i=0}^{H} \frac{(H+i-1)!}{(2i+1)!(H-i)!}\,\lambda^i}.$$

Proof of **P3**: From proposition **P2**, we can rewrite the convexity constraint as follows:

$$1 = \sum_{i=0}^{H} w_{ij} = \sum_{i=0}^{H} P_i(\lambda)\, w_{0j} - \sum_{i=j+1}^{H} R_{i-j}(\lambda).$$

We can now use this to express w_{0j} in terms of $P_i(\lambda)$ and $R_i(\lambda)$, as given in the proposition. We simplify the expression for w_{0j} by substituting the following for $R_i(\lambda)$:

$$\sum_{i=1}^{n} R_i(\lambda) = \sum_{i=1}^{n} \frac{(n+i)!}{(2i)!\,(n-i)!}\,\lambda^i = P_n(\lambda) - 1,$$

which is found by an induction argument. Similarly, we can simplify (27) by noting that

$$\sum_{i=0}^{n} P_i(\lambda) = \sum_{i=0}^{n} \frac{(n+i+1)!}{(2i+1)!\,(n-i)!}\,\lambda^i.$$

Having found the optimal choice of w_{0j}, we obtain the remaining weights by iteratively solving (25). We see immediately from **P3** that for positive λ, w_{0j} is positive; we can similarly show that w_{Hj} is positive. From these facts, we can obtain the following proposition by examining the first differences for the optimal weights.

P4 The optimal weights w_{ij} are positive, increasing and strictly convex over the range $i = 0, 1, \ldots j$. The optimal weights w_{ij} are positive, decreasing and strictly convex over the range $i = j, j+1, \ldots H$. For the optimal set of weights, the largest weight is w_{jj}.

P5 The matrix of optimal weights is symmetric about the off-diagonal. That is, $w_{ij} = w_{H-j,\,H-i}$.

Proof of **P5**: This can be shown by substitution of (27) into (26).

P6 The optimal weights are such that $w_{ij} = w_{H-i,\,H-j}$.

Proof of **P6**: Since the optimal weights satisfy the convexity constraint, we can substitute the convexity constraint into (25) and rewrite, after some rearrangement, as

(28) $w_{i-1,j} = w_{ij} + \lambda\,(w_{ij} + \cdots + w_{Hj} - (1 - u_{i-1,j}))$ for $i = 1, \dots H$.

From (28), by a similar development as used to find (26), we can express the weights as linear functions of w_{Hj}:

(29a) $w_{ij} = P_{H-i}(\lambda)\, w_{Hj}$ for $i = j, \dots H$,

(29b) $w_{ij} = P_{H-i}(\lambda)\, w_{Hj} - R_{j-i}(\lambda)$ for $i = 0, 1, \dots j-1$.

In order for the weights to sum to one, we then find that

(30) $$w_{Hj} = \frac{P_j(\lambda)}{\sum_{i=0}^{H} P_i(\lambda)}.$$

From (29) and (30), we establish the result.

P7 The matrix of optimal weights is symmetric about the diagonal. That is, $w_{ij} = w_{ji}$.

Proof of **P7**: This follows immediately from **P5** and **P6**.

Figure 1 shows the form of the optimal weights for various values of j for $\lambda = 1$ and H=12. Table 1 gives the actual values for the optimal weights. From the table we observe that the matrix of optimal weights is symmetric about both diagonals, as stated in the propositions above. Furthermore, for a fixed index j, the weights increase geometrically to a maximum at w_{jj}, and then decay geometrically over the rest of the column.

Figures 2 and 3 show the form of the optimal weights for $\lambda = 4$ and $\lambda = 0.25$ at H = 12. Intuitively, we would expect that as λ increases to ∞, w_{jj} goes to 1 and w_{ij} goes to 0 for $i \neq j$ (no production smoothing), and as λ decreases to 0, w_{ij} goes to 1/(H+1) (maximum production smoothing). At $\lambda = 4$ and $\lambda = 0.25$ we already begin to observe this behavior.

P8 The optimal objective value for the Lagrangian function in (23) is given by: $L_j(\lambda) = w_{jj}\,\sigma_j^2$ for j = 0, 1, ... H.

Our proof of **P8** involves quite a bit of unattractive and non-intuitive algebra; see Kletter (1994) for the details. The basic structure of the proof is as follows: we rewrite the righthand side of (23) strictly in terms of w_{0j} and λ for a given j by repeatedly applying (26) and factoring out σ_j^2. We then show that this expression equals w_{jj}, where w_{jj} is also expressed in terms of w_{0j} and λ. This is achieved by replacing w_{0j} with the expression given in (27), expressing all terms as polynomials in λ and then manipulating the binomial coefficients until they are shown to be equal.

The value of **P8** is that it provides a relatively quick way to evaluate the objective function of the Lagrangians, namely (21b) and (23). Also, we show next how to get a good approximation of w_{jj}, which will then yield an analytic expression for the objective function of the Lagrangian.

Suppose we define the first difference $\Delta w_{ij} = w_{ij} - w_{i-1,j}$; we can use (25) to express Δw_{ij} by:

$$\Delta w_{ij} = \Delta w_{i-1,j} + \lambda w_{i-1,j} \qquad \text{for } i = 1, 2, \dots H \text{ and } i \neq j+1$$
$$\Delta w_{j+1,j} = \Delta w_{jj} + \lambda w_{jj} - \lambda.$$

To get an approximate solution to these first difference equations, suppose we look at a limiting case where we allow both H and j to grow. In effect, we let the range be i = ... -2, -1, 0, 1, 2, ..., except for i = j + 1. Then in the limit, the solution to these difference equations is

$$(31) \qquad w_{j+k,j} = w_{j-k,j} = \alpha\, [\, (1-\alpha)\ /\ (1+\alpha)\,]^k \qquad \text{for } k = 0, 1, 2, \dots$$

where $\alpha = \sqrt{\lambda / (\lambda + 4)}$.

Furthermore, this solution satisfies the convexity constraint over the weights. From (31) we see that in the limit,

- the optimal weights are symmetric about w_{jj};
- the optimal weights decline geometrically on either side of w_{jj};
- the value of the maximum weight w_{jj} is independent of j; and
- the maximum weight w_{jj} is a simple monotonic function of λ, that approaches 1 as λ increases.

We can see from Figure 1 and Table 1 that for $\lambda = 1$, the optimal weights already begin to approach the limit at H = 12. In particular, we observe that except at the end points j = 0 and j = H, $w_{jj} \approx \alpha = \sqrt{\lambda / (\lambda + 4)} = \sqrt{1/5} \approx 0.4472$ and $w_{j+1,j} = w_{j-1,j} = \alpha\, [\, (1-\alpha)\ /\ (1+\alpha)\,] \approx 0.1708$.

The limit provides a simple approximation to the objective function of the Lagrangian relaxation. Using **P8** and (31), we find that for large values of H we can approximate (21b) by

$$L(\lambda) = \text{MIN Var}\,[\,F_t(t)\,] + \lambda \text{Var}\,[\,I_t(t)\,] - \lambda K^2$$
$$\approx \quad \text{tr}(\,\Sigma\,)\,\sqrt{\lambda/(\lambda+4)}\, - \lambda K^2.$$

This simplification is helpful for finding the value of λ that maximizes the Lagrangian, and thus solves the original optimization problem (21). It can also be helpful in exploring how to set the weights in a multi-stage system.

We end this section with an interesting and perhaps useful result.

P9 The optimal weight matrix is the inverse of a tridiagonal matrix **A**, with $a_{00} = a_{HH} = (\lambda + 1)/\lambda$, $a_{01} = a_{10} = a_{H,H-1} = a_{H-1,H} = -1/\lambda$, and with $(a_{i,i-1},\ a_{ii},\ a_{i,i+1})$ given by $(\,-1/\lambda,\ (\lambda + 2)/\lambda,\ -1/\lambda\,)$ for i = 1, 2, ... H-1.

P9 can be proved by construction through a series of careful matrix operations; see Kletter (1994) for details. Our proof simply shows that inverting the matrix **A** gives **W**, as specified in (29). This is accomplished by first factoring **A** into **LDL'**, where **L** is bidiagonal since **A** is symmetric and tridiagonal, and then inverting to obtain $(\mathbf{LDL'})^{-1} = (\mathbf{L'})^{-1}\,\mathbf{D}^{-1}\,\mathbf{L}^{-1}$. Since the diagonal matrix **D** and the bidiagonal matrix **L** are both easily inverted, we then compute the product and simplify to show that $a_{ij}^{-1} = w_{ij}$ for all i and j.

One significance of **P9** is that it makes the computation of the optimal weight matrix even easier.

i \ j	0	1	2	3	4	5	6
0	0.6180	0.2361	0.0902	0.0344	0.0132	0.0050	0.0019
1	0.2361	0.4721	0.1803	0.0689	0.0263	0.0101	0.0038
2	0.0902	0.1803	0.4508	0.1722	0.0658	0.0251	0.0096
3	0.0344	0.0689	0.1722	0.4477	0.1710	0.0653	0.0250
4	0.0132	0.0263	0.0658	0.1710	0.4473	0.1709	0.0653
5	0.0050	0.0101	0.0251	0.0653	0.1709	0.4472	0.1708
6	0.0019	0.0038	0.0096	0.0250	0.0653	0.1708	0.4472
7	0.0007	0.0015	0.0037	0.0095	0.0249	0.0653	0.1708
8	0.0003	0.0006	0.0014	0.0036	0.0095	0.0249	0.0653
9	1.1E-04	0.0002	0.0005	0.0014	0.0036	0.0095	0.0250
10	4.1E-05	8.2E-05	0.0002	0.0005	0.0014	0.0037	0.0096
11	1.6E-05	3.3E-05	8.2E-05	0.0002	0.0006	0.0015	0.0038
12	8.2E-06	1.6E-05	4.1E-05	1.1E-04	0.0003	0.0007	0.0019

i \ j	7	8	9	10	11	12
0	0.0007	0.0003	1.1E-04	4.1E-05	1.6E-05	8.2E-06
1	0.0015	0.0006	0.0002	8.2E-05	3.3E-05	1.6E-05
2	0.0037	0.0014	0.0005	0.0002	8.2E-05	4.1E-05
3	0.0095	0.0036	0.0014	0.0005	0.0002	1.1E-04
4	0.0249	0.0095	0.0036	0.0014	0.0006	0.0003
5	0.0653	0.0249	0.0095	0.0037	0.0015	0.0007
6	0.1708	0.0653	0.0250	0.0096	0.0038	0.0019
7	0.4472	0.1709	0.0653	0.0251	0.0101	0.0050
8	0.1709	0.4473	0.1710	0.0658	0.0263	0.0132
9	0.0653	0.1710	0.4477	0.1722	0.0689	0.0344
10	0.0251	0.0658	0.1722	0.4508	0.1803	0.0902
11	0.0101	0.0263	0.0689	0.1803	0.4721	0.2361
12	0.0050	0.0132	0.0344	0.0902	0.2361	0.6180

Table 1. Optimal weights for $\lambda = 1$, H = 12

4. Case Study

In this section we describe an industrial application of the Dynamic Requirements Planning (DRP) model from a thesis internship performed by one coauthor (Hetzel) at the Eastman Kodak Company. The internship was conducted as part of MIT's Leaders for Manufacturing Program and ran from June 1992 to December 1992. See Hetzel (1993) for more details on the application.

The general charge for the thesis was to investigate cycle time reduction within the context of the film manufacturing processes at Kodak. As part of the internship Hetzel joined an internal supply chain optimization team that was investigating opportunities for better coordination over a specific supply chain, including issues of cycle time and inventory reduction. One open issue facing the team was that of strategic inventory placement: how much inventory was needed, and where should it be placed across a multi-stage supply chain. Hetzel identified this as an opportunity to apply the DRP model, and the team agreed that it was an appropriate tool for their task of strategic inventory placement.

The goal of the supply chain analysis was to determine the optimal safety stock levels between each stage in the film making supply chain. The underlying concept is that looking at one stage of the supply chain in isolation is inherently sub-optimal. The production speed and quantity of each stage impacts upstream suppliers and downstream customers. All the stages in the supply chain are interconnected by information flows. In short, the inventory and production policies that are best for one stage may not be optimal for the supply chain as a whole.

In the case study, the team was able to address this situation by using the DRP model to consider all stages in the supply chain. Their recommendations challenged the conventional targets and performance measures for individual divisions (stages). For example, an upstream stage, Roll Coating, faced a corporate-wide mandate to lower inventories. However, the DRP model recommended actually *raising* some Roll Coating inventories in order to minimize the overall inventory cost of the supply chain. When Roll Coating holds more inventory, downstream stages can hold less, resulting in a net savings for the corporation. Overall, the analysis determined that inventories for the products of the case study could be reduced by 20%. This example highlights the importance of considering the entire supply chain when setting inventory and production policies.

The rest of this section will describe the supply chain for the case study, provide the results from the DRP model, and comment on implementation issues.

Supply Chain for Case Study

In Figure 4 we give a simplified version of the process for film making. Roll Coating transforms raw chemicals into a roll of film base. Sensitizing coats the film base with a silver halide emulsion. Then Finishing cuts and packages the sensitized rolls into finished products. The structure of this supply chain has three interesting characteristics. First, the number of items grows dramatically from stage to stage; one film base might result in 5 to 10 different sensitized rolls, which might lead to a hundred or more finished goods. Second, there is a rapid growth in the value of the product due to added material (e. g., silver) and nature of the processes. And third, there is a gradual decrease in the lead times across the supply chain.

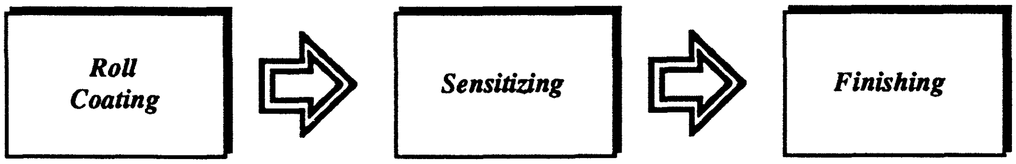
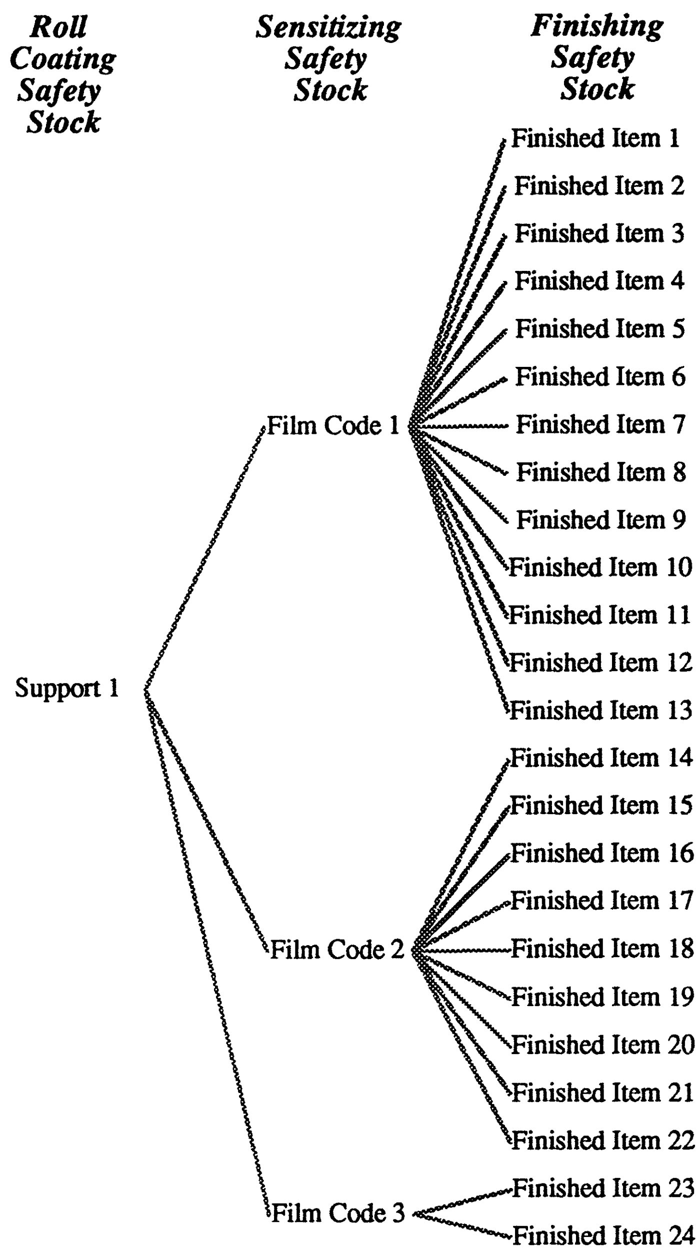


Figure 4. Simplified Version of Film Manufacturing Supply Chain

For the case study, the supply chain optimization team focused on a single film base (called a support). That single base becomes three different sensitized film codes because it can be coated with three different emulsions. Those three film codes can be finished (slit, chopped, and packaged) into 24 different finished good items. Figure 5 illustrates the supply chain for the case study. This particular product "tree" was chosen because it is high volume, it has relatively few end items (24 total), and it represents a "typical product" which the team felt would make a useful pilot program.

It is important to note that the case study does establish arbitrary bounds on the supply chain. The case study starts with the creation of a film base in Roll Coating and excludes the upstream raw material stages such as chemical, gelatin, and polymer production. The case study ends with the Finishing process and arrival at the Central Distribution Center, and ignores the rest of the distribution system. Besides being bounded at both ends, the case study's supply chain is also simplified. In reality, the Sensitizing and Finishing stages have materials flowing into them such as emulsion and packaging components. Even though these materials require inventory management, they are assumed to be available with 100% service and were not explicitly incorporated into the model.

Figure 5: Supply Chain Analysis Case Study



Data Collection

To parameterize the DRP model required an extensive data-collection effort. For each item in the chain, the team gathered data on the item's lead time, unit cost, inventory holding cost, manufacturing frequency, and desired service level. For each end item, they needed the planning horizon, the average demand level, and a time history of forecast process. From this time history, they developed estimates of the covariance matrix (Σ) for the forecast process, as required by the DRP model. Associated with each "branch" linking different stages they calculated a historical "goes into factor" that captures any yield loss or conversion factors.

A side benefit of applying the DRP model was that the data collection effort identified some potential issues along the supply chain. For example, in the course of reviewing the forecast data, the team discovered that the forecasts varied in a systematic way that led to a reevaluation of the forecasting process. In addition, collecting data enhanced supply chain communication and allowed the team to resolve a discrepancy in the annual planned volumes between two of the stages.

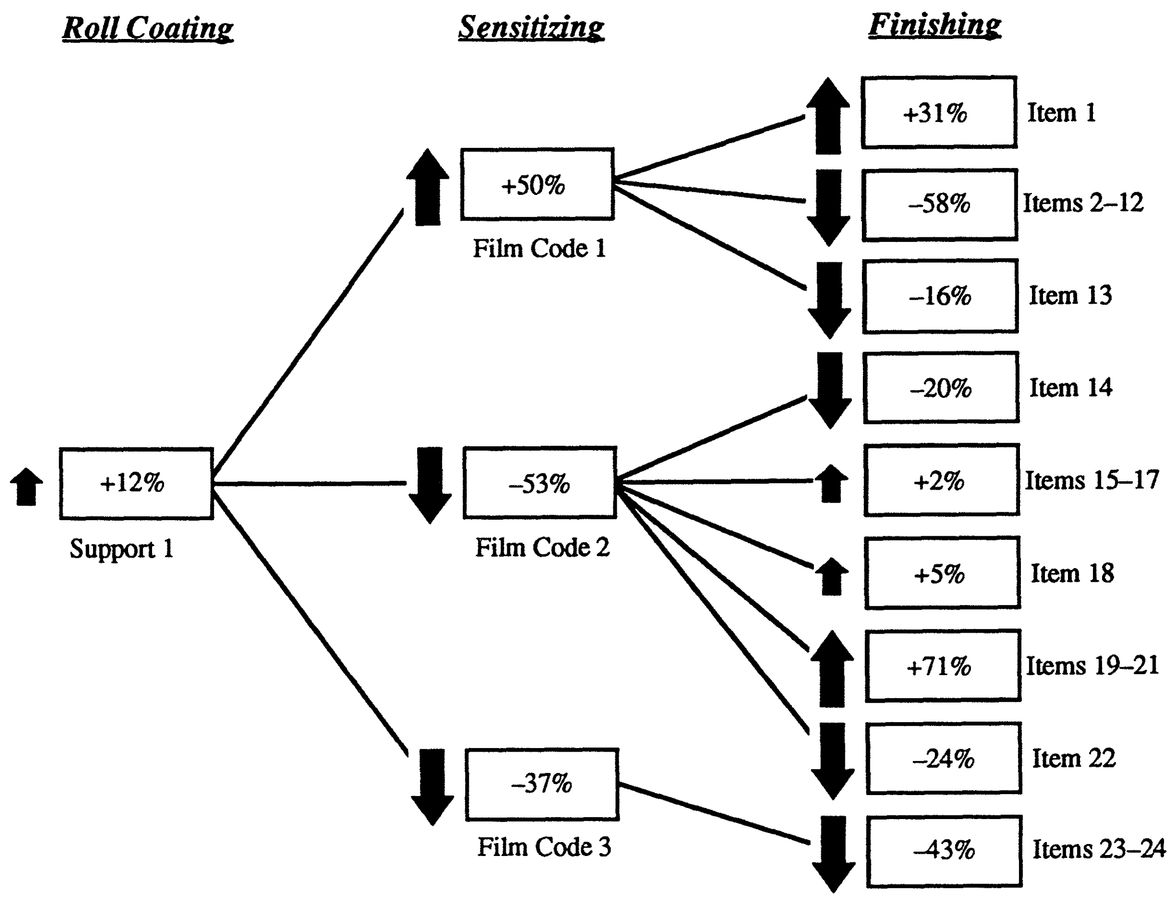
Results

The team used the DRP model to develop a base case recommendation on inventory placements. The twenty-four finished items were grouped into nine product aggregates, where the product aggregates shared common production processes and had similar demand histories. Service levels were set at 95% for each stage. The weight matrices were not optimized and were set to reflect each stage's lead time and manufacturing frequency. In order to reflect Kodak's current scheduling systems, there was no production smoothing across weeks.

The DRP model showed the potential to lower inventory across the case study product "tree" by 20%, as shown in Figure 6. Note that in general, inventories can be pushed upstream where they are in a strategic position because: (i) the inventory is common to the greatest number of finished end items desired by the customer; and (ii) the inventory is at its lowest value added and thus at its lowest carrying cost. In fact, the inventory levels of Roll Coating's "Support 1" actually need to *increase* to provide savings for the supply chain as a whole.

Figure 6: Results from the Supply Chain Analysis – Strategic Inventory Placement

Changes in $ Inventory Values



Note: Excludes emulsion and chemicals inventories. Excludes regional distribution center (RDC) inventories. Includes all WIP, cycle, and in-transit stocks. Finished goods inventories are gross CDC averages. Service levels are at 95% for all stages. There is no production smoothing.

The definition of "inventory" as it is used in this results section is important. The inventory changes and comparisons in Figure 6 represent *average* inventories. Average inventory for each item includes the safety stock calculated by the DRP model, plus the cycle stock due to production batching, plus the pipeline stock from transport needs.

In order to implement the DRP model recommendations, the team needed to develop confidence in the results. Therefore, multiple scenarios were run to test the sensitivity to various parameters, including the service levels; the lead times; and the size (variance) of the forecast errors. See Hetzel (1993) for details.

Besides the required safety stocks, the DRP model also provided information on the variance of the production requirements at each stage of the supply chain. The supply chain optimization team used this variance to determine the "surge" production capability needed for any stage. For instance, they might set the surge capability to be the production level that would cover the production requirements 95% (1.645 standard deviations above the mean, assuming normal forecast errors) of the time.

No model is perfect, and no description of a model is complete without a list of shortcomings. The supply chain optimization team identified three weaknesses: (i) the DRP model does not account for lead time variability; (ii) it assumes stationary average demand over time; and (iii) it cannot accommodate a large product explosion. Whereas the first two are inherent assumptions for the model, the latter concern was due to a limitation in the software that could be easily overcome. However, we expect that in most practical situations a team should probably not be working at any greater level of detail than the case study, say, less than 25 items. Keeping the model at an aggregate level both reinforces the fundamental guiding principles and also makes implementation simpler.

Implementation

Once all of the supply chain analysis requirements were complete, the supply chain optimization team added local intelligence about specific customers and manufacturing issues for each item that could not be captured by the model. After reaching an understanding about how all of the model's proposed changes would impact the supply chain, the team decided to implement a pilot program, with the intention of moving to the more aggressive "basecase" if there were no service problems.

The pilot program only involved the inventories of three items. The plan raised the one Roll Coating item's inventory by 20%, and it lowered two Sensitizing items' inventories, each by 60%. The plan was implemented in early 1993. The savings were captured in the 1993 Annual Operating Plan for the case study's line of business. As of the end of April, 1993, not a single end customer order had been missed on the pilot product due to stockouts or inventory shortages. The team then implemented the remaining recommendations over the course of 1993.

The main barrier that the team had to overcome in reaching the implementation stage was understanding how the DRP model works. This was accomplished by explaining how the model works; by exercising the model for different scenarios, especially conservative ones; by displaying all the input data and its sources for validation; by comparing the model results versus current inventory levels; and by acknowledging the model's shortcomings. Finally, a key success factor was that the model was implemented on a personal computer, provided a graphical interface for representing and visualizing the supply chain, and provided an almost instantaneous response. In addition to the analytic model, we implemented a simulation that could be used, albeit with a longer run time, to confirm the accuracy of the model results.

5. Conclusions

This paper presents a new model of the requirements planning process. We first describe in detail how to model a single production-inventory stage as a linear system and provide the analysis for determining performance measures on production smoothness, production stability and inventory requirements. We then show how to optimize the tradeoff between production smoothness and inventory for a single stage.

To model a multi-stage system, we can use the single-stage model as a building block. The structure of the single-stage model makes it very easy to link single-stage models together to represent the multi-stage system. In particular, each single-stage model takes as input a forecast of demand requirements and coverts this forecast into a production plan. In the context of a network of production stages, the production plan from a downstream stage acts as the demand forecast for an upstream stage. In this way we can cascade the single-stage models to model a multi-stage system.

We also report on an application of the model within the context of a supply chain study. The DRP model was used as a tool to help determine inventory placement across a multi-stage supply chain. This illustration provides some evidence of the value of taking a corporate-wide view by optimizing the supply chain rather than sub-optimizing each of the pieces.

One outgrowth from the case study was a better understanding of industry needs, and where the DRP model was weak. Based on this experience, as well as observations from industry, we identify the following research topics.

- Non-stationary demand: A stationary demand process is not an accurate model for the demand experienced by many products. Common non-stationary effects include seasonal effects, end-of-quarter or end-of-year effects (the 'hockey stick'), and short-product life cycles. Some of these non-stationarities get masked when products are aggregated into families or product groups. Nevertheless, an important enhancement to the model would be to capture, in some way, non-stationary demand processes.

- Service-Level Assumptions: In extending the single-stage model to a multi-stage setting, we assume that there will be sufficient inventory to decouple the stages. In effect, we assume that the service levels will be set to assure a high level of service and in the model analysis, we ignore the downstream consequences of an upstream stockout, i. e., starvation of inputs. These assumptions raise two questions. One is what are the consequences of ignoring the internal stockouts and the second is what should the internal service levels be. Graves (1988a) provides some justification for these assumptions in a related setting. And simulation tests that we have done confirm that ignoring the internal stockouts in the analysis, when service levels are high, does not distort the results of the model. But the issue remains as to how to set the service levels; the literature on multi-echelon distribution systems (e.g., Jackson, 1988; Schwarz, 1989; Graves 1993) suggests that from a system perspective it often may be better to have low levels of internal service.
- Guidelines for Consolidating Stages: On a related note, we conjecture that in some instances the best policy may be to remove the inventory between an upstream and downstream stage and, thus, consolidate these stages for planning purposes (Simpson 1958). Rather than have two stages separated by an inventory buffer, we would have one (combined) stage, albeit with a longer lead time. Within a multi-stage system, depending on the lead times and holding costs, it may be optimal to consolidate some of the stages. We expect it would be helpful to have guidelines for determining what stages are good candidates for consolidation.

- Multi-stage Optimization: The paper describes the optimization of the tradeoff between capacity and inventory in a single stage for a diagonal covariance matrix; it would be interesting to explore how this development extends to non-diagonal covariance matrices, as well as to a multi-stage system. In particular, we would like to develop guidelines for setting the weight matrix W for each stage. Furthermore, one could explore how to choose amongst alternative production release policies, such as pull versus push, in a multi-stage setting.
- Production Assumptions: The model has a highly-simplified model of the production process. The model sets the production outputs, and these outputs are translated into production starts (e. g., by a lead time offset). With this model, we can represent fixed lead times, yield loss factors, batch setup frequencies, as well as uncertainty that can be modeled as an additive factor. Nevertheless, there are issues as to the validity or appropriateness of this representation, and the sensitivity of the model results to these assumptions. It would certainly be useful to have a richer model of the production process. For instance, it would be useful to capture the non-linear congestion effects due to multiple items competing for a shared resource.

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Figure 1: Optimal weights for $\lambda = 1$ and various j

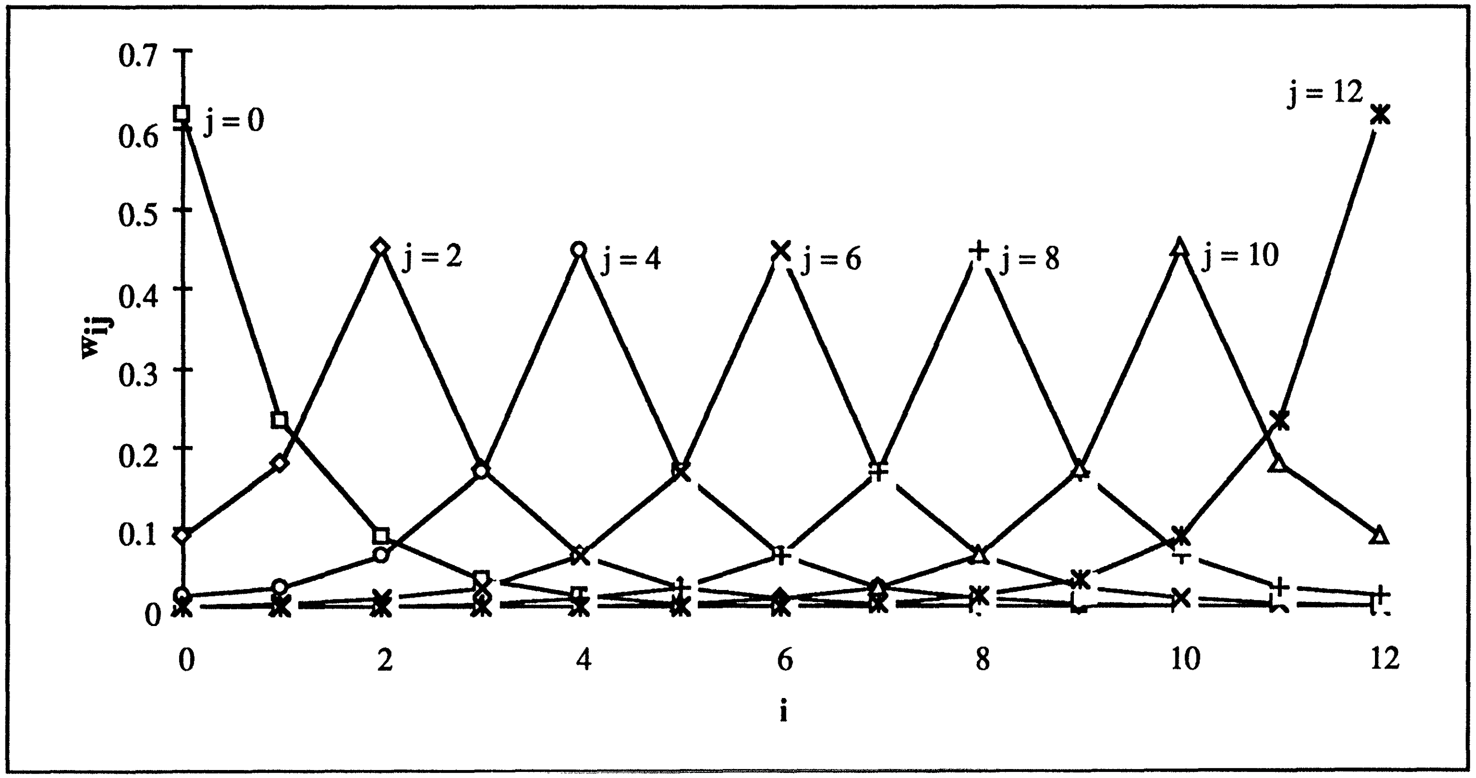

Figure 2: Optimal weights for $\lambda = 4$ and various j

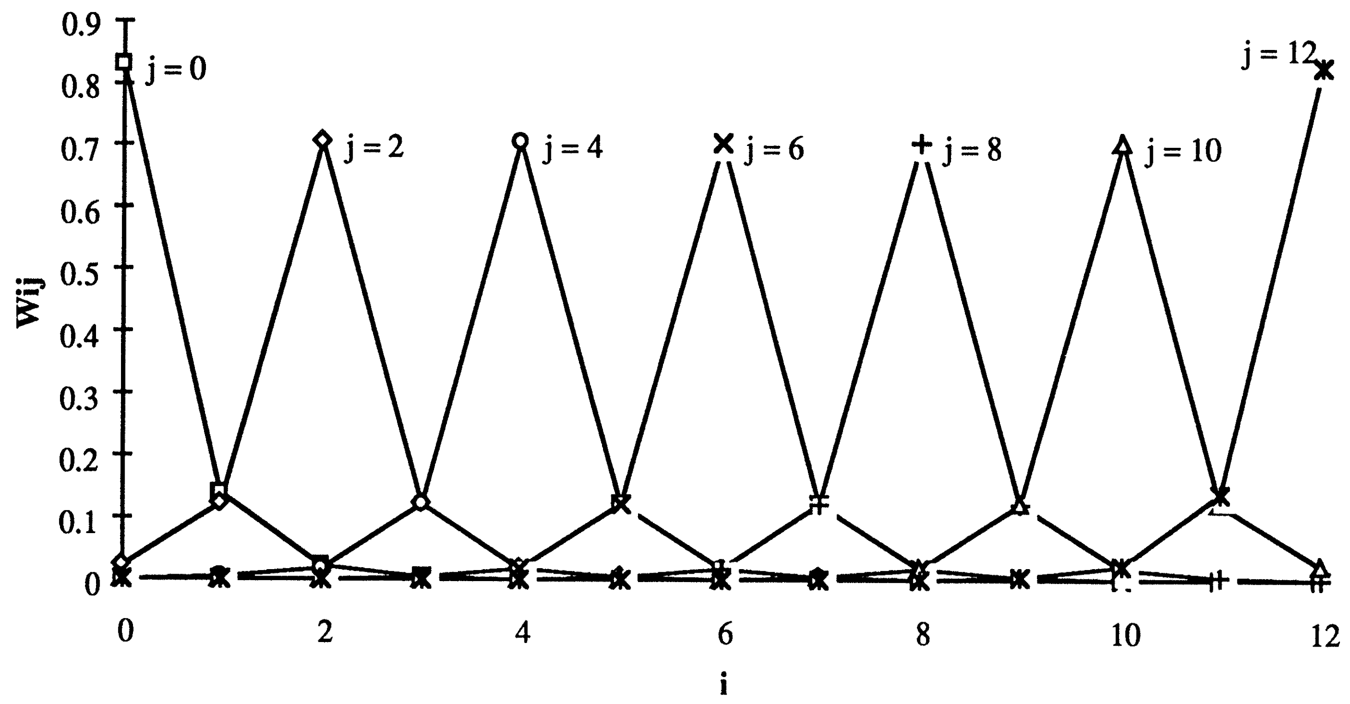

Figure 3: Optimal weights for $\lambda = 0.25$ and various j