Mailing Catalogs: An Optimization Approach

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Abstract

The catalog sales industry is one of the fastest growing business in the U.S. The most important asset a company in this industry has is its list of customers, called the *house list.* Building a house list is expensive, since names of potential customers must be rented. Therefore, limited access to capital plays a central role when these firms plan the strategies by which they will send catalogs.

This paper studies the optimal mailing policies in the catalog industry taking into account cash flow constraints. We consider a stochastic environment given by the random responses of customers and a dynamic evolution of the house list. Given the size of real problems, it is not possible to compute the optimal solutions. Therefore, we develop ad-hoc heuristics based on the optimal solutions of simplified versions of the problem studied. The performance of these heuristics is evaluated by comparing their outcome with upper bounds derived for the original problem. Computational experiments show that these heuristics behave satisfactorily.

1 INTRODUCTION

One of the fastest growing segments within American business is the catalog sales industry. Companies that use catalogs as their sole marketing strategy sent a daily average of 30 million catalogs during 1989, selling 50 billion US dollars during that year (Holtz, 1990).

We define the *catalog sales industry* as those companies doing business by pursuing prospective customers by means of a catalog. The catalog recipient can order a product by mail, telephone, or any other alternative.

Initially, catalog companies were oriented toward rural areas, at a time where access to stores was difficult and expensive. Nowadays, people continue buying via catalogs for several reasons, as for example, the convenience of not having to leave the home or office to buy a product, the popular notion that catalogs offer items more cheaply than retail stores, and the exclusivity of the items offered. On the other hand, customers often hesitate to buy by catalogs because they perceive disadvantages, as for example, the delay experienced in the delivery of the product, the impossibility of examining the product directly before buying it, and the inconvenience of returning an item that, for some reason, is unsatisfactory.

Companies in the catalog sales industry divide the planning horizon into campaigns, which usually concide with the seasons of the year. During a campaign, all the marketing effort is oriented to sell a specific set of products, which have common characteristics that justifies promoting all of them in a same catalog. In what follows, the terms *campaign* and *season* will be used interchangeably.

Companies in the catalog sales industry define a customer as a person that has already bought products from the company; thus her name can be used – in the sense of sending her new catalogs – as often as the company wants (unless the customer explicitly demands the opposite). The list of customers is called the *house list*. The typical behavior of a customer is that after buying a number of times from the company, she stops doing because her taste changes, she switches to the competition, etc. Therefore a company selling by catalog must constantly add new customers to its house list.

A *rental list* is a set of names with certain characteristics in common, such as, for example, age, sex and income. These lists can be rented by the company, usually for a one-time use, paying a price that depends on the number of names rented. These names correspond to potential customers, their response rate — i.e., the percentage of people that respond with a sale — is usually much lower than that of customers on the house list. It follows that the company must use rental lists to obtain new customers, which it then may add to its house list. When a person from a rental list responds with a mail order, she can be incorporated into the house list and from that time onwards the name can be used without paying for doing so.

Ussually, the house list is divided into segments (or states) according to a **R-F-M** classification, where **R**ecency corresponds to the time since the last purchase, **F**requency is equal to the number of purchases in a given period of time and **M**onetary amount is equal to the amount spent in the last purchase. However, recency is by far the most important factor to predict customers' future behavoir (Fleischmann, 1992). Thus, for example, the smaller the recency the larger the response rate, for a given frequency and order size. Therefore, the company faces a stochastic demand that depends directly on the number of catalogs mailed during the season and on the distribution of customers in the states of the house list. The company can also mail several catalogs to the same customers within a season, thereby increasing the response rate. This technique is called *multiple mailing.*

A large house list is one of the main assets a company in the catalog sales industry can have. This is expensive, specially at the beginning, since names of potential customers must be rented. Thus firms encounter serious cash flow restrictions during their first years of existence. Catalog companies usually lose money for two or three years until their house list reaches proportions which contribute more than the costs of acquiring new customers and overhead. When a company starts in the catalog business, it must invest an important fraction of its budget in acquiring customers. We illustrate this idea with the following example: suppose that the company rents a name from a rental list with an average response rate of 1%, and an average size of the order of 80$. If we consider a cost of goods and fulfillment of $40 and a marketing cost (including the catalog and the mail) of $0.50 per customer, the company spends an average of $10 in adding a new customer into its house list. However, people from rental lists that respond with a sale become "good customers" that are expected to generate profits over their lifetime in the house list.

One of the most important decisions that a manager faces in the catalog sales industry is to define the mailing policy, i.e., the fraction of the people in the rental lists and in the states of the house list that receive catalogs. The manager also has to decide the number of catalogs that a customer receives during the season if multiple mailings are allowed.

The purpose of this paper is to determine the optimal mailing policy considering cash flow constraints. We study two tactical models which differ in the way that the supplier interacts with the catalog sales company. In the first model we consider a catalog company that manages the cash flow constraints incorporating the financial impact of carrying inventory. Hence, the model determines the optimal aggregate reordering policy together with the optimal mailing policy. Because the number of people that respond with a sale is a random variable, the manager faces a non-trivial decision when deciding how many products to have in store: having few products may lead to lost sales while having too many products implies spending money that can otherwise be utilized to send additional mailings. In the second model we consider a catalog company that is part of a major retail operation. Thus, all the inventory management decisions are made by the retail store. We also asumme that the catalog sales are a small fraction of the total sales. Therefore, the catalog requests are always satisfied.

Most of the literature related to this topic gives a qualitative analysis describing the main characteristics of the industry, as for example Hill (1989). A good review of the catalog sales industry can be found in Holtz (1990). Bitran and Ramalho (1992) determine the optimal mailing policy in a deterministic environment. In this paper we study optimal mailing and reordering policies in a stochastic environment, where uncertainty originates from customers' random responses.

The remainder of the paper is organized as follows. In Section 2 we introduce two dynamic programming formulations for the problems described above. We also present some properties of the optimal solutions. In Section 3 we develop a methodology to calculate the discounted net profit associated to a customer: *Lifetime value of a customer.* In Section 4 we describe various heuristics to solve the optimization problems. In Section 5 we find upper bounds for the dynamic and stochastic programming formulations, which are useful to measure the performance of the heuristics developed in the previous section. Section 6 contains the computational experiments that show the characteristics of the optimal policies and the performance of the heuristics. Finally, Section 7 presents extensions and conclusions.

2 Mathematical Models

In this section we describe two tactical models to maximize the total expected profit in the catalog sales industry, which differ in the relationship between the company and the supplier. The first model, the **Catalog Mailing Problem with Aggregate Inventory Costs**, corresponds to the case where companies must manage the cash flow incorporating the financial impact of carrying inventory. This model determines the optimal mailing policy together with the optimal reordering policy. We assume that the company can order only at the beginning of the season. Therefore, it must determine the optimal reordering amount taking into account the costs of having unsatisfied customers and carrying inventory from one season to another. The second model, the **Catalog Mailing Problem**, determines the optimal policy for sending catalogs, considering that the catalog company is part of a major retail store and that the catalog sales are a small fraction of the total sales. Therefore, as a good approximation of reality, we assume that all the mail orders received are satisfied. We also assume that the inventory management is carried out by the retail store.

Recency is the main factor considered by the managers in the catalog sales industry to determine the customers' future behavior. Hence, without loss of generality, we only consider the recency to describe the states in the house list. We also consider an average size of the order, common for all the states in the house list and the rental lists.

2.1 Catalog Mailing Problem with Aggregate Inventory Costs

The mathematical model corresponds to a stochastic and dynamic programming formulation where the objective function is to maximize the total expected profit during the planning horizon. The planning horizon is divided into seasons, with, generally, four seasons per year (some companies might consider five seasons including the Christmas sale).

One of the most important decisions that a manager faces in the catalog sales industry is to define the mailing policy, i.e., the fraction of the people in the rental lists and in the states of the house list that receive catalogs. The number of people in the house list that respond with an order is a random variable that depends on the response rate of the corresponding state (the response rate increases with the number of mailings), and on the number of people that receive catalogs. Responses from rental lists have a similar behavior. However, in this case, only single mailing is possible because names are usually rented for one-time use. Therefore, the company faces a stochastic demand that depends directly on the marketing effort, i.e., the number of catalogs mailed in the season.

In practice companies can order a limited number of times from their suppliers. Frequently, they order one month before the beginning of the season and they reorder just once within the first three weeks of the season. Therefore, it is reasonable to asumme that the company orders only once during every season, and that this order takes place at the beginning of the season. Additionally to the mailing policies we have to define the optimal reordering policies.

The model makes the following three assumptions:

- If a customer places an order her recency decreases to one independently of whether or not the request is satisfied.
- There is a monetary cost for not satisfying an order, which represents the company's loss of reputation.
- There is a holding cost for carrying inventory from one season to another.

In what follows we introduce the notation for the parameters, decision variables, and random variables.

PARAMETERS

$p_{i,m}$ = probability that a customer in state i in the house list buys if she receives m catalogs during a season. This parameter is equivalent to the average response rate of state i in the house list.

p_j = probability that a customer in rental list j buys if she receives a catalog. This is equivalent to the average response rate of rental list j.

OS = average size of the order.

c_1 = variable marketing cost per customer in the house list (it includes printing and mailing).

c_2 = variable marketing cost per customer in the rental lists (it includes printing, mailing and renting the name).

g_1 = average cost of goods as a fraction of the average sale per customer.

g_2 = average variable cost associated to a request.

r = penalty for failing an order.

h = holding inventory cost per unit.

A_t = total amount of money that is available for additional investment at the beginning of season t.

$L_{j,t}$ = number of available names in rental list j during season t.

β = discount rate per season.

I = total number of states in the house list.

M = maximum number of mailings within a season.

J = total number of available rental lists.

DECISION VARIABLES

$H_{i,m,t}$ = total number of customers in state i in the house list that receive m catalogs during season t.

$M_{j,t}$ = number of people in rental list j that receive a catalog during season t.

Z_t = Number of units ordered at the beginning of season t.

$S_{i,t}$ = Number of sales associated to customers in state i at season t.

$S_{j,t}$ = Number of sales associated to rental list j at season t.

RANDOM VARIABLES

$X_{i,m,t}$ = total number of responses from people in state i in the house list that receive m catalogs during season t.

$X_{j,t}$ = number of responses from rental list j during season t.

I_t = inventory at the beginning of season t.

Finally, we define the function $F_t(Y_t, I_t, N_t)$ as the maximum discounted expected profit from season t onwards if the company starts with Y_t dollars for investment, I_t products in inventory, and N_t customers in the house list at season t. N_t is a vector with as many elements as the number of states in the house list. Therefore, the element $N_{i,t}$ of N_t corresponds to the number of customers in state i in the house list during season t.

The Model: the objective function at time t is given by the immediate profit during the current season plus the expected profit from the next season onwards. During season t, the manager has to decide the optimal mailing and reordering policies satisfying the cash flow constraints. The set of constraints at season t is given by:

Cash flow constraint.

$$OSg_1Z_t + \sum_{m=1}^{M}\sum_{i=1}^{I} mc_1H_{i,m,t} + \sum_{j=1}^{J} c_2M_{j,t} \le Y_t + A_t. \tag{1}$$

Upper bound for the number of people in each state of the house list.

$$\sum_{m=1}^{M} H_{i,m,t} \le N_{i,t} \qquad i = 1, \ldots, I. \tag{2}$$

Upper bound for the number of available names in the rental lists.

$$M_{j,t} \le L_{j,t} \qquad j = 1, \ldots, J. \tag{3}$$

Updating the number of customers in each house list segment.

- For state with recency equal to 1:

$$N_{1,t+1} = \sum_{m=1}^{M}\sum_{i=1}^{I} X_{i,m,t} \;+\; \sum_{j}^{J} X_{j,t}. \tag{4}$$

- For states with recency larger than 1.

$$N_{i,t+1} = N_{i-1,t} \;-\; \sum_{m=1}^{M} X_{i-1,m,t} \qquad i = 1,\ldots,I. \tag{5}$$

Cash flow balance equation.

$$Y_{t+1} = Y_t \;+\; A_t \;-\; OSg_1Z_t \;-\; (\sum_{m=1}^{M}\sum_{i=1}^{I} mc_1H_{i,m,t} + \sum_{j=1}^{J} c_2M_{j,t}) \;-$$

$$g_2(\sum_{m=1}^{M}\sum_{i}^{I} X_{i,m,t} + \sum_{j}^{J} X_{j,t}) \;+\; OS(\sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t}). \tag{6}$$

Upper bound for the number of sales.

$$\sum_{i=1}^{I} S_{i,t} \;+\; \sum_{j=1}^{J} S_{j,t} \;\leq\; I_t \;+\; Z_t, \tag{7}$$

where,

$$S_{i,t} \;\leq\; \sum_{m=1}^{M} X_{i,m,t} \qquad\qquad \forall m, i, \tag{8}$$

and,

$$S_{j,t} \;\leq\; X_{j,t} \qquad\qquad \forall j. \tag{9}$$

Inventory balance equation.

$$I_{t+1} = I_t \;+\; Z_t \;-\; \sum_{i=1}^{I} S_{i,t} \;-\; \sum_{j=1}^{J} S_{j,t}. \tag{10}$$

The optimization model at time t is given by the following stochastic and dynamic programming formulation:

$$F_t(Y_t, I_t, N_t) = \max_{H_{i,m,t}, M_{j,t}, Z_t \;\; \forall i,m,j} \{-OSg_1Z_t - \sum_{m=1}^{M}\sum_{i=1}^{I} mc_1H_{i,m,t} - \sum_{j=1}^{J} c_2M_{j,t} +$$

$$E_{X_{i,m,t}, X_{j,t}, \forall j,i,m}[G_t(Y_t, I_t, N_T, Z_t, M, H, X)]\}$$

s.t. (1), (2), (3)

and,

$$G_t(Y_t, I_t, N_t, Z_t, X, M, H) = \max_{S_{i,t}, S_{j,t}, \forall i,j} OS(\sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t}) - g_2(\sum_{m=1}^{M}\sum_{i=1}^{I} X_{i,m,t} + \sum_{j=1}^{J} X_{j,t})$$

$$-rl_t - hI_{t+1} + \beta F_{t+1}(Y_{t+1}, I_{t+1}, N_{t+1}).$$

s.t. (4), (5), (6), (7), (8), (9), (10),

where l_t is equal to the amount of unsatisfied demand:

$$l_t = \sum_{m=1}^{M}\sum_{i=1}^{I} X_{i,m,t} + \sum_{j=1}^{J} X_{j,t} - \sum_{i=1}^{I} S_{i,t} - \sum_{j=1}^{J} S_{j,t}.$$

Boundary condition: at the end of the planning horizon the names in state i in the house list have a residual value equal to the lifetime value from T onwards, and the inventory has a residual value equal to v. Therefore, the boundary condition is given by:

$$F_T(Y_T, I_T, N_T) = \sum_{i=1}^{I} LF(i)N_{i,T} + vI_T$$

The first constraint corresponds to the fact that the total marketing cost plus the reordering cost must be less than or equal to the initial budget plus the exogenous investment. The second and the third constraints correspond to the upper bounds for the number of people in the house list and in the rental lists, respectively. In constraints (4) and (5), we update the number of people in the house list at the end of the season. This procedure has implicit a Markov chain representation for the behavior of the customers in the house list, where the states correspond to the recency of the customers. Thus, if a customer receives a catalog, she either decreases her recency to one (if she places an order) or increase her recency by one (if she does not respond with an order). The customer automatically increases her recency if she does not receive a catalog. In constraint (6), we update the budget at the end of the season. Constraint (7) corresponds to the upper bound for the number of sales; the total sale must be less than or equal to the total available inventory. Constraints (8) and (9) correspond to the upper bound in the number of sales in each state of the house list and in each rental list respectively, with respect to the number of requests. Finally, constraint (10) updates the inventory at the end of the season.

2.2 Catalog Mailing Problem

It is not unusual that catalog companies are part of major retail operations and that the inventory levels are managed by the retail stores. The next model assumes that the requests originated from catalogs are always satisfied because they represent a small fraction of the total sales. the model also assumes that all the costs associated to managing inventory are considered in the global planning model for the retail stores.

<u>The Model</u>: the optimization model at time t is given by the following stochastic and dynamic programming formulation (eliminating the presence of the decision variables Z_t in the constraints):

$$F_t(Y_t, N_t) = \max_{H_{i,m,t}, M_{j,t} \forall i,m,j} \{-(\sum_{m=1}^{M}\sum_{i=1}^{I} mc_1 H_{i,m,t} + \sum_{j=1}^{J} c_2 M_{j,t}) +$$

$$\mathrm{Exp}[(OS - g_1 OS - g_2)(\sum_{m}\sum_{i}^{I} X_{i,m,t} + \sum_{j}^{J} X_{j,t}) \; + \; \beta F_{t+1}(Y_{t+1}, N_{t+1})]\}$$

s.t. (1), (2), (3), (4), (5), and

Cash flow balance equation.

$$Y_{t+1} = Y_t \; + \; A_t \; - \; (\sum_{m=1}^{M}\sum_{i=1}^{I} mc_1 H_{i,m,t} \; + \; \sum_{j=1}^{J} c_2 M_{j,t}) \; +$$

$$(OS - g_1 OS - g_2)(\sum_{m=1}^{M}\sum_{i}^{I} X_{i,m,t} \; + \; \sum_{j}^{J} X_{j,t}). \tag{11}$$

Boundary condition:

$$F_T(Y_T, N_T) = \sum_{i=1}^{I} LF(i) N_{i,T}$$

In most applications, the dimensions of the dynamic programming model do not allow to solve optimally the mathematical formulation. Usually, the house list consists of several states with hundreds of customers each. Therefore, in real cases, the dimension of the state space in the mathematical formulation is large. Additionally, the size of the feasible region for the decision variables depends directly on the size of the state space; the larger the state space, the larger the feasible region. Finally, the set of possible outcomes associated with the decision variables also has a dimension that increases with the value of the optimal decision variable, i.e., the larger the number of people that receive catalogs, the larger the size of the set of possible outcomes for the random variable that represents the number of responses.

It is interesting to note that the solutions of the optimization problems described above do not always match the intuition. For instance, it is not always true that the company should spend currently available resources on sending catalogs as long as there remain profitable customers. We have constructed examples where it is better to save part of the current season's budget to spend during the next season on customers whose profitability is larger than that of current customers. We have also found examples where, under the capital constraint, it is better to send a catalog to a customer with larger recency even though there is an "available customer" with a larger response rate. This case happens when the company is going to lose the customer if he is not activated by means of a catalog (this customer is in the last admisible state). Bitran and Ramalho (1992) state that the general practice of the catalog industry for the mailing strategy is focused in short term performance, i.e., it understates the the long-term impact of the current campaign. Hence, the two counterintuitive situations above show that a formal approach can lead to better decisions.

The following proposition shows that the expected number of customers in the house list converges to a constant when the available number of people in the rental lists is constant from a certain period in the planning horizon onwards. The proof of this proposition can be found in Appendix 1.

Proposition 1 *The number of customers in the house list converges to a constant (as time grows). The limit is given by:*

$$\begin{aligned} N_1 &= (\sum_{l=1}^{J} L_l p_l)/(1 - \sum_{j=1}^{i^*} p_j \Pi_{k=1}^{j-1}(1-p_k)), \\ N_i &= N_1(1-p_1)(1-p_2)\dots(1-p_{i-1}) \qquad \forall i = 2,\dots,i^*. \end{aligned}$$

Where:

1. $\Pi_{k=1}^{0} = 1$ (by definition).
2. L_l = number of available names in rental list l, which is constant for $t > t_0$, for some t_0.
3. p_l = probability that a person in rental list l responds with a sale.
4. p_i = probability that a customer in state i in the house list responds with a sale. For the multiple mailing case, it is the corresponding probability associated to the optimal number of mailings according to the lifetime value.
5. i^* = last profitable state in the house list (from state $i^* + 1$ onwards the lifetime value is zero).

PROOF: See Appendix 1 . ∎

3 LIFETIME VALUE OF A CUSTOMER

In what follows we describe a methodology to calculate the discounted net profit associated to a customer in the house list in an infinite planning horizon (*lifetime value of a customer*), with infinite budget. Calculating the lifetime value of a customers is equivalent to finding the optimal mailing policy for a customer when there are unlimited resources. This concept plays a central role in the heuristics that are described in the following section.

We consider a Markov chain to model the behavior of the customers in the house list, where the states are defined by their recency. In this representation there is a trapping state, a maximum admissible recency, that a customer reaches when she leaves the house list. In each state the customer can move to the next state increasing her recency or move to the first state with recency equal to one. The transition probabilities depend on the mailing policy. Therefore, the problem of calculating the lifetime value of a customer is equivalent to determining the optimal policy for sending catalogs to each state of the house list.

We define the additional notation:

1. $LF(i)$ = lifetime value associated to a customer whose current state is i.

2. $\bar{r}_{i,m}$ = immediate expected profit if a customer in state i receives m catalogs during a season.

$$\bar{r}_{i,m} = (OS - g_1 OS - g_2)p_{i,m} - mc_1$$

Hence, the mathematical formulation to determine the lifetime value of a customer in state i is given by:

$$LF(i) = \max \begin{cases} \text{Send m catalogs:} & \bar{r}_{i,m} + \beta p_{i,m} LF(1) + \beta(1 - p_{i,m})LF(i+1) \;\; \forall m \\ \text{Do not send a catalog:} & \beta LF(i+1). \end{cases} \tag{12}$$

Proposition 2 *If it is optimal not to send catalogs to a customer in state i then it is also optimal not to send catalogs to a customer in state* $i + 1$.

PROOF: See Appendix 1 . ∎

The proposition above shows that the first i^* states are profitable, for some state i^*. The remaining states from $i^* + 1$ onwards are not profitable and their lifetime values are equal to zero.

The optimality equation (12) has a unique solution, $LF(i)$, $\forall i$, that corresponds to the expected discount profit when the optimal stationary policy is implemented (see, e.g., Ross, 1983, Chapter II). In order to find the optimal policy and the value of the maximum expected profit it is neccesary to solve the optimality equation (12). This can be done using one of several methods, as for example the Policy Improvement method and the Linear Programming method. In particular, in the computational experiments, we use the Policy Improvement method which is based on successive approximations to the optimal solution. The algorithm starts with a feasible policy and in iteration k computes the left hand side of equation (12) using the solution of iteration $k-1$ in the right hand side (see, e.g., Ross (1983, p. 38) for a complete description of this algorithm). The algorithm converges to the optimal solution under the following assumptions: (i) bounded rewards, (ii) discount factor less than one, and (iii) finite state space; all of them are satisfied in our formulation.

For the single mailing case (where at most one mailing takes place during every season), we develop a simpler algorithm where the maximum number of iterations is bounded by the number of states in the house list. The algorithm is described in Appendix 1 and shows that an implicit expression for the lifetime value in terms of the last profitable state, i^*, is given by (to simplify the notation, the index that represents the number of mailings, m=1, is omitted):

$$LF(1) = (\bar{r}_1 + \beta(1-p1)\bar{r}_2 + \beta^2(1-p_1)(1-p_2)\bar{r}_3 + \ldots + \beta^{i^*-1}(1-p_1)(1-p_2)\ldots(1-p_{i^*-1})\bar{r}_{i^*})/$$
$$(1-\beta p_1 - \beta^2(1-p_1)p_2 - \beta^3(1-p_1)(1-p_2)p_3 - \ldots - \beta^{i^*}(1-p_1)(1-p_2)\ldots(1-p_{i^*-1})p_{i^*}),$$

and,

$$\begin{aligned} LF(i) &= \bar{r}_i + \beta p_i LF(1) + \beta(1-p_i)LF(i+1) \qquad \forall i = 2,\ldots,i^*, \\ LF(i) &= 0 \qquad \forall i = i^*+1,\ldots,I, \end{aligned}$$

where p_i is equal to the response rate if a customer in state i receives one catalog. For the particular case of $\beta = 1$, the expression above simplifies to:

$$LF(1) = \bar{r}_1/(1-p_1)(1-p_2)\ldots(1-p_{i^*}) + \bar{r}_2/(1-p_2)(1-p_3)\ldots(1-p_{i^*}) + \ldots + \bar{r}_{i^*}/(1-p_{i^*}).$$

This expression has the following interesting interpretation: $LF(1)$ is equal to the sum over all the profitable states in the house list of their immediate expected profits times the expected number of visits to those states. In what follows we derive this result: suppose that a customer has a current state equal to 1. Let $\hat{P}_1$ be the probability that this customer returns to state 1 before leaving the house list. This probability is equal to:

$$1 - \hat{P}_1 = (1-p_1)(1-p_2)(1-p_3)\ldots(1-p_{i^*}).$$

Thus, the expected number of visits to state 1 before leaving the house list is given by:

$$\text{Exp[visits to state 1]} = \sum_{k=1}^{\infty} k\hat{P}_1^{k-1}(1-\hat{P}_1) = 1/(1-\hat{P}_1).$$

Hence, the expected profit associated to the visits to state 1 is equal to:

$$\bar{r}_1/(1-\hat{P}_1) = \bar{r}_1/((1-p_1)(1-p_2)\dots(1-p_{i^*})).$$

Similarly, the number of visits to state i before leaving the house list is given by:

$$\text{Exp[visits to state i]} = \sum_{k=1}^{\infty} k\hat{P}_i^{k-1}(1-\hat{P}_i) = 1/(1-\hat{P}_i),$$

with $\hat{P}_i$ equal to the probability to return to state i starting form state i (from state 1 the customer visits state i with probability 1). Thus,

$$1-\hat{P}_i = (1-p_i)(1-p_{i+1})\dots(1-p_{i^*}).$$

Therefore, the total expected profit associated to a customer in state 1 ie equal to:

$$LF(1) = \sum_{i=1}^{i^*} \bar{r}_i/(1-\hat{P}_i)$$

In the case of limited budget, it is possible to show that the lifetime value of a customer is an increasing and concave function of the initial budget. It is possible to show that there is a budget, B^*, such that the lifetime value of a customer starting with a budget greater than B^* is equal to the lifetime value with unlimited budget.

4 Heuristics

In this section we describe the heuristics developed to find "good approximations" to the optimal solutions of the optimization problems described in Section 2. Initially, we describe two heuristics for the catalog mailing problem; later we introduce three heuristics for the general problem that includes the financial impact of inventory.

4.1 Heuristics for the Catalog Mailing Problem

In what follows we describe two heuristics to determine mailing policies for the case where the catalog company is part of a major retail operations.

Heuristic 1.1: HEUR 1.1

This heuristic sorts the states in the house list and the rental lists in increasing order of the lifetime values (the rental lists are considered as particular states where the customer leaves the house list immediately if she does not respond with a sale). Catalogs are then sent according to this order until either the budget constraint is reached or there are no more available customers with positive lifetime value. The number of mailings for every customer that receives a catalog is also determined by the corresponding lifetime value (only single mailing is allowed for the rental lists).

We introduce the following additional notation:

1. $LF(j)$ = Life time value associated to rental list j.

$$LF(j) = \bar{r}_j + \beta p_{j1} LF(1) \qquad \text{and} \quad \bar{r}_j = (OS - g_1 OS - g_2)p_j - c_2$$

2. K = total number of rental lists and states in the house list ($K = I + J$).

3. $d(k)$ = number of people in state k that receive catalogs (k can be a rental list).

4. $dec(k)$ = optimal number of mailings to state k in the house list according to the lifetime value.

5. $N(k)$ = number of people in state k.

6. $c(k)$ = marketing cost associated to state k.

<u>Description of heuristic 1.1 at the beginning of season t.</u>

Step 0: Sorting.

$s(k)$ = state of the k^{th} largest life time value.

Step 1: Initialization.

$k = 1$,
B = Initial budget at season t,
$d(k) = 0 \qquad \forall k$.

Step 2: Stop if the remaining budget is zero.

If ($B = 0$) then GOTO Step 4.

Step 3: Determining the number of catalogs to send to state k.

If $(LF(s(k)) > 0)$ then
  $d(s(k)) = Min\{N(s(k))\ ;\ B/c(s(k))\}$
  $B = B - c(s(k))d(s(k))$
  $k = k + 1$
  if $(k > N)$then
    GOTO Step 4
  else
    GOTO Step 2.

Step 4: STOP.

The following proposition shows that heuristic 1.1 is optimal when there is no budget constraint.

Proposition 3 *Heuristic 1.1 is optimal when there is unlimited amount of money for investment in every season.*

PROOF: In the Catalog Mailing Problem there is only one constraint that involves more than one (in fact, all) clients. This restriction becomes redundant when unlimited amounts of money are available for investment. Therefore, the optimization problem separates by customers; these problems are equal to the optimization problems for calculating lifetime values. Hence, the optimal solution of the global problem is given by the optimal solution of each individual problem. ∎

HEURISTIC 1.2: HEUR 1.2

This heuristic is a modification of HEUR 1.1 to take into account the following effect. Suppose that the optimal decision, given by the lifetime value, is to send 2 catalogs to every customer in state i. If in the current period, we only send one catalog to every customer in state i, the lifetime value is smaller than it would be if the optimal policy is implemented, but we spend half of the money we would spend on marketing under the optimal policy. This additional money can be spent on sending catalogs to additional (twice as many) customers. Therefore, there is a trade off between the decrement in the lifetime value when the optimal decision is not implemented and the additional profit associated to more customers. This modification is only relevant when there is limited amount of money for investment.

We present the following example to clarify the above tradeoff. Suppose there are two states in the house list; the second one is the trapping state. Let us consider a probability of 0.15 that a customer in state 1 responds with a sale if she receives one catalog, and a probability equal to 0.2 if she receives two catalogs, a marketing cost of \$1.0, an average size of the order equal to \$50, a variable cost per sale (including the cost of the good) equal to \$25, and a discount rate of 1. In this case the optimal decision (when computing the lifetime value) is to send two catalogs to state 1 with a lifetime value equal to \$3.75. In what follows assume that we have money to send only two catalogs and there are two customers in state 1. One alternative is to send two catalogs to only one customer with an expected profit equal to \$3.75 or to send one catalog to each customer with a expected profit of \$3.31 for each one (assuming that from the next period onwards we have unlimited budget for investment). Therefore, given the capital constraint, it is better in this case to send catalogs to both customers. The following heuristic captures this effect.

To simplify the notation, we assume that a maximum of two mailings can be sent in every season. We introduce the additional notation:

1. $LF(i,m)$ = life time value associated to state i if we send m catalogs in the current season (from the next season onwards the optimal decision is implemented).
2. $d(i,m)$ = number of people in state i that receive m catalogs.

We redefine the lifetime values for the states in the house list as follows:

If $(dec(i)=0) LF(i,1) = LF(i,2) = -\infty$
If $(dec(i)=1) LF(i,1) = LF(i)$, and $LF(i,2) = -\infty$
If $(dec(i)=2)$ then
$LF(i,1) = -c_1 + (OS - g_1 OS - g_2)p_{i,1} + \beta p_{i,1} LF(1) + \beta(1-p_{i,1})LF(i+1)$
$LF(i,2) = LF(i)$.

The value of the decision variables $d(i,m)$ is given by the solution of the following linear programming problem:

$$\max \sum_{i=1}^{I}\sum_{m=1}^{2} LF(i,m)d(i,m) + \sum_{j=1}^{J} LF(j)d(j)$$

s.t.

$$\sum_{i=1}^{I}\sum_{m=1}^{2} mc_1 d(i,m) + \sum_{j=1}^{J} c_2 d(j) \le B_t$$

$$\sum_{i=1}^{2} d(i,m) \le N_{i,t} \qquad \forall i = 1,\ldots,I$$

$$d(j) \le L_j \qquad \forall j = 1,\ldots,J,$$

$$d(i,m) \ge 0, d(j) \ge 0 \qquad \forall i,m,j.$$

4.2 Heuristics for the Catalog Mailing Problem with Aggregate Inventory Costs

In what follows we describe three heuristics for the problem with cash flow constraints and inventory level constraints. In these heuristics we have to decide simultaneously the mailing and reordering policies.

<u>Heuristic 2.1:</u> HEUR 2.1

This heuristic is based on the News Boy problem for a one period horizon. We first describe the probability distribution for the total demand. The number of requests associated to a state in the house list or to a rental list is a binomial random variable, whose parameters depend on the mailing policy and on the response rate. In general, more than a hundred people receive catalogs in each state of the house list and in each rental list. Therefore, a good approximation for the distribution of the total demand is the normal distribution with mean, μ, and variance, σ^2, equal to:

$$\mu = \sum_{m=1}^{M} \sum_{i=1}^{I} p_{i,m} H_{i,m,t} + \sum_{j=1}^{J} p_j M_{j,t},$$

and,

$$\sigma^2 = \sum_{m=1}^{M} \sum_{i=1}^{I} p_{i,m}(1 - p_{i,m}) H_{i,m,t} + \sum_{j=1}^{J} p_j(1 - p_j) M_{jt}.$$

In what follows we determine the optimal amount of goods to order in a one period problem, considering a normal distribution for the demand. We also consider no residual value for the unsold products. We define $g(Z, I)$ as the total expected profit for a one period problem if we start with I units of products and order Z additional goods. Therefore, the function $g(Z, I)$ is equal to:

$$g(Z, I) = -g_1 OSZ + \int_0^{Z+I} (OSx - h(Z + I - x)) f(x) dx + \int_{Z+I}^{\infty} (OS(Z + I) - r(x - Z - I)) f(x) dx -$$

$$g_2 \int_0^{\infty} x f(x) dx.$$

where the demand has a probability density function equal to $f(x)$. Hence, to obtain the "optimal reordering amount" we set the derivative of $g(Z, I)$ with respect to Z equal to zero, which implies:

$$F(Z + I) = (OS + r - g_1 OS)/(OS + r + h),$$

where $F(x)$ denotes the cumulative distribution function for demand. Therefore, the optimal reordering amount, Z, satisfies the following inequality:

$$\Pr(x \leq Z + I) = (OS + r - g_1 OS)/(OS + r + h).$$

Hence, using the normal distribution for the demand, the expression for the "optimal reordering amount" is given by:

$$Z = \sigma a + \mu - I,$$

and $F_y(a) = (OS + r - g_1 OS)/(OS + r + h)$ where y has a standard normal distribution.

We notice that the reordering amount is a function of the number of catalogs to be sent in the current season, because the mean and the variance of the normal distribution are a function of the mailing policy. Therefore, the reordering amount in the cash flow constraint in the original problem is replaced by this "optimal reordering amount". The rest of the heuristic is similar to heuristic 1.1, i.e., we send catalogs according to the decreasing order of the lifetime values until either the cash flow constraint is binding or there are no more available customers.

Description of heuristic 2.1 at the beginning of season t.

Step 0: Sorting.

$s(k)$ = state of the k^{th} largest life time value.

Step 1: Initialization.

$k = 1,$
B = Initial budget at season t,
$d(k) = 0 \qquad \forall k,$
I=initial inventory at the beginning of season t,
$Z_t = 0.$

Step 2: determining the number of catalogs to send to state k.

d^* = maximum value that satisfies the following 2 inequalities:
$g_1 OS(a\sqrt{(}\sum_{l=1}^{k-1} p(s(l))(1 - p(s(l)))d(s(l)) + p(s(k))(1 - p(s(k)))d^*) +$
$(\sum_{l=1}^{k-1} p(s(l))d(s(l)) + p(s(k))d^* - I) + \sum_{l=1}^{k-1} dec(s(l))c(s(l))d(s(l)) +$
$dec(s(k))c(s(k))d^* \leq B \qquad (i),$
$\sum_{l=1}^{k-1} dec(s(l))c(s(l))d(s(l)) + dec(s(k))c(s(k))d^* \leq B \qquad (ii).$
$d(s(k)) = \min\{d^*, N(s(k))\}.$

Step 3: Stopping criterium.

$k = k + 1$
if $(k > N)$ then
GOTO Step 4

else

GOTO Step 2.

Step 4: Calculating the reordering amount.

$$Z_t = \max\{0, a\sqrt{(}\sum_{l=1}^{K} p(s(l))(1-p(s(l)))d(s(l)) + (\sum_{l=1}^{K} p(s(l))d(s(l)) - I)\}$$

Step 5: STOP.

<u>Heuristic 2.2:</u> HEUR 2.2

This heuristic is similar to heuristic 2.1. The only difference is that, when solving the News Boy problem, it considers a positive residual value for the unsold products. This residual value is equal to the discounted cost of the goods that can be used to satisfy the demand in the next period. Therefore, the modified "optimal reordering amount" is given by the equation:

$$F(Z+I) = (OS + r - g_1 OS)/(OS + r - \beta g_1 OS + h).$$

The heuristic 2.3 is equal to heuristic 2.2 replacing the reordering amount Z by this expression that includes the residual values for the unsold products.

<u>Heuristic 2.3:</u> HEUR 2.3

Finally, in this section we modify the heuristic 2.2 to include the same effect described in heuristic 1.2: the trade off between the decrement in the lifetime value when the optimal mailing policy is not implemented and the additional profit associated to an extra customer that receives a catalog. In this case, before calculating the optimal mailing policy (as in heuristic 2.2, Step 2), we solve the same linear problem as in heuristic 1.2 to determine in which cases we send less catalogs than the optimal number of catalogs determined by the lifetime values.

An heuristic where the reordering amount was based on the expected demand was also studied. However, it is not included in this paper because it was sistematically worse than the previous three heuristics.

5 Upper Bound

In this section we describe an upper bound for the optimization model, which is useful to determine the performance of the heuristics described in the previous section.

Proposition 4 *An upper bound for the optimization problems described in Section 2 is the solution of the deterministic versions of the stochastic models, where the random variables are replaced by their expected values.*

PROOF: The proof is straightforward and the details will be omitted. It is based on successive applications of Jensen's inequality and the concavity of the maximization of a linear programming problem as a function of the right hand side. ∎

Therefore, the upper bound for the catalog mailing problem with aggregate inventory costs is given by:

$$UB = \max\{-OSg_1 \sum_{t=1}^{T} \beta^{t-1} Z_t - c_1 \sum_{t=1}^{T} \sum_{m=1}^{M} \sum_{i=1}^{I} \beta^{t-1} H_{i,m,t} - c_2 \sum_{t=1}^{T} \sum_{j=1}^{J} \beta^{t-1} M_{j,t} +$$

$$OS \sum_{t=1}^{T} \sum_{i=1}^{I} \beta^{t-1} S_{i,t} + \sum_{t=1}^{T} \sum_{j=1}^{J} \beta^{t-1} S_{j,t} - g_2 \sum_{t=1}^{T} \sum_{m=1}^{M} \sum_{i=1}^{I} \beta^{t-1} p_{i,m} H_{i,m,t} -$$

$$g_2 \sum_{t=1}^{T} \sum_{j=1}^{J} \beta^{t-1} p_j M_{j,t} - r \sum_{t=1}^{T} \beta^{t-1} l_t - h \sum_{t=1}^{T} \beta^{t-1} I_t + \sum_{i=1}^{I} \beta^T LF(i) N_{i,T+1}$$

s.t.

$$OSg_1 Z_t + \sum_{m=1}^{M} \sum_{i=1}^{I} c_1 H_{i,m,t} + \sum_{j=1}^{J} c_2 M_{j,t} \leq Y_{t-1} + A_t \qquad \forall t = 1, \ldots, T. \tag{13}$$

$$I_{t+1} - I_t - Z_t + \sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t} = 0 \qquad \forall t = 1, \ldots, T. \tag{14}$$

$$\sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t} - I_t - Z_t \leq 0 \qquad \forall t = 1, \ldots, T. \tag{15}$$

$$S_{i,t} - \sum_{m=1}^{M} p_{i,m} H_{i,m,t} \leq 0 \qquad \forall m, i, t. \tag{16}$$

$$S_{j,t} - p_j M_{j,t} \leq 0 \qquad \forall j, t. \tag{17}$$

$$\sum_{m=1}^{M} H_{i,m,t} - N_{i,t} \leq \qquad \forall i, t. \tag{18}$$

$$M_{j,t} \leq L_{j,t} \qquad \forall j, t. \tag{19}$$

$$N_{1,t+1} - \sum_{m=1}^{M} \sum_{i=1}^{I} p_{i,m} H_{i,m,t} - \sum_{j} p_j M_{j,t} = 0 \qquad \forall t. \tag{20}$$

$$N_{i,t+1} - N_{i-1,t} + \sum_{m=1}^{M} p_{i-1,m} H_{i-1,m,t} \qquad \forall i \neq 1, t. \tag{21}$$

$$Y_{t+1} - Y_t - A_t - OSg_1 Z_t - (\sum_{m=1}^{M} \sum_{i=1}^{I} mc_1 H_{i,m,t} + \sum_{j=1}^{J} c_2 M_{j,t}) +$$

$$- g_2 (\sum_{m=1}^{M} \sum_{i}^{I} p_{i,m} H_{i,m,t} - \sum_{j}^{J} p_j M_{j,t}) - OS(\sum_{i=1}^{I} S_{i,t} + \sum_{j=1}^{J} S_{j,t} = 0 \qquad \forall t. \tag{22}$$

6 Computational Experiments

In this section we study the performance of the heuristics in several computational experiments. We use Monte Carlo simulations to estimate the company's expected profit during the planning horizon under the application of the different heuristics. For the catalog mailing problem the two heuristics described in Section 4 give the mailing policies. We use a binomial distribution to represent the number of people that place an order in each state of the house list and each rental list; the number of trials is equal to the total number of people that receive catalogs in each segment and the probability of success corresponds to the average response rate. For the catalog mailing model with aggregate inventory costs, the three heuristics described in Section 4 give additionally the reordering policies, i.e. the stock level to buy at the beginning of every season.

In the experiments, we implement the mailing and reordering policies given by the heuristics and simulate the number of responses according to each mailing policy. Thus, we compute the profit for each particular outcome of the random variable during the planning horizon. Finally, averaging the profits given by repeated simulations we obtain an estimate of the expected profit. We stop the simulations when the coefficient of variation (standard deviation over expected profit) is less than 0.1%. The heuristics are compared with the corresponding upper bounds described in Section 5.

In the simulations, we use realistic data based on information given by the manager of a catalog sales company and public data obtained from 1990/1991 Statistical Fact Book, Direct Marketing Association.

The initial budget utilized as a reference point, Y_0, is equal to the minimum budget such that from that budget onwards the linear programming problem in the upper bound does not change its objective function.

We use the following set of data in the computational experiments:

Planning horizon: 18 seasons

Number of states in the house list: 12

One rental list with 100000 available names each season.

Marketing cost for the house list: \$0.6

Marketing cost for the rental list: \$0.7

Average size of the order: \$70.

Cost of the goods: 35% of the average sale

Variable cost per request: $1

Penalty cost for rejecting an order: 0

Holding inventory cost: 25% of the good's cost per year

Initial inventory: 0

Discount rate: 0.975

Response rate of the rental list: 1.5%

Response rates for the states in the house list in increasing order of the recency:

Single mailing: 4.4% 4.3% 4.0% 3.6% 3.3% 2.8% 2.5% 2.3% 2.0% 1.8% 1.0% 0.0%

Two mailings: 5.8% 5.6% 5.3% 4.9% 3.7% 3.0% 2.5% 2.3% 2.0% 1.8% 1.0% 0.0%

6.1 Catalog Mailing Problem

This case considers that all the demand is satisfied and there are no costs associated to managing inventory. In the data set, we use a maximum of two mailings per season for customers in the house list and a single mailing for customers in the rental list. The performance of the two heuristics is shown in Table 1.

$(Y/Y_0) * 100$	Y=Inital Budget (US$)	HEUR 1.1 w/r UB	HEUR 1.2 w/r UB
0.1%	80	84.8%	87.7%
1%	800	94.8%	99.8%
10%	8000	96.8%	99.9%
25%	20000	96.7%	99.9%
50%	40000	97.7%	99.7%
100%	80000	99.9%	99.9%

Table 1: Catalog Mailing Model with Two Mailings.

The first column contains the initial budget with respect to the reference budget, Y_0. The second column is the initial budget in dollars. Columns 3 and 4 contain the performance of heuristics 1.1 and 1.2 with respect to the upper bound. We observe that the behavior of the two heuristics improves as long as the initial budget increases, having heuristic 1.2 an excellent performance with an initial budget greater than or equal to 1% of Y_0. We also observe that heuristic 1.2 is always better than heuristic 1.1, with a more significative difference when the initial budget is small. The expected profit associated to each experiment can be found in Table 4 in Appendix 2.

We also observe that the uncertainty in this formulation does not have an important impact when we solve "real problems". In general, companies mail to a large number of customers. Therefore, the fraction of responses converges to a constant in probability. This effect can be specially appreciated when the initial budget is large, because the size of the mailings is also large. Moreover, it is possible to show that, with unlimited budget, the stochastic formulation for the catalog mailing problem is equal to the corresponding deterministic model. The proof is straightforward and is not presented in this paper.

Finally, we remark the simplicity in the implementation of the heuristics. Both are based on a set of indices that are computed once at the beginning of the planning horizon. These indices are associated to each state of the house list and to each rental list. Therefore, to implement heuristic 1.1, the manager has to send catalogs to the customers according to the decreasing order of these indices until either he reaches the budget or there is no more available customers.

6.2 Catalog Mailing Problem with Aggregate Inventory Costs

For the catalog mailing problem with aggregate inventory costs, we present two sets of experiments. In the first set of experiments we consider the single mailing case. In the second group of experiments, we allow a maximum of two mailings for the customers in the house list.

For the single mailing case, the performance of heuristics 2.1, 2.2 is shown in Table 2 (heuristic 2.2 and 2.3 are the same for the single mailing case).

$(Y/Y_0)*100$	Y=Inital Budget (US$)	HEUR 2.1 w/r UB	HEUR 2.2 w/r UB
5%	5900	65.0%	86.7%
10%	11800	75.6%	90.3%
25%	29500	84.0%	94.3%
50%	59000	88.6%	95.6%
75%	88500	92.2%	97.1%
100%	118000	94.0%	98.2%

Table 2: Catalog Mailing Model with Aggregate Inventory Costs: Single Mailing Case.

The first column is the initial budget with respect to the reference budget Y_0. The second column is the initial budget in dollars. Columns 3 and 4 show the behavior of the heuristics 2.1 and 2.2 with respect to the upper bound respectively. We observe that heuristic 2.2 has a better performance, with an achievement of more than 95% with respect to the upper bound when the budget is greater than 50% of Y_0. Heuristic 2.2 is better than heuristic 2.1 because it considers that the unsold products in a season can be sold in the next period with the corresponding opportunity cost of the capital. However, the reordering policy in heuristic 2.1 considers that the unsold products in a season have a residual value equal to zero. The detailed information about the expected profits can be found in Appendix 2, Table 5.

In the second set of experiments, we allow a maximum of two mailings within a season for the house list. The performance of the four heuristics is shown in Table 3.

$(Y/Y_0) * 100$	Y=Inital Budget (US$)	HEUR 2.1 w/r UB	HEUR 2.2 w/r UB	HEUR 2.3 w/r UB
5%	6150	63.9%	83.4%	87.5%
10%	12300	72.9%	87.4%	90.8%
25%	30750	81.4%	89.2%	94.1%
50%	61500	86.0%	92.9%	94.1%
75%	92250	90.9%	95.7%	95.8%
100%	123000	94.1%	98.3%	98.4%

Table 3: Catalog Mailing Model with Aggregate Inventory Costs:
Two Mailings.

In this case we observe the same pattern of behavior for the three heuristics as in the previous set of experiments. In this case heuristic 2.3 gives better results than heuristics 2.2, because when the budget is small (specially during the first seasons in the planning horizon) it is better to activate more customers than to send the optimal number of catalogs to few of them.

In the Catalog Mailing Model with Aggregate Inventory Costs, it is still true that with unlimited budget it is optimal to send catalogs to all the profitable customers. However, we cannot guarantee that the reordering policy given for the heuristics is the optimal reordering policy. Even, for the single period problem, the optimal mailing and reordering policy cannot be determined in a close form solution.

Finally, we remark that heuristics 2.2 and 2.3 have a very good performance, with solutions that are close to the solutions given by the upper bounds. The quality of the solutions improves as long as the initial budget increases. With a budget from 25% onwards with respect to the reference budget, heuristic 2.3 reaches more than 94% of the upper bound. It is reasonable to assume that in practice companies have access to loans when they have a profitable business. Therefore, they can finance at least a 25% of the total possible investment.

In what follows, we present a set of computational experiments to compare the effect of multiple mailings versus a single mailing. The following parameters are considered: 60 seasons, 12 states in the house list with zero initial customers, one rental list with 100000 names each season, an initial budget of US$60000, a marketing cost of $0.6 for the house list and $0.7 for the rental list, an average size of the order equal to $70, a variable cost per sale (including the cost of the good) equal to 35% of the average sale, a variable cost per sale equal to $1, a rejection cost of zero, and a discount rate of 0.975. The initial inventory is equal to zero.

The average response rate of the rental list is 1.5%. The response rates for the states in the house list in increasing order of the recency is equal to:
Single mailing :3.0% 2.8% 2.6% 2.3% 2.0% 1.8% 1.5% 1.3% 1.0% 0.8% 0.1% 0.0%.
Two mailings :5.8% 5.6% 5.3% 4.9% 4.7% 4.4% 3.5% 3.3% 2.5% 1.8% 1.0% 0.0%.

We compare the accumulated cash at the beginning of every season, the total number of people in the house list at the end of each season, and the number of catalogs mailed every season. The results are shown in figures 1, 2, 3 respectively. We observe that during the first five seasons the company loses money with the single mailing strategy; only after season thirteen it recovers the initial investment. However, with the two mailings strategy, the company recovers its investment in only six seasons, facing loses during the first three campaigns. The acceleration in the investment recovery period is due to the increment in the customer response rates produced by the multiple mailings. We remark that implementing the multiple mailing strategy, the company can always send a single mailing to some (or all) states during the planning horizon. Therefore, the multiple mailing strategy is at least as good as the single mailing strategy. We also observe that the number of people in the house list converges to a constant in the multiple mailing case after approximately season forty five. This effect is not observed in the single mailing case because we need a larger planning horizon to capture this asymptotic bahavior. Finally, the number of catalogs mailed every season also converges to a constant, because the profitable customers do.

Informal evidence suggests that firms in the catalog sales industry often go bankrupt. The methodology developed in this paper allow us to do risk analysis to determine what fraction of the times the company would run out of cash (at the beginning of some period in the planning horizon the company has no money left for new mailings). With this purpose, after each simulation, we compute an index equal to one if the company runs out of cash for that specific outcome of the random demand or equal to zero otherwise. Averaging the indices given by repeated simulations we obtain an estimate the probability of going bankrupt.

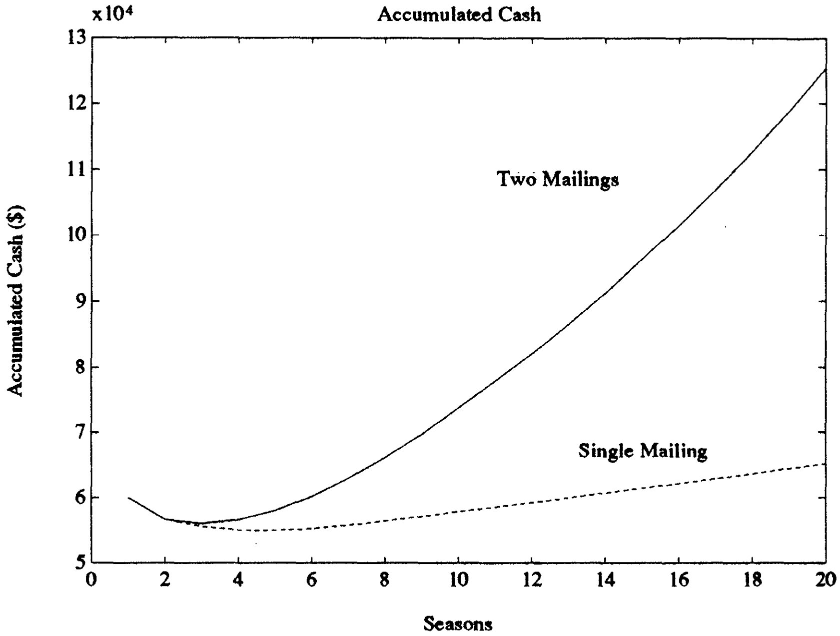

Figure 1: Accumulated Cash at the End of Every Season

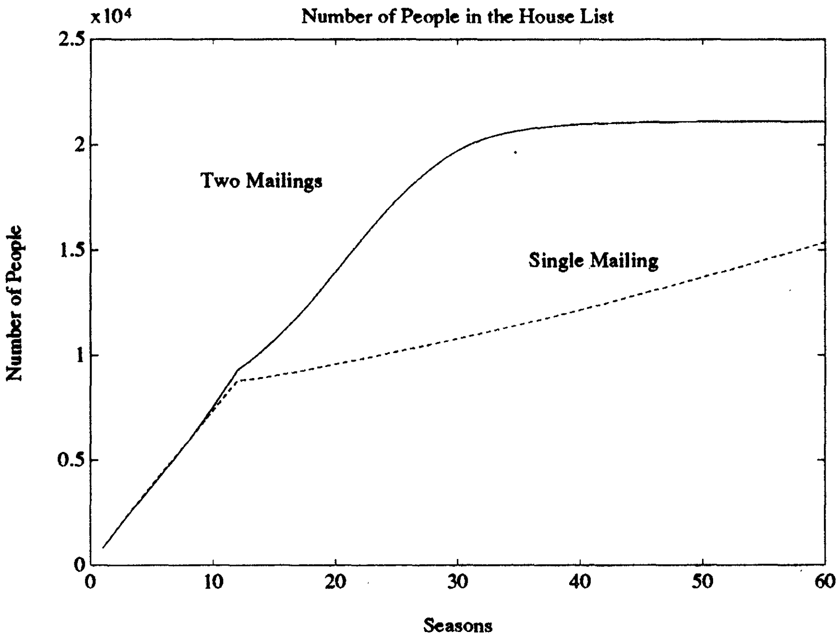

Figure 2: Total Number of People in the House List

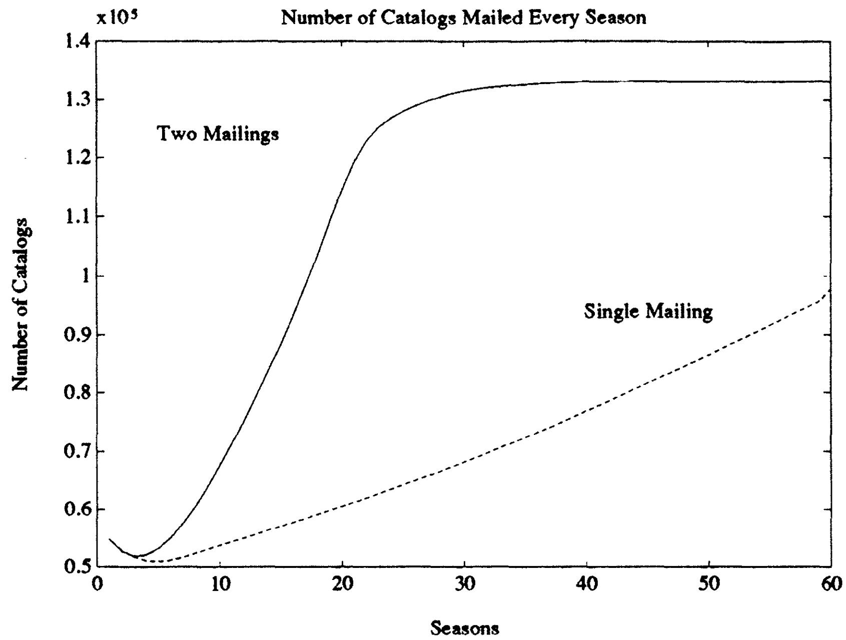

Figure 3: Number of Catalogs Mailed Every Season

7 Conclusions and Extensions

This paper has presented two mathematical models, which vary in the way suppliers interact with the catalog companies, to find the mailing and reordering policies that maximize the expected profit of a company in the catalog sales industry. Stochastic demand and dynamic evolution of the customers within the house list were considered. Optimal solutions are hard to compute for real size problems. Therefore, ad-hoc heuristics were implemented based on the solutions of simplified versions of this problem. Computational experiments showed that these heuristics give satisfactory results. Without loss of generality, we only use recency to describe the states in the house list, which is by far the most important factor to predict customers' response rate.

The models proposed in this paper allow us to study how the optimal solution changes with market conditions, as for example price, cost of goods, response rates, and mailing cost. The models can also be used to do risk analysis, i.e. to study what fraction of the times the company runs out of cash. Informal evidence indicates that firms in the catalog industry often go bankrupt. Running simulations, we can easily compute if the company runs out of cash for every outcome of the stochastic demand during the planning horizon. With this information, we can estimate the probability of going bankrupt. This suggests that it would be interesting to incorporate the probability of bankruptcy into the firm's objective function; we leave this topic for future research.

The tactical model presented in this paper establishes the optimal aggregated levels of reordering and the optimal number of catalogs to be sent to each segment in the house list and to each rental list at the beginning of every season. As a topic of future research, a hierarchical approach could be pursued to disaggregate the total reordering amount into individual items and to schedule the mailings during the corresponding season.

The current formulation considers a penalty cost for the unsatisfied demand. An altenative approach that could be studied is to replace the penalty cost by a service level constraint that assures the demand is satisfied with a given probability.

Appendix 1

Proof of Proposition 1

In the long run, the average number of customers in each state of the house list is given by (the optimal mailing policy is to send catalogs to all the profitable customers):

$$\begin{aligned} N_{1,t} &= \sum_{j=1}^{J} p_j M_j + \sum_{i=1}^{i^*} p_i N_{i,t-1}, \\ N_{i,t} &= (1-p_i) N_{i-1,t-1} \qquad \forall i = 2,\ldots,i^*. \end{aligned}$$

In matrix notation, the above system is equivalent to:

$$N_t = AN_{t-1} + b, \tag{23}$$

where,

$$A = \begin{matrix} p_1 & p_2 & p_3 & \cdots p_{i^*-1} & p_{i^*} \\ 1-p_1 & 0 & 0 \ldots & 0 & 0 \\ 0 & 1-p_2 & 0 \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots & 1-p_{i^*-1} & 0 \end{matrix}$$

and,

$$b = [\textstyle\sum_{j=1}^{J} p_j M_j, 0, 0, \ldots, 0].$$

Successive applications of equation (23) lead to:

$$N_t = A^t N_0 + \sum_{i=0}^{t-1} A^i b.$$

Therefore, the limit of the customers in the house list is equal to:

$$\lim_{t\to\infty} N_t = \lim_{t\to\infty}[A^t N_0 + \sum_{i=0}^{t-1} A^i b] = \lim_{t\to\infty}[A^t N_0 + (I - A)^{-1}(b - A^t b)].$$

Therefore, the limit can be written as:

$$\lim_{t\to\infty} N_t = \lim_{t\to\infty} A^t N_0 - \lim_{t\to\infty}(I - 1)A^t b + \lim_{t\to\infty}(I - A)^{-1} b.$$

Finally we prove that A is a linear contraction, i.e.:

$$\lim_{t\to\infty} A^t x = 0 \qquad \forall x \in R^n.$$

Lemma 1 *The matrix A is a linear contraction.*

PROOF: A is a linear contraction iff all their eigen values have absolute values less than 1 (See Hirsch and Smale(1974)). By definition an eigen value, λ, satisfies the equation:

$$AN = \lambda N \qquad \text{for some } N \neq 0.$$

Replacing the value of A in the above equation, we obtain the following system of equation:

$$\begin{aligned}
p_1 N_1 + p_2 N_2 + \ldots + p_{i^*} N_{i^*} &= \lambda N_1 && (a1)\\
(1 - p_1)N_1 &= \lambda N_2 && (a2)\\
(1 - p_2)N_2 &= \lambda N_3 && (a3)\\
\vdots \qquad &\vdots && \\
(1 - p_{i^*-1})N_{i^*-1} &= \lambda N_{i^*} && (ai^*).
\end{aligned}$$

Replacing $(a2), (a3), \ldots, (ai^*)$ in (a1) we obtain:

$$LHS = p_1/\lambda + p_2(1 - p_1)/\lambda^2 + p_3(1 - p_1)(1 - p_2)/\lambda^3 + p_{i^*}(1 - p_1)(1 - p_2)\ldots(1 - p_{i^*-1})/\lambda^{i^*} = 1.$$

For the purpose of contradiction we assume that there exists a real eigen value greater than or equal to 1. Hence, assuming that all the probabilities are strictly less than 1:

$$LHS \leq p_1 + p2(1 - p_1) + p_3(1 - p_1)(1 - p_2) + p_{i^*}(1 - p_1)(1 - p_2)\ldots(1 - p_{i^*-1}) < 1,$$

which leads to a contradiction. Therefore, the largest real eigen values is strictly less than one. Using the fact that A is a positive matrix (all its elements are non negative with at least one strictly positive element) and the Frobenius theorem (Karlin and Taylor, page 547), we know that the absolute value of all its eigen value are less than or equal to the largest real eigen value.

Therefore, all the eigen values of A have absolute values less than 1, or equivalently A is a linear contraction ∎

Therefore, using lemma 1, we conclude that the limit of the number of customers in the house list is given by:

$$\lim_{t \to \infty} N_t = (I - A)^{-1} b. \quad \blacksquare$$

Proof of Proposition 2

Consider that the optimal decision for state i is not to send catalogs. For the purpose of contradiction, suppose that it is optimal to send m mailings to state $i+1$. Therefore,

$$LF(i) = \beta LF(i+1),$$

and

$$LF(i+1) = \bar{r}_{i+1,m} + \beta p_{i+1,m} LF(1) + \beta(1 - p_{i+1,m}) LF(i+2)$$

By definition of $LF(i)$, we have the following inequality:

$$LF(i) = \beta LF(i+1) \geq \bar{r}_{i,m} + \beta p_{i,m} LF(1) + \beta(1 - p_{i,m}) LF(i+1)$$

Hence, we obtain:

$$LF(i+1) \geq (\bar{r}_{i,m} + \beta p_{i,m} LF(1))/(\beta p_{i,m}). \tag{24}$$

By definition of $LF(i+1)$, we have:

$$LF(i+1) = \bar{r}_{i+1,m} + \beta p_{i+1,m} LF(1) + \beta(1 - p_{i+1,m}) LF(i+2) \geq \beta LF(i+2).$$

Therefore, we obtain the following inequality:

$$LF(i+2) \leq (\bar{r}_{i+1,m} + \beta p_{i+1,m} LF(1))/(\beta p_{i+1,m}). \tag{25}$$

Using (24) and the definition of $LF(i+1)$ we obtain:

$$LF(i+2) \geq (\bar{r}_{i,m}/\beta p_{i,m} - \bar{r}_{i+1,m} + (1 - \beta p_{i+1,m}) LF(1))/\beta(1 - p_{i+1,m}). \tag{26}$$

Finally, (25) and (26) together lead to the following inequality:

$$\bar{r}_{i+1,m}/p_{i+1,m} \geq \bar{r}_{i,m}/\beta p_{i,m} + LF(1)(1 - \beta) \geq \bar{r}_{i,m}/\beta p_{i,m},$$

which leads to a contradiction. ∎

Algorithm to Compute the Lifetime Value

In what follows we describe the algorithm to compute the lifetime value of a customer for the single mailing case. From proposition 2, we know that there is a last profitable state that we denote by i^*. Therefore, the lifetime value from $i^* + 1$ onwards is equal to zero.

$$LF(i) = 0 \qquad \forall i = i^* + 1, \ldots, I.$$

The optimal decision in states 1 to i^* is to send one catalog. Hence,

$$LF(1) = \bar{r}_1 + \beta p_1 LF(1) + \beta(1 - p_1)LF(2),$$

or equivalently,

$$LF(1)(1 - \beta p_1) = \bar{r}_1 + \beta(1 - p_1)LF(2).$$

Replacing $LF(2)$ by its optimal value, we get:

$$LF(1)(1 - \beta p_1 - \beta(1 - p_1)p_2) = \bar{r}_1 + \beta(1 - p_1)\bar{r}_2 + \beta^2(1 - p_1)(1 - p_2)LF(3)$$

Replacing successively $LF(3), LF(4), \ldots, LF(i^*)$ by their optimal values, we obtain the following expression for $LF(1)$:

$$LF(1) = (\bar{r}_1 + \beta(1-p_1)\bar{r}_2 + \beta^2(1-p_1)(1-p_2)\bar{r}_3 + \ldots + \beta^{i^*-1}(1-p_1)(1-p_2)(1-p_3)\ldots(1-p_{i^*-1})\bar{r}_{i^*})/$$
$$(1-\beta p_1 - \beta^2(1-p_1)p_2 - \beta^3(1-p_1)(1-p_2)p_3 - \ldots - \beta^{i^*}(1-p_1)(1-p_2)(1-p_3)\ldots(1-p_{i^*-1})p_{i^*}).$$

The last profitable state must be the last states satisfying the condition:

$$LF(i^*) = \bar{r}_{i^*} + \beta p_{i^*} LF(1) > 0.$$

Therefore, the algorithm to compute the optimal lifetime value of a customer works as follows:

Step 1: Initialization

$i^* = N - 1.$

Step 2: Computing the value of $LF(1)$.

$$\text{LF}(1) = (\bar{r}_1 + \beta(1 - p1)\bar{r}_2 + \ldots + \beta^{i^*-1}(1 - p_1)(1 - p_2)(1 - p_3)\ldots(1 - p_{i^*-1})\bar{r}_{i^*})/$$
$$(1 - \beta p_1 - \beta^2(1 - p_1)p_2 - \ldots - \beta^{i^*}(1 - p_1)(1 - p_2)(1 - p_3)\ldots(1 - p_{i^*-1})p_{i^*}).$$

Step 3: Checking if state i^* is the last profitable state.

If $(\bar{r}_{i^*} + \beta p_{i^*} LF(1) > 0)$then

GOTO Step 4.

else

$i^* = i^* - 1$

GOTO Step 2.

Step 4: Computing the lifetime values.

$LF(i) = 0 \quad \forall i = i^* + 1, \ldots, I$

$LF(i) = \bar{r}_i + \beta p_i LF(1) + \beta(1 - p_i) LF(i+1) \quad \forall i = 2, \ldots, i^*.$

STOP.

APPENDIX 2

$(Y/Y_0) * 100$	Initial Budget Y($)	HEUR 1.1	HEUR 1.2	BOUND
.1%	80	179	185	211
1%	800	2000	2106	2110
10%	8000	20417	21080	21101
25%	20000	51010	52720	52753
50%	40000	97320	99241	99583
100%	80000	138370	138415	138546

Table 4: Expected Profits for the Catalog Mailing Problem

$(Y/Y_0) * 100$	Initial Budget Y($)	HEUR 2.1	HEUR 2.2	BOUND
5%	5900	5572	7433	8575
10%	11800	12967	15479	17148
25%	29500	36003	40433	42871
50%	59000	75950	81949	85741
75%	88500	109487	115329	118758
100%	118000	127364	133114	135544

Table 5: Expected Profits for the Catalog Mailing Model with Aggregate Inventory Costs. Single Mailing Case

$(Y/Y_0) * 100$	Initial Budget Y($)	HEUR 2.1	HEUR 2.2	HEUR 2.3	BOUND
5%	6150	5742	7495	7859	8984
10%	12300	13101	15703	16314	17968
25%	30750	36580	40047	42261	44921
50%	61500	77244	83448	84557	89821
75%	92250	112181	118160	118232	123427
100%	123000	130306	136172	136287	138546

Table 6: Expected Profits for the Catalog Mailing Model with Aggregate Inventory Costs. Two Mailings Case

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