

**INSIDER TRADING AND MARKET MANIPULATIONS:  
EXISTENCE AND UNIQUENESS OF EQUILIBRIUM**

by

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## ABSTRACT

In this paper, we consider a non-competitive rational expectations model in the line of Kyle (1985). We show that: i) There exists a unique equilibrium independently of the distribution of uncertainty; ii) This equilibrium minimizes the expected gains of the informed agent under incentive compatibility constraints. We extend our results to models of market manipulations and to a class of signalling games.

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## I. INTRODUCTION

Understanding the mechanism by which information gets incorporated into market prices is central to the analysis of their informational content. The competitive noisy rational expectations literature<sup>1</sup> argues that when equilibrium prices are partially revealing, the marginal benefit to information acquisition is positive and decreasing with the economy wide amount of information. Hence, an equilibrium is attained where marginal benefit equals marginal cost. However, this mechanism has the unpleasant feature that, when privately informed agents are either risk neutral or perfectly informed, they do not profit from their information<sup>2</sup>. This shortcoming of competitive noisy rational expectations models has forced economists to recognize the nonconvexities inherent to costly information acquisition<sup>3</sup>, and to take non-competitive behavior explicitly into account.

In his seminal paper, Kyle (1985) shows how a risk neutral and perfectly informed insider can profit from his private information by trading strategically and by hiding his trade behind the activity of "noise traders". In his model, Kyle makes specific assumptions about the distribution of uncertainty (normality) and restricts himself to a specific class of equilibria, the linear ones<sup>4</sup>. The argument that the linear equilibria are more tractable than the nonlinear ones

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<sup>1</sup>See for instance: Grossman and Stiglitz (1980), Diamond and Verrecchia (1981), Hellwig (1980) and Admati (1985).

<sup>2</sup> See Grossman and Stiglitz (1980).

<sup>3</sup>The presence of fixed costs of information acquisition is a central theme in economics of information (see Grossman and Stiglitz (1980) for instance).

<sup>4</sup>These assumptions are commonplace in the abundant literature that has built upon Kyle's model: See among many others Admati and Pfleiderer (1988 and 1990), Chowdhry and Nanda (1990), Gorton and Pennachi (1989), Pagano (1989), Pagano and Roell (1990), Roell (1989), Subrahmanyam (1990a and 1990b) and Vives (1990). An interesting exception can be found in Battacharya and Spiegel (1989) who consider the possibility of nonlinear equilibria with normal distributions. Another important exception, which is more directly related to the present paper, can be found in Back (1991). We shall return to the relationship between Back's work and ours in the conclusion.

is not convincing. The signalling literature (see Laffont and Maskin (1989), (1990) among others) abounds in examples where multiple equilibria are the rule.<sup>5</sup> In the context of market trading with asymmetric information, it is important to understand whether the uniqueness of an equilibrium comes from the ad-hoc linearity assumption, from the distributional assumption or from the nature of the strategic interaction.

The existence of a unique equilibrium enables the economist to answer the following questions: What are the welfare properties of non-competitive rational expectations models? What is the equilibrium amount of information revealed by prices in the presence of non-competitive behavior?

The purpose of this paper is to address the existence and uniqueness issue in a one-shot trading game in the spirit of Kyle (1985). We show that: i) There exists a unique perfect Bayesian equilibrium independently of the distribution of uncertainty; ii) This equilibrium minimizes the expected gains of the insider under incentive compatibility constraints. The first result justifies the use of the linearity assumption in the case of normal uncertainty. The second result is interpreted as a weak invisible hand property: The decentralized market minimizes the expected cost borne by the noise traders under incentive compatibility constraints. It is the key ingredient for the uniqueness result.

Of course this property does not hold in general, as can be anticipated by the multiplicity of equilibria in similar models (Laffont and Maskin (1989), 1990)). However, we show that it is valid for a whole class of signalling games, those games in which the surplus is independent of the second player's action. It is also true for certain models of market manipulations in which the value of the assets depends on an action taken by the informed player.

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<sup>5</sup>See also Bhattacharya and Spiegel (1989).

The paper is organized as follows: In section II, we present our central result in a relatively informal manner. Section III is dedicated to the rigorous derivation of the uniqueness result and the invisible hand property. In section IV, the uniqueness property is shown to hold for a whole class of signalling games. Section V is dedicated to the existence question, while section VI presents extensions and concluding remarks.

## II. AN HEURISTIC PRESENTATION

The rigorous derivation of our results involves a somewhat delicate apparatus. In this section, we shall set aside some difficult technical issues in order to focus on the central intuition. These important technical issues will be dealt with in subsequent sections.

The model is a one-shot trading game inspired from Kyle (1985). A single asset is traded by three types of traders: Noise traders, market makers and the informed trader (the insider). The ex-post liquidation value of the asset is denoted by  $v$ , the quantity traded by noise traders by  $u$ , the quantity traded by the informed trader by  $x$ , and the price of the asset by  $p$ . In this model, noise traders are passive players. They buy a quantity  $u$  which is the realization of an exogenously given random variable  $\tilde{u}$ . Their motives of trade are not explicitly modeled: One may think of the noise trading as either tax related trading<sup>6</sup> or liquidity trading.

The game unfolds as follows: In stage one, the values of  $u$  and  $v$  are realized. The informed trader observes both  $u$  and  $v$ , and submits a market order  $x(u,v)$ . In stage two, market makers observe aggregate demand  $y(u,v)=x(u,v)+u$  but do not directly observe  $u$ ,  $v$  or  $x$ . Having observed aggregate demand,  $y$ , they engage in a competitive auction à la Bertrand to supply  $y$ . The price resulting from this auction is denoted by  $p(y)$ . The profits of the informed trader are given by  $\pi(x,u,v) = (v-p(x+u))x$ . For simplicity, we shall assume that  $u$  and  $v$  have compact supports. Without loss of generality, we can then normalize  $u$  and  $v$  so that  $0 \leq u \leq 1$  and  $0 \leq v \leq 1$ .

A market equilibrium is characterized by a trading strategy and a pricing

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<sup>6</sup>See Constantinides (1983).

rule  $p$  such that: i) Given the pricing rule, for every  $(u,v)$ , the trading strategy  $x(u,v)$  maximizes the informed trader's profits  $\pi(x,u,v)$ . ii) Given the trading strategy, the pricing rule  $p(\cdot)$  is semi-strong efficient in the sense that the price  $p$  is set equal to the expected value of  $v$  conditional upon public information  $y$ :  $p(y) = E[v|y(u,v)]$ . This condition is implied by our assumption of Bertrand competition between market makers.

At this stage, it is not clear that such an equilibrium exists and more importantly that it is uniquely defined. Furthermore, we do not know whether an equilibrium pricing rule, if any, is differentiable or even continuous<sup>7</sup>. We shall deal with the questions of existence, uniqueness and regularity later in the paper. For now, we consider a differentiable pricing rule and write the first order condition for the insider's problem:

$$-p'(y)x + v - p(y) = 0. \quad (2.1)$$

Taking conditional expectations of (2.1) with respect to  $y$  yields that:

$$-p'(y)E(x|y) + E(v|y) - p(y) = 0.$$

Using the efficient pricing condition  $p = E(v|y)$ , and assuming that  $p'(y) \neq 0$  we obtain that, in an equilibrium, the expected insider trading  $x$  conditional upon the public information  $y$  is zero, i.e.

$$E(x|y) = 0. \quad (2.2)$$

Condition (2.2) is easy to understand: If the market estimate of insider trading was say positive, then a risk neutral market maker would want to "piggyback" on the insider. He would then buy, which in turn would raise  $p$ . Hence condition (2.2) is necessary in equilibrium. We will prove in proposition 5.2 that condition (2.2) is also sufficient.

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<sup>7</sup>The regularity issue is not merely a technical one. In fact, non-continuous pricing rules play an important role in the signalling literature (see Laffont and Maskin (1990)).

Assuming normality of the distribution of  $(u,v)$ , Kyle (1985) has shown the existence of a unique linear equilibrium, and obtained interesting comparative static properties for this particular equilibrium. For example, the sensitivity of price to quantity is well explained by the variances of  $u$  and  $v$ , and information is transmitted only partially (or gradually in the dynamic version of the model) by prices.

The relevance of this analysis depends on two questions: What happens when uncertainty is not normally distributed, and do there exist other equilibria? In similar models with asymmetric information, a large multiplicity of (Perfect Bayesian) Equilibria is usually the rule. For instance, Laffont and Maskin consider two models (1989), (1990) (the first one in the context of a monopoly producing a good of uncertain quality, the second one in the same context as Kyle's) in which any amount of information transmission is compatible with one of the equilibria.

The central result of the paper, which we present below, establishes that nothing of the sort can happen in the Kyle model. For any distribution of uncertainty, equilibrium is unique. The key ingredient to this result is a weak invisible hand property. To understand its features, imagine for the moment that the price function, instead of being determined by the market, is chosen by a Social Planner. The objective of this Social Planner, who can precommit himself to any price function  $p(\cdot)$ , is to minimize the expected cost borne by uninformed traders, market makers and noise traders, because of the presence of the insider. Since we have a zero sum game, this is equivalent to minimizing the expected profit of the insider.

We define an optimal pricing rule as a pricing rule which minimizes the expected profit of the insider i.e. which solves the problem



$$\min_{p(\cdot)} E [ \max_x (v-p(x+u))x ] . \quad (2.3)$$

Given an optimal pricing rule, the reaction of the insider is given by a function  $x(u,v)$ . We claim that condition (2.2) is a necessary and sufficient condition for an optimal pricing rule. The intuition is as follows: Consider an optimal pricing rule  $p^*(\cdot)$  and a given realization of  $y$ . If the Social Planner changes  $p^*(y)$  to  $p^*(y)+\Delta p$  the change in the insider's expected profit is:

$$\Delta p \cdot E[x|y].$$

This quantity must be non-negative for any value of  $\Delta p$ , which yields:  $E(x|y)=0$ . More formally, consider an alternative pricing rule  $p^*(\cdot)+\epsilon h(\cdot)$  and let  $F(\epsilon)$  be the insider profit's under this alternative pricing rule:

$$F(\epsilon) = E [ \max_x [v-p^*(x+u)-\epsilon h(x+u)]x ] . \quad (2.4)$$

From the optimality of  $p^*(\cdot)$ ,  $F'(0)=0$ . We now use the envelope theorem to compute  $F'(\epsilon)|_{\epsilon=0}$  and we obtain:

$$E [ x(u,v)h(y(u,v)) ] = 0, \quad \forall h(\cdot). \quad (2.5)$$

From a standard result in probability theory<sup>8</sup>, (2.5) is equivalent to (2.2). It therefore follows that an equilibrium pricing rule is an optimal pricing rule and vice versa: This is precisely the weak invisible hand property (propositions 3.2 and 5.2).

From the invisible hand property, we draw three implications. First, since an equilibrium pricing rule must be optimal, all equilibrium pricing rules give the same expected profit to the insider (proposition 3.2). In addition, we will show in proposition 3.3, that all equilibrium pricing rules generate the same insider strategy  $x(u,v)$  and the same equilibrium prices  $p(x(u,v)+v)$ . In this sense the equilibrium is unique. Second, proving the existence of an optimal

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<sup>8</sup>See for instance, Chung (1974), p. 298.

pricing rule (an optimization problem) is easier than proving the existence of an equilibrium pricing rule (a fixed point problem). In section V, we establish the existence of an optimal pricing rule and then use the weak invisible hand property to show that an equilibrium exists. Third and finally, we interpret the weak invisible hand property as a weak welfare property: Even in the presence of asymmetric information and a monopsonistic insider, the decentralized market minimizes the expected losses born by noise traders. Without a utility-based model of noise trading, the issue of Pareto optimality cannot be addressed so that our welfare property is a weak one.

In what follows, we present the rigorous derivation of the invisible hand property and show that this property can be extended to a whole class of signalling games as well as to certain models of market manipulations.

### III. UNIQUENESS OF EQUILIBRIUM AND THE INVISIBLE HAND PROPERTY

In this section, we reconsider the model of section II:  $u$  is the noise trading,  $v$  the asset's liquidation value,  $x$  the insider's trade,  $y=x+u$  the aggregate demand and  $p$  the price. Formally, we define our equilibrium as follows:

**Definition 3-1:** A market equilibrium is defined as a pair  $(P(\cdot), Z(\cdot, \cdot))$  such that:

- i)  $P(\cdot)$  is a upper hemicontinuous convex valued correspondence from  $\mathbb{R}$  to  $\mathbb{R}$ .
- ii)  $Z(\cdot, \cdot) = (y(\cdot, \cdot), p(\cdot, \cdot))$  is a measurable function from  $[0,1] \times [0,1]$  to  $\mathbb{R}^2$ .
- iii) For almost every  $(u, v)$ ,  $Z(u, v)$  belongs to  $\text{ArgMax}_{(y, p); p \in P(y)} (v-p)(y-u)$  (3.1)
- iv) For almost every  $(u, v)$ ,  $p(u, v) = E[v | Z(u, v)]$ . (3.2)

Technical comments on the definition: a) For technical reasons (see section V), we have allowed the pricing rule to be multivalued for some values of  $y$ , with the convention that the insider can choose whatever price maximizes his profits in the set  $P(y)$  of market prices at  $y$ . If the correspondence  $P(\cdot)$  is a function then our equilibrium concept is standard. As it turns out, any equilibrium price correspondence  $P(\cdot)$  is single valued at  $y(u, v)$  with probability 1 (see proposition 3.1 below) and thus condition iv) is equivalent to Kyle's condition:

$$P(y) = E [ v | y(u, v) = y ] .$$

- b) The reader will note the notational difference between the pricing rule  $P(y)$  and the resulting price  $p(u, v)$ .
- c) The reader will also note that for convenience, we have expressed the strategy of the insider in terms of the aggregate demand  $y$  as opposed to the insider trade  $x$ .

Differences between our model and Kyle's: a) As opposed to Kyle, we do not assume

any specific distribution of  $(u,v)$ . For instance, we do not rule out the possibility that  $(u,v)$  be correlated. The compact support assumption is essentially technical and can be relaxed.

b) We assume that the informed trader observes the noise trade  $u$ . This is equivalent to a situation where the informed trader does not observe  $u$ , but is allowed to submit limit orders  $x(v,p)$ <sup>9</sup>.

As we shall see later in section IV, existence of such a (pure strategy) equilibrium is not guaranteed in general, except with a continuous distribution of uncertainty. Thus we assume:

**Assumption 1:** The distribution  $\nu$  of  $(u,v)$  has a continuous density w.r.t. Lebesgue measure.

One of the simplifications introduced by Assumption 1 stems from the following lemma:

**Lemma 3.1:** For any price correspondence  $P(\cdot)$ , the set of  $(u,v)$  such that the insider's problem (3.1) has several maxima, is Lebesgue negligible.

Proof: See appendix A. ■

Another important consequence of Assumption 1 is the fact that we can restrict our attention to price correspondences which are (1) non-decreasing and (2) almost surely single valued continuous and differentiable at  $y(u,v)$ . The latter property enables us to write the first order condition for (3.1).

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<sup>9</sup>In the one shot version of the Kyle (1985) model, the insider submits market order  $x(v)$  and does not observe the price at which trade occurs. If we modify Kyle's (1985) in the following way: (i) Allow the insider to submit limit orders  $x(v,p)$ ; (ii) Define a market equilibrium with limit orders as in Kyle (1989), then there is an equivalence between our equilibrium and a market equilibrium with limit orders (see Vila (1988), (1989)). An interesting analogy can be drawn between our model and the continuous time version of Kyle (1985). Indeed in the continuous time Kyle model, since the price process has continuous sample paths, the insider does indeed observe the current price (see Back (1991), see also sections 6.3 and 6.4.)

**Definition 3.2:** A market equilibrium  $(P,Z)$  is supported by a price correspondence  $P_1$  if and only if, for a.e.  $(u,v)$ ,  $Z(u,v) \in \underset{P_1 \in P_1(y)}{\text{ArgMax}}(v-p_1)(y-u)$ .

The next proposition shows that any equilibrium can be supported by a 'well behaved' price correspondence:

**Proposition 3.1.** Under assumption 1, every market equilibrium  $(P,Z)$  can be supported by a non-decreasing correspondence  $P_1$  such that:

- (i) For a.e.  $(u,v)$ ,  $P_1$  is single valued, continuous and differentiable at  $y(u,v)$  and  $P_1'(y(u,v)) > 0$ .
- (ii)  $P_1(y) = 0$  for  $y \leq 0$ ,  $P_1(y) = 1$  for  $y \geq 1$ .

Proof: See appendix A. ■

Proposition 3.1 yields the following simplifications:

a) Since  $P_1$  is a non-decreasing correspondence which is single valued at  $y(u,v)$ , we shall, when convenient<sup>10</sup>, think of  $P(\cdot)$  as a function as opposed to a correspondence. However, in section V, we shall prove the existence of an equilibrium pricing correspondence. Our proof uses the local compactness of the space of closed graph correspondences endowed with the Hausdorff topology (see section V).

b) Since  $P(\cdot)$  is differentiable at  $y(u,v)$ , we can write the first order condition for the insider's maximization problem (3.1).

c) From (ii) above and for further reference, we will take  $P(y) = 0$  for  $y \leq 0$  and  $P(y) = 1$  for  $y \geq 1$ .

Let  $C$  be the set of compact graph correspondences from  $[0,1]$  to  $\mathbb{R}$  endowed

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<sup>10</sup>That is for any economic application.

with the Hausdorff topology. We define the functional  $W(\cdot)$  by:

$$\forall P \in C, W(P) = E \left[ \text{Max}_{p \in P(y)} (v-p)(y-u) \right].$$

**Definition 3.3:** An optimal pricing rule is any element of  $C$  that minimizes  $W$ :  
 $\forall P \in C, W(\bar{P}) \leq W(P)$ .

The next lemma shows that an optimal pricing rule can be taken to be non-decreasing.

**Lemma 3.2:** For any optimal pricing rule  $P(\cdot)$ , there exists a non-decreasing correspondence  $P_1(\cdot)$  such that:

- i)  $W(P) = W(P_1)$
- ii)  $\text{ArgMax}_{p \in P(y)} (v-p)(y-u) \cap \text{ArgMax}_{p \in P_1(y)} (v-p)(y-u) \neq \emptyset$ , almost surely.

Proof: See appendix A. ■

The next proposition shows that an equilibrium pricing rule is an optimal pricing rule.

**Proposition 3.2:** Under assumption 1, let  $(P, Z)$  be a market equilibrium. Then  $P$  is an optimal pricing rule. As a consequence, all market equilibria give the same expected profit to the insider.

Proof: See appendix A. ■

We now prove a stronger uniqueness result, namely that equilibrium prices and quantities are uniquely determined.

**Proposition 3.3:** Under assumption 1, let  $(P_1, Z_1)$  and  $(P_2, Z_2)$  be two market equilibria. Then for a.e.  $(u, v)$ ,  $Z_1(u, v) = Z_2(u, v)$ .

Proof: See appendix A. ■

The insider trading model thus possesses something specific that prevents the inconvenient multiplicity of Bayesian equilibria that is usual in similar contexts, like for instance, in Laffont-Maskin (1989) and (1990). In order to understand this specificity, we will study the uniqueness question in a more general set up, namely that of signalling games à la Cho-Kreps (1987). This is the objective of the next section.

## IV. THE UNIQUENESS PROPERTY FOR A PARTICULAR CLASS OF SIGNALLING GAMES

In this section, we consider signalling games à la Cho-Kreps (1987), i.e. two-player games with the following structure: Pay-offs depend on a random variable  $\omega$  (the state of nature) that is observed only by the first player (the informed player). This informed player sends a signal  $s$  to the second player, who then reacts by choosing some action  $r$ . The only difference with the original formulation of Cho-Kreps is that the sets  $\Sigma$  (possible signals) and  $\Gamma$  (possible reactions) are not finite, but are instead compact convex subsets of a linear space  $E$ . This will allow us to focus on pure strategy equilibria. The parameter  $\omega$  takes its values in a set  $\Omega$ , in accord with a probability measure  $\nu$  that is common knowledge. Finally, the pay-off functions are denoted  $U(\omega, s, r)$  for the informed player and  $V(\omega, s, r)$  for the other player. We shall assume that these pay-off functions are continuous w.r.t.  $(s, r)$  and  $\nu$ -measurable w.r.t.  $\omega$ .

A strategy for the informed player is a  $\nu$ -measurable mappings  $S(\cdot)$  from  $\Omega$  to  $\Sigma$ . The set of such strategies will be denoted  $\mathcal{S}$ , it is a convex subset of  $L^\infty(\Omega, E)$  the linear space of bounded  $\nu$ -measurable mapping from  $\Omega$  to  $E$ .

Similarly a strategy for the uninformed player is a continuous mapping  $R$  from  $\Sigma$  to  $\Gamma$ . The set  $\mathcal{R}$  of such strategies is a convex subset of  $C(\Sigma, E)$ , the linear space of continuous mapping from  $\Sigma$  to  $E$ .

The expected pay-offs that are associated to a couple of strategies  $(S, R)$  are given by:

$$\begin{aligned}\bar{U}(S, R) &= \int U(\omega, S(\omega), R(S(\omega))) d\nu(\omega); \\ \bar{V}(S, R) &= \int V(\omega, S(\omega), R(S(\omega))) d\nu(\omega).\end{aligned}$$



**Definition 4.1:** A Bayesian Nash Equilibrium is composed of a couple of strategies  $(S^*, R^*)$  and a family  $\nu(\cdot|s)$  of probability measures on  $\Omega$ , indexed by  $s \in \Sigma$  such that:

i) For a.e  $\omega$  in  $\Omega$ ,  $S^*(\omega) \in \underset{s}{\text{ArgMax}} U(\omega, s, R^*(s))$

ii) For all  $s$  in  $\Sigma$ ,  $R^*(s) \in \underset{r}{\text{ArgMax}} \int V(\omega, s, r) d\nu(\omega|S^*(\omega)=s)$ .

iii) For all  $s$  in the support of  $S^*(\omega)$ ,  $\nu(\cdot|s)$  is deduced from  $\nu$  by application of Bayes' rule.

In particular, for all such equilibria, we have:

$$\text{For all } S \in \mathcal{S}, \bar{U}(S, R^*) \leq \bar{U}(S^*, R^*) \quad (4.1)$$

$$\text{For all } R \in \mathcal{R}, \bar{V}(S^*, R) \leq \bar{V}(S^*, R^*) \quad (4.2)$$

Thus for any Bayesian Nash Equilibrium  $(S^*, R^*, \nu(\cdot|\cdot))$ ,  $(S^*, R^*)$  is a Nash Equilibrium of the game  $(\bar{U}, \bar{V})$  but the converse is not true in general.

It is well known that signalling games typically admit a large multiplicity of Bayesian equilibria<sup>11</sup>. However, we shall restrict our attention to a particular class of such games, in which the total payoff is independent of the uninformed player's action:

**Assumption 2:**  $U(\omega, s, r) + V(\omega, s, r) = \phi(\omega, s)$ <sup>12</sup>.

The main result of this section is that all Bayesian equilibria of such games give the same expected payoff to the informed agent. Before we state and prove this result, we need two definitions:

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<sup>11</sup>See, for instance, Laffont and Maskin (1989)

<sup>12</sup>Assumption 2 generalizes the concept of zero sum game; the invisible hand property generalizes the min-max property.

Definition 4.2: Let  $(\bar{S}, \bar{R})$  be a couple of strategies. We say that  $\bar{S}$  is a best response to  $\bar{R}$  if and only if:  $\forall S \in \mathcal{S}, \bar{U}(S, \bar{R}) \leq \bar{U}(\bar{S}, \bar{R})$ .

Definition 4.3: Let  $\bar{R}$  be strategy of the uninformed agent. We say that  $\bar{R}$  is an optimal reaction function if and only if:  $\forall R \in \mathcal{R}, W(\bar{R}) \leq W(R)$  where  $W(R) = \max_S \bar{U}(S, R)$ .

Next, we prove that a Bayesian equilibrium involves an optimal reaction function.

Proposition 4.1: Under assumption 2, for all Bayesian equilibrium  $(S^*, R^*)$ :

- (i)  $R^*$  is an optimal reaction function and
- (ii)  $S^*$  is a best response to  $R^*$ .

As a consequence, all Bayesian equilibria give the same expected pay-off to the informed player.

Proof: See appendix B. ■

Under an additional assumption, one can prove that all Bayesian equilibria also give the same payoff to the uninformed player.

Assumption 3:  $V$  is concave w.r.t.  $r$

Proposition 4.2: Under assumptions 2 and 3, let  $(S_1, R_1)$  and  $(S_2, R_2)$  be two Bayesian equilibria. Then  $S_1 = S_2$  almost everywhere and  $\bar{V}(S_1, R_1) = \bar{V}(S_2, R_2)$ .

Proof: See appendix B. ■

So far, we have not considered the problem of existence of a Nash Equilibrium of the game  $(\bar{U}, \bar{V})$  (a slightly more general concept than that of Bayesian equilibrium). Under additional regularity assumptions, this problem is

solved by establishing the converse to proposition 4.3.

**Assumption 4:**  $V$  is differentiable w.r.t.  $r$ ,  $\frac{\partial V}{\partial r}(\omega, s, r)$  is continuous w.r.t.  $(s, r)$  and measurable w.r.t.  $\omega$ .

**Proposition 4.3:** Under assumptions 2, 3, 4, let  $\bar{R}$  be an optimal reaction function and  $\bar{S}$  be a best response to  $\bar{R}$ . Consider the following subset of  $\Omega$ :

$$\Omega_0 = \{\omega: s \rightarrow U(\omega, s, \bar{R}(s)) \text{ has at least two maxima}\}$$

Then, if  $\nu(\Omega_0) = 0$ ,  $(\bar{S}, \bar{R})$  is a Nash equilibrium of the game  $(\bar{U}, \bar{V})$ .

**Proof:** See appendix B. ■

**Remark:** When  $\nu(\Omega_0) \neq 0$ , then in general pure strategy equilibria fail to exist. However there exists a (unique) mixed strategy equilibrium like in the example that we study in Section V.

We are now in a position to clarify the key difference between the Kyle model, and other models (like the Laffont-Maskin models) that do not possess the uniqueness property. For that purpose, we will construct artificial signalling games that possess the same Bayesian Equilibria as the models under study.

**Example 1: The Kyle model:** Surprisingly, it turns out that the strategy of the insider (the informed player) has to be a price (and not a quantity). This is related to the fact that the second player is supposed to represent the aggregate behavior of competitive market makers. Using the notations introduced above, we must set:

$$\omega = (u, v)$$

information of the 1st player

$s = P(u, v)$	strategy of the 1st player (price)
$r = Y(p)$	strategy of the 2nd player (quantity)
$U(\omega, s, r) = (v-s)(r-u)$	utility of the 1st player
$V(\omega, s, r) = (s-v)r$	utility of the 2nd player

If  $(P, Y)$  is a perfect Bayesian Equilibrium of this game, then one must have:

$$\forall p, Y(p) \in \underset{y}{\text{ArgMax}} (E[v|p] - p)y, \text{ and}$$

$$\forall \omega \quad P(\omega) \in \underset{p}{\text{ArgMax}} (v-p)(Y(p) - u).$$

The first condition implies semi-strong efficiency:  $E[v|p] = p$ . Likewise, if we define  $P_1 = Y^{-1}$ , and  $Z(p) = (Y(p), p)$ , the second condition can also be stated as:

$$\forall (u, v), \quad Z(u, v) \in \underset{(y, p), p \in P_1(y)}{\text{ArgMax}} (v-p)(y-u)$$

So that  $(P_1, Z)$  is a market equilibrium. The uniqueness property comes from assumption 2:  $U(\omega, s, r) + V(\omega, s, r) = u(s-v) = \phi(\omega, s)$ .

**Example 2: The Laffont-Maskin monopoly model (1990):** In this model the informed player is a monopolist and  $\omega$  is a quality parameter unknown to the buyer (the second player). The signal sent by the monopolist is a price and the response of the second player is a quantity. Thus we have to set:

$$s = P(\omega) \text{ and } r = Q(p),$$

$$U(\omega, s, r) = (s - v(\omega))r,$$

$$V(\omega, s, r) = (\omega - s)r$$

where  $v(\omega)$  and  $\omega$  denote respectively marginal cost and marginal utility of the good. The multiplicity of Perfect Bayesian Equilibria comes from the fact that assumption 2 is not satisfied:

$$U(\omega, s, r) + V(\omega, s, r) = (\omega - v(\omega))r$$

which is independent of  $s$ , but not of  $r$ !

**Example 3: The Laffont-Maskin insider trading model (1989):** This model presents

close similarities to the Kyle model: The informed player is an insider who possesses some (imperfect) information on the liquidation value of some risky asset. The two main differences with the Kyle model are: a) There is no uncertainty on aggregate demand (there are no noise traders) and b) market makers are risk averse. Because of a), the case where market makers are risk neutral is not interesting: The unique equilibrium is completely revealing, and there is no trade. But then, when market makers are risk averse, assumption 2 ceases to be true for the corresponding signalling games and there may exist many equilibria (Laffont-Maskin (1989)). Again, because of a) this result does not apply to the Kyle model with risk aversion, and we do not know if the uniqueness property in the Kyle model is preserved when market makers are risk averse. However, one can show that equilibrium is unique in the Laffont-Maskin model when risk aversion is small.

## V. EXISTENCE OF THE EQUILIBRIUM IN THE KYLE MODEL

As far as we know, no general existence result is available for a market equilibrium in the Kyle model. It turns out that our invisible hand model gives the key ingredient in that direction. We begin by proving the existence of an optimal pricing rule.

**Proposition 5.1:** Under assumption 1, there exists an optimal price correspondence function which by lemma 3.2 is non-decreasing.

Proof: The proof is immediate by continuity of  $W(\cdot)$  and compactness of  $C$  when  $C$  is endowed with the Hausdorff topology. ■

We now establish the converse to proposition 3.2.

**Proposition 5.2.:** Under assumption 1, let  $\bar{P}$  be an optimal pricing rule (in the sense of definition 3.3) and  $\bar{Y}$  be a best response to  $\bar{P}$ . For a.e  $(u,v)$ ,  $\bar{P}$  is differentiable at  $\bar{Y}(u,v)$  and  $(\bar{P}, \bar{Z})$  is a market equilibrium where  $\bar{Z}$  is defined by:  $\bar{Z}(u,v) = (\bar{Y}, \bar{P}(\bar{Y}(u,v)))$ .

Proof: See appendix C. ■

## VI. EXTENSIONS AND CONCLUSIONS

## 6.1. The case of discrete distributions

One may wonder whether the uniqueness result does not come from the absolute continuity of measure  $\nu$  (assumption 1). In fact, it is only the existence of a pure strategy equilibrium that poses problems. With a discrete distribution, the invisible hand result can be shown to hold: There is a one-to-one relation between optimal pricing rules and mixed strategy equilibria. As a consequence, there exists a unique mixed strategy equilibrium.

For the purpose of this paper, we will only exhibit an example where a pure strategy equilibrium fails to exist.

Let us assume that  $\nu$  is supported by  $\{A=(0,1), B=(1,0), C=(\frac{2}{3}, \frac{2}{3})\}$ . Since a market equilibrium is necessarily an increasing function, the only possible informational structure at equilibrium is  $\{A,B\}, \{C\}$ . In other words, the price function  $\bar{P}$  has to be such that A and B choose the same  $(y,p)$ . The equilibrium conditions give:

$$\begin{aligned} (y_A, p_A) &= (y_B, p_B) = \left( \frac{\nu(B)}{\nu(A) + \nu(B)}, \frac{\nu(A)}{\nu(A) + \nu(B)} \right) \\ (y_C, p_C) &= \left( \frac{2}{3}, \frac{2}{3} \right) \end{aligned}$$

This is indeed a market equilibrium if and only if the self selection constraints are fulfilled:

$$\begin{aligned} y_A(1-p_A) &\geq y_C(1-p_C) = \frac{2}{9} \\ p_B(1-y_B) &\geq p_C(1-y_C) = \frac{2}{9} \end{aligned}$$

which are equivalent to:  $\frac{\sqrt{2}}{3-\sqrt{2}} \leq \frac{\nu(A)}{\nu(B)} \leq \frac{3}{\sqrt{2}} - 1$ . If these inequalities are not

satisfied, then there exists no pure strategy equilibrium<sup>13</sup>.

## 6.2. Endogenous value games: Market manipulations

In this final part, we consider situations where an agent (called the manipulator) has the possibility to affect the value of an asset by taking a costly action. As was pointed out by Hirshleifer (1971), our agent's incentive to take this action will be affected by the position of the asset in his portfolio. Furthermore, if the agent can secretly change his position prior to taking the action, he will do so and his trading decision will affect his subsequent action.

Several examples of a situation like the one above can be given. A first example involves a firm investing in a research and development project which will affect the financial market (Hirshleifer (1971)). A second example involves a raider who has the potential to improve the value of a target firm by engineering a takeover. Kyle and Vila (1991) describe how the raider will acquire shares on the open market prior to the takeover. A third example concerns a futures market manipulator who can corner the market by taking a large long position (see Kyle (1984) and Vila (1988)). A final example involves an agent who commits a criminal act against a corporation and who simultaneously takes a short position in the corporation's stock (see Vila (1987), (1989)).

In such situations, the informational advantage of the large player is somewhat different from a Kyle-type situation. Indeed, the informational advantage of the large player lies in his trading decision which will affect his incentive to change the asset's value. Despite this difference, the analysis in section 2 can be extended.

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<sup>13</sup>See also Vila (1988).



For this purpose let  $c(v)$  be the cost of setting the value of the asset equal to  $v$ . By normalization, we take  $c(0)=0$ . We assume that  $c(\cdot)$  is a convex function. Let  $u$  be the noise trading,  $x$  the larger trader's trade. The large trader's profits are now equal to:  $\pi(x,u,v)=(v-p(x+u))x-c(v)$ .

An equilibrium is defined by a trading strategy  $x(u)$ , an action  $v(u)$  and a pricing rule  $p(\cdot)$  such that:

i)  $(x(u),v(u))$  maximizes  $\pi(x,v,u)$ .

ii)  $p(y) = E(v(u) | y=u+x(u))$

Assuming differentiability of  $p(\cdot)$ , the market efficiency condition is equivalent to the condition  $E(x(u) | y)=0$ . Defining an optimal pricing rule as a function  $p(\cdot)$  which minimizes the expected gains of the large trader, we obtain that the condition  $E(x(u) | y)=0$  characterizes the optimal pricing rule. The invisible hand result follows.

### 6.3. The Case of Non-Nested Information Sets

The version of the Kyle (1985) model that we have considered assumes that the insider observes the noise trading,  $u$ , or equivalently is able to submit limit orders. If, as in the original Kyle model the insider does not observe  $u$  then neither the market makers who observe  $p$  but not  $v$ , nor the insider who observe  $v$  but not  $p$ , have superior information. In this case, the uniqueness issue become more delicate as we shall see.

First, if the support of  $u$  is bounded, say  $[0,1]$ , then, for every  $x \neq 0$ , the probability that insider trading will be detected is positive. Indeed if  $x$  is positive and  $x+u$  is greater than 1, the market makers know that the insider is buying. Hence, there exists a no trade nash equilibrium where:

$x(v)=0$ ;  $p(y)=-\infty$  if  $y<0$ ;  $p(y)=+\infty$  if  $y>1$  and  $p(y)=E(v)$  otherwise.

The no trade equilibrium<sup>14</sup> is not a Bayesian equilibrium since  $+\infty$  and  $-\infty$  do not belong to the support of  $v$ . However, we see that we already need a refinement (support restriction) of definition 3.1 to rule out this trivial equilibrium which was not the case in section III.

Second, even with the support restriction uniqueness is not guaranteed in general as can be seen from the example displayed in appendix D.

#### 6.4. Concluding remarks

The main result of this paper is that the equilibrium is always unique in the Kyle (1985) model of insider trading (with limit orders), for any distribution of noise and asset returns. This contrasts markedly with similar models of asymmetric information (like Laffont-Maskin (1989), (1990)) which admit a large multiplicity of perfect Bayesian Equilibria. The key ingredient to our uniqueness result is a weak invisible hand property: The equilibrium price function is the one that minimizes expected profits of the insider under incentive compatibility constraints.

We also establish this uniqueness result for a particular class of signalling games, in which, once the informed player has played, the interests of the two players are completely opposed.

The invisible hand property also allows us to prove existence of a pure strategy equilibrium in the case of continuous uncertainty, and of a mixed strategy equilibrium in the case of a discrete distribution.

Finally, our uniqueness result is also shown to hold in the context of market manipulations, where the value of the asset is endogenous.

Several extensions of these results can be thought of: For instance, it

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<sup>14</sup>Of course the no trade equilibrium minimizes the insider's expected profits.

is reasonable to conjecture that the uniqueness result passes to the dynamic version of the Kyle model. Indeed if trading occurs continuously and if the price process has continuous sample paths then the insider knows at which price he is trading and therefore the information sets of market makers and the insider become nested. Recent work by Back (1991) shows that if noise trading  $u_t$  follows a Brownian process, then there exists a unique equilibrium within a general class of pricing rule for any distribution of  $v$ . Back's result differs from ours in the sense that: (i) It applies to the continuous time version of Kyle (1985) (ii) The distribution of noise trading is not arbitrary (it must be a Brownian motion) (iii) The techniques that are used are quite different. However, the key condition in Back's paper is that the aggregate cumulative trade  $y_t = x_t + u_t$  be a martingale from the point of view of market makers, i.e.:

$$E(y_{t_2} | y_s; s < t) = y_s. \quad (6.1)$$

Given that future values of noise trading flows are unpredictable, condition (6.1) means that future trades of the insider are unpredictable which is analogous to our condition (2.2). It is therefore reasonable to conjecture that Back's uniqueness result (and ours) will hold in continuous time for any noise trading process.

By contrast, it is not clear to us whether it would also pass in the case of risk aversion, since our invisible hand property does not hold anymore. Similarly, we do not know what happens when there are several insiders, as in Kyle (1989).

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## APPENDIX A

Proof of lemma 3.1: Let  $P(\cdot)$  be any price correspondence and  $\pi(P,u,v)$  denote the maximum in (3.1), i.e.  $\pi(P,u,v) = \max_{(y,p); p \in P(y)} (v-p)(y-u)$ . If we add  $uv$  to both sides of this equality, the expression to be maximized becomes linear with respect to  $(u,v)$ , which implies that  $G(P,u,v) = \pi(P,u,v) + uv$  is a convex function of  $(u,v)$ . Moreover, if the maximum is attained at  $(y(u,v), p(u,v))$  then by the envelope theorem, we have

$$\text{For a.e. } (u,v), \begin{pmatrix} p(u,v) \\ y(u,v) \end{pmatrix} \in \partial G(P,u,v)$$

where  $\partial G$  denotes the subdifferential<sup>15</sup> of the convex function  $G$ . A convex function on a finite dimensional space being differentiable except on a Lebesgue negligible set, we obtain our uniqueness result. More precisely:

$$\text{For a.e. } (u,v), \quad p(u,v) = \frac{\partial G}{\partial u}(P,u,v); \quad y(u,v) = \frac{\partial G}{\partial v}(P,u,v). \quad \blacksquare$$

Proof of Proposition 3.1

The proof builds upon the following steps:

\* Step 1: For every pricing correspondence  $P(\cdot)$ , the insider's profits  $\pi(P,u,v)$  are positive for almost every  $(u,v)$ . Indeed, for  $\pi(P,u,v)$  to be zero it must be that:

$$p > v \text{ for all } y > u \text{ and } p \in P(y) \text{ and}$$

$$p < v \text{ for all } y < u \text{ and } p \in P(y).$$

which occurs only for  $(u,v)$  in a set of measure zero.

\* Step 2: The price is almost surely non fully revealing:  $p(u,v) \neq v$  for almost every  $(u,v)$ . Indeed, if  $p(u,v) = v$  then  $\pi(P,u,v) = 0$  which happens only with probability zero (step 1.)

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<sup>15</sup>See for instance Aubin (1979) p.105.

\* Step 3: Every buyer pools with at least one seller and vice versa. For almost every  $(u,v)$ , there exists  $(u',v')$  such that:

$$Z(u,v)=Z(u',v') \text{ and } (y(u,v)-u)(y(u',v')-u')<0.$$

Indeed since  $p(u,v) \neq v$  and  $p(u,v)=E(v|Z(u,v))$  it follows that the signal  $Z(u,v)$  is sent by both buyers and sellers.

\* Step 4: For almost every  $(u,v)$ ,  $P(\cdot)$  is differentiable at  $y(u,v)$ .

Using the previous step, we take  $u < y(u,v) = y(u',v') < u'$ ;  $p(u,v) = p(u',v')$ . Both  $(u,v)$  and  $(u',v')$  send the same signal  $(y,p)$ . It follows that the graph of  $P(\cdot)$  must separate the sets:

$$A = \{(y,p): y > u \text{ and } (y-u)(v-p) > (y(u,v)-u)(v-p(u,v))\} \text{ and}$$

$$B = \{(y,p): y < u' \text{ and } (u'-y)(p-v') > (u'-y(u',v'))(p(u',v')-v')\}.$$

(see figure 1)

Hence,  $P(\cdot)$  must be single valued, continuous and differentiable<sup>16</sup> at  $y(u,v)$ .

Furthermore  $P(\cdot)$  satisfies the following monotonicity property:

For all  $y > y(u,v)$  and  $p \in P(y)$ ,  $p > p(u,v)$  and

for all  $y < y(u,v)$  and  $p \in P(y)$ ,  $p < p(u,v)$ .

\* Step 5:  $P(\cdot)$  can be taken to be non-decreasing. For this purpose, let

$$f_p(y) = \max_{p \in P(z); z \leq y} p.$$

The function  $f_p(\cdot)$  is non-decreasing and left continuous. Let  $P_1(\cdot)$  be the convex closure of  $f_p(\cdot)$  i.e.

$$P_1(y) = [f_p(y); \inf_{z \geq y} f_p(z)].$$

It follows from the monotonicity property above that  $P_1(\cdot)$  supports the equilibrium  $(P,Z)$  (details are left to the reader.)

\* Step 6: From the differentiability of  $P(\cdot)$  and the first order condition

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<sup>16</sup>In the sense that every selection is differentiable.

in (3.1). It follows that:

$$v - P(y(u,v)) - (y-u)P'(y(u,v)) = 0.$$

We know that  $v \neq P(y(u,v))$  a.s. (step 2). Hence:  $P'(y(u,v)) \neq 0$  a.s. and since  $P(\cdot)$  is non-decreasing  $P'(y(u,v)) > 0$ .

\* Step 7: The proof of 3.1 (ii) is immediate and left to the reader. ■

Proof of lemma 3.2:

An optimal pricing rule can be taken to be non-decreasing. Indeed, let  $P(\cdot)$  be an optimal pricing rule and  $P_1(\cdot)$  defined as in the proof of proposition 3.1, step 5.

Let:  $(p_1(u,v), y_1(u,v))$  belong to  $\underset{P_1 \in P_1(y_1)}{\text{ArgMax}}(v-p_1)(y_1-u)$  and

$(p(u,v), y(u,v))$  belong to  $\underset{p \in P(y)}{\text{ArgMax}}(v-p)(y-u)$ .

$P_1(y)$  is by construction greater than  $P(y)$  for every  $y$ . Therefore, buyers make smaller profits with  $P_1(\cdot)$  than with  $P(\cdot)$ . Formally, if  $y_1(u,v) \geq u$  then  $(v-p_1(u,v))(y_1(u,v)-u) \leq (v-p)(y_1(u,v)-u)$  for some  $p$  in  $P(y_1(u,v))$ .

Thus,  $\pi(P, u, v) \geq \pi(P_1, u, v)$  (see the proof of lemma 3.1 for a definition of  $\pi$ ).

Now, suppose that  $y_1(u,v) < u$ . By construction, there exists  $z \leq y_1(u,v)$  and  $p \in P(z)$  such that  $p \geq p_1(u,v)$ . Therefore,

$$\pi(P_1, u, v) = (p_1(u,v) - v)(u - y_1(u,v)) \leq (p - v)(u - z) \leq \pi(P, u, v).$$

Hence,  $W(P) \geq W(P_1)$  so that  $P_1(\cdot)$  is also an optimal pricing rule.

Furthermore, since  $P(\cdot)$  is optimal, it must be that  $\pi(P, u, v) = \pi(P_1, u, v)$  a.s. which implies that the inequalities above must be equalities. Lemma 3.2 follows. ■

Proof of proposition 3.2: Let  $(P, Z)$  be a market equilibrium. By proposition 3.1 this equilibrium can be supported by a nondecreasing  $P_1$  such that, for a.e.  $(u,v)$ ,  $P_1$  is differentiable at  $y(u,v)$  and  $P'_1(y(u,v)) > 0$ . Thus we can write the



first order condition corresponding to condition (3.1)):

$$\text{for a.e. } (u,v) \quad v - P_1(y(u,v)) - P_1'(y(u,v))(y(u,v)-u) = 0$$

By taking expectations conditional upon  $y(u,v)$ , and using the fact that  $P_1'(y(u,v))$  is positive, it follows that the market efficiency condition (3.2) is equivalent to:

$$E [u|y(u,v)] = y(u,v) \quad (\text{A.1})$$

We have to prove that  $P$  (or  $P_1$ ) is a minimum of  $W$ : Let  $H$  be any bounded function from  $[0,1]$  to  $\mathbb{R}$ . Condition (3.1) above implies:

$$E [H(y(u,v))(y(u,v)-u)] = 0 .$$

$$\text{Thus: } W(P) = E [ \{v - (P+H)(y(u,v))\} \{y(u,v) - u\} ] \leq \text{Max}_{0 \leq y \leq 1} [ \{v - (P+H)(y)\} \{y - u\} ] = W(P+H) .$$

Finally, for all bounded  $H$ :  $W(P) \leq W(P+H)$ . Hence  $P$  minimizes  $W$  in the space of bounded functions. By lemma 3.1 it follows that  $P$  minimizes  $W$  in the space of compact graph correspondences. ■

Proof of proposition 3.3: By proposition 3.1, we can assume without loss of generality that  $P_1$  and  $P_2$  are nondecreasing functions. Proposition 3.2 then implies:

$$W(P_1) = W(P_2) = \text{Min}_{P \in C} W(P) .$$

$W$  being convex, this implies in turn:

$$W(P_1) = W(P_2) = W(\frac{1}{2}P_1 + \frac{1}{2}P_2) . \quad (\text{A.2})$$

For any measurable  $Y: [0,1] \times [0,1] \rightarrow \mathbb{R}$  let us define

$$B(P,Y) = E [ \{v - P(Y(u,v))\} \{Y(u,v) - u\} ] .$$

We have by definition:  $W(P) = \text{Max}_Y B(P,Y)$  and by linearity of  $B$  w.r.t.  $P$

$$B(\frac{P_1+P_2}{2}, Y) = \frac{1}{2}B(P_1, Y) + \frac{1}{2}B(P_2, Y) .$$

But then condition (A.2) implies:

$$\text{Max}_Y [\frac{1}{2}B(P_1, Y) + \frac{1}{2}B(P_2, Y)] = \frac{1}{2}\text{Max}_Y B(P_1, Y) + \frac{1}{2}\text{Max}_Y B(P_2, Y).$$

This is only possible if the two maxima in the right hand side are attained for the same  $Y$ , i.e. if:  $Y_1(u, v) = Y_2(u, v)$ , for a.e.  $(u, v)$ . Using again condition (3.2),  $p(u, v) = E[v | Y(u, v)]$ , this implies:  $P_1(Y_1(u, v)) = P_2(Y_2(u, v))$  for a.e.  $(u, v)$ . ■

## APPENDIX B

Proof of proposition 4.1: Let  $(S^*, R^*)$  be a Bayesian equilibrium. ii) is simply a reformulation of condition (4.1). We have to prove that  $R^*$  is a minimum of  $W$ . By Assumption 2 we have:  $W(R^*) = \bar{U}(S^*, R^*) = \int \phi(\omega, S^*(\omega)) d\nu(\omega) - \bar{V}(S^*, R^*)$ .

Now we use condition (4.2):

$$\forall R, W(R^*) \leq \int \phi(\omega, S^*(\omega)) d\nu(\omega) - \bar{V}(S^*, R) \leq \bar{U}(S^*, R) \leq \text{Max}_{S \in \mathcal{S}} \bar{U}(S, R) = W(R).$$

As a consequence, if  $(S_1, R_1)$  and  $(S_2, R_2)$  are two Bayesian equilibria, we have:

$$\bar{U}(S_1, R_1) = \bar{U}(S_2, R_2) = \text{Min}_{R \in \mathcal{R}} W(R). \quad \blacksquare$$

Proof of proposition 4.2: Assumptions 2 and 3 imply that  $U$  is convex w.r.t.  $R$  and thus that  $W$  is also convex, as a supremum of convex mappings. Let  $(S_1, R_1)$  and  $(S_2, R_2)$  be two Bayesian equilibria. By proposition 1, we know that:  $W(R_1) = W(R_2) = \text{Min}_{R \in \mathcal{R}} W(R)$ . By convexity of  $W$  this is also equal to  $W(\frac{1}{2}R_1 + \frac{1}{2}R_2)$ . Thus, for almost every  $\omega$ , we have:

$$\text{Max}_s U(\omega, s, (\frac{1}{2}R_1 + \frac{1}{2}R_2)(s)) = \text{Max}_s U(\omega, s, R_1(s)) = \text{Max}_s U(\omega, s, R_2(s)), \text{ a.s. .}$$

Using again the convexity of  $U$  w.r.t.  $r$ , we have:

$$\text{Max}_s \{U(\omega, s, R_1(s)) + U(\omega, s, R_2(s))\} = \text{Max}_s U(\omega, s, R_1(s)) + \text{Max}_s U(\omega, s, R_2(s)), \text{ a.s. .}$$

This is only possible if the maximum is attained for the same  $s$ :

$$S_1(\omega) = S_2(\omega), \quad \text{for a.e. } \omega.$$

Thus: 
$$\int \phi(\omega, S_1(\omega)) d\nu(\omega) = \int \phi(\omega, S_2(\omega)) d\nu(\omega).$$

Using again assumption 2 and proposition 4.1, we conclude:  $\bar{V}(S_1, R_1) = \bar{V}(S_2, R_2)$  ■

Proof of proposition 4.3: Let  $R$  be any element of  $\mathcal{R}$  and define, for all  $t > 0$  and  $\omega$  in  $\Omega$ :  $a(t, \omega) = \text{Max}_{s \in \Sigma} U(\omega, s, (1-t)\bar{R}(s) + tR(s))$ . By a standard result of convex analysis<sup>17</sup>, assumptions 3 and 4 imply that

$$\lim_{t \rightarrow 0^+} \frac{a(t, \omega) - a(0, \omega)}{t} = \sup_{s \in S_0(\omega)} \left\{ \frac{\partial U}{\partial r}(\omega, s, \bar{R}(s)) (R(s) - \bar{R}(s)) \right\}$$

where

$$S_0(\omega) = \text{ArgMax}_{s \in \Sigma} U(\omega, s, \bar{R}(s)).$$

Since  $\nu(\Omega_0) = 0$  and  $\bar{S}$  is supported by  $\bar{R}$  we have for almost every  $\omega$ :

$$\lim_{t \rightarrow 0^+} \frac{a(t, \omega) - a(0, \omega)}{t} = \frac{\partial U}{\partial r}(\omega, \bar{S}(\omega), \bar{R}(\bar{S}(\omega))) (R(\bar{S}(\omega)) - \bar{R}(\bar{S}(\omega)))$$

Since  $U$  is convex w.r.t.  $r$ ,  $t \rightarrow \frac{1}{t}(a(t, \omega) - a(0, \omega))$  is non-decreasing, and we can apply Lebesgue's monotone convergence result:

$$\lim_{t \rightarrow 0^+} \int \frac{a(t, \omega) - a(0, \omega)}{t} d\nu(\omega) = \int \frac{\partial U}{\partial r}(\omega, \bar{S}(\omega), \bar{R}(\bar{S}(\omega))) (R(\bar{S}(\omega)) - \bar{R}(\bar{S}(\omega))) d\nu(\omega).$$

Moreover,  $\int a(t, \omega) d\nu(\omega) = W((1-t)\bar{R}(s) + tR(s))$ , so that

$$\int \frac{a(t, \omega) - a(0, \omega)}{t} d\nu(\omega) = \frac{1}{t} [W((1-t)\bar{R}(s) + tR(s)) - W(\bar{R}(s))],$$

$\bar{R}$  being a minimum of  $W$ , this quantity is non-negative, thus we have proved:

$$\forall R \in \mathcal{R}, \quad 0 \leq \int \frac{\partial U}{\partial r}(\omega, \bar{S}(\omega), \bar{R}(\bar{S}(\omega))) [R(\bar{S}(\omega)) - \bar{R}(\bar{S}(\omega))] d\nu(\omega).$$

By assumption A2, this implies:

$$\forall R \in \mathcal{R}, \quad 0 \geq \int \frac{\partial V}{\partial r}(\omega, \bar{S}(\omega), \bar{R}(\bar{S}(\omega))) [R(\bar{S}(\omega)) - \bar{R}(\bar{S}(\omega))] d\nu(\omega).$$

$V$  being concave w.r.t.  $r$  we also have:

$$\frac{\partial V}{\partial r}(\omega, \bar{S}(\omega), \bar{R}(\bar{S}(\omega))) [R(\bar{S}(\omega)) - \bar{R}(\bar{S}(\omega))] \geq V(\omega, \bar{S}(\omega), R(\bar{S}(\omega))) - V(\omega, \bar{S}(\omega), \bar{R}(\bar{S}(\omega))).$$

By integrating over  $\omega$  we deduce:  $0 \geq V(\bar{S}, R) - V(\bar{S}, \bar{R})$

which implies that  $(\bar{S}, \bar{R})$  is indeed a Nash Equilibrium of the game  $(U, V)$ . ■

<sup>17</sup> Aubin [1979], proposition 6 pp. 118-119.

## APPENDIX C

Proof of proposition 5.2: The proof of proposition 5.2 combines arguments from the proofs of propositions 3.1 and 4.3. Recall the definition of the functional  $W$ :  $W(P) = E [\text{Max}_{p \in P(y)} (v-p)(y-u)]$ . Because of assumption 1, lemma 3.1 implies that the maximum above is attained almost everywhere at a single  $y(u,v)$ . Thus  $W$ , which is convex as a supremum of a linear functional, is also Gateaux-differentiable<sup>18</sup>. For any  $H$ , element of  $C$ , one has:

$$\lim_{t \rightarrow 0} \frac{W(\bar{P} + tH) - W(\bar{P})}{t} = \int H(\bar{Y}(u,v) - u) dv(u,v).$$

Since  $\bar{P}$  is a minimum of  $W$ , this quantity has to be zero for all  $H$ . This is equivalent to:

$$E[u|\bar{Y}] = \bar{Y}. \quad (C.1)$$

If  $\bar{P}$  is differentiable at  $\bar{Y}(u,v)$  for a.e.  $(u,v)$ , we can write the first order condition to the insider maximization problem:

$$\text{for a.e. } (u,v), \quad v - \bar{P}(\bar{Y}(u,v)) - \bar{P}'(\bar{Y}(u,v))(\bar{Y}(u,v) - u) = 0.$$

By taking expectations conditional on  $\bar{Y}$ , we get:

$$E[v|\bar{Y}] - \bar{P} = \bar{P}'(\bar{Y})(\bar{Y} - E(u|\bar{Y}))$$

which is zero by condition (C.1).

It now remains to be seen that  $\bar{P}$  is differentiable. The proof follows the reasoning in appendix A (proof of proposition 3.1). First, we recall that

- i)  $\pi(P,u,v) > 0$  a.s.
- ii)  $E(u|Z(u,v)) = y(u,v)$  a.s. (C.1)

It follows from i) that  $u \neq y$  a.s.. Using ii), we get that every buyer must pool with a seller and differentiability follows (see proof of proposition 3.1 step 4). ■

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<sup>18</sup>Aubin (1979) p. 111.

## APPENDIX D

In this appendix, we show that, if the insider does not observe the noise trading, then multiple Bayesian equilibria can exist. For this purpose, we consider the following case:

(i) The distribution of  $u$  is discrete and take the values  $+1$  and  $-1$  with equal probability  $\frac{1}{2}$ .

(ii) The value  $v$  takes the values  $+2$ ,  $+1$ ,  $-1$  and  $-2$  with probability  $\frac{1}{4}$ .

(iii)  $u$  and  $v$  are independent.

With these assumptions, we can construct equilibria  $(x(v), p(y))$  with the following features:

(i)  $x(+2) = 1+a$ ,  $x(+1) = 1-a$ ,  $x(-v) = -x(+v)$  for every  $v$ ,  $0 < a < 1$ .

(ii)  $p(-y) = -p(+y)$  for every  $y$ .

In this equilibrium, the  $(u=-1; v=+2)$  and  $(u=+1; v=-1)$  (respectively  $(u=+1; v=-2)$  and  $(u=-1; v=+1)$ ) are pooling in the sense that  $x(+2)-1 = x(-1)+1$  (respectively  $x(+1)-1 = x(-2)+1$ ). It follows that in this equilibrium, the following equalities must hold:

$$p(2+a)=2; p(2-a)=1; p(a)=\frac{1}{2}; p(-a)=-\frac{1}{2}; p(a-2)=-1 \text{ and } p(-a-2)=-2.$$

We next define the function:  $\phi(x) = \frac{1}{2}p(x+1) + \frac{1}{2}p(x-1)$ .  $\phi(x)$  is the average price paid for  $x$ . It is easy to check that:

$$\phi(1+a)=\frac{5}{4}; \phi(1-a)=\frac{1}{4}; \phi(a-1)=-\frac{1}{4}; \phi(-1-a)=-\frac{5}{4}.$$

Next, we verify the incentive compatibility constraints. The only one that may cause problem is the constraint that  $v=+2$  ( $v=-2$ ) must not imitate  $v=+1$  ( $v=-1$ ), i.e.

$$(2-\phi(1+a))(1+a) \geq (2-\phi(1-a))(1-a) \text{ i.e.}$$

$$a \geq \frac{2}{5}.$$

The final step is to define  $p(y)$  for every value of  $y$ . This can be done for

$l > a \geq \frac{1}{5}$  by following the construction in Kyle and Vila (1991) (details left to the reader.)

An interesting question remains, namely the multiplicity of equilibria for the case where  $u$  and  $v$  are normal as in Kyle (1985). At this point this question is, as far as we know, still open. ■

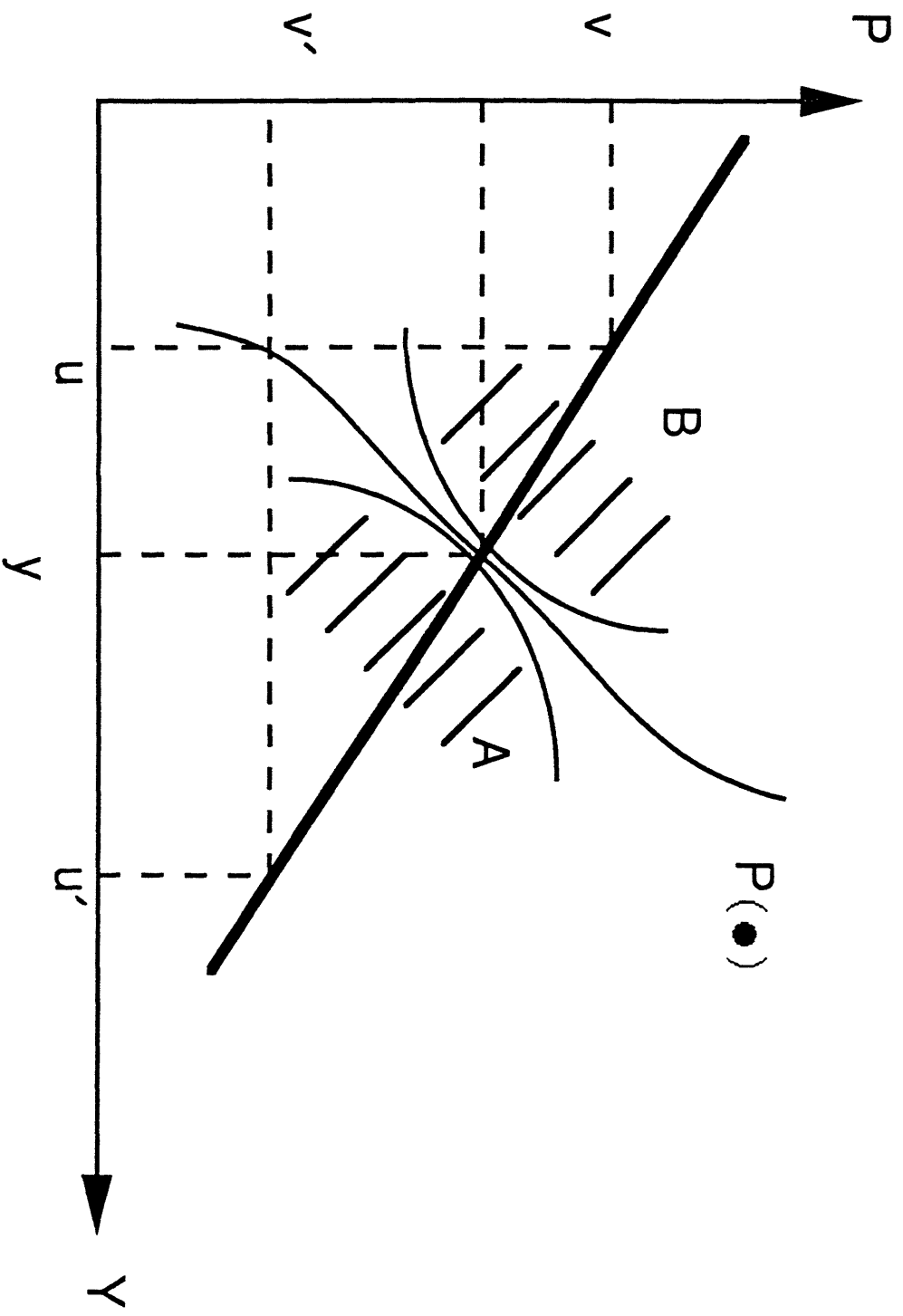


Figure 1