NEWTON'S METHOD FOR THE GENERAL PARAMETRIC CENTER PROBLEM WITH APPLICATIONS

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Abstract

The general parametric center problem is to trace the (analytic) center of a linear inequality system $Ax \leq b$ as the data $(A,\ b)$ of the system is parametrically deformed. We propose an algorithm, which is based on Newton's method, for generating a piecewise-linear path of approximate centers as the deformation parameter varies over a prespecified range. We then apply this algorithm and methodology to four mathematical programming problems. Our algorithm when applied to the generalized linear fractional programming problem (GLFP) requires $O((m+k)k)$ iterations to achieve a fixed improvement in the objective functional value, where m is the total number of constraints, and k is the number of linear fractional functionals in the objective function. When applied to the linear programming problem, our algorithm specializes to Renegar's path-following algorithm.

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1 Introduction

The *(analytic) center* of a system of m linear inequalities in $\mathbf{R}^n$ of the form $Ax \leq b$, assuming the polyhedral set $\{x \in \mathbf{R}^n | Ax < b\}$ is nonempty and bounded, is the unique optimal solution $\hat{x}$ of the nonlinear program, which we call the *center problem*,

$$\begin{array}{lll} CP: & \max_{x,s} & \sum_{i=1}^{m} \ln s_i \\ & s.\ t. & Ax + s = b \\ & & s > 0. \end{array}$$

(See Sonnevend [23,24].)

Since the seminal work of Karmarkar [13], it has been shown by various researchers that the concept of centers plays an important role in the development of efficient interior-point algorithms for linear and convex quadratic programming problems. (See Anstreicher [2], Bayer and Lagarias [3,4], Freund [5], Jarre [12], Mehrotra and Sun [17], Renegar [22], Sonnevend [23,24] and Vaidya [27] among many others.) For example, in Karmarkar's projective scaling algorithm, at each iteration, the problem is transformed so that the current iterate is mapped onto the center of the transformed feasible set before a projected gradient step is taken to reduce a logarithmic potential function. The potential function was used intelligently by Karmarkar to monitor the progress of the algorithm.

In this paper, we are primarily interested in *tracing* the path of centers as the system of linear inequalities is parametrically deformed. Specifically, let $\hat{x}_\alpha$ denote the center of the system $(A + \alpha B)x \leq b + \alpha d$, where A and B are $m \times n$ matrices, b and d are m-vectors, and α is a scalar parameter. That is, $\hat{x}_\alpha$ is the (optimal) solution to the nonlinear program

$$\begin{array}{lll} CP(\alpha): & \max_{x,s} & \sum_{i=1}^{m} \ln s_i \\ & s.\ t. & (A + \alpha B)x + s = b + \alpha d \\ & & s > 0. \end{array}$$

As the parameter α increases (or decreases), we are interested in generating a piece-wise linear path of approximate centers $\bar{x}_\alpha$ such that $\bar{x}_\alpha$ is close to $\hat{x}_\alpha$ for all values of α over a specified range. (In the appendix, we present three closely related measures of closeness to the center.) We call this the *General Parametric Center Problem*, as opposed to the Right-hand-side (RHS) Parametric Center Problem considered in [8].

Suppose there are k nonzero rows in the matrix B and $l \geq 1$ nonzero rows in the matrix $[B,\ d]$. That is, we allow l linear inequalities to vary with the parameter, $(l-k)$ of which involve varying only the right-hand-side. We propose an algorithm, which is based on Newton's method, for generating a piecewise-linear path of approximate solutions $\bar{x}_\alpha$ to $CP(\alpha)$ as the parameter α is increased strictly monotonically over a prespecified range and analyze its algorithmic performance. To achieve a fixed increase in the parametric value, our algorithm requires $O(m(\sqrt{l}+k))$ iterations, where each iteration involves the solution of an $n \times n$ system of linear equations. We then apply the same methodology to four mathematical programming problems; namely, the linear programming problem (LP), the linear fractional programming problem (LFP), the von Neumann model of economic expansion (EEP) and the generalized linear fractional programming problem (GLFP).

Notation

For any vector $s \in \mathbf{R}^k$, we let the corresponding upper-case letter S denote the $k \times k$ diagonal matrix with i^{th} diagonal entry equals to s_i. We write $S := diag\ (s)$. If Q is a positive definite matrix, the *Q-norm* $||v||_Q$ is given by

$$||v||_Q = \sqrt{v^T Q v}.$$

The usual l_1-, l_2- and l_∞-norms will be denoted by $||\cdot||_1$, $||\cdot||$ and $||\cdot||_\infty$ respectively. Given a matrix M, we let M_i denote the i^{th} row of M and M_i^T denote the transpose of M_i. The usual (Euclidean) matrix norm of M is given by

$$||M|| = \sup_{||x||=1} ||Mx||.$$

Note that if M is a diagonal matrix, then $||M|| = \max_i\{|m_{ii}|\}$. Similarly, the Q-norm of M is given by

$$||M||_Q = \sup_{||x||=1} ||Mx||_Q.$$

The vector of all ones (of appropriate dimension) shall be denoted by e, that is, $e := (1, 1, \ldots, 1)^T$.

We shall let $\hat{x}$ denote the center of the system $Ax \leq b$ and let $\hat{x}_\alpha$ denote the center of the system $(A + \alpha B)x \leq b + \alpha d$. With respect to the system $(A + \alpha B)x \leq b + \alpha d$, for x satisfying

$$s_\alpha := (b + \alpha d) - (A + \alpha B)x > 0,$$

we let $Q_\alpha(x)$ denote the negative of the Hessian of the logarithmic barrier function

$$f_\alpha(x) = \sum_{i=1}^{m} \ln[(b + \alpha d) - (A + \alpha B)x]_i$$

for the center problem $CP(\alpha)$ on the system $(A + \alpha B)x \leq b + \alpha d$. We note that the Karush-Kuhn-Tucker conditions for $CP(\alpha)$, which characterize the center $\hat{x}_\alpha$ of the system $(A + \alpha B)x \leq b + \alpha d$, are

$$\hat{s}_\alpha = b + \alpha d - (A + \alpha B)\hat{x}_\alpha > 0, \tag{1}$$

$$(A + \alpha B)^T \hat{S}_\alpha^{-1} e = 0. \tag{2}$$

Given an interior point $\bar{x}$ satisfying $\bar{s} = b - A\bar{x} > 0$, we let

$$\bar{\alpha} = \alpha_{\bar{x}} := 1/||\bar{S}^{-1}(B\bar{x} - d)||_\infty \tag{3}$$

and, for each $\alpha \in [0,\ \bar{\alpha})$, we let

$$\bar{s}_\alpha := (b + \alpha d) - (A + \alpha B)\bar{x} > 0 \tag{4}$$

and

$$Q_\alpha(\bar{x}) := (A + \alpha B)^T \bar{S}_\alpha^{-2} (A + \alpha B). \tag{5}$$

Assumptions

We shall make the following assumptions in this paper. Let $\mathcal{X} = \{x \in \mathbf{R}^n | Ax \leq b\}$ and $\mathcal{X}^+ = \{x \in \mathbf{R}^n | Ax < b\}$.

Assumption 1. The set $\mathcal{X}^+$ is nonempty and bounded.

Assumption 2. For every $x \in \mathcal{X}$, we have $Bx \geq d$ and $Bx \neq d$.

Assumption 1 ensures that the center of the system $Ax \leq b$ exists uniquely, and Assumption 2 ensures that the polyhedral set

$$\mathcal{X}_\alpha := \{x \in \mathcal{X} |\ (A + \alpha B)x \leq b + \alpha d\}, \tag{6}$$

is shrinking for increasing values of $\alpha \geq 0$, i.e. $\alpha_1 > \alpha_2$ implies that $\mathcal{X}_{\alpha_1}$ is properly contained in $\mathcal{X}_{\alpha_2}$. Therefore, there exists a maximal α such that the system $(A + \alpha B)x \leq b + \alpha d$ is feasible. Let α^* denote the maximal α such that the system $(A+\alpha B)x \leq b+\alpha d$ is feasible. Then under these Assumptions, it is easy to see that $0 < \alpha^* < \infty$, and for all $\alpha \in [0,\ \alpha^*)$, the interior of the set $\mathcal{X}_\alpha$ is nonempty and bounded. Finally, note that the Assumptions imply that the number l of nonzero rows in the matrix $[B,\ d]$ is at least one. [Of course, if $l = 0$ then the system does not change when α is varied, and the general parametric center problem is only as difficult as finding the center of the system $Ax \leq b$.]

Remark: The general parametric center problem is closely related to the GLFP problem. It is easy to see that, under the above Assumptions, α^* equals the maximal value of the following GLFP program.

$$\begin{aligned} \alpha^* := \max_x \min_i \quad & \left\{ \frac{b_i - A_i x}{B_i x - d_i} \right\} \\ s.\ t. \quad & Ax \leq b. \end{aligned} \tag{7}$$

Note that we may express $\bar{\alpha}$ as

$$\bar{\alpha} = \alpha_{\bar{x}} = \frac{1}{||\bar{S}^{-1}(B\bar{x} - d)||_\infty} = \min_i \left\{ \frac{b_i - A_i \bar{x}}{B_i \bar{x} - d_i} \right\}.$$

Therefore, $\bar{\alpha}$ corresponds to the value of the program (7) evaluated at $\bar{x}$. Also, $\bar{\alpha}$ is the value of α such that the boundary of the polytope $\mathcal{X}_\alpha$ just touches the point $\bar{x}$, so that $\bar{x} \in \mathcal{X}_\alpha^+ := \{x |\ (A + \alpha B)x < b + \alpha d\}$ for all $\alpha \in [0,\ \bar{\alpha})$.

The organization of the rest of this paper is as follows. In Section 2, we state the main results of this paper and present our algorithm for the general parametric center problem.

[That is, $\bar{x}_{new}$ is a Newton iterate from $\bar{x}$ in center problem $CP(\beta)$.] Then $\bar{x}_{new}$ is again a δ-approximate center of the new system $(A+\beta B)x \leq b+\beta d$.

Hence, with β given by (8), we may repeat the procedure with $\bar{x}_{new}$ replacing $\bar{x}$, $A+\beta B$ replacing A and $b+\beta d$ replacing b. Note that the increase in α is a fraction of $\bar{\alpha}$, the value of program (7) at $\bar{x}$. It makes good sense that the fraction depends on, and is inversely proportional to, the total number of varying linear inequalities through the quantities k and l.

Next, we have the following result which allows us to extend and generate a piecewise linear path of approximate centers. In the following theorem, we increase α from $\alpha = 0$ to $\alpha = \beta$, take a step from $\bar{x}$ to $\bar{x}_{new} = \bar{x} + \eta$, where η is the Newton step from $\bar{x}$ for the center problem $CP(\beta)$ and then extend the path of approximate centers by linearly interpolating between $\bar{x}$ and $\bar{x}_{new}$.

Theorem 2.2 (Path Extension Theorem) .

Under the same conditions and definitions as Theorem 2.1, define $\bar{x}_\alpha$, for all $\alpha \in [0,\ \beta]$, by

$$\bar{x}_\alpha := \bar{x} + (\frac{\alpha}{\beta})(\bar{x}_{new} - \bar{x}).$$

Then

$$||\bar{x}_\alpha - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} \leq 0.38\ .$$

Remark: Note that by Lemma A.3 in the Appendix, Theorem 2.2 implies that $\bar{x}_\alpha$ is a δ-approximate center of system $(A+\alpha B)x \leq b+\alpha d$ with $\delta = 0.62$ for all $\alpha \in [0,\ \beta]$. The proof of Theorems 2.1 and 2.2 are given in Section 4.

2.1 A Parametric Center Algorithm

Based on Theorems 2.1 and 2.2, we propose the following Newton method-based algorithm for tracing the path of approximate centers of systems $(A+\alpha B)x \leq b+\alpha d$ as α varies over a given range $\alpha \in [0,\ \alpha^{upper}]$. The input of the algorithm include the $m \times n$ matrices A and B, where we know that k of the rows of B are nonzero, the m-vectors b and d, such that there are l nonzero rows in the matrix $[B,\ d]$, a scalar $\alpha^{upper} > 0$, and a δ-approximate center with $\delta = 1/21$. Such an approximate center can be obtained by using an algorithm for the center problem (see Vaidya [26] or Freund [5]). The output includes a sequence of breakpoints $\{(\bar{x}^j,\ \alpha^j)\}$, $\bar{x}^j = \bar{x}_{\alpha^j}$, and a piecewise linear path of δ-approximate centers $\bar{x}_\alpha$ with $\delta = 0.62$ for $\alpha \in [0,\ \alpha^{upper}]$. We note that, of course, the algorithm may not terminate if $\alpha^{upper} > \alpha^*$.

Algorithm PCP

INPUT: $A,\ B,\ b,\ d,\ \alpha^{upper},\ k,\ l,\ x^0$ (with $||x^0 - \hat{x}||_{Q_0(x^0)} \leq 1/21$).

INITIALIZATION:

Set $j = 0,\ \ \alpha^0 = 0,\ \ \bar{x} = x^0,\ \ \bar{A} = A$ and $\bar{b} = b$.

ITERATION: **Repeat** the following steps **until** $\alpha^j \geq \alpha^{upper}$.

Step 1. Set $\bar{s} = \bar{b} - \bar{A}\bar{x}$, and

$$\bar{\alpha} = \frac{1}{||\bar{S}^{-1}(B\bar{x} - d)||_\infty}, \qquad \beta = \frac{\bar{\alpha}}{88(\sqrt{l} + k)}.$$

Step 2. Compute the Newton step $\bar{\eta}$ from $\bar{x}$ in problem $CP(\beta)$:

$$\bar{s}_\beta = (\bar{b} + \beta d) - (\bar{A} + \beta B)\bar{x};$$

$$Q_\beta(\bar{x}) = (\bar{A} + \beta B)^T \bar{S}_\beta^{-2}(\bar{A} + \beta B);$$

$$\bar{\eta} = Q_\beta^{-1}(\bar{x})(\bar{A} + \beta B)^T \bar{S}_\beta^{-1} e.$$

Step 3. Set $\bar{x}_{new} = \bar{x} + \bar{\eta},\ \ \alpha^{j+1} = \alpha^j + \beta,$ and **for all** $\alpha \in [\alpha^j,\ \alpha^{j+1}]$, **define**

$$\bar{x}_\alpha = \bar{x} + \left(\frac{\alpha - \alpha^j}{\beta}\right)(\bar{x}_{new} - \bar{x}).$$

Step 4. Update $j \leftarrow j+1$, $\bar{x} \leftarrow \bar{x}_{new}$, $\bar{A} \leftarrow \bar{A} + \beta B$, $\bar{b} \leftarrow \bar{b} + \beta d$, and go to Step 1.

2.2 Complexity Analysis of the Algorithm

Next, we shall analyze the complexity of Algorithm PCP. We shall show that Algorithm PCP requires $O(m(\sqrt{l}+k))$ iterations to achieve a fixed increase in the parameter value. We shall first show that the increase (β given by (8)) in the parameter value at each iteration of Algorithm PCP is a fraction of $O(\frac{1}{m(\sqrt{l}+k)})$ of the maximal value α^* of α such that the system $(A+\alpha B)x \leq b+\alpha d$ is feasible. Since the increase in the parameter value is a fraction of $\bar{\alpha}$, this will follow if we can obtain an upper bound on α^* in terms of $\bar{\alpha}$ and m.

Now, from a property of the center of the system $Ax \leq b$ (Corollary A.1 in the Appendix) and the Assumptions given in Section 1, we obtain an upper bound on α^* in terms of $\bar{\alpha}$ and m as follow. Let u and v be the following constants.

$$u = \max_i \max_x \{B_i x - d_i \mid Ax \leq b\} \tag{11}$$

$$v = \min_x \{\max_i B_i x - d_i \mid Ax \leq b\} \tag{12}$$

[Note that u can be obtained by solving k linear programs, and v can be obtained by solving one linear program.] Since $\{x \mid Ax \leq b\}$ is compact under Assumption 1, we have

$$\max_x \{B_i x - d_i \mid Ax \leq b\} < \infty$$

for each $i = 1, 2, \ldots, m$. Thus, $u < \infty$. By Assumption 2, for each x satisfying $Ax \leq b$, there exists an i such that $B_i x > d_i$, therefore

$$\max_i \{B_i x - d_i\} > 0.$$

Since $\{x \mid Ax \leq b\}$ is compact under Assumption 1, we have $v > 0$. It is easy to see that for all x satisfying $Ax \leq b$, we have $B_i x - d_i \leq u$ for all $i = 1, 2, \ldots, m$ and

$$0 < v \leq \max_i \{B_i x - d_i\} \leq u < \infty.$$

Now, define the constant c_0 by

$$c_0 := \frac{u}{v}. \tag{13}$$

Then we see that $0 < c_0 < \infty$ and we have the following.

Theorem 2.3 *Suppose $\hat{x} = \hat{x}_0$ is the center of the system $Ax \leq b$. Let $\hat{\alpha} = \alpha_{\hat{x}}$ be given by (3). Then $\alpha^* \leq c_0 m \hat{\alpha}$, where c_0 is the constant given by (11)-(13).*

Proof: For $i \in \{1, 2, \ldots, m\}$ and $x \in \{x | \, Ax \leq b\}$, we have

$$0 \leq b_i - A_i x \leq m\hat{s}_i,$$

where $\hat{s} = b - A\hat{x}$, from Corollary A.1. Therefore,

$$\frac{b_i - A_i x}{B_i x - d_i} \leq \left(\frac{m\hat{s}_i}{B_i\hat{x} - d_i}\right)\left(\frac{B_i\hat{x} - d_i}{B_i x - d_i}\right).$$

Hence,

$$\begin{aligned}
\alpha^* &= \max_x \left\{ \min_i \frac{b_i - A_i x}{B_i x - d_i} \mid Ax \leq b, \right\} \\
&\leq \max_x \left\{ \min_i \left(\frac{m\hat{s}_i}{B_i\hat{x} - d_i}\right)\left(\frac{B_i\hat{x} - d_i}{B_i x - d_i}\right) \mid Ax \leq b \right\} \\
&\leq \frac{m(u/v)}{||\hat{S}^{-1}(B\hat{x} - d)||_\infty} \\
&= c_0 m \hat{\alpha}.
\end{aligned}$$

Q.E.D.

Now, taking $\delta = 1/21$, let c_1 be the following constant.

$$c_1 := (1.1)(\frac{u}{v}). \tag{14}$$

Note that $c_1 = (1.1)c_0$, where c_0 is the constant defined by (13). Also note that if L denotes that number of bits in a binary encoding of the given data $(A,\ B,\ b,\ d)$, then $c_0 \leq 2^{2L}$. From Theorem 2.3, we have the following.

Corollary 2.1 *Suppose $\bar{x}$ is a δ-approximate center of the system $Ax \leq b$ with $\delta = 1/21$. Let $\bar{\alpha}$ be given by (3). Then $\alpha^* \leq c_1 m \bar{\alpha}$, where c_1 is the constant defined by (11)-(14).*

Proof: In Lemma 4.9 of the next section, we show that $\hat{\alpha}$ of Theorem 2.3 satisfies

$$\hat{\alpha} \leq \left(\frac{1+\delta}{1-\delta}\right)\bar{\alpha} = \left(\frac{1+\frac{1}{21}}{1-\frac{1}{21}}\right)\bar{\alpha} = (1.1)\bar{\alpha}.$$

The proof then follows from Theorem 2.3. *Q.E.D.*

The complexity of Algorithm PCP is thus given in the following.

Theorem 2.4 *Suppose α^j is the value of the parameter α at the start of the j^{th} iteration of Algorithm PCP. Then, for any given $\varepsilon > 0$, we have $\alpha^* - \alpha^j \leq \varepsilon$ after at most $K = \lceil 88c_1 m(\sqrt{l}+k)(\ln(\alpha^* - \alpha^0) - \ln\varepsilon)\rceil$ iterations, where c_1 is the constant defined by (11)-(14).*

Proof: From Theorem 2.1, we have

$$\alpha^{j+1} - \alpha^j = \frac{\bar{\alpha}}{88(\sqrt{l}+k)},$$

and from Corollary 2.1,

$$\alpha^* - \alpha^j \leq c_1 m\bar{\alpha}.$$

Therefore,

$$\begin{aligned}
\alpha^{j+1} - \alpha^j &\geq \frac{1}{88c_1 m(\sqrt{l}+k)}(\alpha^* - \alpha^j),\\
\alpha^* - \alpha^{j+1} &\leq \left(1 - \frac{1}{88c_1 m(\sqrt{l}+k)}\right)(\alpha^* - \alpha^j),\\
\alpha^* - \alpha^j &\leq \left(1 - \frac{1}{88c_1 m(\sqrt{l}+k)}\right)^j(\alpha^* - \alpha^0).
\end{aligned}$$

Hence for $j \geq K$,

$$\begin{aligned}
\ln(\alpha^* - \alpha^j) &\leq j\ln\left(1 - \frac{1}{88c_1 m(\sqrt{l}+k)}\right) + \ln(\alpha^* - \alpha^0)\\
&\leq \frac{-K}{88c_1 m(\sqrt{l}+k)} + \ln(\alpha^* - \alpha^0)\\
&\leq -\ln(\frac{\alpha^* - \alpha^0}{\varepsilon}) + \ln(\alpha^* - \alpha^0)\\
&= \ln\varepsilon,
\end{aligned}$$

whereby $\alpha^* - \alpha^j \leq \varepsilon$. *Q.E.D.*

3 Applications

In this section, we show how Algorithm PCP may be applied to four mathematical programming problems; namely, the linear programming problem (LP), the linear fractional programming problem (LFP), the von Neumann model of economic expansion (EEP), and the generalized linear fractional programming problem (GLFP).

3.1 Linear Programming

Suppose we are interested in solving a linear program in the following format:

$$\widetilde{LP}: \qquad z^* = \max\ c^T x \\ \qquad\qquad s.t.\ \tilde{A}x \leq \tilde{b},$$

where $x \in \mathbf{R}^n$ and $\tilde{A}$ is an $m \times n$ matrix. Non-negativity constraints and lower and upper bounds are not distinguished from other inequalities. We assume that $c \neq 0$, for otherwise any feasible solution will be optimal.

Then it is easy to see that $\widetilde{LP}$ is equivalent to the following program.

$$LP: \quad z^* = \max_{x,\,\alpha}\ \alpha \\ \qquad s.t.\ Ax \leq b + \alpha d,$$

where A, b and d are the following $2m \times n$ matrix and $2m$-vectors

$$A = \begin{bmatrix} \tilde{A} \\ -c^T \\ \vdots \\ -c^T \end{bmatrix}, \qquad b = \begin{bmatrix} \tilde{b} \\ -v \\ \vdots \\ -v \end{bmatrix}, \qquad d = \begin{bmatrix} 0 \\ -1 \\ \vdots \\ -1 \end{bmatrix}, \tag{15}$$

for some lower bound $v < z^*$ such that the set $\{x|Ax < b\}$ is nonempty and bounded. It can be shown that any linear program can be reformulated such that these assumptions are satisfied (see Renegar [22], for example). We may then assume that we have a δ-approximate center $\bar{x}$ of $Ax \leq b$ with $\delta = 1/21$, perhaps after applying a centering algorithm of Vaidya [26] or Freund [5].

We may apply Algorithm PCP to solve LP in the following way. Note that LP corresponds to a parametric center problem where the total number of linear inequalities is $2m$, the number of varying linear inequalities $l = m$, and the matrix $B = 0$ (therefore $k = 0$ in Theorem 2.1 for this case). Suppose at the start of iteration j the value of α is α^j and we have a δ-approximate center $\bar{x}^j$ of the system $Ax \leq b + \alpha^j d$ with $\delta = 1/21$. (Recall that $\bar{x}$ is a δ-approximate center of a system $Ax \leq b$ if $||\bar{x} - \hat{x}||_{Q_0(\bar{x})} \leq \delta$.) We observe that for this case, $\bar{\alpha}$ according to (3) is $\bar{\alpha} = c^T\bar{x}^j - \alpha^j - v$. We therefore set $\beta = \frac{c^T\bar{x}^j - \alpha^j - v}{88\sqrt{m}}$, according to Theorem 2.1. Then we change the right-hand-side to $(b + \alpha^j d) + \beta d$. Then we take a Newton iterate from $\bar{x}^j$ in the problem to find the center of system $Ax \leq b + \alpha^{j+1}d$, where $\alpha^{j+1} = \alpha^j + \beta$. Let $\bar{x}^{j+1}$ be the Newton iterate. Then according to Theorem 2.1, $\bar{x}^{j+1}$ is again a δ-approximate center of the system $Ax \leq b + \alpha^{j+1}d$, and we enter iteration $(j+1)$ and repeat the procedure.

Complexity Analysis

Note that

$$\alpha^{j+1} - \alpha^j = \beta = \frac{1}{88\sqrt{m}}(c^T\bar{x}^j - \alpha^j - v). \tag{16}$$

On the other hand, using a property of the center (Lemma A.2 in the Appendix), we have the following upper bounds.

Lemma 3.1 *Suppose $\hat{x}$ is the center of $Ax \leq b$, where A, b are given by (15). For any x satisfying $Ax \leq b$, we have*

$$0 \leq c^Tx - v \leq 2(c^T\hat{x} - v).$$

Proof: Let $\hat{s} = b - A\hat{x}$ and $s = b - Ax$. Then from Lemma A.2, we have

$$2m = e^T\hat{S}^{-1}s = m\left(\frac{c^Tx - v}{c^T\hat{x} - v}\right) + \sum_{i=1}^{m}\frac{\tilde{b}_i - \tilde{A}_ix}{\tilde{b}_i - \tilde{A}_i\hat{x}} \geq m\left(\frac{c^Tx - v}{c^T\hat{x} - v}\right).$$

The Lemma follows immediately. *Q.E.D.*

Corollary 3.1 *Suppose $\bar{x}$ is a δ-approximate center of $Ax \le b$, with $\delta = 1/21$. For all x satisfying $Ax \le b$, we have*

$$0 \le c^T x - v \le (2.2)(c^T \bar{x} - v).$$

Proof: Suppose $\hat{x}$ is the center of $Ax \le b$. From Lemma 4.9 in the following section, we see that

$$(0.9)(c^T \bar{x} - v) \le c^T \hat{x} - v \le (1.1)(c^T \bar{x} - v).$$

The Corollary then follows immediately from the previous Lemma. *Q.E.D.*

Therefore (replacing b with $b + \alpha^j d$ and $\bar{x}$ with $\bar{x}^j$ in the above Corollary),

$$0 \le z^* - \alpha^j - v \le (2.2)(c^T \bar{x}^j - \alpha^j - v). \tag{17}$$

Combining (16) and (17), we have

$$\alpha^{j+1} - \alpha^j \ge \frac{1}{194\sqrt{m}}(z^* - \alpha^j - v). \tag{18}$$

Rearranging terms, we get

$$z^* - \alpha^{j+1} - v \le \left(1 - \frac{1}{194\sqrt{m}}\right)(z^* - \alpha^j - v). \tag{19}$$

Hence, at each iteration, the gap $(z^* - \alpha - v)$ decreases by at least a factor of $(1 - \frac{1}{194\sqrt{m}})$. Note that, at each iteration j,

$$0 \le z^* - c^T \bar{x}^j \le z^* - \alpha^j - v.$$

Therefore, we can show as in [22] that, Algorithm PCP can be used to solve LP in $O(\sqrt{m}L)$ iterations. Summarizing the discussion above, we have the following.

Theorem 3.1 *Algorithm PCP, if properly initiated, can be used to solve the LP problem in $O(\sqrt{m}L)$ iterations, where m is the number of constraints and L is the number of bits in a binary encoding of the problem instance.*

Remark: Note that this implementation of Algorithm PCP to solve the LP problem structurally duplicates Renegar's algorithm [22].

3.2 Linear Fractional Programming

Suppose we are interested in solving a linear fractional program in $\mathbf{R}^n$ of the following format

$$LFP: \qquad \alpha^* = \max_x \quad \frac{f - \tilde{c}^T x}{\tilde{d}^T x - h} \qquad s.\,t. \quad \tilde{A}x \le \tilde{b},$$

where $x \in \mathbf{R}^n$, $\tilde{A}$ is an $m \times n$ matrix, $\tilde{b}$ is an m-vector, $\tilde{c}$ and $\tilde{d}$ are n-vectors and f and h are scalars. We assume that

(1) the set $\mathcal{X}_0^+ := \{x | \tilde{A}x < \tilde{b},\ \tilde{c}^T x < f\}$ is nonempty and bounded,

(2) $\tilde{d}^T x - h \ge 0$ for every $x \in \mathcal{X} := \{x | \tilde{A}x \le \tilde{b}\}$, and

(3) $\tilde{d}^T x - h > 0$ for every $x \in \mathcal{X}_0 := \{x | \tilde{A}x \le \tilde{b},\ \tilde{c}^T x \le f\}$.

Note that under these assumptions, $0 < \alpha^* < \infty$, and if $\tilde{d} = 0$ then LFP is just a linear program, so we may assume that $\tilde{d} \ne 0$. (We note that Anstreicher [1] has a similar set of assumptions.)

It is easy to see that LFP is equivalent to the following program in $\mathbf{R}^{n+1}$:

$$\begin{array}{llrcl} \overline{LFP}: & \alpha^* = \max\limits_{x,\alpha} & \alpha & & \\ & s.\,t. & \tilde{A}x & \le & \tilde{b} \\ & & (\tilde{c} + \alpha\tilde{d})^T x & \le & f + \alpha h. \end{array}$$

We may then apply Algorithm PCP to solve $\overline{LFP}$ by tracing the centers of the parametric family of systems $(A + \alpha B)x \le b + \alpha d$, where α is taken as a parameter and is increased strictly monotonically, and

$$A = \begin{bmatrix} \tilde{A} \\ \tilde{c} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \tilde{d} \end{bmatrix}, \qquad b = \begin{bmatrix} \tilde{b} \\ f \end{bmatrix}, \qquad d = \begin{bmatrix} 0 \\ h \end{bmatrix}.$$

Note that in this case the total number of linear inequalities is $(m+1)$, and since $[\tilde{d},\ h] \ne 0$ by our assumptions and the number of nonzero rows in the matrix B is $k = 1$, the number of varying linear inequalities is $l = 1$. We can assume that we have an interior point $\bar{x}^0$ satisfying the starting criterion of Algorithm PCP, that is, $\bar{x}^0$ is a δ-approximate center of the system $Ax \leq b$ with $\delta = 1/21$, perhaps by using an algorithm (of Vaidya [26] or Freund [5]) for finding the center of $Ax \leq b$.

When we apply Algorithm PCP to trace the parametric center of $(A + \alpha B)x \leq b + \alpha d$ as α increases strictly monotonically over the range $\alpha \in [0,\ \alpha^* - \varepsilon]$, we have the following. At the start of iteration j, the value of α is α^j and the current iterate is $\bar{x} = \bar{x}^j$, which is a δ-approximate center of the system $(A + \alpha^j B)x \leq b + \alpha^j d$ with $\delta = 1/21$. We observe that $\bar{\alpha}$, as defined by (3), is

$$\bar{\alpha} = \frac{(f + \alpha^j h) - (\tilde{c} + \alpha^j \tilde{d})^T \bar{x}^j}{\tilde{d}^T \bar{x}^j - h},$$

since the only varying linear inequality is the last one, and in the j^{th} iteration, the last inequality is

$$(\tilde{c} + \alpha^j \tilde{d})^T x \leq f + \alpha^j h.$$

Therefore, β is set, in accordance with Theorem 2.1, to be

$$\beta = \frac{1}{176}\bar{\alpha} = \frac{(f + \alpha^j h) - (\tilde{c} + \alpha^j \tilde{d})^T \bar{x}^j}{176(\tilde{d}^T \bar{x}^j - h)}.$$

Next α^{j+1} is set to be $\alpha^{j+1} = \alpha^j + \beta$, and a Newton iterate from $\bar{x}^j$ is taken (in the program to find the center of the system $(A + \alpha^{j+1} B)x \leq b + \alpha^{j+1} d$). The next iterate $\bar{x}^{j+1}$ is set to be the Newton iterate from $\bar{x}^j$. Then according to Theorem 2.1, $\bar{x}^{j+1}$ is again a δ-approximate center of the system $(A + \alpha^{j+1} B)x \leq b + \alpha^{j+1} d$ with $\delta = 1/21$, and we enter iteration $(j+1)$. Note that

$$\bar{\alpha} = \frac{(f + \alpha^j h) - (\tilde{c} + \alpha^j \tilde{d})^T \bar{x}^j}{\tilde{d}^T \bar{x}^j - h} = \frac{f - \tilde{c}^T \bar{x}^j}{\tilde{d}^T \bar{x}^j - h} - \alpha^j > 0. \tag{20}$$

Therefore, α^j is a strict lower bound on the objective value of LFP at $\bar{x}^j$.

Complexity Analysis

We shall now analyze the complexity of Algorithm PCP when applied to the LFP problem. First, we have

$$\alpha^{j+1} - \alpha^j = \beta = \frac{1}{176}\bar{\alpha}. \tag{21}$$

Next, using a property of the center of a linear inequality system [see Lemma A.2 in the Appendix], we obtain an upper bound on α^* in term of $\bar{\alpha}$ and m as follows. We have [Note that the total number of inequalities in the system $(A+\alpha B)x \leq b+\alpha d$ is $(m+1)$.]

Lemma 3.2 *Suppose $\hat{x}_\alpha$ is the center of the system $(A+\alpha B)x \leq b+\alpha d$. Then, for all $x \in \mathcal{X}_\alpha := \{x \mid (A+\alpha B)x \leq b+\alpha d\}$,*

$$0 \leq (f+\alpha h) - (\tilde{c}+\alpha\tilde{d})^T x \leq (m+1)[(f+\alpha h) - (\tilde{c}+\alpha\tilde{d})^T \hat{x}_\alpha].$$

Next, by Lemma A.3 of the Appendix and the above Lemma, we have

Lemma 3.3 *Suppose $\bar{x}$ is a δ-approximate center of $(A+\alpha B)x \leq b+\alpha d$. Then, for all $x \in \mathcal{X}_\alpha := \{x \mid (A+\alpha B)x \leq b+\alpha d\}$,*

$$0 \leq (f+\alpha h) - (\tilde{c}+\alpha\tilde{d})^T x \leq (1+\delta)(m+1)[(f+\alpha h) - (\tilde{c}+\alpha\tilde{d})^T \bar{x}].$$

Let u' and v' be the following constants.

$$u' := \max \{\tilde{d}^T x - h \mid \tilde{A}x \leq \tilde{b}\} \tag{22}$$

$$v' := \min \{\tilde{d}^T x - h \mid \tilde{A}x \leq \tilde{b},\ \tilde{c}^T x \leq f\} \tag{23}$$

Define the constant c_2 by

$$c_2 := (\frac{22}{21})(\frac{u'}{v'}). \tag{24}$$

Under the assumptions on LFP, for all x satisfying $Ax \leq b$, we have

$$0 < v \leq \tilde{d}^T x - h \leq u < \infty.$$

Therefore, we see that $0 < c_2 < \infty$. Note that if L is the number of bits in a binary encoding of the given data $(\tilde{A},\ \tilde{b},\ \tilde{c},\ \tilde{d},\ f,\ h)$, then $c_2 \leq 2^{2L}$. We have the following.

Lemma 3.4 *Let $\bar{\alpha}$ be given by (20), and let c_2 be the constant defined by (22)-(24). Then* $\alpha^* - \alpha^j \leq c_2(m+1)\bar{\alpha}$.

Proof: Note that $\bar{x} = \bar{x}^j$ is a δ-approximate center of the system $(A + \alpha^j B)x \leq b + \alpha^j d$ with $\delta = 1/21$. For any x satisfying $Ax \leq b$,

$$\begin{aligned}
\frac{f - \tilde{c}^T x}{\tilde{d}^T x - h} - \alpha^j &= \frac{(f + \alpha^j h) - (\tilde{c} + \alpha^j \tilde{d})^T x}{\tilde{d}^T x - h} \\
&\leq (\frac{22}{21})(m+1)\frac{(f + \alpha^j h) - (\tilde{c} + \alpha^j \tilde{d})^T \bar{x}}{\tilde{d}^T x - h} \\
&= (\frac{22}{21})(m+1)\left(\frac{\tilde{d}^T \bar{x} - h}{\tilde{d}^T x - h}\right)\frac{(f + \alpha^j h) - (\tilde{c} + \alpha^j \tilde{d})^T \bar{x}}{\tilde{d}^T \bar{x} - h} \\
&\leq (\frac{22}{21})(m+1)(\frac{u}{v})\frac{(f + \alpha^j h) - (\tilde{c} + \alpha^j \tilde{d})^T \bar{x}}{\tilde{d}^T \bar{x} - h} \\
&= c_2(m+1)\bar{\alpha},
\end{aligned}$$

where the first inequality follows from Lemma 3.3. Also,

$$\begin{aligned}
\alpha^* := \max\left\{\frac{f - \tilde{c}^T x}{\tilde{d}^T x - h} \mid \tilde{A}x \leq \tilde{b}\right\} &= \max\left\{\frac{f - \tilde{c}^T x}{\tilde{d}^T x - h} \mid \tilde{A}x \leq \tilde{b},\ \tilde{c}^T x \leq f\right\} \\
&= \max\left\{\frac{f - \tilde{c}^T x}{\tilde{d}^T x - h} \mid Ax \leq b\right\}
\end{aligned}$$

Hence, $\alpha^* - \alpha^j \leq c_2(m+1)\bar{\alpha}$. *Q.E.D.*

Lemma 3.4 and (21) implies that

$$\alpha^{j+1} - \alpha^j \geq (\frac{1}{176c_2(m+1)})(\alpha^* - \alpha^j). \quad (25)$$

Rearranging terms, we get

$$\alpha^* - \alpha^{j+1} \leq (1 - \frac{1}{176c_2(m+1)})(\alpha^* - \alpha^j). \quad (26)$$

Therefore, the gap $(\alpha^* - \alpha)$ decreases geometrically with a rate of at least $(1 - O(\frac{1}{m+1})) = (1 - O(\frac{1}{m}))$. Hence, we can show that the algorithm requires $O(m)$ iterations to decrease the optimality gap $(\alpha^* - \alpha)$ by a fixed quantity. Summarizing the discussion above, we have the following.

Theorem 3.2 *Suppose Algorithm PCP is applied to solve LFP. Then it will produce a feasible solution* $\bar{x}$ *such that*

$$\alpha^* - \varepsilon < \frac{f - \tilde{c}^T \bar{x}}{\tilde{d}^T \bar{x} - h} \leq \alpha^*$$

after at most $K = \lceil 176 c_2 (m+1)(\ln \alpha^* - \ln \varepsilon) \rceil$ *iterations, where* c_2 *is a constant defined by (22)-(24).*

Proof: The proof, similar to that of Theorem 2.4, follows from (26) and (20). *Q.E.D.*

3.3 von Neumann Model of Economic Expansion

In 1932, the mathematician John von Neumann developed a linear model of an expanding economy which was published in 1937, and an English translation was published in 1945 [20]. The model involves n productive processes $P_1, P_2, \ldots, P_n$ producing m economic goods $G_1, G_2, \ldots, G_m$. At unit intensity of operation, each process P_j will consume an amount $a_{ij} \geq 0$ and produce an amount $b_{ij} \geq 0$ of each good G_i. The non-negative $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are respectively called the *input* and *output matrices* of the model. (See Gale [9].)

Suppose each process P_j is operated at an intensity $x_j \geq 0$, and let the vector $x := (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n_+$ denote the intensity vector. Then the components of the vector Ax give the amounts of inputs used up in production, and the components of the vector Bx give the amounts of outputs produced, during a unit time period. The model with input matrix A and output matrix B is referred to symbolically as (A, B).

Given an intensity vector $x \geq 0$, let α_x be defined by

$$\begin{aligned} \alpha_x &:= \max \{\alpha \mid Bx \geq \alpha Ax\} \\ &= \min_i \left\{ \frac{B_i x}{A_i x} \mid A_i x > 0 \right\}, \end{aligned}$$

where A_i denotes the i-th row of A. Then α_x represents the *expansion factor* of the economy operating at intensity x. Thus the output of each good G_i is at least α_x times as great as its input. The *technological expansion problem (TEP)* for an economic model (A, B) is to find an intensity vector x such that α_x is maximal. We may write this as a nonlinear program:

$$\begin{aligned} \alpha^{max} := \max \quad & \alpha \\ s.\,t. \quad & (B - \alpha A)x \geq 0 \\ & x \geq 0,\ x \neq 0. \end{aligned} \tag{27}$$

(For an interpretation, see Gale [9].)

Observe that problem (27) is homogeneous, so that if $\tilde{x}$ is optimal then also so is any positive multiple of $\tilde{x}$. Therefore, (27) may be expressed equivalently as the following. (Remember that e denotes the vector of all ones of the appropriate dimension.)

$$\begin{aligned} \alpha^{max} := \max \quad & \alpha \\ s.\,t. \quad & (B - \alpha A)x \geq 0 \\ & e^T x = 1,\ x \geq 0. \end{aligned} \tag{28}$$

It is clear that in order for the model to correspond to economic reality, some conditions must be imposed on the input and output coefficients. Therefore, the following conditions are assumed. (See Gale [9], Kemeny et. al. [14].)

Assumption 3.1: For every (good) i, there exists some (process) j such that $b_{ij} > 0$. That is, every good G_i is produced by some process P_j.

Assumption 3.2: For every (process) j, there exists some (good) i such that $a_{ij} > 0$. That is, every productive process P_j consumes some input G_i.

Under these assumptions, the problem becomes very structured. For example, it is easy to see that the set of feasible intensity vectors with expansion rate α

$$\mathcal{X}_\alpha := \{x \in \mathbf{R}^n \mid (B - \alpha A)x \geq 0,\ e^T x = 1,\ x \geq 0\}$$

is shrinking as the expansion factor α increases. That is, $\mathcal{X}_{\alpha_1} \subset \mathcal{X}_{\alpha_2}$ whenever $\alpha_1 > \alpha_2$. (See Gale [9] for a more detailed description of the model.)

Now, we shall describe a method for solving the von Neumann model using parametric centers. Suppose we are interested in solving the following technological expansion problem for an economic model $(\tilde{A},\ \tilde{B})$.

$$\begin{aligned} \alpha^{max} := \max \quad & \alpha \\ s.\,t. \quad & (\tilde{B} - \alpha\tilde{A})\tilde{x} \geq 0 \\ & e^T\tilde{x} = 1 \\ & \tilde{x} \geq 0, \end{aligned} \tag{29}$$

where $\tilde{A}$ and $\tilde{B}$ are non-negative $m \times n$ matrices, and $\tilde{x} \in \mathbf{R}^n$. Without loss of generality, we assume that $\tilde{A} = [\bar{A},\ \tilde{A}_n]$, $\tilde{B} = [\bar{B},\ \tilde{B}_n]$, where $\tilde{A}_n$ and $\tilde{B}_n$ are column vectors, and $\tilde{x}^T = [x^T,\ \tilde{x}_n]$, with $\bar{A},\ \bar{B} \in \mathbf{R}_+^{m\times(n-1)}$, $\tilde{A}_n,\ \tilde{B}_n \in \mathbf{R}_+^m$ and $x \in \mathbf{R}^{n-1}$. Then, by using the constraint $e^T\tilde{x} = 1$ to eliminate the n^{th} variable $\tilde{x}_n$, we see that, for each α, the following two linear systems are equivalent:

$$\left.\begin{aligned} (\tilde{B} - \alpha\tilde{A})\tilde{x} &\geq 0 \\ e^T\tilde{x} &= 1 \\ \tilde{x} &\geq 0 \end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned} ([\tilde{B}_ne^T - \bar{B}] + \alpha[\bar{A} - \tilde{A}_ne^T)x &\leq \tilde{B}_n + \alpha(-\tilde{A}_n) \\ e^Tx &\leq 1 \\ -x &\leq 0 \end{aligned}\right\} \tag{31}$$

in the sense that there exists $\tilde{x} \in \mathbf{R}^n$ satisfying system (30) if and only if there exists $x \in \mathbf{R}^{n-1}$ satisfying system (31), where the obvious transformations are

$$x^T = (\tilde{x}_1,\ \tilde{x}_2,\ \ldots,\ \tilde{x}_{n-1})^T \tag{32}$$

$$\tilde{x}^T = (x_1,\ x_2,\ \ldots,\ x_{n-1},\ 1 - e^Tx)^T. \tag{33}$$

Now, system (31) may be expressed as $(A + \alpha B)x \leq b + \alpha d$, where $A,\ B \in \mathbf{R}^{(m+n)\times(n-1)}$ and $b,\ d \in \mathbf{R}^{m+n}$ are given by

$$A = \begin{bmatrix} \tilde{B}_ne^T - \bar{B} \\ e^T \\ -I \end{bmatrix}, \quad B = \begin{bmatrix} \bar{A} - \tilde{A}_ne^T \\ 0 \\ O \end{bmatrix}, \quad b = \begin{bmatrix} \tilde{B}_n \\ 1 \\ 0 \end{bmatrix}, \quad d = \begin{bmatrix} -\tilde{A}_n \\ 0 \\ 0 \end{bmatrix}. \tag{34}$$

Hence, (29) is equivalent to the following problem.

$$\begin{aligned} \alpha^{max} := \max \quad & \alpha \\ s.\,t. \quad & (A+\alpha B)x \le b+\alpha d. \end{aligned} \tag{35}$$

Observe that the set $\{x|Ax \le b\}$ is bounded because of the existence of the constraints $e^T x \le 1,\ x \ge 0$. Also, it is straightforward to verify that $x = \frac{1}{n}e \in \{x|Ax < b\}$. Therefore, $\{x|Ax < b\}$ is nonempty and bounded. Next, we see that the system $Ax \le b$, under the transformations (32) and (33), is equivalent to the system

$$\tilde{B}\tilde{x} \ge 0,\ \ e^T\tilde{x} = 1,\ \ \tilde{x} \ge 0,$$

and for $\tilde{x}$ satisfying $e^T\tilde{x} = 1,\ \ \tilde{x} \ge 0$, the system

$$\tilde{A}\tilde{x} \ge 0,\ \ \tilde{A}\tilde{x} \ne 0$$

is equivalent to the system

$$Bx \ge d,\ \ Bx \ne d.$$

Therefore, if the model $(\tilde{A},\ \tilde{B})$ satisfies Assumptions 3.1 and 3.2, then it is straightforward to verify that the parametric system of linear inequalities $(A+\alpha B)x \le b+\alpha d$, where $(A,\ B,\ b,\ d)$ are given by (34), satisfies Assumptions 1 and 2 of Section 1.

Hence, we may apply Algorithm PCP to solve problem (35) by tracing the center of system $(A+\alpha B)x \le b+\alpha d$ as α is increased strictly monotonically. Starting from $\frac{1}{n}e$, we use a center finding algorithm of Vaidya [26] or Freund [5] to get an approximate center $\bar{x}$ (satisfying $||\bar{x}-\hat{x}||_{Q_0(\bar{x})} \le 1/21$) for the system $Ax \le b$. Next, we apply Algorithm PCP with $\alpha^{upper} = \alpha^{max} - \varepsilon$, where ε is a given error tolerance.

Complexity Analysis

We analyze next the complexity of Algorithm PCP as an algorithm for TEP. Let α^j be the value of α and let $\bar{x}^j$ be the iterate in iteration j. Observe that the total number of constraints in problem (35) is $(m+n)$ and the numbers of non-zero rows in the matrices B

and $[B,\ d]$ are both equal to m, whereby we can set $k = l = m$. From Theorem 2.1, since $\sqrt{m} + m \leq 2m$,

$$\alpha^{j+1} - \alpha^j = \frac{1}{88(\sqrt{m} + m)})\bar{\alpha} \geq \frac{1}{176m}\bar{\alpha}, \tag{36}$$

where $\bar{\alpha}$ is defined by (3), and from Corollary 2.1,

$$\alpha^{max} - \alpha^j \leq c_1(m+n)\bar{\alpha}, \tag{37}$$

where $c_1 < \infty$ is the constant defined by (11)-(14). Combining (36) and (37),

$$\alpha^{j+1} - \alpha^j \geq \frac{1}{176c_1(m+n)m}(\alpha^{max} - \alpha^j). \tag{38}$$

Rearranging terms, we see that the optimality gap $(\alpha^{max} - \alpha^j)$ at the j^{th} iteration of Algorithm PCP satisfies the following.

$$\alpha^{max} - \alpha^{j+1} \leq (1 - \frac{1}{176c_1(m+n)m})(\alpha^{max} - \alpha^j) \tag{39}$$

Therefore, we have the following complexity result.

Theorem 3.3 *Suppose Algorithm PCP is applied to solve problem (35). Then it will produce an intensity vector $\bar{x}$ and an expansion factor $\bar{\alpha}$ such that $\alpha^{max} - \varepsilon \leq \bar{\alpha} \leq \alpha^{max}$ after at most $K = \lceil 176c_1(m+n)m(\ln\alpha^{max} - \ln\varepsilon)\rceil$ iterations, where c_1 is a constant defined by (11)-(14).*

Proof: The proof follows from (39) as in Theorem 2.4. *Q.E.D.*

3.4 Generalized Linear Fractional Programming

Suppose we are interested in solving the following GLFP program.

$$\alpha^* = \max_x \min_i \quad \left\{\frac{f_i - C_i x}{D_i x - h_i}\right\} \tag{40}$$

$$s.\,t. \quad \tilde{A}x \leq \tilde{b}, \tag{41}$$

where $\tilde{A} \in \mathbf{R}^{m\times n}$, $\tilde{b} \in \mathbf{R}^m$, C, $D \in \mathbf{R}^{k\times n}$ and f, $h \in \mathbf{R}^k$, and $[D_i,\ h_i] \neq 0$ for $i = 1, 2, \ldots, k$. Let A, $B \in \mathbf{R}^{(m+k)\times n}$ and b, $d \in \mathbf{R}^{m+k}$ be the following matrices.

$$A = \begin{bmatrix} \tilde{A} \\ C \end{bmatrix}, \quad B = \begin{bmatrix} O \\ D \end{bmatrix}, \quad b = \begin{bmatrix} \tilde{b} \\ f \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ h \end{bmatrix}. \tag{42}$$

Then it is easy to see that (40) is equivalent to the following program.

$$\alpha^* = \max_{x,\alpha} \qquad \alpha \tag{43}$$

$$s.\ t. \qquad (A + \alpha B)x \leq b + \alpha d, \tag{44}$$

where $(A,\ B,\ b,\ d)$ are given by (42).

We may then apply Algorithm PCP to solve (43) by tracing the parametric centers of the family of systems $(A + \alpha B)x \leq b + \alpha d$, where $(A,\ B,\ b,\ d)$ are given by (42) while α, taken as a parameter, is increased strictly monotonically over the range $\alpha \in [0,\ \alpha^* - \varepsilon]$, where ε is the given error tolerance. We note that the total number of inequalities in this case is $(m + k)$ and the number of inequalities that vary with α is equals to k.

Suppose α^j is the value of α at the start of iteration j. From Theorem 2.1 we get

$$\alpha^{j+1} - \alpha^j = \left(\frac{1}{88(\sqrt{k} + k)}\right)\bar{\alpha}, \tag{45}$$

where $\bar{\alpha}$ is defined by (3). Also from Corollary 2.1 we get

$$\alpha^* - \alpha^j \leq c_1(m + k)\bar{\alpha}, \tag{46}$$

for some constant c_1 defined by (11)–(14). Therefore, combining (45) and (46), we get

$$\alpha^{j+1} - \alpha^j \geq \frac{1}{88c_1(m + k)(\sqrt{k} + k)}(\alpha^* - \alpha^j) \geq \frac{1}{176c_1(m + k)k}(\alpha^* - \alpha^j), \tag{47}$$

since $\sqrt{k} + k \leq 2k$. Rearranging terms, we get

$$\alpha^* - \alpha^{j+1} \leq \left(1 - \frac{1}{176c_1(m + k)k}\right)(\alpha^* - \alpha^j). \tag{48}$$

Therefore, as in the preceding applications, we can show the following.

Theorem 3.4 *Suppose Algorithm PCP is applied to solve problem (43). Then it will produce a feasible solution* $\bar{x}$ *such that*

$$\alpha^* - \varepsilon \leq \min_i \left\{ \frac{f_i - C_i \bar{x}}{D_i \bar{x} - h_i} \right\} \leq \alpha^*$$

after at most $K = \lceil 176 c_1 (m + k) k (\ln \alpha^* - \ln \varepsilon) \rceil$ *iterations, where* c_1 *is a constant defined by (11)-(14).*

4 Proofs of Main Theorems

In this section, we shall prove Theorem 2.1 and Theorem 2.2. Let us begin with some preliminary lemmas.

Lemma 4.1 *Let* Q *be a (symmetric) positive definite matrix, and* d *be a given nonzero* n*-vector. Suppose* $\bar{x} \in \mathbf{R}^n$ *satisfies* $\bar{h} = d^T \bar{x} - h > 0$*. Then we have*

$$\min_x \{ \, ||x - \bar{x}||_Q^2 \mid d^T x = h \, \} = \frac{\bar{h}^2}{d^T Q^{-1} d}.$$

Proof: Follows directly from the fact that

$$x = \bar{x} - \left(\frac{\bar{h}}{d^T Q^{-1} d} \right) Q^{-1} d$$

is the solution of the minimization program. *Q.E.D.*

That is, the distance (in the Q-norm) of the point $\bar{x}$ to the plane $\{x | d^T x = h\}$ is inversely proportional to the Q^{-1}-norm of the vector $\frac{1}{\bar{h}} d$. Under Assumption 2, each (interior) point of the polytope $\{x | \, Ax < b\}$ is at least some positive distance away from each of the planes $\{x | B_i^T x = d_i\}$, $i = 1, 2, \ldots, m$. Therefore, each B_i should be bounded in some sense which we will make precise shortly in the next lemma.

Let $\bar{x}$ satisfying $\bar{s} = b - A\bar{x} > 0$ be a given (interior) point. Define $\bar{\alpha}$ by (3). Let $\bar{s}_\alpha$ be given by (4) and let $Q_\alpha(\bar{x})$ be defined by (5).

Lemma 4.2 *Under the Assumptions of Section 1, suppose* $0 \le \alpha < \bar{\alpha}$*, then*

$$(i) \qquad \|(\bar{S}^{-1}B)_i^T\|_{Q_{\alpha}^{-1}(\bar{x})} \le \|\bar{S}^{-1}(B\bar{x}-d)\|_{\infty}, \quad i = 1, 2, \ldots, m;$$

$$(ii) \qquad \|(\bar{S}_{\alpha}^{-1}B)_i^T\|_{Q_{\alpha}^{-1}(\bar{x})} \le \|\bar{S}_{\alpha}^{-1}(B\bar{x}-d)\|_{\infty}, \quad i = 1, 2, \ldots, m;$$

$$(iii) \qquad \|B^T\bar{S}_{\alpha}^{-1}e\|_{Q_{\alpha}^{-1}(\bar{x})} \le k\|\bar{S}_{\alpha}^{-1}(B\bar{x}-d)\|_{\infty},$$

where k *is the number of non-zero rows in the matrix* B.

Proof: (i) Let $Q_\alpha = Q_\alpha(\bar{x})$. We first note that Q_α is positive definite for each α satisfying $0 \le \alpha < \bar{\alpha}$, so that the norm $\|\cdot\|_{Q_{\alpha}^{-1}}$ is well-defined. Let $F_{in} = \{x \mid \|x-\bar{x}\|_{Q_\alpha} \le 1\}$ and let $\mathcal{X}_\alpha$ be given by (6). Then, from Lemma A.9 in the Appendix, $F_{in} \subset \mathcal{X}_\alpha$. Under the Assumptions, $B_i x < d_i$ implies that $x \notin \mathcal{X}_\alpha$, whereby $x \notin F_{in}$. Therefore, using Lemma 4.1, we have

$$\frac{(B_i\bar{x}-d_i)^2}{B_iQ_{\alpha}^{-1}B_i^T} = \min\{\|x-\bar{x}\|_{Q_\alpha}^2 \mid B_ix = d_i\} \ge 1.$$

Therefore,

$$B_iQ_{\alpha}^{-1}B_i^T \le (B_i\bar{x}-d_i)^2. \tag{49}$$

Hence

$$\begin{aligned} \|(\bar{S}^{-1}B)_i^T\|_{Q_{\alpha}^{-1}} &= \left(\frac{B_iQ_{\alpha}^{-1}B_i^T}{\bar{s}_i^2}\right)^{1/2} \\ &\le \left|\frac{(B_i\bar{x}-d_i)}{\bar{s}_i}\right| \\ &\le \|\bar{S}^{-1}(B\bar{x}-d)\|_{\infty}. \end{aligned}$$

Part (ii) follows similarly from (49). Using the triangle inequality, part (iii) follows from part (ii) immediately. *Q.E.D.*

The next lemma shows the close relationship between the two norms $\|\cdot\|_Q$ and $\|\cdot\|_{Q^{-1}}$.

Lemma 4.3 *Let* Q_1 *and* Q_2 *be two (symmetric) positive definite* $n \times n$ *matrices, and let* $\|\cdot\|_{Q_i}$ *and* $\|\cdot\|_{Q_i^{-1}}$ *denote the norms defined by* Q_i *and* Q_i^{-1} *respectively. Suppose there exists a constant* $\kappa > 0$ *such that* $\|v\|_{Q_1} \le \kappa\|v\|_{Q_2}$ *for all* $v \in \mathbf{R}^n$*. Then, for any* $p \in \mathbf{R}^n$*, we have* $\|p\|_{Q_2^{-1}} \le \kappa\|p\|_{Q_1^{-1}}$.

Proof: Let $p \in \mathbf{R}^n$ be given. Without loss of generality, we may assume that $p \neq 0$. Then there exists $\hat{u} \in \mathbf{R}^n$ such that $p^T \hat{u} = 1$. Therefore, using Lemma 4.1 and by the hypothesis, we have

$$\begin{aligned} 1/||p||_{Q_1^{-1}} = (p^T Q_1^{-1} p)^{-1/2} &= \min_u \{||u - \hat{u}||_{Q_1} \mid p^T u = 0\} \\ &\leq \min_u \{\kappa ||u - \hat{u}||_{Q_2} \mid p^T u = 0\} \\ &= \kappa (p^T Q_2^{-1} p)^{-1/2} = \kappa / ||p||_{Q_2^{-1}}. \end{aligned}$$

Hence, $$||p||_{Q_2^{-1}} \leq \kappa ||p||_{Q_1^{-1}}.$$ *Q.E.D.*

The next lemma shows that we can bound the norm of a matrix in term of a bound on the transposes of the rows of the matrix (considered as vectors).

Lemma 4.4 *Let Q be a positive definite $n \times n$ matrix and let $||\cdot||_Q$ denote the norm defined by Q. Suppose M is a $k \times n$ matrix. Let M_i^T denotes the transpose of the i^{th} row of M. Suppose there exists a constant $c > 0$ such that $||M_i^T||_Q \leq c$ for all $i = 1, 2, \ldots, k$. Then*

$$||M^T||_Q := \max_{||y||=1} ||M^T y||_Q \leq c\sqrt{k}.$$

Proof: Using the triangle inequality,

$$||M^T y||_Q = ||\sum_{i=1}^{k} y_i M_i^T||_Q \leq \sum_{i=1}^{k} |y_i| \; ||M_i^T||_Q \leq \sqrt{k}||y||c,$$

where the last inequality follows from the fact that

$$\sum_{i=1}^{k} |y_i| = ||y||_1 \leq \sqrt{k}||y||.$$ *Q.E.D.*

Remark: It can be easily shown that $||M^T||_Q = ||MQM^T||^{1/2}$, and if $M^T = [N^T, 0]$, then $||M^T||_Q = ||N^T||_Q$.

The following lemma is an immediate corollary of Lemma 4.4 and Lemma 4.2.

Lemma 4.5 *Under the same conditions and definitions as Lemma 4.2, we have*

$$||B^T\bar{S}^{-1}||_{Q_\alpha^{-1}(\bar{x})} \leq \sqrt{k}||\bar{S}^{-1}(B\bar{x}-d)||_\infty = \sqrt{k}/\bar{\alpha}.$$

Proof: Follows immediately from Lemma 4.2 and Lemma 4.4. *Q.E.D.*

Next, we want to show the relationship between the two norms $||\cdot||_{Q_0(\bar{x})}$ and $||\cdot||_{Q_\alpha(\bar{x})}$, where $||\cdot||_{Q_\alpha(\bar{x})}$ is the norm defined using the Hessian at $\bar{x}$ of the logarithmic barrier function in $CP(\alpha)$. Before that, we need the following.

Lemma 4.6 *Suppose* $0 \leq \alpha \leq \xi\bar{\alpha},\ \ 0 \leq \xi < 1$. *Then*

$$(i) \qquad ||\bar{S}^{-1}\bar{S}_\alpha|| \leq 1; \quad \textit{and}$$

$$(ii) \qquad ||\bar{S}_\alpha^{-1}\bar{S}|| \leq 1/(1-\xi).$$

Proof: (i) Follows easily from the following

$$0 \leq \bar{S}^{-1}\bar{s}_\alpha = e - \alpha\bar{S}^{-1}(B\bar{x}-d) \leq e.$$

(ii) We have $\bar{S}_\alpha^{-1}\bar{s} = e + \alpha\bar{S}_\alpha^{-1}(B\bar{x}-d)$. Therefore,

$$\begin{aligned} ||\bar{S}_\alpha^{-1}\bar{S}|| &\leq 1 + \alpha||\bar{S}_\alpha^{-1}(B\bar{x}-d)||_\infty \\ &\leq 1 + \alpha||\bar{S}_\alpha^{-1}\bar{S}||\ \ ||\bar{S}^{-1}(B\bar{x}-d)||_\infty \\ &= 1 + (\alpha/\bar{\alpha})||\bar{S}_\alpha^{-1}\bar{S}||. \end{aligned}$$

Rearranging, we get $||\bar{S}_\alpha^{-1}\bar{S}|| \leq \frac{1}{1-\alpha/\bar{\alpha}} \leq \frac{1}{1-\xi}$. *Q.E.D.*

Now we can show the relationship between the two norms $||\cdot||_{Q_0(\bar{x})}$ and $||\cdot||_{Q_\alpha(\bar{x})}$.

Lemma 4.7 *Under the Assumptions of Section 1, suppose* $0 \leq \alpha \leq \xi\bar{\alpha},\ \ 0 \leq \xi < 1$, *where* $\bar{\alpha}$ *is given by (3). Let* $Q_\alpha(\bar{x})$ *be defined by (4)-(5). Then, for all* $v \in \mathbf{R}^n$,

$$(i) \qquad ||v||_{Q_\alpha(\bar{x})} \leq \left(\frac{1+\xi\sqrt{k}}{1-\xi}\right)||v||_{Q_0(\bar{x})};$$

$$(ii) \qquad ||v||_{Q_0(\bar{x})} \leq (1+\xi\sqrt{k})||v||_{Q_\alpha(\bar{x})}.$$

Proof: (i) Let $Q_\alpha = Q_\alpha(\bar{x})$. Using Lemma 4.6, the triangle inequality, the Cauchy-Schwartz inequality and Lemma 4.5, we have

$$\begin{aligned}
||v||_{Q_\alpha} &= ||\bar{S}_\alpha^{-1}(A+\alpha B)v|| \\
&\le ||\bar{S}_\alpha^{-1}\bar{S}|| \cdot ||\bar{S}^{-1}(A+\alpha B)v|| \\
&\le \left(\frac{1}{1-\xi}\right)\left(||\bar{S}^{-1}Av|| + \alpha||\bar{S}^{-1}Bv||\right) \\
&\le \left(\frac{1}{1-\xi}\right)(||v||_{Q_0} + \alpha||B^T\bar{S}^{-1}||_{Q_0^{-1}}\ ||v||_{Q_0}) \\
&\le \left(\frac{1}{1-\xi}\right)(1+\alpha\sqrt{k}/\bar{\alpha})||v||_{Q_0} \\
&\le \left(\frac{1}{1-\xi}\right)(1+\xi\sqrt{k})||v||_{Q_0}.
\end{aligned}$$

(ii) Similarly, we have

$$\begin{aligned}
||v||_{Q_0} &= ||\bar{S}^{-1}Av|| \\
&\le ||\bar{S}^{-1}(A+\alpha B)v|| + \alpha||\bar{S}^{-1}Bv|| \\
&\le ||\bar{S}^{-1}\bar{S}_\alpha|| \cdot ||\bar{S}_\alpha^{-1}(A+\alpha B)v|| + \alpha||B^T\bar{S}^{-1}||_{Q_\alpha^{-1}}\ ||v||_{Q_\alpha} \\
&\le (1+\alpha\sqrt{k}/\bar{\alpha})||v||_{Q_\alpha} \\
&\le (1+\xi\sqrt{k})||v||_{Q_\alpha}.
\end{aligned}$$

Q.E.D.

As an immediate corollary of Lemma 4.7 and Lemma 4.3, we have the next lemma.

Lemma 4.8 *Under the same conditions and definitions as Lemma 4.7, for all* $p \in \mathbf{R}^n$,

(i) $$||p||_{Q_\alpha^{-1}(\bar{x})} \le (1+\xi\sqrt{k})||p||_{Q_0^{-1}(\bar{x})};$$

(ii) $$||p||_{Q_0^{-1}(\bar{x})} \le \left(\frac{1+\xi\sqrt{k}}{1-\xi}\right)||p||_{Q_\alpha^{-1}(\bar{x})}.$$

Proof: Follows immediately from Lemma 4.3 and Lemma 4.7. *Q.E.D.*

Finally, suppose $\hat{x} = \hat{x}_0$ is the center of $Ax \le b$ and suppose $\bar{x}$, satisfying $\bar{s} = b - A\bar{x} > 0$, is a δ-approximate center of $Ax \le b$. Let $\hat{\alpha} = \alpha_{\hat{x}}$ and $\bar{\alpha} = \alpha_{\bar{x}}$ be defined by (3). The next lemma shows the relationship between $\hat{\alpha}$ and $\bar{\alpha}$.

Lemma 4.9 *Suppose* $\hat{x} = \hat{x}_0$ *is the center and* $\bar{x}$, *satisfying* $\bar{s} = b - A\bar{x} > 0$, *is a* δ-*approximate center of the system* $Ax \leq b$. *[i.e.* $||\bar{x} - \hat{x}||_{Q_0(\bar{x})} \leq \delta < 1$, *where* $Q_0(\bar{x})$ *is given by (4)—(5).] Let* $\hat{\alpha} = \alpha_{\hat{x}}$ *and* $\bar{\alpha} = \alpha_{\bar{x}}$ *be defined by (3). Then*

$$\left(\frac{1-\delta}{1+\delta}\right)\bar{\alpha} \leq \hat{\alpha} \leq \left(\frac{1+\delta}{1-\delta}\right)\bar{\alpha}.$$

Proof: First we note that $||\bar{S}^{-1}\hat{S}|| \leq 1 + \delta$ (from Lemma A.3 in the Appendix), and for $i = 1, 2, \ldots, m$,

$$|[\hat{S}^{-1}B(\bar{x} - \hat{x})]_i| \leq ||(\hat{S}^{-1}B)_i^T||_{Q_0^{-1}(\hat{x})} \;\; ||\bar{x} - \hat{x}||_{Q_0(\hat{x})} \leq \frac{1}{\hat{\alpha}}\left(\frac{\delta}{1-\delta}\right),$$

where the first inequality is a Cauchy-Schwartz inequality and the last inequality follows from Lemma 4.2 and Lemma A.3. Therefore

$$||\hat{S}^{-1}B(\bar{x} - \hat{x})||_\infty \leq \frac{1}{\hat{\alpha}}\left(\frac{\delta}{1-\delta}\right),$$

and hence

$$\begin{aligned} 1/\bar{\alpha} &= ||\bar{S}^{-1}(B\bar{x} - d)||_\infty \\ &\leq ||\bar{S}^{-1}\hat{S}||\left(||\hat{S}^{-1}(B\hat{x} - d)||_\infty + ||\hat{S}^{-1}B(\bar{x} - \hat{x})||_\infty\right) \\ &\leq (1+\delta)\left(1 + \frac{\delta}{1-\delta}\right)\frac{1}{\hat{\alpha}} \\ &= \left(\frac{1+\delta}{1-\delta}\right)\frac{1}{\hat{\alpha}}. \end{aligned}$$

This proves the first part. The second part is proved in a similar manner. *Q.E.D.*

We are now finished with the preliminary lemmas. The proofs of Theorems 2.1 and 2.2 will follow from the following. For simplicity, we shall consider increasing the parameter from $\alpha = 0$ to some $\alpha > 0$. In the following, we shall use another measure of closeness to the center, which we call the τ-*measure.* (See the Appendix.)

Theorem 4.1 *Suppose* $\bar{x}$ *satisfying* $\bar{s} = b - A\bar{x} > 0$ *is near the center of the system* $Ax \leq b$ *in the sense that*

$$\tau = \tau(\bar{x}) := \sqrt{e^T\bar{S}^{-1}A[A^T\bar{S}^{-2}A]^{-1}A^T\bar{S}^{-1}e} \leq \varepsilon.$$

Let $\bar{\alpha}$, $\bar{s}_\alpha$ and $Q_\alpha(\bar{x})$ be given by (3)—(5). Suppose $0 \le \alpha \le \frac{\bar{\alpha}}{80(\sqrt{l}+k)}$. Then $\bar{x}$ is near the center of the system $(A+\alpha B)x \le b + \alpha d$ in the sense that

$$\tau_\alpha(\bar{x}) := ||(A+\alpha B)^T \bar{S}_\alpha^{-1} e||_{Q_\alpha^{-1}(\bar{x})} \le \varepsilon_1 := \frac{81}{80}\varepsilon + \frac{1}{78}.$$

Proof: Let $Q_\alpha = Q_\alpha(\bar{x})$. First we note that

$$\begin{aligned} A^T \bar{S}_\alpha^{-1} e - A^T \bar{S}^{-1} e &= A^T \bar{S}_\alpha^{-1} \bar{S}^{-1}(\bar{s} - \bar{s}_\alpha) \\ &= \alpha A^T \bar{S}_\alpha^{-1} \bar{S}^{-1}(B\bar{x} - d). \end{aligned}$$

Therefore, using the triangle inequality, we get

$$\begin{aligned} ||A^T \bar{S}_\alpha^{-1} e||_{Q_0^{-1}} &\le ||A^T \bar{S}_\alpha^{-1} e - A^T \bar{S}^{-1} e||_{Q_0^{-1}} + ||A^T \bar{S}^{-1} e||_{Q_0^{-1}} \\ &= \alpha ||A^T \bar{S}_\alpha^{-1} \bar{S}^{-1}(B\bar{x} - d)||_{Q_0^{-1}} + \tau \\ &\le \alpha ||\bar{S}_\alpha^{-1}(B\bar{x} - d)|| + \varepsilon \\ &\le \alpha(80/79)\sqrt{l}/\bar{\alpha} + \varepsilon, \end{aligned}$$

where the second inequality follows from the fact that (since $Q_0 = A^T \bar{S}^{-2} A$)

$$||A^T \bar{S}^{-1}||^2_{Q_0^{-1}} = ||\bar{S}^{-1} A (A^T \bar{S}^{-2} A)^{-1} A^T \bar{S}^{-1}|| \le 1,$$

because the matrix is a projection matrix, and the last inequality follows from Lemma 4.6 (with $\xi = 1/80$) and the fact that if there are l non-zero rows in the matrix $[B,\ d]$ (i.e. the number of varying constraints is l), then

$$||\bar{S}_\alpha^{-1}(B\bar{x} - d)|| \le \sqrt{l}||\bar{S}_\alpha^{-1}(B\bar{x} - d)||_\infty.$$

Hence, by Lemma 4.8 (with $\xi = \frac{1}{80(\sqrt{l}+k)}$), we have

$$||A^T \bar{S}_\alpha^{-1} e||_{Q_\alpha^{-1}} \le \frac{81}{80}\left(\frac{80}{79}\frac{\sqrt{l}\alpha}{\bar{\alpha}} + \varepsilon\right) = \frac{81}{79}\frac{\sqrt{l}\alpha}{\bar{\alpha}} + \frac{81}{80}\varepsilon.$$

On the other hand, we have from Lemma 4.2 and Lemma 4.6

$$||B^T \bar{S}_\alpha^{-1} e||_{Q_\alpha^{-1}} \le k||\bar{S}_\alpha^{-1}(B\bar{x} - d)||_\infty \le k(\frac{80}{79})||\bar{S}^{-1}(B\bar{x} - d)||_\infty = (\frac{80}{79})(\frac{k}{\bar{\alpha}}).$$

Using the triangle inequality again, we get

$$\begin{aligned}\tau_\alpha(\bar{x}) := \|(A+\alpha B)^T\bar{S}_\alpha^{-1}e\|_{Q_\alpha^{-1}} &\le \|A^T\bar{S}_\alpha^{-1}e\|_{Q_\alpha^{-1}} + \alpha\|B^T\bar{S}_\alpha^{-1}e\|_{Q_\alpha^{-1}} \\ &\le \frac{81}{79}\sqrt{l}(\alpha/\bar{\alpha}) + \frac{81}{80}\varepsilon + \frac{80}{79}k(\alpha/\bar{\alpha}) \\ &\le \frac{81}{79}(\sqrt{l}+k)(\alpha/\bar{\alpha}) + \frac{81}{80}\varepsilon \\ &\le \frac{1}{78} + \frac{81}{80}\varepsilon. \qquad\qquad Q.E.D.\end{aligned}$$

As an immediate corollary of the above result, we have the following, which shows that if the increase in parameter value is not too large, then the two successive centers of the corresponding systems will be sufficiently near to each other such that Newton's method will work well (i.e., converge quadratically). First, we fix the notation.

Suppose $\hat{x} = \hat{x}_0$ is the center of system $Ax \le b$. That is, we have

$$\hat{s} = b - A\hat{x} > 0, \tag{50}$$

$$A^T\hat{S}^{-1}e = 0. \tag{51}$$

Define $\hat{\alpha} = \alpha_{\hat{x}}$, $\hat{s}_\alpha$ and $Q_\alpha(\hat{x})$ by (3)—(5). That is,

$$\hat{\alpha} = \alpha_{\hat{x}} = 1/\|\hat{S}^{-1}(B\hat{x} - d)\|_\infty \tag{52}$$

and, for $0 \le \alpha < \hat{\alpha}$,

$$\hat{s}_\alpha := (b + \alpha d) - (A + \alpha B)\hat{x} > 0, \tag{53}$$

$$Q_\alpha(\hat{x}) = (A+\alpha B)^T\hat{S}_\alpha^{-2}(A+\alpha B). \tag{54}$$

Theorem 4.2 *Suppose* $0 \le \alpha \le \dfrac{\hat{\alpha}}{80(\sqrt{l}+k)}$, *where* $\hat{\alpha}$ *is given by (52). Then* $\hat{x}$ *is near the center of the system* $(A+\alpha B)x \le b + \alpha d$, *in the sense that*

$$\tau_\alpha(\hat{x}) := \|(A+\alpha B)^T\hat{S}_\alpha^{-1}e\|_{Q_\alpha^{-1}(\hat{x})} \le \frac{1}{78},$$

where $\hat{s}_\alpha$ *and* $Q_\alpha(\hat{x})$ *are given by (53)-(54).*

Proof: Since $A^T\hat{S}^{-1}e = 0$, we have $\tau = 0$ and the result follows immediately from Theorem 4.1. *Q.E.D.*

Remark: Note that $y \equiv y_\alpha := (A+\alpha B)^T\hat{S}_\alpha^{-1}e$ is the gradient and $Q_\alpha(\hat{x})$ is the negative of the Hessian of the logarithmic barrier function for the center problem $CP(\alpha)$,

$$f_\alpha(x) = \sum_{i=1}^{m} \ln\, [(b+\alpha d) - (A+\alpha B)x]_i,$$

evaluated at $\hat{x}$.

Theorem 4.2 implies (by Lemma A.7 in the Appendix) that $\hat{x}$ is a δ-approximate center of the system $(A+\alpha B)x \le b+\alpha d$ with $\delta = 1/12$, which we state formally as:

Corollary 4.1 *Under the same conditions and definitions as Theorem 4.2, we have* $||\hat{x} - \hat{x}_\alpha||_{Q_\alpha(\hat{x})} \le 1/12$, *where $\hat{x}_\alpha$ denotes the center of $(A+\alpha B)x \le b+\alpha d$.*

Proof: Follows directly from Lemma A.7 of the Appendix. *Q.E.D.*

By Corollary 4.1 and Lemma 4.9, we have the following.

Theorem 4.3 *Suppose $\bar{x}$ satisfying $\bar{s} = b - A\bar{x} > 0$ is a δ-approximate center of the system $Ax \le b$ with $\delta = 1/21$. [i.e. $||\bar{x} - \hat{x}||_{Q_0(\bar{x})} \le 1/21$, where $Q_0(\bar{x})$ is given by (4)—(5).] Let $\bar{\alpha}$ be defined by (3). Suppose $0 \le \alpha \le \dfrac{\bar{\alpha}}{88(\sqrt{l}+k)}$. Then $\bar{x}$ is near the center of system $(A+\alpha B)x \le b+\alpha d$ in the sense that*

$$||\bar{x} - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} \le 0.148\ ,$$

where $Q_\alpha(\hat{x}_\alpha)$ is defined as in (4)—(5).

Proof: First note that by Lemma 4.9, $0 \le \alpha \le \dfrac{\bar{\alpha}}{88(\sqrt{l}+k)}$ implies that $0 \le \alpha \le \dfrac{\hat{\alpha}}{80(\sqrt{l}+k)}$. Therefore, by Corollary 4.1,

$$||\hat{x} - \hat{x}_\alpha||_{Q_\alpha(\hat{x})} \le 1/12 \tag{55}$$

and, by Lemma A.3(iv) (in the Appendix),

$$||\hat{x} - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} \leq 1/11. \tag{56}$$

Also, Lemma 4.7 (with $\xi = \frac{1}{88(\sqrt{l}+k)} \leq \frac{1}{88}$) implies that

$$||\bar{x} - \hat{x}||_{Q_\alpha(\bar{x})} \leq (\frac{88}{87})(\frac{89}{88})||\bar{x} - \hat{x}||_{Q_0(\bar{x})} < 0.049. \tag{57}$$

and therefore, by Lemma A.3(vi),

$$||\bar{x} - \hat{x}||_{Q_\alpha(\hat{x})} \leq \frac{0.049}{1 - 0.049} < 0.052. \tag{58}$$

Now, Lemma A.3(iv), together with (55), implies that

$$||\bar{x} - \hat{x}||_{Q_\alpha(\hat{x}_\alpha)} \leq \frac{12}{11}||\bar{x} - \hat{x}||_{Q_\alpha(\hat{x})} < 0.057. \tag{59}$$

Hence (using the triangle inequality) we have

$$||\bar{x} - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} \leq ||\bar{x} - \hat{x}||_{Q_\alpha(\hat{x}_\alpha)} + ||\hat{x} - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} < 0.057 + 1/11 < 0.148. \qquad Q.E.D.$$

As a consequence of Theorem 4.3 and Lemma A.10 of the Appendix, we have the following theorem which states that $\bar{x}$, which is a δ-approximate center of system $Ax \leq b$, is sufficiently near to $\hat{x}_\alpha$, the center of $(A + \alpha B)x \leq b + \alpha d$, such that we may apply a Newton step to solve for $\hat{x}_\alpha$ starting from $\bar{x}$, provided that $0 \leq \alpha \leq \frac{\bar{\alpha}}{88(\sqrt{l}+k)}$.

Theorem 4.4 *Suppose $\bar{x}$ satisfying $\bar{s} = b - A\bar{x} > 0$ is a δ-approximate center of the system $Ax \leq b$ with $\delta = 1/21$. Let $\bar{\alpha}$ be defined by (3). Suppose $0 \leq \alpha \leq \frac{\bar{\alpha}}{88(\sqrt{l}+k)}$. Let $\eta = Q_\alpha^{-1}(\bar{x})(A + \alpha B)^T \bar{S}_\alpha^{-1} e$, where $\bar{s}_\alpha$ and $Q_\alpha(\bar{x})$ are defined as in (4)—(5) [η be a Newton step from $\bar{x}$ in the center problem $CP(\alpha)$] and let $\bar{x}_{new} = \bar{x} + \eta$. Then*

$$||\bar{x}_{new} - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} \leq 0.034 \ .$$

Proof: We have, from Theorem 4.3,

$$\varepsilon := ||\bar{x} - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} < 0.148. \tag{60}$$

Therefore, by Lemma A.10,

$$||\bar{x}_{new} - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} \le \frac{(1+0.148)^2(0.148)^2}{1-0.148} < 0.034 \ . \qquad Q.E.D.$$

Corollary 4.2 *Under the same definitions and conditions as Theorem 4.4,*

$$||\bar{x}_{new} - \hat{x}_\alpha||_{Q_\alpha(\bar{x}_{new})} \le 1/21 \ .$$

Proof: From Lemma A.3(vi),

$$||\bar{x}_{new} - \hat{x}_\alpha||_{Q_\alpha(\bar{x}_{new})} \le \frac{0.034}{1-0.034} < \frac{1}{21}. \qquad Q.E.D.$$

Now, we are ready to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1: Follows immediately from Corollary 4.2. *Q.E.D.*

Proof of Theorem 2.2: Using the triangle inequality, and by Theorem 4.3 and Theorem 4.4,

$$||\bar{x}_{new} - \bar{x}||_{Q_\beta(\hat{x}_\beta)} \le ||\bar{x}_{new} - \hat{x}_\beta||_{Q_\beta(\hat{x}_\beta)} + ||\bar{x} - \hat{x}_\beta||_{Q_\beta(\hat{x}_\beta)} \le 0.034 + 0.148 = 0.182.$$

Also from Corollary 4.1, $||\hat{x} - \hat{x}_\alpha||_{Q_\alpha(\hat{x})} \le 1/12$, which in turn implies [by applying Lemma A.3 of the Appendix to the system $(A + \alpha B)x \le b + \alpha d$]

$$||v||_{Q_\alpha(\hat{x})} \le \frac{13}{12}||v||_{Q_\alpha(\hat{x}_\alpha)} \tag{61}$$

and

$$||v||_{Q_\alpha(\hat{x}_\alpha)} \le \frac{12}{11}||v||_{Q_\alpha(\hat{x})}. \tag{62}$$

Next, using Lemma 4.7 (note that $\xi \leq 1/80$), we have

$$||v||_{Q_\alpha(\hat{x})} \leq \frac{81}{79}||v||_{Q_0(\hat{x})} \tag{63}$$

and

$$||v||_{Q_0(\hat{x})} \leq \frac{81}{80}||v||_{Q_\alpha(\hat{x})}. \tag{64}$$

Finally, observe that $\bar{x}_\alpha - \bar{x} = (\alpha/\beta)(\bar{x}_{new} - \bar{x})$. Hence, using (61)—(64), we have

$$\begin{aligned}
||\bar{x}_\alpha - \bar{x}||_{Q_\alpha(\hat{x}_\alpha)} &\leq \left(\frac{12}{11}\right) ||\bar{x}_\alpha - \bar{x}||_{Q_\alpha(\hat{x})} \\
&\leq \left(\frac{12}{11}\right)\left(\frac{81}{79}\right) ||\bar{x}_\alpha - \bar{x}||_{Q_0(\hat{x})} \\
&\leq \left(\frac{12}{11}\right)\left(\frac{81}{79}\right)\left(\frac{81}{80}\right) ||\bar{x}_\alpha - \bar{x}||_{Q_\beta(\hat{x})} \\
&\leq \left(\frac{12}{11}\right)\left(\frac{81}{79}\right)\left(\frac{81}{80}\right)\left(\frac{13}{12}\right) ||\bar{x}_\alpha - \bar{x}||_{Q_\beta(\hat{x}_\beta)} \\
&= \left(\frac{12}{11}\right)\left(\frac{81}{79}\right)\left(\frac{81}{80}\right)\left(\frac{13}{12}\right)\left(\frac{\alpha}{\beta}\right) ||\bar{x}_{new} - \bar{x}||_{Q_\beta(\hat{x}_\beta)} \\
&< 0.23.
\end{aligned}$$

Thus,

$$\begin{aligned}
||\bar{x}_\alpha - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} &\leq ||\bar{x}_\alpha - \bar{x}||_{Q_\alpha(\hat{x}_\alpha)} + ||\bar{x} - \hat{x}_\alpha||_{Q_\alpha(\hat{x}_\alpha)} \\
&< 0.23 + 0.148 < 0.38.
\end{aligned}$$

Q.E.D.

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A Appendix

Recall that the *center problem* on a system of linear inequalities $Ax \leq b$ is the following optimization problem.

$$\begin{array}{lll} CP: & maximize & \sum_{i=1}^{m} \ln s_i \\ & subject\ to & Ax + s = b \\ & & s > 0. \end{array}$$

In this Appendix, we present some known results concerning the center problem CP, and give three measures of closeness to the center which we use in this paper.

A.1 The Analytic Center

Assuming that the set $\{x | Ax < b\}$ is nonempty and bounded, the solution $\hat{x}$ of the center problem CP exists uniquely, and is called the *analytic center* of the system $Ax \leq b$. (Sonnevend [23,24].) We shall refer to it simply as the *center*. Since the objective function is strictly concave, the center $\hat{x}$ is uniquely characterised by the Karush-Kuhn-Tucker conditions

$$\hat{s} = b - A\hat{x} > 0 \tag{65}$$

$$A^T \hat{S}^{-1} e = 0. \tag{66}$$

For any x satisfying $s = b - Ax > 0$, let $Q(x)$ be the negative Hessian of the barrier function for the center problem CP at x, that is,

$$Q(x) = A^T S^{-2} A. \tag{67}$$

Properties of the Analytic Center

One particularly important property of the center is the following.

Lemma A.1 **([23,24], [5] Theorem 2.1)** *Let $\hat{x}$ denote the center of the linear inequality system $Ax \leq b$. Let $\mathcal{X} := \{x \in \mathbf{R}^n \mid Ax \leq b\}$, and define the ellipsoids E_{in} and E_{out} by*

$$\begin{aligned} E_{in} &:= \{ x \in \mathbf{R}^n \mid \|x - \hat{x}\|_{Q(\hat{x})} \leq \sqrt{\frac{m}{m-1}} \}, \\ E_{out} &:= \{ x \in \mathbf{R}^n \mid \|x - \hat{x}\|_{Q(\hat{x})} \leq \sqrt{m(m-1)} \}. \end{aligned}$$

Then $E_{in} \subset \mathcal{X} \subset E_{out}$.

That is, we can construct contained and containing ellipsoids centered at the center. Note that E_{out} is an enlargement of E_{in} with an enlargement factor of $(m-1)$, i.e., $(E_{out} - \hat{x}) = (m-1)(E_{in} - \hat{x})$. Next, the following lemma shows that the slacks of all feasible points of the linear inequality system $Ax \leq b$ are contained in a simplex. Therefore, as a corollary, we can bound the slacks of any feasible point $x \in \mathcal{X}$.

Lemma A.2 **([22] Proposition 3.1, [5] Proposition 2.1)** *Suppose $\hat{x}$ is the center of the linear inequality system $Ax \leq b$. Let $\hat{s} = b - A\hat{x}$. For any x satisfying $Ax \leq b$, let $s = b - Ax$. Then $e^T \hat{S}^{-1} s = m, \ s \geq 0$.*

Corollary A.1 *With the same conditions and definitions as Lemma A.2, $0 \leq s_i \leq m\hat{s}_i$ for all $i = 1, 2, \ldots, m$.*

A.2 Approximate Centers and Measures of Closeness

There are various ways to measure the closeness of a point $\bar{x}$ to the center $\hat{x}$ of an inequality system $Ax \leq b$. We shall give three closely related measures in this subsection. A direct way is to use some norm. In fact, this way of measure was used by many authors of path-following algorithms (Renegar [22], Gonzaga [10], Kojima et al. [15], Monteiro and Adler [18,19], Jarre [12], and Mehrotra and Sun [17], among others). The first measure of closeness to the center is defined in a similar way as follows.

A.2.1 First Measure of Closeness

For all $v \in \mathbf{R}^n$, define the $Q(x)$-norm (Hessian norm) of v by

$$||v||_{Q(x)} := \sqrt{v^T Q(x) v}. \tag{68}$$

Definition: We say that $\bar{x}$ is a δ-*approximate center* of the system $Ax \leq b$ if $||\bar{x} - \hat{x}||_{Q(\bar{x})} \leq \delta$.

We have the following lemma which gives some basic inequalities.

Lemma A.3 ([8] Lemma 3.2) *Suppose $\bar{x} \in \mathbf{R}^n$ is given such that $\bar{s} = b - A\bar{x} > 0$ and let $Q(\bar{x})$ be defined by (67). Then for any $\tilde{x} \in \mathbf{R}^n$ such that $||\bar{x} - \tilde{x}||_{Q(\bar{x})} \leq \delta < 1$, we have*

$$
\begin{array}{ll}
(i) & \tilde{s} = b - A\tilde{x} > 0, \\
(ii) & ||\tilde{S}^{-1}\bar{S}|| \leq 1/(1-\delta), \\
(iii) & ||\bar{S}^{-1}\tilde{S}|| \leq 1+\delta, \\
(iv) & ||v||_{Q(\tilde{x})} \leq \dfrac{1}{1-\delta}||v||_{Q(\bar{x})}, \quad \text{for all } v \in \mathbf{R}^n, \\
(v) & ||v||_{Q(\bar{x})} \leq (1+\delta)||v||_{Q(\tilde{x})}, \quad \text{for all } v \in \mathbf{R}^n, \\
(vi) & ||\bar{x} - \tilde{x}||_{Q(\tilde{x})} \leq \dfrac{\delta}{1-\delta},
\end{array}
$$

where $Q(\tilde{x})$ is defined by (67).

Proof: Observe that

$$
\begin{aligned}
||\bar{x} - \tilde{x}||_{Q(\bar{x})} &= ||(\bar{x} - \tilde{x})^T A^T \bar{S}^{-2} A(\bar{x} - \tilde{x})||^{1/2} \\
&= ||(\tilde{s} - \bar{s})^T \bar{S}^{-2} (\tilde{s} - \bar{s})||^{1/2} \\
&= ||\bar{S}^{-1}(\tilde{s} - \bar{s})|| \leq \delta < 1.
\end{aligned}
$$

Therefore, for each $i = 1, 2, \ldots, m$,

$$\left| \frac{\bar{s}_i - \tilde{s}_i}{\bar{s}_i} \right| \leq \delta,$$

and hence,

$$(1-\delta)\bar{s}_i \leq \tilde{s}_i \leq (1+\delta)\bar{s}_i.$$

Parts (i)–(iii) follow immediately. To prove Part (iv), we observe that

$$||v||_{Q(\bar{x})} = \sqrt{v^T A^T \tilde{S}^{-2} A v} = ||\tilde{S}^{-1} A v||.$$

Therefore, from Part (ii),

$$||v||_{Q(\bar{x})} = ||\tilde{S}^{-1} A v|| \le ||\tilde{S}^{-1} \bar{S}|| \; ||\bar{S}^{-1} A v|| \le \frac{1}{1-\delta} ||v||_{Q(\bar{x})}.$$

The proof of Part (v) is the same as Part (iv) but uses Part (iii), and Part (vi) follows from Part (iv) immediately. *Q.E.D.*

A.2.2 Second Measure of Closeness

The second measure of closeness to the center is defined as follows. For $\bar{x} \in \{x | Ax < b\}$, let $\bar{S} := diag(b - A\bar{x})$ be the diagonal matrix of the slacks at $\bar{x}$. Define

$$Q := (1/m) A^T \bar{S}^{-2} A, \tag{69}$$

$$y = y(\bar{x}) := (1/m) A^T \bar{S}^{-1} e, \tag{70}$$

and

$$\gamma = \gamma(\bar{x}) := \sqrt{\frac{(m-1) y^T Q^{-1} y}{1 - y^T Q^{-1} y}}. \tag{71}$$

Note that $Q := (1/m)Q(\bar{x})$ and $y^T Q^{-1} y < 1$ for $\bar{x} \in \{x | Ax < b\}$, so that $\gamma(\bar{x})$ ((71)) is well-defined, and from the Karush-Kuhn-Tucker conditions ((65)–(66)), $y(\hat{x}) = 0$ and so $\gamma(\hat{x}) = 0$. In [5], the scalar $\gamma(\bar{x})$ is used to measure the closeness of $\bar{x}$ to the center $\hat{x}$. We found this measure to be very convenient because we do not need to know the exact center. To show the relationship between the two measures of closeness, we have the following lemmas. The proofs can be found in the Appendix of [8]. First we need to refer to two functions defined in [5] (equations (6.3) and (6.4) of [5] respectively). Define, for $h > 0$,

$$p(h) := \frac{h - \ln(1+h)}{h^2}, \tag{72}$$

$$q(h) := \frac{1}{2}\left(1 + hp(h) - \sqrt{1 + (hp(h))^2}\right). \tag{73}$$

Lemma A.4 ([5] Lemma 8.1, [8] Lemma 3.3) .

Let $\hat{x}$ denote the center of the system $Ax \le b$. Let $h > 0$ be a given parameter. Suppose $\gamma = \gamma(\bar{x}) \le q(h)$, where $q(h)$ is given in (72)—(73). Then

$$||\bar{x} - \hat{x}||^2_{Q(\bar{x})} \le \left(\frac{m}{m-1}\right) \frac{h^2(1+\gamma^2)}{(1-h\gamma)^2}.$$

That is, if $\gamma(\bar{x})$ is small then $\bar{x}$ is a δ-approximate center for some small δ. For example, if $\gamma(\bar{x}) \le .0072$ (taking $h = 0.03$), then $||\bar{x} - \hat{x}||_{Q(\bar{x})} \le 1/21$.

On the other hand, if $\bar{x}$ is a δ-approximate center then we have the following lemma which says that $\gamma(\bar{x})$ should be small.

Lemma A.5 ([8], Lemma 3.4) *Suppose $||\bar{x} - \hat{x}||_{Q(\bar{x})} \le \delta < 1/2$. Then $\gamma(\bar{x}) \le a + \sqrt{2a}$, where $a = \dfrac{\delta^2}{2(1-\delta)(1-2\delta)}$.*

For example, when $\delta = 1/21$, then $\gamma(\bar{x}) \le 0.0527$.

A.2.3 Third Measure of Closeness

Observe that in the definition of $\gamma(\bar{x})$ ((71)), $y^T Q^{-1} y$ may be expressed as

$$\begin{aligned} y^T Q^{-1} y &= \frac{1}{m} e^T \bar{S}^{-1} A Q(\bar{x})^{-1} A^T \bar{S}^{-1} e \\ &= \frac{1}{m} ||A^T \bar{S}^{-1} e||^2_{Q(\bar{x})^{-1}}. \end{aligned}$$

We note that $A^T \bar{S}^{-1} e$ is the gradient of the logarithmic barrier function in the center problem CP at $\bar{x}$. Therefore, when $y^T Q^{-1} y$ is sufficiently small, $\gamma(\bar{x})$ is almost the same as the size of the gradient of the logarithmic barrier function measured in the norm of the Hessian inverse. We therefore define the third measure of closeness as follow.

For any $\bar{x} \in \mathbf{R}^n$ satisfying $\bar{s} = b - A\bar{x} > 0$, let $Q(\bar{x})$ be defined by (67). Define $\tau = \tau(\bar{x})$ by

$$\tau = \tau(\bar{x}) := ||A^T \bar{S}^{-1} e||_{Q(\bar{x})^{-1}}. \tag{74}$$

It is easy to see that $\tau(\bar{x}) < \sqrt{m}$ for $\bar{x} \in \{x|Ax < b\}$, and that γ and τ are related as follow.

Lemma A.6

$$(i) \quad \gamma^2 = \frac{(m-1)\tau^2}{m-\tau^2}; \qquad (ii) \quad \tau^2 = \frac{m\gamma^2}{m-1+\gamma^2}.$$

Therefore, the following corollary follows easily.

Corollary A.2 *Let $\gamma = \gamma(\bar{x})$ be defined by (69)—(71) and $\tau = \tau(\bar{x})$ be defined by (74).*

$$(i) \qquad \text{if } \tau \le 1, \text{ then } \gamma \le \tau,$$

$$(ii) \qquad \tau \le \gamma\sqrt{m/(m-1)}.$$

We observe that the factor of $\sqrt{\frac{m}{m-1}}$ in Corollary A.2 is almost equal to 1, even for moderate values of m. For example, $\sqrt{\frac{m}{m-1}} \le 1.05$ if $m > 10$. From Lemma A.4 and Corollary A.2, we have

Lemma A.7 *Let $\tau = \tau(\bar{x})$ be defined by (74). Suppose $\tau \le 1/76$. Then $||\bar{x} - \hat{x}||_{Q(\bar{x})} \le 1/12$.*

Proof: From Lemma A.6(i), we have $\gamma \le \tau \le 1/76$. Thus from Lemma A.4 with $h = 1/18$,

$$||\bar{x} - \hat{x}||^2_{Q(\bar{x})} \le \frac{2h^2(1+\gamma^2)}{(1-h\gamma)^2} < (\frac{1}{12})^2. \qquad Q.E.D.$$

Lemma A.8 *Assume that $m > 10$. Let $\tau = \tau(\bar{x})$ be defined by (74). If $||\bar{x} - \hat{x}||_{Q(\bar{x})} \le 1/21$, then $\tau \le 0.056$.*

Proof: From Lemma A.5 we conclude that $\gamma = \gamma(\bar{x}) \le a + \sqrt{2a}$, where $a = \frac{(1/21)^2}{2(1-(1/21))(1-2(1/21))} = \frac{1}{760}$. Thus by Lemma A.6(ii), we have $\tau \le \sqrt{m(m-1)}\gamma \le (1.05)(0.053) < 0.056$. *Q.E.D.*

A.2.4 Property of An Approximate Center

Analogous to Lemma A.1, we have the following property of an approximate center.

Lemma A.9 ([5] Theorem 8.1, [8] Lemma 5.2) .
Let $\hat{x}$ denote the center and suppose $\bar{x}$ is a δ-approximate center of the linear inequality system $Ax \leq b$ with $\delta < 1$. Define the ellipsoids F_{in} and F_{out} by

$$\begin{aligned} F_{in} &:= \{ x \in \mathbf{R}^n : ||x - \bar{x}||_{Q(\bar{x})} \leq 1 \}, \\ F_{out} &:= \{ x \in \mathbf{R}^n : ||x - \bar{x}||_{Q(\bar{x})} \leq (1+\delta)\sqrt{m(m-1)} + \delta \}. \end{aligned}$$

Then $F_{in} \subset \mathcal{X} \subset F_{out}$.

That is, we can construct contained and containing ellipsoids centered at a δ-approximate center. Note that F_{out} is an enlargement of F_{in} with an enlargement factor of $O(m)$. The elliptical bounds of Lemma A.9 are used in the derivation of complexity bounds.

A.3 Newton's Method for the Center Problem

The following important useful result, giving a region and rate of convergence of Newton's method for the center problem CP, is due to Renegar [22].

Lemma A.10 ([22] Theorem 3.2) *Let $\hat{x}$ denotes the center of the system $Ax \leq b$ and suppose $\bar{x}$ satisfies $\bar{s} = b - A\bar{x} > 0$ and $\varepsilon := ||\bar{x} - \hat{x}||_{Q(\hat{x})} < 1$. Let $\bar{\eta} := Q(\bar{x})^{-1}A^T\bar{S}^{-1}e$ be the Newton step from $\bar{x}$ in the center problem CP, and let $\bar{y} := \bar{x} + \bar{\eta}$. Then*

$$||\bar{y} - \hat{x}||_{Q(\hat{x})} \leq \frac{(1+\varepsilon)^2}{1-\varepsilon}\varepsilon^2.$$

Remarks: The Newton step $\bar{\eta}$ is the solution of an unconstrained quadratic approximation to the center problem CP,

$$\bar{\eta} = \arg\max \{e^T\bar{S}^{-1}A\eta - \frac{1}{2}\eta^T Q(\bar{x})\eta \mid \eta \in \mathbf{R}^n\},$$

where $A^T\bar{S}^{-1}e$ and $Q(\bar{x})$ are the gradient and the negative of the Hessian of the logarithmic barrier function $\sum_{i=1}^{m}\ln(b_i - A_i x)$ at $\bar{x}$. The solution $\bar{\eta}$ can be obtained by solving an $n \times n$ system of linear equations

$$Q(\bar{x})\eta = A^T\bar{S}^{-1}e.$$

With respect to the third measure of closeness to the center, we see from (74) that

$$\tau(\bar{x}) = ||\bar{\eta}||_{Q(\bar{x})}.$$

In other words, $\bar{x}$ is close to the center if the Newton step from $\bar{x}$ for the center problem CP is "small", that is, measured in an appropriate norm – the Hessian norm.