Hotel Sales and Reservations Planning†

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Abstract

The profitability of a hotel depends largely on how well it uses its capacity. However, managing this operation is immensely difficult. Reservations and the other major sources of room demand--stay extensions and walk-ins--have associated uncertainties. Hotel operators must determine how to allocate rooms to guests who are willing to pay different rates and, at the same time, manage a reservation operation with these uncertainties.

This study is motivated by the description of an actual hotel sales and reservations planning problem. In our problem, stays are not limited to single days and there are multiple room-types. We introduce the concept of guest-classes. Each class corresponds to a market segment: people who want a particular room-type, want to pay no more than a particular rate, and have similar cancellation and show behaviors.

We study how hotels should plan reservations and manage sales under fairly general conditions. The model can be used to support decision-making by providing an analytical approach for setting targets and rates for rooms occupancy, and marketing and sales planning.

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1. Introduction

Hotels take room reservations from a few months to one day in advance. Prospective guests can cancel their reservations anytime before the day the rooms are required; cancellations are made with no penalty. Prospective guests, without informing the hotel, may even fail to show up for their reservations. The number of cancellations and no-shows can be highly variable. Though expected no-show rate is around 15%, Rothstein [1974] quoted estimates from hotel executives that no-show rates in excess of 25% are common, indicating the problem's magnitude and difficulty.

Other major sources of room demand are stayover and walk-in. Occasionally, trips must be taken on short notice, forcing the traveler to seek accommodations as a walk-in, a prospective guest with no reservation. Even when a guest makes and honors a reservation, the estimated length of stay may be inaccurate. A business executive who planned a three day visit, for example, may take four days to settle her affairs, thus making it necessary to extend the room occupation. Conversely, she may finish in two days, permitting early departure. Therefore, these room demands are also random.

Even though the major sources of demand are random, some types of demand can be controlled. Reservation demand is controlled by limiting the number of reservations to accept. Stayovers cannot exceed the number of rooms currently occupied. This, in turn, depends on the number of reservations previously accepted. Some hotels, in policy, honor all requests for stay extension. But most hotels, depending on capacity available, may or may not extend a stay beyond what was scheduled. (When this general practice is resisted, hotels will usually back-off to avoid unnecessary negative publicity. Occurrences like these are rare and may be neglected.) As such, the hotels have some control over stay extensions. Similarly, walk-in demands can be selectively rejected when there is insufficient capacity. Premature departures, on the other hand, cannot be directly controlled. So as to get enough time to adapt, most operators set rules on the amount of pre-checkout notice their guests must give.

In sharp contrast to the consumer's right of cancellation without penalty, a hotel, on the other hand, is obligated to live up to its reservation commitments. To remain competitive and profitable, it is prudent that hotels plan how they run the reservation operation. We propose that they plan the booking of reservations, to complement the other demands. We aim towards maximizing expected profit subject to service constraints for meeting the demand from booked reservations. We believe that this is a novel formulation for the hotel problem.

The problem is related to the production planning problem with stochastic yields. The number of reservations to accept corresponds to the production lot size and no-shows correspond to rejects. Reservations accepted and guests present are equivalents of stocking items. These stocks "perish" when there are cancellations or premature departures. Unlike manufacturing of products, services such as hotel room "rentals" cannot be produced ahead of time and stocked in anticipation of seasonal demand. Hence capacity not utilized is lost forever; pre-emptive production is not possible. Furthermore, since there is no backordering, demands not met are also lost forever. From this comparison, we see that the hotel reservation problem is richer and more interesting than the production planning problem.

This paper is organized as follows. We review in section 2 the literature related to the hotel reservation problem. Section 3 describes the problem that we intend to solve. In section 4, we formulate the problem as linear programs and present the main results. Additional comments and extensions are given in section 5. We end the paper with a summary and conclusions.

2. Literature Review

Rothstein [1974] claimed that he found no published model directed specifically to the hotel problem and provided one. His model is an extension of the airline overbooking problem examined previously by Rothstein [1968, 1971a, 1971b]. He used the Markovian sequential decision process to generate booking policies for hotels with one room-type and single-day stays. This problem differs from the airline problem by allowing double occupancy--more than one guest per room.

Ladany [1976] extended Rothstein's airline work to provide a hotel model where there are two room-types: single and double rooms. Stay durations are still limited to single-days only. The author claimed that the model may be extended for many room-types and multiple-day stays. The state space for this dynamic program will be huge. One study that explicitly model stays of more than one period is [Kinberg, Rao, and Sudit 1980]. In this model, there are two categories of demand: package (subscription) and spot. The model determines how the fixed resource capacity should be allocated to the two demand categories. Subscriptions are sold with price discounts, but are paid in advance; the trade-off is between degree of demand uncertainty and expected total revenue. The problem is fundamentally different from ours in that tickets sold are paid; no-shows do not create problems. Glover et al. [1982] and Pfeifer [1989] studied how airlines should allocate capacity to different fare classes. Again, these problems do not consider cancellations and show uncertainties.

Liberman and Yechiali [1978] allow hotels to cancel confirmed reservations or acquire additional reservations. Both are done with penalties to the hotel. With identical rooms and focusing on a single target date, they showed that the optimal policy consists of 3 regions demarcated by 2 threshold numbers. The regions are where the options--(a) accept all new requests and acquire additional reservations, (b) do nothing, and (c) cancel some confirmed reservations--are appropriate. This model is essentially an extension of the well-known newsvendor problem. Buying and selling of reservations may be viewed as an indirect approach of incorporating the multiple room-types feature in a one room-type model.

William's [1977] model is the most complete, considering practically all the major sources of demand. However, his model assumes that there is only one type of room. He evaluated the problem on three separate criteria: expected cost, expected underbook and number of walks, and expected occupancy rate and number of walks. Walks are people who have made reservations but cannot check-in because of room shortages; they walk away dissatisfied. The most interesting outcome from William's work is a set of histograms and smoothed approximations constructed from data obtained from two hotels. He showed that reservations, scheduled stayovers, and unscheduled stayovers show-rates can be approximated by Beta distributions; and walk-ins follow the Gamma distribution. Scheduled stayover show-rate is one minus premature departure rate.

Even though the works mentioned studied service operations, they and most others do not incorporate explicit measures on service performance. Exceptions include the work by Thompson [1961], Taylor [1962], Shlifer and Vardi [1975], and Jennings [1981]. Thompson, who initiated the approach, studied control issues in airline reservations. He provides feasible solutions to the problem with two seat-classes that has constraints on the risk of exceeding capacity. No cost parameter or objective function is present in this problem or in the problems in the other papers mentioned in this paragraph. Single flight-leg problems, in these papers, are similar to one period hotel problems; multiple flight-legs problems are similar to multiple periods problems.

In general, the airline problem has a lot of features in common with the hotel problem. The interested reader should refer to [Rothstein 1985] for a review of that problem. Other related problems include hospital admissions and bed allocations ([Kao and Tung 1981]), clinic appointment systems ([Rising, Baron and Averill 1973]), and car or equipment rentals ([Tainter 1964] and [Whisler 1967]).

In this paper, we draw upon the parallel between the hotel problem and the manufacturing problem solved in [Bitran and Leong 1989]. The problem considered in that paper has random production yield and substitutable product demand. Unlike previous hotel reservation studies, the formulation we provide has multiple periods, room-types, and guest-classes. New features addressed, not found in the manufacturing problem, include perishability of inventory, no pre-emptive production, and multiple recourse opportunities. Also, in manufacturing terminology, the related production model backorders when there are shortages whereas hotels has lost-sales.

We alluded to the first two features in the introduction. We now mention briefly what multiple recourse opportunities mean. Reservations, made in advance, may be cancelled by the guest before the required day. However, as long as that day is still in the future, additional reservations can be accepted, to make up for those cancelled. So the hotel model, unlike the manufacturing analogue we mentioned, has multiple opportunities to respond to a demand--room-type for a certain day.

3. Problem description

Hotel rooms are frequently classified into types: suite, deluxe, and standard rooms, to suit different lifestyles and budgets. When a prospective guest with reservation, arriving in good time, finds no available room in the hotel, an oversale is said to have taken place. Oversale occurs because hotels sometimes overbook reservations to keep occupancy levels high. When oversale of a particular room-type occurs, hotel operators can choose between turning away the prospective guest or giving her, at no additional cost, a better room. The first option must be mitigated with an offer of alternative accommodation--at a competing hotel--and freebies, for example, a free dinner at the hotel's restaurant. In addition to loss of revenue and extra costs, the fear of goodwill loss makes hotel management desire to see this happen as rarely as possible. "Downgrading" a room, on the other hand, adds a contribution to profit though smaller than what it is potentially capable of. Nevertheless, the downgraded room may have remained vacant and contributed nothing.

We classify hotel rooms into ordered types $s \in \{1,..,m\}$ where 1 is the most luxurious and m the least. A room from each room-type may be offered at more than one rate. The rates are different because of the nature of occupancy (single/double/with children), discounts, commissions, and costs of extra promotion. We also classify the market into ordered classes $i \in \{1,..,n\}$. Now, we let $a(s)$, $s=1,..,m$, be the indices of classes such that $1=a(1) < a(2),.., < a(m) \leq n$ and guests in classes $a(s),..,a(s+1)-1$ request room-type s.

Class i guests pay c_i per room for each night of occupancy. The guest-classes for the same room-type are labeled in descending order of the rates charged; guests for room-type s may be charged any of the rates c_i, i

ϵ {a(s),..,a(s+1)-1}. The highest rate for each room-type is often referred to as the rack rate for that room-type. We assume that guests of more luxurious rooms always pay more for their rooms than guests of less luxurious rooms; that is, $c_i \geq c_j$ if $i < j$.

The reader should note that classes are not necessarily defined according to rates alone: market segments that compete for the same room-type and pay the same rates may be classified as different classes. The classes defined, however, must not be discriminatory against individuals and, at the time of receiving a reservation request, the hotel operator should be able to distinguish which class the request belongs to. For example, Shlifer and Vardi [1975] mention that, because of the significant differences in their cancellation and show behaviors, reservations from different geographical origins have been classified into different classes.

Figure 1 demonstrates, with an example, the relationship among the room-types and guest-classes. Each vertex represents a guest-class. A directed edge leading from vertex i to vertex j represents the possibility that a room allocated to class i can be offered to class j. By virtue of the labeling order of room-types and guest-classes, there is a directed edge from every class i to i+1. That is, a class j guest paying class j rate, but offered a room that is acceptable to class i guests will not be dissatisfied if $i < j$.

	Suite	Deluxe	Standard
ROOM-TYPE:	1	2	3
GUEST-CLASS:	1 ------> 2 ------>	3 ------> 4 ------> 5 ------>	6 ------> 7

Figure 1. Room-types and Guest-classes--An example

Set aside for prospective class i guests are N_{it}, $i=1,..,n$ $t=1,..,T$, number of reservations for period t at the start of period t. T is the length of the planning horizon. The number of class i guests who will show up in period t is q_{it} N_{it} where q_{it} is the class i reservation show-rate for period t. Show-rate $q_{it} \varepsilon [0,1]$ is a continuous random variable. The yield or show size is given as a product of the show-rate and the size of the reservation. This assumes that the yield rate distribution is independent of the reservation size. Liberman and Yechiali [1978] made the same assumption and William [1977] provided empirical evidence that this assumption is reasonable. We will also assume that reservation and show sizes are sufficiently large that the requirement for the decision variables to be integers may be relaxed.

4. Model

The purpose of our model is to assist in the planning for the optimal level of reservations and in appropriating the hotel's capacity to market segments. These decisions support both the sales and operations functions. We propose to solve the problem in two stages: (a) reservations planning, and (b) walk-in control. We end this section with additional guidelines for managing sales and setting room rates.

RESERVATIONS PLANNING

For a given horizon, we first work toward getting the optimal reservation target levels. Operators are then "authorized" to accept reservations up to these levels. The target levels suggest how the capacity of the hotel should be allocated to the guest-classes; the reservations targets are attainable only when there is sufficient demand. We do not consider any specific assignment of rooms to the reservations since the reservations may be cancelled or may not show. By default, the capacity remaining is for walk-in guests. Walk-in guests, usually charged rack rates, may have a significant portion of the rooms set aside for them. Unlike the airline reservations problem, hotels do not always need to overbook reservations because the walk-in demand, fetching high returns, can be substantial.

In figures 1 and 2 of his paper, William [1977] fitted Beta distributions to reservations and scheduled stayover show-rate data and showed that the fits are excellent. The mean and coefficient of variation of the fitted reservations show-rate distribution are 0.83 and 0.083 respectively. The corresponding statistics for scheduled stayover show-rate are 0.86 and 0.083. It is reasonable to assume that the two show-rates are statistically independent. These evidences suggest that scheduled stayovers and reservations have show-rate probability distributions that are practically identical.

As such, when a prospective guests make reservations for period t, to stay for s days ($s \geq 1$), we record the reservation as s separate individual reservations for periods $t, .., t+s-1$. When a guest with multiple-days booking does not show on the first day of the intended stay or cuts short the scheduled stay, the bookings for the remaining days are considered cancelled. For the rest of this paper, we refer to the combined show-rate distribution of reservations and scheduled stayovers as simply the show-rate distribution.

Booked reservations, being commitments, are given the highest priority when conflict arises. The second priority goes to walk-ins. Stay extensions have the lowest priority: hotels are not bound to satisfy stay extension requests. No service performance limits are set for meeting stay extension requests; stay extension inquiries will be treated as if they are new reservation requests. When stay extension "reservations" requests have show behaviors that are different from the normal reservations requests, we create new guest classes for them.

We define MU_{st} as the number of type s rooms available in period t and M_{it} to be the number of rooms initially allocated to guest-class i for period t. We set $M_{it} = MU_{st}$ for $i = a(s)$ and $M_{it} = 0$ otherwise, $s=1,..,m$, $t=1,..,T$. In this way, we allocate the rooms to the highest guest-class possible and we make the rooms available indirectly to the other classes through W_{it}, the number of rooms from those allocated to class i to downgrade to class i+1 during period t.

We define NS_{it} as the random variable for the demand of guest-class i reservations in period t and YS_{it} as the number of class i prospective guests that will walk into the hotel during period t without reservations. NS_{it} and YS_{it}, $i=1,..,n$ and $t=1,..,T$, have finite mean and variance, and are random variables in $[0,\infty)$. Figure 2 shows the sources of demand by class i prospects for rooms in period t.

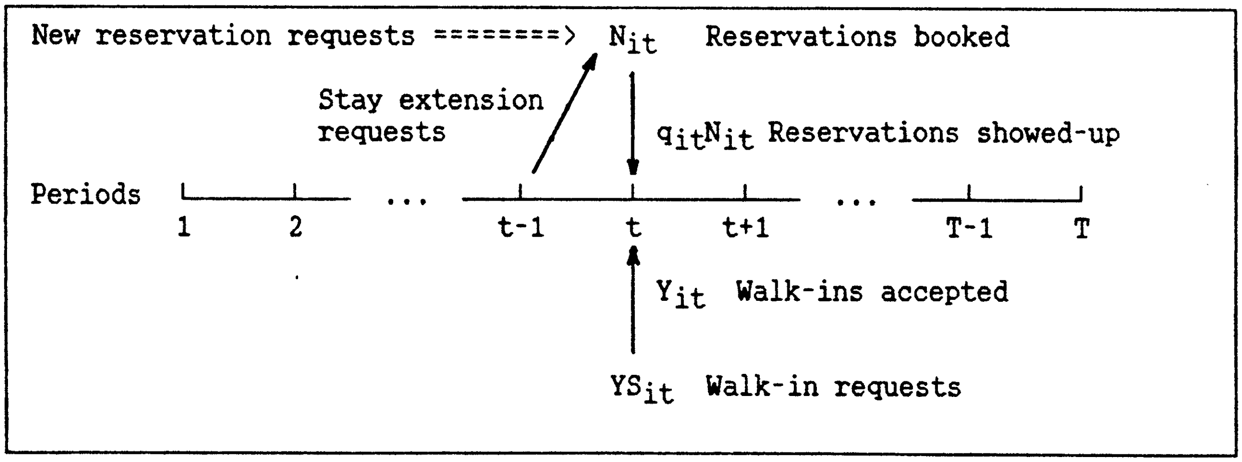

Figure 2. Demand by class i prospects for rooms in period t.

For simplicity of presentation, we let $A(s) = \{a(s),..,a(s+1)-1\}$ and $AU(s) = \{1,..,a(s+1)-1\}$, $s=1,..,m$. $A(s)$ is the set of all guest classes for type s rooms and $AU(s)$ is the set of all guest-classes that can be offered type s rooms. We present, below, a stochastic linear programming formulation of the reservations problem.

(SPa)

$$Z_{SPa} = \text{Max } \{E_q[E_{NS}[\Sigma^n_{i=1}\Sigma^T_{t=1}\ c_{it}\ (M_{it} + W_{i-1,t} - W_{it} - J_{it}^+)] \quad (1)$$

$$+ \text{ Max } \Sigma^n_{i=1}\Sigma^T_{t=1}\ c_{it}\ E_{YS}[Y_{it}]] \quad (2)$$

subject to

$$Y_{it} \le YS_{it}, \quad i=1,..,n,\ t=1,..,T, \quad (3)$$

$$0 \le Y_{it} \le J_{it}^+, \ i=1,..,n,\ t=1,..,T \ \} \quad (4)$$

subject to

$$N_{it} \le NS_{it}, \quad i=1,..,n,\ t=1,..,T, \quad (5)$$

$$\text{Prob}(J_{it} \ge 0,\ i=1,..,n) \ge \alpha, \quad t=1,..,T, \quad (6)$$

$$W_{it},\ N_{it} \ge 0, \quad i=1,..,n,\ t=1,..,T, \quad (7)$$

where $E_x[.]$ is the expectation function over the random vector x; and q, NS, YS are the vectors of random variables q_{it}, NS_{it}, YS_{it} respectively. Also, $W_{0t} = 0$, t=1,..,T,

$$J_{it} = M_{it} + W_{i-1,t} - W_{it} - q_{it} N_{it}, \quad i=1,..,n,\ t=1,..,T, \quad (8)$$

and

$$J_{it}^+ = \text{Max}(0,\ J_{it}), \quad i=1,..,n,\ t=1,..,T. \quad (9)$$

In the reservations problem (SPa), we optimize the total expected revenue, by allocating rooms among reservation and walk-in prospects. This is subjected to service constraints to ensure that revenues are not increased by making reservation promises that the hotel cannot usually keep.

The first term in the objective function, (1), is obtained as follows: the revenue from class i guests in period t equals $c_{it}\ q_{it}\ N_{it}$ when $J_{it} \ge 0$, and $c_{it}\ (M_{it} + W_{i-1,t} - W_{it})$ when $J_{it} < 0$; with some algebraic manipulation and taking expectation results in (1). We call (2) to (4), the sub-problem in (SPa), (RPa): (RPa) is the walk-in recourse problem. Constraint (3) states that walk-ins accepted cannot exceed walk-in requests. Constraint (4) ensures the capacity of the hotel is not exceeded and walk-ins cannot take negative values. Constraint (5) makes certain that the reservations booked cannot exceed reservations requested. The service constraint, (6), guarantees that oversale occurs with less than $100(1-\alpha)\%$ probability. $\alpha \in [0,1]$ is the service performance target for booked reservations and, set according to management policy, should be close to 1. Constraint (7) are non-negativity constraints for the decision variables in the main problem. The other equations, self-explanatory, are introduced to simplify the presentation.

Notice that constraints (3) and (5) have stochastic right-hand-side terms that must not be violated. Therefore, other than the trivial zero reservations and zero walk-ins solution, there is no other feasible solution to (SPa). As such problem (SPa) has no meaningful solutions; we will reformulate the problem slightly. Before proceeding further, we present some important results of the reservations planning problem.

Theorem 1 [Time period separation]: Problems (SPa) separates into T one-period problems. ▪

This theorem suggests that reservations planning can be executed by focussing on one period at a time. In view of this, the results of earlier papers that focus on single-period problems may be valid. Therefore, by theorem 1, we drop the period index, t, and focus on a particular period of interest--referred to, from here on, as the target period. All subsequent reference to equations will be made as if index t does not exist.

Theorem 2 [Joint chance constraint separation]: Constraint (6) is equivalent to $\mathrm{Prob}(\Sigma^{i}_{j=1}\, q_j\, N_j \le \Sigma^{i}_{j=1}\, M_j) \ge \alpha$, $i=1,..,n$.

Proof of theorem 2: Constraint (6) => $\mathrm{Prob}(\Sigma^{i}_{j=1}\, J_j \ge 0) \ge \alpha$, $i=1,..,n$. By equation (8), $\mathrm{Prob}(\Sigma^{i}_{j=1}\, (M_j - W_j - q_j\, N_j) \ge 0) \ge \alpha$, $i=1,..,n$. By the non-negativity constraint (7), $W_i \geq 0$. Hence the result. The converse is true using similar arguments and downgrading when necessary. ▪

Theorem 2 provides an alternative way of expressing the service constraint. The resulting separation of the original joint chance constraint into individual chance constraints makes the problem more tractable.

We now reformulate the problem by incorporating constraints (3) and (5) into the objective function but first we introduce more notation. We define $f(x;y)$ and $F(x;y)$ to be the value of the probability density and the cumulative density functions respectively for any random variable x evaluated at y. We let $YS_i(y) = YS_i$ for $YS_i \leq y$ and $YS_i(y) = y$ otherwise, $i=1,..,n$. Therefore, $YS_i(y)$, $i=1,..,n$, are random variables which are the same as YS_i, $i=1,..,n$, except that all its densities beyond y is concentrated at y. Using this new variable, we can insert (3) into the $E_{YS}[Y_i]$ term in the objective function of (RPa) to give (RPb).

(RPb)

$Z_{RPb} = \text{Max } \Sigma^n_{i=1}\ c_i\ E_{YS}[YS_i(Y_i)]$	(2a)
subject to	
$0 \leq Y_i \leq J_i^+$, $\quad$ $i=1,..,n$.	(4a)

<u>Theorem 3</u>: $E_{YS}[YS_i(y)]$, $i=1,..,n$, is non-decreasing, concave in y.

<u>Proof of theorem 3</u>: The first derivative of $E[YS_i(y)] = 1 - F(YS_i;y) \geq 0$. Also, the second derivative of $E[YS_i(y)] = - f(YS_i;y) \leq 0$. Therefore, $E[YS_i(y)]$, $s=1,..,m$, is non-decreasing and concave in y. ▪

<u>Theorem 4</u>: (RPb) separates into m sub-problems, one for each guest-class i, and it has the optimal solution $Y_i = \text{Max}(0, J_i) \equiv J_i^+$, $i=,..,n$. ▪

Using the results of the theorems above, we re-write (SPa) to give (SPb).

(SPb)

Z_{SPb}=Max $E_q[E_{NS}[\Sigma^n_{i=1}\ c_i(M_i+W_{i-1}-W_i-J_i^+)] + \Sigma^n_{i=1}\ c_i\ E[YS_i(J_i^+)]]$		(1a)
subject to		
$N_i \le NS_i$,	$i=1,..,n$,	(5a)
$\text{Prob}(\Sigma^i_{j=1}\ q_j\ N_j \le \Sigma^i_{j=1}\ M_j) \ge \alpha$,	$i=1,..,n$,	(6a)
$W_i,\ N_i \ge 0$,	$i=1,..,n.$	(7)

We mentioned, in the first paragraph of section 3, that oversales are usually mitigated with offers of alternative accommodations. Up to now, we have not included the cost of oversales into the problem. This cost, except for the more explicit components, is usually quite difficult to quantify. We will assume, from here on, that the cost of oversale for each guest-class is its room rate. This is an attempt to capture as much of the quantifiable costs as possible. Of course, we already have constraints to ensure that the service goals are met--an indirect way of acknowledging the more esoteric costs. The resulting program differs from (SPb) by the absence of the $(.)^+$ function for the second term in the objective function. We also repeat the approach used to reformulate (RPa) to incorporate constraint (5) into the objective function. We present the final formulation as (SPc).

(SPc)

$$Z_{SPc}=\text{Max } E_q[E_{NS}[\Sigma^n{}_{i=1}\ c_i(M_i+W_{i-1}-W_i-J_i(N_i)^+)] + \Sigma^n{}_{i=1}\ c_i\ E[YS_i(J_i)]] \quad (1b)$$

subject to

$$\text{Prob}(\Sigma^i{}_{j=1}\ q_j\ N_j \le \Sigma^i{}_{j=1}\ M_j) \ge \alpha, \quad i=1,..,n, \quad (6a)$$

$$W_i,\ N_i \ge 0, \quad i=1,..,n, \quad (7a)$$

where $NS_i(y) = NS_i$ for $NS_i \le y$ and $NS_i(y) = y$ otherwise, $i=1,..,n$, (10)

$$J_i(y) = M_i + W_{i-1} - W_i - q_i\ NS_i(y), \quad i=1,..,n, \quad (11)$$

and

$$J_i(y)^+ = \text{Max}(0,\ J_i(y)), \quad i=1,..,n. \quad (12)$$

APPROXIMATIONS

We propose two approximations: stochastic and deterministic. Each approximation leads us progressively towards a tractable problem. (SP), the stochastic program, approximates (SPc) by linearizing the feasible region of (SPc). The user chooses how accurate the approximation should be. At the expense of doing an infinite amount of work, (SP) becomes (SPc). In our experience, a simple approximation like the one we are presenting has small relative errors. The deterministic approximation (DP1) approximates (SP) by simplifying the objective function. An upper bound on the relative error between (SP) and (DP1) is presented in theorem 6. Lastly, we linearize the separable convex program (DP1) into deterministic linear program (DP2).

To construct (SP), we replace, for each i, the service constraint (6a) by a set of linear constraints. This set of linear constraints is uniformly tighter than the original constraint it replaces; any solution feasible to the set of linear constraints is also feasible to the original constraint. The detail of this inner-linearization approach is discussed in [Bitran and Leong 1989]. The approach is as follows: We define $\phi(a_1,..,a_n) = F^{-1}(\Sigma^n{}_{i=1}\ a_i\ q_i;\ \alpha)$ where $a_i \ge 0$, $i=1,..,n$ and unit vector u_i =

$(a_1,..,a_n)$ where $a_j = 1$ for $j = i$ and $a_j = 0$ otherwise. We let vector $[\Omega_{1k},..,\Omega_{nk}]$ be such that $\Omega_{ik} = \phi(\Sigma^n{}_{j=1}\, u_j) - (\Sigma^n{}_{j=1}\, \phi(u_j) - \phi(u_k))$ for $i = k$ and $\Omega_{ik} = \phi(u_i)$ otherwise, $i = 1,..,n$, $k = 1,..,K1(i)$, and $K1(i) = n$. The vectors are the coefficients of the decision variables in the service constraints. (SP) is presented below.

(SP)

Z_{SP}=Max $E_q[E_{NS}[\Sigma^n{}_{i=1}\, c_i\, (M_i+W_{i-1}-W_i-J_i(N_i)^+)] + \Sigma^n{}_{i=1}\, c_i\, E[YS_i(J_i)]]$	(1b)
subject to	
$\Sigma^i{}_{j=1}\, \Omega_{jk}\, N_j \le \Sigma^i{}_{j=1}\, M_j$, $k=1,..,K1(i)$, $i=1,..,n$	(6b)
W_i, $N_i \ge 0$, $i=1,..,n$	(7a)

<u>Theorem 5</u>: $E_{NS}[NS_i(y)]$, $i=1,..,n$, is non-decreasing, concave in y.

<u>Proof of theorem 5</u>: Same as in theorem 3. ▪

For α sufficiently close to 1, by constraints (6), (6a), or (6b) and the presence of a recourse problem, the capacity allocation guarantees that oversale seldom happen: $J_i \ge 0$ most of the time. As an approximation, we will assume that $J_i \ge 0$ for all i. Next, we remove the outer most expectation function and take expectation of variable J_i.

(DP1)

Z_{DP1} = Max $\Sigma^n{}_{i=1}\, c_i\, (E[q_i]\, E_{NS}[NS_i(N_i)] + E_{YS}[YS_i(E_q[J_i])]])$	(1c)
subject to	
$\Sigma^i{}_{j=1}\, \Omega_{jk}\, N_j \le \Sigma^i{}_{j=1}\, M_j$, $k=1,..,K1(i)$, $i=1,..,n$	(6b)
W_i, $N_i \ge 0$, $i=1,..,n$	(7a)

<u>Theorem 6</u> [Upper bound on the relative error between the value of the optimal solutions to (SP) and (DP1)]: Let vector N^* be the optimal solution to (DP1) and vector W^* be such that (N^*,W^*) is a feasible solution in (SP). The relative error between the values of the optimal solutions to (DP1) and

(SP) is bounded from above by $(Z_{DP1} - ZU(N^*,W^*))/ZU(N^*,W^*)$ where $ZU(N^*,W^*)$ is the value of (N^*,W^*) in (SP).

Proof of theorem 6: We call upon the convex properties of functions $(.)^+$, and theorems 3 and 5 to apply Jensen's inequality. ▪

By theorems 3 and 5, using a standard approach in separable convex programming, we linearize the objective function: (a) the first term in (1c) is replaced by $\Sigma^n_{i=1}\Sigma^{K2(i)}_{k=1} d_{ik}\, x_{ik}$ where d_{ik}, $d_{i1} > .. > d_{i,K2(i)}$, are new cost coefficients and x_{ik}, $0 \le x_{ik} \le xu_{ik}$, i=1,..,n, k=1,..,K2(i) are the new variables; (b) the second term in (1c) is replaced by $\Sigma^n_{i=1}\Sigma^{K3(i)}_{k=1} e_{ik}\, y_{ik}$ where e_{ik}, $e_{i1} > .. > e_{i,K3(i)}$, are new cost coefficients and y_{ik}, $0 \le y_{ik} \le yu_{ik}$, i=1,..,n, k=1,..,K3(i) are the new variables. Note that $N_i = \Sigma^{K2(i)}_{k=1} x_{ik}$ and each x_{ik} is contained in a given partition where the expected marginal return is approximately d_{ik}. Similarly, $E[J_i] = \Sigma^{K3(i)}_{k=1} y_{ik}$ and each y_{ik} is contained in a partition where the expected marginal return is approximately e_{ik}. $K_{2(i)}$ and $K_{3(i)}$ are the number of piecewise-linear segments used to approximate each of the corresponding functions. After making the approximations, we simplify and present the new problem as (DP2).

(DP2)

$Z_{DP2} = \text{Max } \Sigma^n_{i=1} (\Sigma^{K2(i)}_{k=1} d_{ik}\, x_{ik} + \Sigma^{K3(i)}_{k=1} e_{ik}\, y_{ik})$		(1d)
subject to		
$\Sigma^i_{j=1} \Omega_{jk} N_j \le \Sigma^i_{j=1} M_j$,	k=1,..,K1(i), i=1,..,n	(6b)
$\Sigma^{K2(i)}_{k=1} x_{ik} = N_i$,	i=1,..,n	(13)
$0 \le x_{ik} \le xu_{ik}$,	k=1,..,K2(i), i=1,..,n	(14)
$\Sigma^i_{j=1}\Sigma^{K3(j)}_{k=1} y_{jk} \le \Sigma^i_{j=1} (M_j - E[q_j]\, N_j)$,	i=1,..,n	(15)
$0 \le y_{ik} \le yu_{ik}$,	k=1,..,K3(i), i=1,..,n	(16)
$N_i \ge 0$,	i=1,..,n	(17)

In practice, hotels designate some capacity for walk-ins and then, basing on the remaining capacity, estimate how many reservations to accept. (DP2) does the same thing but achieve it with an analytical approach.

Given the reservation targets, the desired operational response is to control the external and stay-extension requests for reservations, by reacting to cancellations. This aim of the exercise is to have the reservation levels, for each day at the start of that day, hit their respective targets. This is impossible when there are insufficient requests. Even when there are enough requests, it is difficult to attain these targets, using the approaches currently practiced, because the cancellations are random. The approaches in use usually accept reservations, for periods far into the future, up to some authorization level. The authorization level is usually given as a fixed percentage above available capacity. In reality because of cancellations, authorization levels, rather than being flat over time, should be larger the further away the current period is from the target period.

Accepting early bookings increases the certainty of getting enough business. Examples of early booking sources are package tour operators and convention organizers. These early bookings tend to fetch lower rates and, therefore, hotels may refuse some of them in the hope of getting more lucrative business later. The demand from the later market segments may be very uncertain and hence the need to trade-off. To include this trade-off into our model, so as to give better authorization levels, we broaden the concept of show-rate.

Show-rate was defined in conjunction with the definition of N_i: it was defined as the fraction of reservations still 'alive' at the start of the target period that will show up by the end of that period. There are two time-points of reference here: an end point and a start point. The end point is the end of the target period and the start point is the point the reservation targets are set for. Since we are usually concerned about the reservation targets for the beginning of the current period, we will call the start time-point the current period.

The broader concept, the survival rate, introduced now, involves both the cancellation and the show characteristics of reservations. We say a reservation survived if it has not been cancelled or failed to show. For a target period the survival-rate, q_i, is the fraction of reservations that will survive from among the reservations that were "alive" now (at the current period) plus those to be accepted from now until the target period. With this amendment, the reservation targets obtained from the programs will be the authorization levels for the current period--and not, as previously defined, for the start of the target period. The earlier definition is a special case of this extended definition.

WALK-IN CONTROL

Walk-ins targets are not explicitly specified in the solution of our problem. In this sub-section, to assist in the control of walk-in demand, we present a decision rule. This rule helps hotel operators decide how to allocate rooms to the requests by different class of walk-ins and, in particular, suggests when rooms should be downgraded for walk-ins. Consider the problem (C1).

(C1)

$Z_{C1}(Y_i, Y_j)$

$$= c_i \int_0^{Y_i} y\, f(YS_i;y)dy + c_i\, Y_i \int_{Y_i}^{\infty} f(YS_i;y)dy$$

$$+ c_j \int_0^{Y_j} y\, f(YS_j;y)dy + c_j\, Y_j \int_{Y_j}^{\infty} f(YS_j;y)dy$$

$$+ \mu_i\ (LY_i - Y_i)$$

$$+ \mu_j\ (LY_j - Y_j)$$

where

$i < j$, $j=2,..,n$,

Y_i is the number of rooms to offer to class i walk-ins, i=1,..,n,

LY_i is the capacity available for class i walk-in, i=1,..,n,

and

μ_i is the associated dual (shadow) price, i=1,..,n.

This problem considers the total expected return associated with accepting walk-in requests for two guest-classes. We take first and second derivatives to show that $Z_{C1}(Y_i, Y_j)$ is concave and has an optimal solution such that $c_i\ [1-F(YS_i;Y_i)] = \mu_i$ and $c_j\ [1-F(YS_j;Y_j)] = \mu_j$. For $i < j$, the capacity allocated to class i can be downgraded to class j. So since we can always downgrade--but not upgrade--we want to keep $\mu_i \le \mu_j$ and hence we get decision-rule (WALCON).

(WALCON)

For $i < j$, $j=2,..,n$, $[1-F(YS_i;YA_i)]/[1-F(YS_j;YA_j)] \le c_j/c_i$

where

YS_i is the random variable for the number of walk-in's for the time remaining in the target period, i=1,..,n, and

YA_i is the limit on the number of class i walk-ins to accept, i=1,..,n. ▪

(WALCON) gives only limits on the relative sizes of walk-in request to accept. The absolute limits depend on net quantity of rooms available for walk-ins. This is deduced, with subjective judgements and given the service performance requirements, from the total capacity available, the number of booked reservations that remains on record, and the probability that they will show.

SALES MANAGEMENT AND RATES SETTING

We had assumed that the room rates are determined by competitive market forces. This is often true only for rack rates. To increase occupancy, hotels offer discounts to tour operators, convention organizers, and others. The hotel operators, therefore, have some discretion in setting the rates. The next rule provides some guidance on the relative value of rates for the guest-classes. It points out that the important contributors to rates differentials are the relative magnitudes of their reservation demand and survival characteristics.

We assume that the survival-rate distributions are independent of the demand distributions and consider the following problem.

(C2)

$$
\begin{aligned}
&Z_{C2}(N_i,N_j) \\
&= c_i \int_0^{N_i} \int_0^1 x\ N\ f(q_i;x) f(NS_i;N)\,dx\,dN + c_i\ N_i \int_{N_i}^{\infty} \int_0^1 x\ f(q_i;x) f(NS_i;N)\,dx\,dN \\
&+ c_j \int_0^{N_j} \int_0^1 x\ N\ f(q_j;x) f(NS_j;N)\,dx\,dN + c_j\ N_j \int_{N_j}^{\infty} \int_0^1 x\ f(q_j;x) f(NS_j;N)\,dx\,dN \\
&+ \pi_i(L_i - E[q_i]N_i) + \pi_j(L_j - E[q_j]N_j)
\end{aligned}
$$

where L_i is a given allocation of capacity to guest-class i, i=1,..,n

and π_i is the dual (shadow) price associated with the allocation, i=1,..,n.

This problem gives the total expected return associated to allocating the available capacities to two guest-classes. So we have a problem similar to the one for walk-in control. By taking first and second derivatives, it is easy to show that $Z_{C2}(N_i,N_j)$ is concave and has an optimal solution where $c_i\ [1-F(NS_i;N_i)] = \pi_i$ and $c_j\ [1-F(NS_j;N_j)] = \pi_j$.

Theorem 7: In the optimal solution for (C2), $\pi_i \geq \pi_j$ for $i < j$, $i=1,..,n-1$.

Proof of theorem 7: Suppose the theorem is false and $\pi_i < \pi_j$ for $i < j$. Then, we downgrade rooms from those allocated to class i to class j and gain an additional return of $(\pi_j - \pi_i)$ per unit downgraded. ▪

By the result presented in theorem 7, we give below the decision-rule for setting rates or granting discounts.

(RATESET)

For $i = 2,..,n$, $c_{i+1} \leq Q_i\ c_i$

$$\text{where} \quad Q_i = \frac{[1-F(NS_i;\ N_i]}{[1-F(NS_{i+1};\ N_{i+1})]}, \quad i=1,..,n-1. \qquad ▪$$

Here, we assumed that the relative values of the reservation targets, N_i, $i=1,..,n$ are given. (RATESET) suggests how market segmentation should be exploited: market should be segmented according to the strength of its demand relative to the availability of rooms. It also gives limits that will guide pricing negotiations with tour and convention groups. From above, for $i < j$, c_i is not always greater or equal to c_j. However, by our labelling convention, $c_i \leq c_j$ for $i < j$ when classes i and j are for the same room-type. But across room-types, guest classes in a room-type can have rates lower than the rack rate of a less luxurious room-type.

5. Comments and Extensions

The creation of the guest-class concept helps hotels earn more revenue by exploiting market segmentation. It does so by controlling spills and diversions. Glover et al. [1982] gave the definition of spill and diversion for the airline context: "Spill is the movement of passengers to other flights, either the same or competing carriers. Diversion occurs when a passenger who would have stayed with the same carrier at the original higher fare takes advantage of a discount fare which was offered to stimulate increased occupancy, thus generating less revenue for the carrier." Spill, in our problem, refers to walk-in or reservation requests that the hotel has to turn away. We reduce spills from high-revenue guest-classes by controlling the number of low revenue requests to accept. Diversions are managed through better understanding of the characteristics of the market segments and applying to guests from these segments the appropriate rates.

The Parker house hotel in Boston actually created "service product" packages for different groups of customers that corresponds to what we have called guest-classes. The hotel's sales department pursue and develop the demand from these groups through direct contact. The capacity for tour group reservations are allocated after the capacity targeted to the higher paying groups have been accounted for, consistent with the outcome suggested by our analysis. The marketing strategy of Parker house, as well as many other hotels, requires that rooms are usually available for the higher-paying walk-in guests. For these cases, additional service constraints may be added to our formulation to ensure that most walk-ins are accepted as guests. This extension can be done easily.

Airlines have been using authorization levels for reservations booking. The methods they used to obtain the authorization level are different from ours and they also do not account explicitly for downgrading effects. The airline reservations problem also deviates fundamentally from the hotel problem in that (except shuttle flights) it has fewer walk-ins.

The alternatives available to the air-traveller are also restricted: the air traveller cannot just change to another flight when it has an oversale--there are very few flights that have the same destination and take off within a short time of each other. Simple extensions can be made to apply our approach to the airline reservations problem.

On the other extreme, restaurants, like those famous seafood places in Boston, have so much demand that some do only walk-in business: they do not typically accept reservations. It is not difficult to provide a plausible explanation using the results of our analysis of the hotel problem: assuming other things being equal, holding reservations runs the additional risk of cancellations, late arrivals, and no-shows. Therefore, not only would there be situations when walk-in customers wait in frustration while tables lie idle, but the burden of management also increases.

New variations in the circumstances surrounding the problems like the penalty schemes to discourage no-shows: non-refundable sales, first day deposits, etc. are appearing. These present new challenges for extending our model which we leave for future research. Another area of future research is to explore the possible use of heuristics to solve the hotel problem. (DP2) has an interesting structure that suggests how one might work: a "knapsack" filling approach where we increase the values of decision variables that have the higher marginal returns first until the constraints are binding.

Finally, we will mention briefly how hotels measure their performance relative to each other. A common measure of operational efficiency for hotels is percent occupancy. One way of achieving high occupancy is to give large discounts and overbook excessively. Operating this way, the hotel fills up easily but reaps low revenue and, in violation of good practice, leaves many prospective reserved guests without rooms. Therefore, the level of occupancy does not fully reflect how well the hotel is managed.

Merliss and Lovelock [1980] highlighted an alternative performance measure (being used by the Parker House) called the room sales efficiency (RSE). RSE is the total room sales revenue over a period divided by the potential revenue that might be obtained if, during the same period, all available rooms were sold at rack rates. Maximizing expected return also maximizes expect RSE. This is an excellent measure for comparing hotels of different sizes and measuring how well they serve their market segments.

6. Summary and Conclusions

Previous studies consider the capacity allocation and the yield management problems independently. In this paper, we show how they can be coordinated. We also showed how the profitability of a hotel can be optimized by careful utilization of its accommodation resources--not merely by increasing occupancy. The model we provide allows us to solve hotel reservations and sales planning problems that have multiple-day stays, multiple room-types, multiple guest-classes, and service constraints. We show that the problem can be separated into single-period problems. Using inner-linearization approximations, we can obtain near-optimal solution for the reservation targets. We also provide rules to assist in accepting walk-ins and in setting room rates. The rules can be applied to aid sales management and control discount offers. The model demonstrates, through the use of guest-classes, how the market segmented effectively can increase profits.

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APPENDIX

NOTATIONS

n,m,T: Number of guest-classes, number of room-types, and length of planning horizon respectively.

a(s): Smallest guest-class label for room-type s, s=1,..,m.

A(s): Set of all guest classes for type s rooms. A(s) = {a(s),..,a(s+1)-1}.

AU(s): Set of all guest-classes that can be offered type s rooms. AU(s) = {1,..,a(s+1)-1}, s=1,..,m.

c_{it}: Rate, per room per period, charged for guest-class i, 1=1,..,n in period t, t=1,..,T.

MU_{st}: Number of type s rooms available in period t.

M_{it}: Number of rooms initially allocated to guest-class i for period t. ($M_{it} = MU_{st}$ for i=a(s) and $M_{it} = 0$ otherwise, s=1,..,m, t=1,..,T.)

W_{it}: Number of rooms from those allocated to class i to downgrade to class i+1 during period t and $W_{0t} = 0$, t=1,..,T.

q_{it}: Class i reservation show-rate (or survival-rate) for period t.

N_{it}: Number of reservations for class i guest in period t.

NS_{it}: Random variable for the demand of guest-class i reservations in period t.

$NS_{it}(y)$: $NS_{it}(y) = NS_{it}$ for $NS_{it} \le y$ and $NS_{it}(y) = y$ otherwise, $i=1,..,n$ and $t=1,..,T$.

YS_{it}: Random variable for the number of class i prospective guests that will walk into the hotel during period t without reservations. $YS_{it} \in [0,\infty)$, $i=1,..,n$ and $t=1,..,T$, have finite mean and variance.

$YS_i(y)$: $YS_i(y) = YS_i$ for $YS_i \le y$ and $YS_i(y) = y$ otherwise, $i=1,..,n$.

q,NS,YS: The vectors of random variables q_{it}, NS_{it}, YS_{it} respectively.

$f(x;y)$: Probability density function of any random variable x evaluated at y.

$F(x;y)$: Cumulative density function of any random variable x evaluated at y.

Prob(.): Probability of the event argument.

$E_x[.]$: Expectation over the random vector x.

α: Service performance target for booked reservations; probability target for meeting reservation demand. (Typically, $\alpha \in [0,1]$ is close to 1.)

$\phi(.)$: $\phi(a_1,..,a_n) = F^{-1}(\Sigma^n_{i=1} a_i q_i; \alpha)$ where $a_i \ge 0$, $i=1,..,n$.

J_{it}: $J_{it} = M_{it} + W_{i-1,t} - W_{it} - q_{it} N_{it}$, $i=1,..,n$ and $t=1,..,T$.

J_{it}^+: $J_{it}^+ = \mathrm{Max}(0, J_{it})$, $i=1,..,n$ and $t=1,..,T$.

$J_{it}(y)$: $J_{it}(y) = M_{it} + W_{i-1,t} - W_{it} - q_{it} NS_{it}(y)$, $i=1,..,n$ and $t=1,..,T$.

$J_{it}(y)^+$: $J_{it}(y)^+ = \mathrm{Max}(0, J_{it}(y))$, $i=1,..,n$ and $t=1,..,T$.

u_i: Unit vector $u_i = (a_1,..,a_n)$ where $a_j = 1$ for $j = i$ and $a_j = 0$ otherwise.