PROJECTIVE TRANSFORMATIONS FOR INTERIOR-POINT ALGORITHMS, AND A SUPERLINEARLY CONVERGENT ALGORITHM FOR THE W-CENTER PROBLEM

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This paper is a revision of the two papers "Projective Transformations for Interior Point Methods, Part I: Basic Theory and Linear Programming," O.R. Working Paper 179-88 and "Projective Transformations for Interior Point Methods, Part II: Analysis of An Algorithm for finding the Weighted Center of a Polyhedral System," O.R. Working Paper 180-88, M.I.T.

Abstract

The purpose of this study is to broaden the scope of projective transformation methods in mathematical programming, both in terms of theory and algorithms. We start by generalizing the concept of the analytic center of a polyhedral system of constraints to the w-center of a polyhedral system, which stands for weighted center, where there are positive weights on the logarithmic barrier terms for reach inequality constraint defining the polyhedron X . We prove basic results regarding contained and containing ellipsoids centered at the w-center of the system X . We next shift attention to projective transformations, and we exhibit an elementary projective transformation that transforms the polyhedron X to another polyhedron Z , and that transforms the current interior point to the w-center of the transformed polyhedron Z . We work throughout with a polyhedral system of the most general form, namely both inequality and equality costraints.

This theory is then applied to the problem of finding the w-center of a polyhedral system X . We present a projective transformation algorithm, which is an extension of Karmarkar's algorithm, for finding the w-center of the system X . At each iteration, the algorithm exhibits either a fixed constant objective function improvement, or converges superlinearly to the optimal solution. The algorithm produces upper bounds on the optimal value at each iteration. The direction chosen at each iteration is shown to be a positively scaled Newton direction. This broadens a result of Bayer and Lagarias regarding the connection between projective transformation methods and Newton's method. Furthermore, the algorithm specializes to Vaidya's algorithm when used with a line-search, and so shows that Vaidya's algorithm is superlinearly convergent as well. Finally, we show how the algorithm can be used to construct well-scaled containing and contained ellipsoids at near-optimal solutions to the w-center problem.

Key Words: analytic center, w-center, projective transformation, linear program, ellipsoid, barrier penalty, Newton method.

Running header: Projective Transformations.

I. Introduction.

The purpose of this study is to broaden the scope of projective transformation methods in mathematical programming, both in terms of theory and algorithms. We start by generalizing the concept of the analytic center of a polyhedral system of constraints to the w-center of a polyhedral system, which stands for weighted center, where there are positive weights on the logarithmic barrier terms for reach inequality constraint defining the polyhedron X . We prove basic results regarding contained and containing ellipsoids centered at the w-center of the system X . We next shift attention to projective transformations, and we exhibit an elementary projective transformation that transforms the polyhedron X to another polyhedron Z , and that transforms the current interior point to the w-center of the transformed polyhedron Z . We work throughout with a polyhedral system of the most general form, namely both inequality and equality constraints. This theory is then applied to the problem of finding the w-center of a polyhedral system X . We present a projective transformation algorithm, which is an extension of Karmarkar's algorithm, for finding the w-center of the system X . At each iteration, the algorithm exhibits either a fixed constant objective function improvement, or converges superlinearly to the optimal solution.

The W-Center of a Polyhedral System. In [15], Karmarkar simultaneously introduced ideas regarding the center of a polyhedral system, a projective transformation that centers a given point, and a linear programming algorithm that uses this methodology to decrease a potential function involving an objective function component and a centering component. Karmarkar's ideas have since been generalized along a umber of lines, both theoretical and computational. Herein, we expand on Karmarkar's methodology in at least two ways. First we analyze the w-center of a polyhedron system $X = \{x \in R^n \mid Ax \le b, Mx = g\}$, defined as the solution $\hat{x}$ to the following optimization problem:

$$
\begin{aligned}
P_w: \quad & \text{maximize} \quad \sum_{i=1}^{m} w_i \ln s_i \\
& \text{s.t.} \qquad Ax + s = b \\
& \qquad\qquad Mx \quad = g, \\
& \qquad\qquad\qquad s > 0 .
\end{aligned}
$$

Note that P_w is a generalization of the analytic center problem first analyzed by Sonnevend [22], [23]. This problem has had numerous applications in mathematical programming, see Renegar [19], Gonzaga [13], and Monteiro and Adler [17, 18], among others. Also note the P_w is defined for the most general polyhedral representation, namely inequality as well as equality constraints of arbitrary form. In P_w, the weights w_i can be arbitrary positive scalars, and for convenience they are normalized so that $\sum_{i=1}^{m} w_i = 1$. Let $\overline{w}$ be the smallest weight, i.e., $\overline{w} = \min_i \{w_i\}$. The main result for the w-center problem is that if $\hat{x}$ is the w-center, then there exist well-scaled contained and contained ellipsoids at $\hat{x}$ as follows. Let $X = \{x \in R^n \mid Ax \le b, \ Mx = g\}$. Then there exist ellipsoids E_{IN} and E_{OUT} centered at $\hat{x}$, for which $E_{IN} \subset X \subset E_{OUT}$, and $(E_{OUT} - \bar{x}) = ((1 - \overline{w})/\overline{w})\,(E_{IN} - \bar{x})$, i.e., the outer ellipse is a scaled copy of the inner ellipse, with scaling factor $(1 - \overline{w})/\overline{w}$. When the weights are identical, $w = (1/m)e$, and $(1 - \overline{w}/\overline{w}) = (m - 1)$. Essentially, the scaling factor $(1 - \overline{w})/\overline{w}$ is (almost exactly) inversely proportional to the smallest (normalized weight w_i.

Projective W-Centering for Polyhedra in Arbitrary Form. Numerous researchers have extended Karmarkar's projective transformation methodology, and this study broadens this methodology as well. Gay [10] has shown how to apply Karmarkar's algorithm to linear programming problems in standard form (i.e., "$Ax = b$, $x \geq 0$"), and how to process inequality constraints by implicitly converting them to standard form. Later, Gay [11] shows how to process problems in standard form with upper and lower bounds, as does Rinaldi [20]. Bayer and Lagarias [4] have added to the theoretical foundations for linear programming by showing that for inequality constrained problems, there exists a class of projective transformation for centering a polyhedron about a given point $\bar{x}$. Anstreicher [2] has shown a different methodology for processing linear programming problems in standard form, and in [7] the author gives a simple projective transformation that constructively uses the result of Bayer and Lagarias for linear programming problems with inequality constraints. Even though linear programs in any one form (e.g., standard primal form) can be either linearly of projectively transformed into another form, such transformations can be computationally bothersome and awkward, and lack aesthetic appeal. Herein, we work throughout with the most general polyhedral system, namely $X = \{x \in R^n \mid Ax \leq b, Mx = g\}$. This system contains all of the above as special cases, without transformations, addition or elimination of variables, etc. In Sections III and IV of this paper, we present an elementary projective transformation that projectively transforms a general polyhedral system $X = \{x \in R^n \mid Ax \leq b, Mx = g\}$ to an equivalent system $Z = \{z \in R^n \mid \tilde{A}x \leq \tilde{b}, Mx = g\}$, and that results in a given point $\bar{x}$ (in the relative interior of X) being the w-center of the polyhedral system Z . The approach taken is based on classical polarity theory for convex sets, see Rockafellar [21] and Grünbaum [14].

A Canonical Optimization Problem. The results on the w-center problem are applied to the following canonical optimization problem:

$$\text{CP:} \quad \underset{x}{\text{minimize}} \quad F(x) = \ln(U - c^T x) - \sum_{i=1}^{m} w_i \ln(b_i - A_i x)$$

$$\begin{aligned} \text{s.t.} \quad & Ax + s = b \\ & s > 0 \\ & Mx = g \\ & c^T x < U \,. \end{aligned}$$

where $X = \{x \in R^n \mid Ax \le b,\ Mx = g\}$ is given. Note that problem CP has two important special cases: linear programming and the w-center problem itself. If $p = c$ is the objective function of a linear program maximization problem defined on $X = \{x \in R^n \mid Ax \le b,\ Mx = g\}$, and if U is an appropriate upper bound on the optimal objective function value, then CP is just the problem of minimizing Karmarkar's potential function (generalized to nonuniform weights w_i on the constraints). If $c = 0$ and $U = 1$, then CP is just the w-center problem P_w. In Section V of this paper, we present a local improvement algorithm for program CP that is analogous to and is a generalization of Karmarkar's algorithm.

An Algorithm for the W-Center Problem. In Sections V and VI, the methodology and theory regarding the w-center, projecting to the w-center, and the local improvement algorithm for the canonical optimization problem CP , are applied to an algorithm to solve the w-center problem P_w. Other algorithms for this problem have been developed by Censor and Lent [5] and by Vaidya [26]. We present a projective transformation algorithm for finding the w-centerthat is an extension of the ideas of Karmarkar's algorithm applied to the program CP .

This algorithm produces upper bounds on the optimal objective value at each iteration, and these bounds are used to prove that the algorithm is superlinearly convergent. We also show that the direction chosen at each iteration is a positively scaled Newton direction. Thus, if the algorithm is augmented with a line-search, it specializes to Vaidya's algorithm. Although Vaidya has shown that his algorithm exhibits linear convergence, our approach and analysis demonstrate that his algorithm is actually superlinearly convergent, verifying a conjecture of Vaidya [27] that his algorithm might exhibit stronger convergence properties. We also show that after a fixed number of iterations of the algorithm, that one can construct "well-scaled" containing and contained ellipsoids at the current iterate of the algorithm. If $X = \{x \in R^n \mid Ax \leq b, Mx = g\}$ is the current iterate, one can easily construct ellipsoids F_{IN} and F_{OUT} centered at $\bar{x}$, with the property that $F_{IN} \subset \mathbf{X} \subset F_{OUT}$, and $(F_{OUT} - \bar{x}) = (2.9 / \overline{w})(F_{IN} - \bar{x})$. When all weights are identical, then this scale factor is $(2.9m)$ which is $O(m)$. In general, the order of this scale factor is $O(1/\overline{w})$, which is the same as for the ellipses E_{IN} and E_{OUT} centered at the optimal solution to P_w, whose scale factor is $(1 - \overline{w})/\overline{w} = 1/\overline{w} - 1$.

The paper is organized as follows. Section II presents notation, definitions and a characterization of the properties of the w-center. Section III presents general results regarding properties of projective transformations of polyhedra. In Section IV, we exhibit an elementary projective transformation for transforming the current point $\bar{x}$ to the w-center of the transformed polyhedral system. In Section V, we introduce the canonical optimization program CP, and present a projective transformation algorithm for the w-center program P_w. In Section VI, the performance of this algorithm is analyzed, and we demonstrate superlinear convergence. In Section VII, we show that the direction generated by the algorithm at each iterate is a positively-scaled Newton direction, and we discuss consequences of this result. In Section VIII, we show how to construct inner and outer ellipsoids at points near the w-center.

II. Notation and Characterization at the w-Center.

Throughout this paper, we will be concerned with a system of constraints of the form

$$\begin{aligned} Ax &\le b \\ Mx &= g \end{aligned} \tag{2.1}$$

where A is $m \times n$, M is $k \times n$, $x \in R^n$, $b \in R^m$, and $g \in R^k$.

One can think of the constraint system as given by the data (A, b, M, g), and so we denote

$$X = (A, b, M, g) \tag{2.2}$$

as the symbolic representation of the constraint system of (2.1). In many contexts, however, it will be particularly convenient to represent the polyhedron determined by all solution x of (2.1) and so we write

$$X = \{x \in R^n \mid Ax \le b \ , \ Mx = g\} \ . \tag{2.3}$$

Although the notation (2.2) and (2.3) are different, when referring to X in this paper the denotation will be clear. For convenience we assume that A has rank m and M has rank k , and so $m \ge n$ and $k \le n$.

If X is given, we denote the slack space of X by

$$S = \{ s \in R^m \mid s \geq 0, s = b - Ax \text{ for some } x \text{ satisfying } Mx = g \} , \tag{2.4}$$

i.e., S is the space of all slack vectors $s = b - Ax$ of the constraint system X . We say X has an interior if and only if there exists x for which $Ax < b$ and $Mx = g$, and we write int $X \neq \phi$. Likewise, if there is a vector $s \in S$ for which $s > 0$, then S has an interior and we write int $S \neq \phi$. Obviously int $X \neq \phi$ if and only if int $S \neq \phi$.

Also, we use the following standard notation for diagonal matrices: if w , s , $\bar{s}$, are vectors in R^m , then W , S , $\bar{S}$ denote the diagonal matrices whose diagonal entries correspond to the vectors w , s , $\bar{s}$. Let $e = (1, ..., 1)^T$ denote the column of ones of appropriate dimension. Let e_i denote the i^{th} unit vector.

Let w be a vector in R^m for which $w > 0$ and w has been normalized so that

$$e^T w = 1 , w > 0 . \tag{2.5}$$

Consider the problem

$$\begin{aligned} P_w: \quad & \text{maximize} \quad F_w(x) = \sum_{i=1}^{m} w_i \ln (b_i - A_i x) \\ & \text{s.t.} \quad Ax + s = b \\ & \qquad\quad Mx \qquad = g \\ & \qquad\qquad\quad s > 0 . \end{aligned} \tag{2.6}$$

This problem is a (weighted) generalization of the analytic center problem, posed by Sonnevend [22, 23], and used extensively in interior point algorithms for solving linear programming problems, see Renegar [19], Gonzaga [13], and Monteiro and Adler [17, 18], among others.

Under the assumption that X is bounded and $\operatorname{int} X \neq \phi$, then P_w will have a unique solution, $\bar{x}$, which we denote as the *w-center* of the constraint system X. The Karush-Kuhn-Tucker (K-K-T) conditions are necessary and sufficient for optimality in P_w, and thus $\bar{x}$ is the w-center of X if and only if $\bar{x}$ satisfies

$$
\begin{aligned}
&\text{(2.7a)} \quad && A\bar{x} + \bar{s} = b \\
&\text{(2.7b)} \quad && M\bar{x} = g \\
&\text{(2.7c)} \quad && \bar{s} > 0 \\
&\text{(2.7d)} \quad && w^T \bar{S}^{-1} A = \bar{\pi}^T M \quad \text{for some} \quad \bar{\pi} \in R^k .
\end{aligned}
\tag{2.7}
$$

Let $\bar{w}$ denote the smallest component of w, i.e.

$$\bar{w} = \min\{w_1, \ldots, w_m\} , \tag{2.8a}$$

and define

$$r = \sqrt{\frac{\bar{w}}{1-\bar{w}}} , \qquad R = \sqrt{\frac{1-\bar{w}}{\bar{w}}} . \tag{2.8b}$$

Generalizing Sonnevend [22, 23], we have the following properties of the w-center of X, that characterize inner and outer ellipsoids centered and $\bar{x}$.

Theorem 2.1. Let $X = \{x \in R^n \mid Ax \leq b\ ,\ Mx = g\}$, and let $\bar{x}$ be the w-center of X , and let $\bar{s} = b - A\bar{x}$. Let

$$E_{IN} = \{x \in R^n \mid Mx = g\ ,\ (x-\bar{x})^T A^T \bar{S}^{-1} W\ \bar{S}^{-1} A\,(x-\bar{x}) \leq r\}$$

and $E_{OUT} = \{x \in R^n \mid Mx = g\ ,\ (x-\bar{x})^T A^T \bar{S}^{-1} W\, \bar{S}^{-1} A\,(x-\bar{x}) \leq R^2\}$, where r and R are defined in (2.8).

Then $E_{IN} \subset X \subset E_{OUT}$.

Before proving this theorem, we make the following remark.

Remark 2.1. $(E_{OUT} - \bar{x}) = (R/r)\,(E_{IN} - \bar{x})$, i.e., the outer ellipse is a scaled copy of the inner ellipse, with scaling factor $R/r = (1-\overline{w})/\overline{w}$. If $w = (1/m)\,e$, then $\overline{w} = 1/m$, and so the scaling factor is $R/r = (m-1)$.

The proof of Theorem 2.1 is aided by the following three propositions:

Proposition 2.1. If $\bar{x}$ is the w-center of X , and $\bar{s} = b - A\bar{x}$, then S is contained in the simplex $\Delta = \left\{s \in R^m \;\middle|\; s \geq 0\ ,\ w^T \bar{S}^{-1} s = 1\right\}$.

Proof. If $s \in S$, then $w^T \bar{S}^{-1} s = w^T \bar{S}^{-1} (b - Ax)$ for some $x \in X$, and so $w^T \bar{S}^{-1} s = w \bar{S}^{-1} (\bar{s} + A\bar{x} - Ax) = w^T \bar{S}^{-1} \bar{s} + w^T \bar{S}^{-1} A\,(\bar{x} - x)$. From (2.7d), this latter expression equals $w^T \bar{S}^{-1} \bar{s} + \bar{\pi}^T M\,(\bar{x} - x) = w^T \bar{S}^{-1} \bar{s} = w^T e = 1$, since $M\,(x - \bar{x}) = g - g = 0$. ■

Proposition 2.2. Suppose $v \in R^m$ and v satisfies $w^T v = 0$ and

$v^T Wv \le r^2$. Then $|v_i| \le 1$ for each $i = 1, \ldots, m$.

Proof. If suffices to show that $v_i \le 1$, $i = 1, \ldots, m$. For each i , consider the program

$$\begin{aligned} \max \quad & v_i \\ \text{s.t.} \quad & v^T Wv \le w_i / (1 - w_i) && (\alpha) \\ & w^T v = 0 \,. && (\beta) \end{aligned}$$

The optimal solution to this program is

$v^* = (1/1 - w_i))\,(-w_i\, e + e_i)$, with K-K-T multipliers $\alpha = (1 - w_i) / (2w_i)$ and $\beta = 1$, which satisfy the K-K-T conditions $e_i = 2\alpha\, Wv + \beta w$. Notice that $v_i^* = 1$. Thus if $v^T Wv \le r^2 \le w_i / (1 - w_i)$ and $w^T v = 0$ then $v_i \le 1$. ■

Proposition 2.3. Let $\bar{x}$ be the w-center of X . If $s \in R^m$ satisfies $w^T \bar{S}^{-1} s = 1$ and $(s - \bar{s})^T \bar{S}^{-1} W \bar{S}^{-1} (s - \bar{s}) \le r^2$, then $0 \le s_i \le 2\bar{s}_i$, $i = 1, \ldots, m$.

Proof. Let s be as given in the proposition. Let $v = \bar{S}^{-1}(s - \bar{s})$. Then v satisfies the hypotheses of Proposition 2.2, and hence $|v_i| \le 1$, $i = 1, \ldots, m$. Thus $0 \le s_i \le 2s_i$, $i = 1, \ldots, m$. ■

Proof of Theorem 2.1. We first prove that $X \subset E_{OUT}$. By Proposition 2.1, $S \subset \Delta$. The extreme points of Δ are $(\bar{s}_i / w_i)\, e_i$, $i = 1, \ldots, m$. Note that each extreme point satisfies $((\bar{s}_i / w_i) e_i - \bar{s})^T \bar{S}^{-1} W \bar{S}^{-1} ((\bar{s}_i / w_i) e_i - \bar{s}) = (1 - w_i) / w_i \le R^2$. Thus, because Δ is a convex set, every $s \in S$ satisfies $(s - \bar{s})^T \bar{S}^{-1} W \bar{S}^{-1} (s - \bar{s}) \le R^2$. But $(s - \bar{s}) = - A (x - \bar{x})$, so $(x - \bar{x}) A^T \bar{S}^{-1} W \bar{S}^{-1} A (x - \bar{x}) \le R^2$. This shows that $X \subset E_{OUT}$.

We next show that $E_{IN} \subset X$. Let $x \in E_{IN}$, and let s be the slack corresponding to x , i.e., $s = b - Ax$. Then $(s - \bar{s})^T \bar{S}^{-1} W \bar{S}^{-1} (s - \bar{s}) = (x - \bar{x}) A^T \bar{S}^{-1} W \bar{S}^{-1} A (x - \bar{x}) \le r^2$. Also, similar to Proposition 2.1, it is straightforward to show that $w^T \bar{S}^{-1} s = 1$. Thus by Proposition 2.3, $s \ge 0$. Thus $Ax \le b$, and since $x \in E_{IN}$, $Mx = g$. Thus $x \in X$. ■

The next proposition shows how the w-center can be used to construct an upper bound on the slack $s_i = (b - Ax)_i$ of any constraint of X , $i = 1, \ldots, m$.

Proposition 2.4. Let $\bar{x}$ be the w-center of X . For each $i = 1, \ldots, m$, for any $x \in X$, $(b_i - A_i x) \le \bar{s}_i / w_i$.

Proof. For any $x \in X$, let $s = b - Ax$. By Proposition 2.1, $w^T \bar{S}^{-1} s = 1$, $s \ge 0$, so $s_i \le \bar{s}_i / w_i$, i.e., $b_i - A_i x \le \bar{s}_i / w_i$. ■

The last result of this section characterizes the behavior of the weighted-logarithmic function $\sum_{i=1}^{m} w_i \ln (b_i - A_i x)$ near the w-center $\bar{x}$ of X . This lemma parallels similar results for the uniformly weighted center in Karmarkar [15] and Vaidya [26].

Lemma 2.1. Let $\bar{x}$ be the w-center of X, let $\bar{s} = b - A\bar{x}$, and let $d \in R^n$ be a direction that satisfies $Md = 0$, and $d^T A^T \bar{S}^{-1} W \bar{S}^{-1} Ad \le r^2$. Then for all α satisfying $0 \le \alpha < 1$,

$$\sum_{i=1}^{m} w_i \ln (b_i - A_i (\bar{x} + \alpha d)) \ge \sum_{i=1}^{m} w_i \ln (\bar{s}_i) - \frac{\alpha^2}{2(1-\alpha)} r^2 .$$

Proof. Let $v = \bar{S}^{-1} Ad$. Then $w^T v = w^T \bar{S}^{-1} Ad = \bar{\pi}^T Md = 0$ for some $\bar{\pi} \in R^k$ by (2.7d). Furthermore $v^T Wv \le r^2$. Thus by Proposition 2.2, $|v_i| \le 1$, $i = 1, \ldots, m$. Therefore

$$\sum_{i=1}^{m} w_i \ln (b_i - A_i (\bar{x} + \alpha d)) = \sum_{i=1}^{m} w_i \ln (\bar{s}_i (1 - \alpha v_i))$$

$$= \sum_{i=1}^{m} w_i \ln (\bar{s}_i) + \sum_{i=1}^{m} w_i \ln (1 - \alpha v_i)$$

$$\ge \sum_{i=1}^{m} w_i \ln (\bar{s}_i) + \sum_{i=1}^{m} w_i (- \alpha v_i) - \sum_{i=1}^{m} w_i \frac{(\alpha v_i)^2}{2(1-\alpha)}$$

(by Proposition A.2)

$$= \sum_{i=1}^{m} w_i \ln \bar{s}_i - \alpha w^T v - \frac{\alpha^2 v^T W v}{2(1-\alpha)}$$

$$\ge \sum_{i=1}^{m} w_i \ln \bar{s}_i - \frac{\alpha^2}{2(1-\alpha)} r^2 . \quad \blacksquare$$

III. Projective Transformations.

Let X be the polyhedron defined by (2.2) or (2.3) and let S be the slack space of X defined in (2.4). This section develops a class of projective transformations of X and S into image sets Z and T .

Let $\bar{x}$ satisfy $A\bar{x} < b$ and $M\bar{x} = g$, i.e., $\bar{x} \in \text{int } X$, and let $\bar{s} = b - A\bar{x}$ be the slack vector corresponding to $\bar{x}$. Our interest lies in properties of a projective transformation of X of the form

$$z = g(x) = g_{y,\bar{x}}(x) = \bar{x} + \frac{x - \bar{x}}{1 - y^T(x - \bar{x})} \tag{3.1}$$

for a suitable choice of the vector parameter $y \in R^n$ appearing in the denominator of the transformation. The criterion of suitability that we impose is that the denominator $1 - y^T(x - \bar{x})$ remains positive for all $x \in \text{int } X$. If y is chosen so that

$$y \in \text{int } Y_{\bar{x}} = \left\{ y \in R^n \;\middle|\; y = A^T \bar{S}^{-1} \lambda \quad \text{for some } \lambda > 0 \quad \text{satsfying } \lambda^T e = 1 \right\}, \tag{3.2}$$

then it is elementary to verify that $y^T(x - \bar{x}) < 1$ for all $x \in \text{int } X$, so that the projective transformation $g(x)$ given in (3.1) is well defined for all $x \in X$. Note that $g(x)$ is more formally denoted as $g_{y,\bar{x}}(x)$ because the transformation is parameterized by y and $\bar{x}$. Also note that $\bar{x}$ is a fixed point of $g(\cdot)$, i.e., $\bar{x} = g(\bar{x})$. If $x \in \text{int } X$ and $z = g(x)$, then it is straightforward to verify that z satisfies the constraint system

$$\begin{aligned} \tilde{A}z &\le \tilde{b} \\ Mz &= g \end{aligned} \tag{3.3}$$

where

$$\tilde{A} = A - \bar{s}\,y^T \tag{3.4a}$$

$$\tilde{b} = b - \bar{s}\,y^T\bar{x} \ . \tag{3.4b}$$

Analogous to (2.2), (2.3) and (2.4), we thus can define the image of $g(\cdot)$ as

$$Z = Z_{y,\bar{x}} = \left(\tilde{A}\,,\,\tilde{b}\,,\,M\,,\,g\right) = \left(A - \bar{s}\,y^T\bar{x}\,,\ b - \bar{s}\,y^T\bar{x}\,,\ M\,,\ g\right) \tag{3.5}$$

as a constraint system or

$$Z = Z_{y,\bar{x}} = \left\{z \in R^n \,\middle|\, \tilde{A}z \le \tilde{b}, Mz = g\right\} = \left\{z \in R^n \,\middle|\, \left(A - \bar{s}y^T\right)z \le \left(b - \bar{s}y^T\bar{x}\right), Mz = g\right\} \tag{3.6}$$

and the slack space of Z as

$$T = T_{y,\bar{x}} = \left\{t \in R^m \,\middle|\, t \ge 0\,,\ t = \tilde{b} - \tilde{A}z \quad \text{for some } z \text{ satisfying } Mz = g\right\} \tag{3.7}.$$

The inverse of $g(\cdot)$ is given by the function

$$x = h(z) = h_{y,\bar{x}}(z) = g^{-1}_{y,\bar{x}}(z) = \bar{x} + \frac{z - \bar{x}}{1 + y^T(z - \bar{x})} \ . \tag{3.8}$$

The transformations developed in (3.1) – (3.8) are illustrated in Figure 3.1. Finally, we can extend $g(\cdot)$ and $h(\cdot)$ to the slack spaces S and T as follows. Let

$$(X; S) = \left\{(x,s) \in R^n x R^m \,\middle|\, Ax + s = b\,,\, s \ge 0\,,\, Mz = g\right\}$$

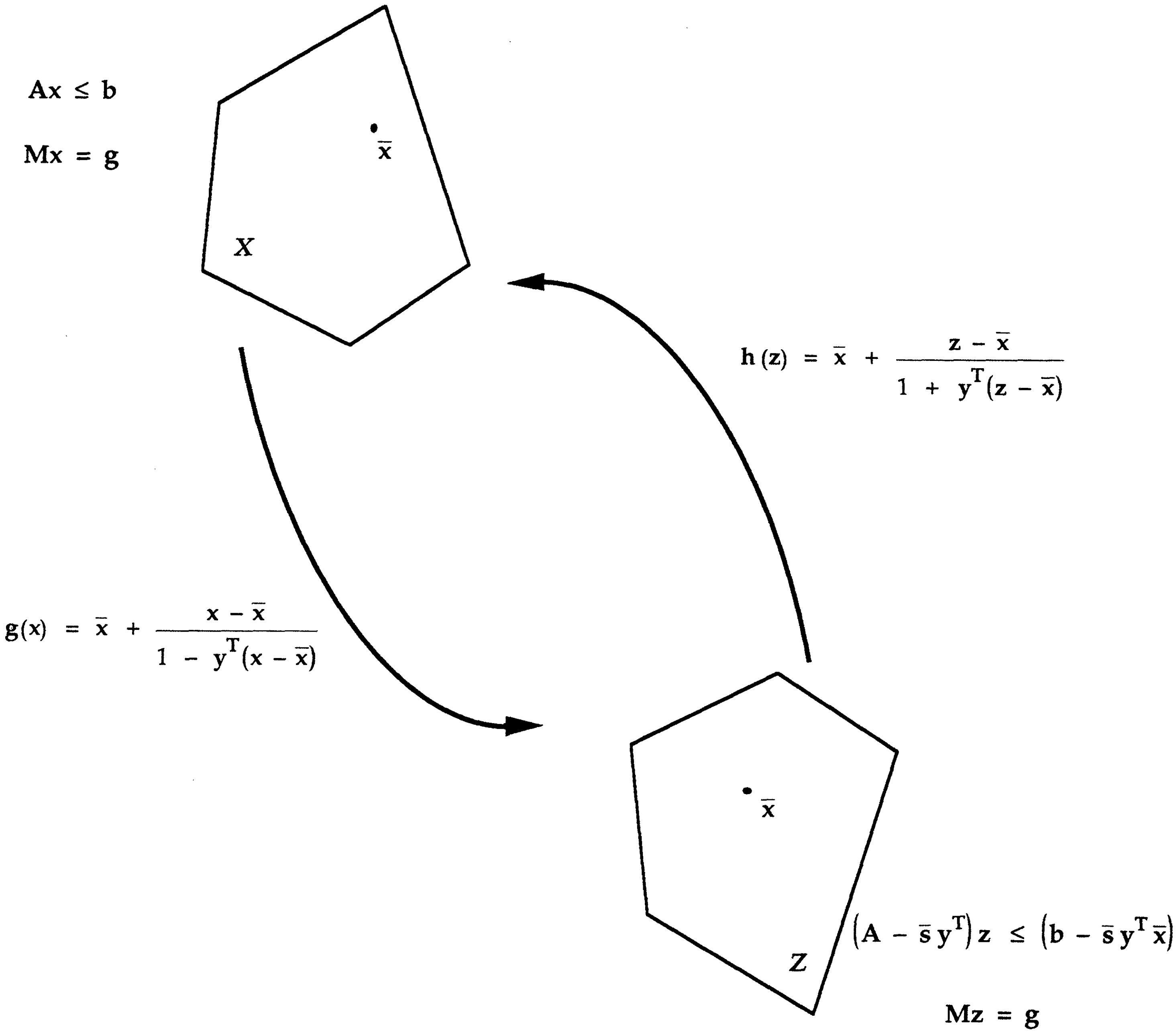

Figure 3.1. Projective Transformation

and

$$(Z, T) = \{(z,t) \in R^n x R^m \mid \tilde{A}z + t = \tilde{b},\ t \geq 0,\ Mz = g\} \text{ , and define}$$

$$(z, t) = g_{y,\bar{x}}(x, s) = \left(\bar{x} + \frac{x - \bar{x}}{1 - y^T(x - \bar{x})},\ \frac{s}{1 - y^T(x - \bar{x})} \right) \text{ for } (x, s) \in (X, S), \quad (3.9a)$$

$$(x, s) = h_{y,\bar{x}}(z, t) = \left(\bar{x} + \frac{z - \bar{x}}{1 + y^T(z - \bar{x})},\ \frac{t}{1 + y^T(z - \bar{x})} \right) \text{ for } (z, t) \in (Z, T), \quad (3.9b)$$

To formally identify the properties of the transformation $g(\cdot) = g_{y,\bar{x}}(\cdot)$, we consider the two cases when X is bounded or not as separate.

Lemma 3.1. Let X be given by (2.2) or (2.3), and suppose X is bounded. Let $\bar{x} \in \text{int}\, X$ be given, let $\bar{s} = b - A\bar{x}$, let y , $g(\cdot)$, $h(\cdot)$, Z , and T satisfy (3.2) – (3.9). Then

(i) $g(\cdot)$ maps X onto Z and S onto T.

(ii) $h(\cdot)$ maps Z onto X and T onto S .

(iii) X and Z are the same combinatorial type, and $g(\cdot)$ maps faces of X onto corresponding faces of Z .

Proof. It suffices to show that (i) $y^T(x - \bar{x}) < 1$ for all $x \in X$ and (ii) $-y^T(z - \bar{x}) < 1$ for all $z \in Z$.

(i) Suppose $x \in X$ and $y^T(x-\bar{x}) \geq 1$. Then from (3.2),

$$1 \leq y^T(x-\bar{x}) = \lambda^T \bar{S}^{-1} A(x-\bar{x}) = \lambda^T \bar{S}^{-1}(Ax - b + \bar{s}) = \lambda^T \bar{S}^{-1}(Ax-b) + 1 ,$$

and so $\lambda^T \bar{S}^{-1}(Ax-b) \geq 0$. Thus $Ax = b$, since $\lambda > 0$ and $Ax \leq b$.

Therefore $v = \bar{x} - x$ satisfies $Av = -\bar{s} < 0$ and $Mv = 0$, and so X is unbounded, which is a contradiction. Therefore $y^T(x-\bar{x}) < 1$.

(ii) Now suppose $z \in Z$ and $-y^T(z-\bar{x}) \geq 1$. Then define $v = z - \bar{x}$ and note that $v \neq 0$. We have

$$Av = Az - A\bar{x} = Az - b + \bar{s} \leq Az - b - \bar{s}\left(y^T(z-\bar{x})\right) = \tilde{A}z - \tilde{b} \leq 0$$

from (3.4) – (3.6). Also $Mv = 0$, so that v is a ray of X, which contradicts the boundedness of X. ■

In the case when X is unbounded, we no longer can guarantee that the projective transformation $g(\cdot)$ is onto and invertible, unless we assume that the system $Ax \leq b$ has been appended to include a trivial constraint of the form $0^T x \leq 1$. We then have:

Lemma 3.2. Let X be given by (2.2) or (2.3), and suppose that the last row of the inequality constraints $Ax \leq b$ is of the form $0^T x \leq 1$. Let $\bar{x} \in \text{int}\, X$ be given, let $\bar{s} = b - A\bar{x}$, and let y, $g(\cdot)$, $h(\cdot)$, Z and T satisfy (3.2) – (3.9). Then

(i) $g(\cdot)$ is well defined for all $x \in X$.

$g(\cdot)$ maps int X onto int Z and int S onto int T.

$g(\cdot)$ maps bounded faces F of X onto those faces G of Z that do not meet the hyperplane $H = \{z \in R^n \mid -y^T(z-\bar{x}) = 1\}$.

(ii) $h(\cdot)$ is well defined for all $z \in Z$.

$h(\cdot)$ maps int Z onto int X and int T onto int S.

$h(\cdot)$ maps faces G of Z that do not meet H onto bounded faces F of X.

(iii) If $z \in Z$ and $z \in H$, then $r = z - \bar{x}$ is a ray of X.

Proof. (i) Let $x \in X$, and let $s = b - Ax$. Then $s \geq 0$, and from (3.2), $y^T(x - \bar{x}) = \lambda^T \bar{S}^{-1}(\bar{s} - s) = 1 - \lambda^T \bar{S}^{-1} s < 1$, because the last constraint of $Ax \leq b$ is $0^T x \leq 1$, $\bar{s}_m = s_m = 1$, and $\lambda > 0$. Thus $g(\cdot)$ is well-defined for all $x \in X$. If $z = g(x)$, then it remains to show that $-y^T(z-\bar{x}) < 1$. We have

$$-y^T(z - \bar{x}) = \frac{-y^T(x - \bar{x})}{1 - y^T(x - \bar{x})} < 1,$$

because $y^T(x-\bar{x}) < 1$.

(ii) If $z \in Z$ then the last constraint of $\tilde{A}z \leq \tilde{b}$ is $-y^T z \leq 1 - y^T \bar{x}$, from (3.4). If $z \notin H$, then $-y^T(z - \bar{x}) < 1$, and so $h(z) = g^{-1}(z)$ is well-defined. ■

As a corollary to both Lemma 3.1 and 3.2 we have:

Corollary 3.1. Let X be given by (2.2) or (2.3), and suppose that X satisfies the condition A:

(A) Either X is bounded or the last row of the inequalities $Ax \le b$ is of the form $0^T x \le 1$. (3.10)

Then the mappings $g(\cdot)$ and $h(\cdot)$ of Lemmas 3.1 or Lemma 3.2 are well-defined for all $x \in \operatorname{int} X$ and $z \in \operatorname{int} Z$. ■

Properties of the projective transformation $g(\cdot) = g_{y,\bar{x}}(\cdot)$ for general constraint systems X (bounded or not, without the restriction on the last row of (A, b)) are developed further in [8]. Also, in [8], it is shown that the projective transformation $g(\cdot)$ is quite general, in that any projective transformation $g(\cdot)$ that leaves $\bar{x}$ fixed and preserves directions from $\bar{x}$ can be written in a form satisfying (3.1) and (3.2). The projective transformation $g(x) = g_{y,\bar{x}}(x)$ can also be developed through convex polarity theory. The set $Y_{\bar{x}}$ of (3.2) is the polar of $(X - \bar{x})$, see Grünbaum [14], and Rockafellar [21]. The set $(Z - \bar{x})$ then is the polar of the translation of $Y_{\bar{x}}$ by y , i.e., $Z = \left((X - \bar{x})^{\circ} - y\right)^{\circ} + \bar{x}$, see [8].

IV. Projective Transformations to w-Center a Given Interior Point.

Let X be the constraint system defined by (2.2) or (2.3) and let S be the slack space of X defined in (2.4). Let $\bar{x}$ satisfy $A\bar{x} < b$ and $M\bar{x} = g$, i.e., $\bar{x} \in \operatorname{int} X$, and let $\bar{s} = b - A\bar{x}$ be the slack vector corresponding to $\bar{x}$. Suppose we wish to find a projection parameter $y \in Y_{\bar{x}}$ so that $\bar{x}$ is the w-center of the projectively transformed constraint system $Z = Z_{y,\bar{x}}$ under the projective transformation $g(x) = g_{y,\bar{x}}(x)$.

Theorem 4.1. Let $w > 0$ be an m-vector satisfying $e^T w = 1$. Let X be a constraint system of the form (2.2), (2.3), let $\bar{x} \in \text{int}\, X$, $\bar{s} = b - A\bar{x}$, and let

$$y = A^T \bar{S}^{-1} w \,. \tag{4.1}$$

Then $y \in Y_{\bar{x}}$ given in (3.2), and $\bar{x}$ is the w-center of the projectively transformed constraint system $Z = Z_{y,\bar{x}}$ given by (3.3) – (3.6), under the projective transformation $g(x) = g_{y,\bar{x}}(x)$ of (3.8) – (3.9).

Proof. By setting $\lambda = w$, we see that $y \in \text{int}\, Y_{\bar{x}}$. Note that $g(\bar{x}, \bar{s}) = (\bar{x}, \bar{s})$, so that $(\bar{x}, \bar{s}) \in (Z;\, T)$, i.e., $\tilde{A}\bar{x} + \bar{s} = \tilde{b}$, $M\bar{x} = g$. From (2.7), it remains to show 17 that $w^T \bar{S}^{-1} \tilde{A} = \pi^T M$ for some $\pi \in R^k$. Let $\pi = 0$. Then $w^T \bar{S}^{-1} \tilde{A} = w^T \bar{S}^{-1}\left(A - \bar{s} y^T\right) = w^T \bar{S}^{-1}\left(A - \bar{s} w^T \bar{S}^{-1} A\right) = 0 = \pi^T M$. Thus (2.7) is satisfied, completing the proof. ■

Theorem 4.1 is a generalization of a theorem of Lagarias [16] which asserts the existence of a projective transformation that will result in $\bar{x}$ being the w-center of a full-dimensional polytope X in the case of $w = (1/m)e$. Theorem 4.1 covers a general linear system of both inequality and equality constraints, and covers the case of non-uniform weights w . It also generalizes the projective transformation construction in [7]. Although the projective transformation $g(x) = g_{y,\bar{x}}(x)$ defined in Theorem 4.1 using (4.1) does not appear to resemble Karmarkar's projective transformation [15] for centering in simplex, it is shown in [8] that Theorem 4.1

specializes to Karmarkar's projective transformation when viewed in the slack space S .

V. Local Improvement of a Canonical Optimization problem, and an Algorithm for the w-Center Problem.

In this section, we consider the following canonical optimization problem:

$$\begin{array}{llll} \text{CP:} & \underset{x}{\text{minimize}} & F(x) = \ln(U - c^T x) - \sum_{i=1}^{m} w_i \ln(b_i - A_i x) & \\ & \text{s.t.} & Ax + s = b & (5.1) \\ & & s > 0 & \\ & & Mx = g & \\ & & c^T x < U \, . & \end{array}$$

The data for the problem is the data for the constraint set $X = (A, b, M, g)$, plus the m-vector of positive weights $w = (w_1, \ldots, w_m)^T$ which satisfy the normalization $e^T w = 1$, plus the data for constraint $c^T x < U$. Note that the linear programming problem:

$$\begin{array}{llll} \text{LP:} & \text{maximize} & c^T x & \\ & \text{s.t.} & Ax \le b & (5.2) \\ & & Mx = g \, , & \end{array}$$

can be cast as an instance of CP . By setting c to be the LP objective function and U to be an upper bound on the optimal LP objective value, CP becomes the potential function minimization problem for LP , as in Karmarkar [15], see also [7]. This problem instance has already been treated in [7] and also [8].

The problem of finding the w-center, namely problem P_w defined in (2.6), is also an instance of CP . By setting

$$c = 0 \quad \text{and} \quad U = 1 \, , \tag{5.3}$$

problem CP specializes to problem P_w . In this section, as well as in Sections 6 and 7, we present an analysis of problem P_w viewed through the canonical optimization problem CP .

Returning now to problem CP directly, suppose we wish to solve CP , and that we have on hand a feasible solution $\bar{x}$ of CP , i.e., $\bar{x} \in \operatorname{int} X$ and $c^T\bar{x} < U$. If $\bar{x}$ happens to be the w-center of X , then $\bar{x}$ has optimized the second part of the objective function F(x) of CP . If $\bar{x}$ is not the w-center of X , we can perform the projective transformation of Theorem 4.1 in order to ensure that $\bar{x}$ is the w-center of the transformed constraint set $Z = Z_{y,\bar{x}}$ (where $y = A^T\bar{S}^{-1}w$ is given in (4.1)) under the projective transformation $z = g(x) = g_{y,\bar{x}}(x)$ of (3.9). Under this projective transformation, the constraints of X are mapped into the constraints of Z , which are given by (3.3) and (3.4). Furthermore, if $\bar{x} \in X$ and x satisfies $c^Tx \le U$, it is then elementary to show $z = g_{y,\bar{x}}(x)$ will satisfy

$$\tilde{c}^T z \le \tilde{U} \tag{5.4}$$

where $$\tilde{c} = c - \left(U - c^T\bar{x}\right)y \, , \quad \tilde{U} = U - \left(U - c^T\bar{x}\right)y^T\bar{x} \, . \tag{5.5}$$

The next lemma shows that under the projective transformation $g_{y,\bar{x}}(\cdot)$, that program CP is transformed into program

$$\widetilde{CP} = \widetilde{CP}_{y,\bar{x}} : \qquad \underset{z}{\text{minimize}} \quad \tilde{F}(z) = \ln(\tilde{U} - \tilde{c}^T z) - \sum_{i=1}^{m} w_i \ln(\tilde{b}_i - \tilde{A}_i z) \quad (5.6)$$

$$\begin{aligned} \text{s.t.} \quad & \tilde{A}z + t = \tilde{b} , \ t > 0 , \\ & Mz = g \\ & \tilde{c}^T z < \tilde{U} \end{aligned}$$

where $\tilde{U}$, $\tilde{c}$ are given by (5.5) and $(\tilde{A} , \tilde{b})$ is given by (3.4).

Lemma 5.1 (Equivalence of CP and $\widetilde{CP}_{y,\bar{x}}$). Suppose $y \in \text{int}\, Y_{\bar{x}}$ of (3.2) and define the projective transformation $g(\cdot) = g_{y,\bar{x}}(\cdot)$ as in (3.9a) and its inverse $h(\cdot) = h_{y,\bar{x}}(\cdot)$ as in (3.9b). If X satisfies condition (A) of (3.10), then programs CP and $\widetilde{CP}$ are equivalent, i.e.,

(i) if x is feasible for CP , $z = g(x)$ is feasible for $\widetilde{CP}$ and $F(x) = \tilde{F}(z)$.

(ii) if z is feasible for $\widetilde{CP}$, $x = h(z)$ is feasible for CP and $\tilde{F}(z) = F(x)$.

Proof. (i) If x is feasible for CP , then $x \in \text{int}\, X$ and so from Corollary 3.1, $z = g(x)$ is well-defined and $z \in \text{int}\, Z$. The equality $F(x) = \tilde{F}(z)$ follows by direct substitution. Parallel logic also demonstrates assertion (ii). ■

Lemma 5.1 implies that in optimizing CP we can instead optimize program $\widetilde{CP}$. If $y \in \text{int}\, Y_{\bar{x}}$ is chosen as $y = A^T \bar{S}^{-1} w$ (from 4.1), then from Theorem 4.1, $\bar{x}$ is the w-center of the constraint set Z . In this case the second part of the objective function $\tilde{F}(z)$ of (5.6) has already been optimized, because this second part is simply the objective function of the w-center problem. Therefore, in analyzing the program CP , we can presume without loss of generality that the current point $\bar{x}$ is the w-center of the constraint set X , by performing the projective transformation $g_{y,\bar{x}}(x)$ of (3.9), where y is given by (4.1).

We therefore suppose, without loss of generality, that we have on hand a feasible solution $\bar{x}$ of CP , i.e., $\bar{x} \in X$, and $c^T\bar{x} < U$, and that $\bar{x}$ is the w-center of X . Then the inner ellipsoid E_{IN} at the w-center is contained in X (from Theorem 2.1), and $F(x)$ can be improved by optimizing c^Tx over the inner ellipsoid E_{IN} . From Theorem 2.1, the problem of finding the direction d that maximizes $c^T(\bar{x} + d)$ over the ellipsoid E_{IN} is

$$\begin{array}{lll} \text{maximize} & c^Td & \\ \text{s.t.} & d^TA^T\bar{S}^{-1}W\bar{S}^{-1}Ad \le r^2 & \\ & Md = 0 , & \end{array} \tag{5.7}$$

where r is defined in (2.8), and $\bar{s} = b - A\bar{x}$. Under the assumption that A and M have full rank, program (5.7) has a unique solution given by

$$\bar{d} = \frac{rGc}{\sqrt{c^TGc}} \tag{5.8a}$$

where

$$Q = A^T\bar{S}^{-1}W\bar{S}^{-1}A \quad \text{and} \quad G = \left[Q^{-1} - Q^{-1}M^T\left(MQ^{-1}M^T\right)^{-1}MQ^{-1}\right] . \tag{5.8b}$$

It is straightforward to check that G is positive semi-definite, and so $c^T G c \geq 0$. Furthermore, $c^T G c = 0$ if and only if c^T lies in the row space of M , which implies that $\bar{x}$ solves CP , due to the supposition that $\bar{x}$ is the w-center of the system X . Therefore, unless $\bar{x}$ solves CP , the denominator of (5.8a) is well-defined and $\bar{d}$ given in (5.8) is the unique solution to program (5.7).

The extent of improvement in optimizing F(z) of (5.1) by moving from $\bar{x}$ in the direction $\bar{d}$ of (5.8) is presented in the following theorem.

Theorem 5.1 (Improvement of CP from the w-center $\bar{x}$). Suppose $\bar{x}$ is the w-center of X , $\bar{s} = b - A\bar{x}$, and let $\bar{d}$ be the solution to (5.7) given in (5.8). Define the quantity

$$\gamma = \frac{c^T \bar{d}}{\left(U - c^T \bar{x}\right) r^2} \tag{5.9}$$

Then (i) if $\gamma \geq 1/r^2$, program CP is unbounded from below.

(ii) if $\gamma < 1/r^2$, then $F(\bar{x} + \alpha\bar{d}) \leq F(\bar{x}) - r^2\left(\gamma\alpha - \frac{\alpha^2}{2(1 - \alpha)}\right)$ for all $\alpha \in [0, 1)$.

Before proving the theorem, we offer the following comments. The optimal objective value of the inner ellipsoid maximization program (5.7) is $c^T \bar{d}$, and so γ is just a rescaling of this value by the quantity $(U - c^T \bar{x}) r^2$. In (ii) of the theorem, the extent of improvement in the objective function CP is proportional to the function

$$f(\alpha) = \gamma\alpha - \frac{\alpha^2}{2(1 - \alpha)} \tag{5.10}$$

The value of α that maximizes $f(\alpha)$ over $\alpha \in [0, 1)$ is

$$\alpha = 1 - \frac{1}{\sqrt{1 + 2\gamma}} \tag{5.11}$$

which yields the value of $f(\alpha)$ of

$$k(\gamma) = \left(1 + \gamma - \sqrt{1 + 2\gamma}\right). \tag{5.12}$$

Summarizing, we have

Corollary 5.1. If α is given in (5.11) in Theorem 5.1 (ii), then

$$F(\bar{x} + \alpha\bar{d}) - F(\bar{x}) \le -r^2\left(1 + \gamma - \sqrt{1 + 2\gamma}\right) = r^2 k(\gamma). \tag{5.13}$$

Proof of Theorem 5.1. (i) Suppose $\gamma \ge 1/r^2$. Then from (5.9), $c^T\bar{d} \ge U - c^T\bar{x}$, i.e., $c^T(\bar{x} + d) \ge U$. Thus, as $\alpha \to 1$, $\ln\left(U - c^T(\bar{x} + \alpha d)\right) \to -\infty$. As a consequence of (5.7), (5.8), and Theorem 2.1, $\bar{x} + \alpha d \in X$ for all $\alpha \in [0, 1)$. If CP is not bounded from below, then $A_i(\bar{x} + \alpha\bar{d}) \to b_i$ for every $i = 1, \ldots, m$, as $\alpha \to 1$, i.e., $A\bar{d} = \bar{s}$, which implies X is unbounded, which in turn implies that the w-center of X cannot exist, contradicting the hypothesis of the Theorem. Thus CP is unbounded from below. (ii) Suppose $\gamma < 1/r^2$. Then

$$F(\bar{x} + \alpha\bar{d}) - F(\bar{x}) = \ln\left(\frac{U - c^T(\bar{x} + \alpha\bar{d})}{U - c^T\bar{x}}\right) - \sum_{i=1}^{m} w_i \ln\left(\frac{b_i - A_i(\bar{x} + \alpha\bar{d})}{b_i - A_i\bar{x}}\right)$$

$$= \ln\left(1 - \alpha r^2 \gamma\right) - \sum_{i=1}^{m} w_i \ln\left(1 - \alpha\left(\bar{S}^{-1} A\bar{d}\right)_i\right) \qquad \text{(from 5.9)}$$

$$\leq - r^2 \alpha\gamma + \frac{r^2 \alpha^2}{2(1 - \alpha)} \qquad \text{(from Proposition A.1 of the Appendix and Lemma 2.1)}$$

$$= r^2\left(- \gamma\alpha + \frac{+ \alpha^2}{2(1 - \alpha)}\right). \quad \blacksquare$$

Lemma 5.1 and Theorem 5.1 suggest an algorithm for solving CP as follows: At each iteration, CP is projectively transformed to $\widetilde{CP} = \widetilde{CP}_{y,\bar{x}}$ of (5.6) where $y = A^T\bar{S}^{-1}w$ (of (4.1)), which transforms the current point $\bar{x}$ to the w-center of the transformed constraint set (Theorem 4.1). Then the algorithm steps a length α in the direction $\bar{d}$ of (5.7) – (5.8) that maximizes the transformed objective function vector $\tilde{c}$ over the inner ellipsoid E_{IN}, where α is given by (5.11). This basic algorithm methodology can then be used to solve a linear program (5.2) or to solve P_w (2.6), which are each special instances of CP. The specialization of the algorithmic methodology of this section to solving LP is detailed in [8]. The remainder of this section treats the specialization of this methodology to solve the w-center program P_w.

Recall that program P_w (2.6) is the special case of CP (5.1) where $c = 0$ and $U = 1$ (5.3). The algorithm for solving P_w then is as follows:

Algorithm WP $(A, b, M, g, w, \overset{\circ}{x}, \varepsilon)$.

Step 0 (Initialization). $\bar{x} = \overset{\circ}{x}$, $\bar{w} = \min_i \{w_1, \ldots, w_m\}$, $r = \sqrt{\bar{w}/(1-\bar{w})}$,
$R = \sqrt{(1-\bar{w})/\bar{w}}$, $F^* = +\infty$.

Step 1 (Projective Transformation to w-center).

$$\bar{s} = b - A\bar{x}$$
$$y = A^T\bar{S}^{-1}w$$
$$\tilde{A} = A - \bar{s}y^T , \tilde{b} = b - \bar{s}y^T\bar{x} ,$$
$$\left(\tilde{c} = -y , \tilde{U} = 1 - y^T\bar{x}\right)$$

Step 2 (Optimization over inner ellipsoid). Solve the program:

$$\begin{aligned} \text{EP:} \quad & \text{maximize} \quad -y^T d \\ & \text{s.t.} \quad d^T\tilde{A}^T\bar{S}^{-1}W\bar{S}^{-1}\tilde{A}d \le r^2 \\ & \qquad Md = 0 . \end{aligned} \tag{5.14}$$

The optimal solution is given by

$$\bar{d} = \frac{-\tilde{G}y}{\sqrt{y^T\tilde{G}y}} ,$$

where $\tilde{Q} = \tilde{A}^T\bar{S}^{-1}W\bar{S}^{-1}\tilde{A}$ and $\tilde{G} = \left[\tilde{Q}^{-1} - \tilde{Q}^{-1}M^T\left(M\tilde{Q}^{-1}M^T\right)^{-1}M\tilde{Q}^{-1}\right]$. (5.15)

If EP is unbounded from above, stop. P_w is unbounded. (5.16)

Step 2a (Update upper bound on F^*).

Set $\gamma = \gamma(\bar{x}) = \left(-y^T\bar{d}\right)/r^2$ (5.17)

If $\gamma \geq 1/r^2$, stop. Problem P_w is unbounded, and $\bar{d}$ is a ray of X . (5.18)

If $\gamma < 1$, $F^* \leftarrow \min\left\{F^*, F_w(\bar{x}) + \gamma + \dfrac{\gamma^2}{2(1-\gamma)}\right\}$ (5.19)

If $\gamma \leq \frac{1}{8}$, $F^* \leftarrow \min\left\{F^*, F_w(\bar{x}) + (0.82)\, r^2\gamma^2\right\}$ (5.20)

Step 3 (Take step in the set Z).

$$\alpha = 1 - \frac{1}{\sqrt{1 + 2\gamma}}$$

$$z_{NEW} = \bar{x} + \alpha\bar{d}$$

Step 4 (Transform back to the set X).

$$x_{NEW} = \bar{x} + \frac{z_{NEW} - \bar{x}}{1 + y^T(z_{NEW} - \bar{x})}$$

Step 5 (Stopping Criterion).

Set $\bar{x} = x_{NEW}$. If $F_w(\bar{x}) \geq F^* - \varepsilon$, stop. (5.21)

Otherwise, go to Step (1).

The data for the problem is the data (A, b, M, g) of the constraint set X, the vector w of positive weights that satisfy $e^T w = 1$, an initial feasible solution x° of P_w, and an optimality tolerance $\varepsilon > 0$. We can assume without loss of generality that the constraint set X satisfies condition (A) of (3.10) by prior knowledge of the boundedness of X or by adding the null constraint $0^T x \le 1$ to the system (A, b). In Step 0, the value of $\bar{x}$ is initialized and the constants $\overline{w}$, r, and R of (2.8) are computed. In Step 1, the value of y of (4.1) is computed, and the constraint set data is transformed according to (3.4). In addition, we have from (5.3) and (5.5) that

$$\tilde{c} = -y \text{ and } \tilde{U} = 1 - y^T \bar{x} . \tag{5.22}$$

In Step 2, the inner ellipsoid program of (5.7) is solved via (5.8) for the transformed data. In Step 2a, the upper bound F^* is updated. The bounds given in (5.19) and (5.20) will be proven in Section 6. (The unboundedness criteria of (5.16) and (5.18) will be proven below in Lemma 5.3). In Step 3, the stepsize α is computed according to (5.9) and (5.11). Note that the computation of γ from (5.9) is

$$\gamma = \frac{\tilde{c}^T \bar{d}}{\left(\tilde{U} - \tilde{c}^T \bar{x}\right) r^2} = \frac{-y^T \bar{d}}{r^2} ,$$

as is stated in (5.17). In Step 4, the new value of $z = z_{NEW}$ (in the transformed set Z) is transformed back to the set X via the projective transformation $h(z) = h_{y,\bar{x}}(z)$ of (3.9). In Step 5, the optimality tolerance criterion is checked.

Lemma 5.1, Theorem 5.1, and Corollary 5.1 combine to yield the following:

Lemma 5.2 (Performance of Algorithm WP). At each iteration of Algorithm WP ,

$$F_w(x_{NEW}) \geq F_w(\bar{x}) + r^2\left(1 + \gamma - \sqrt{1 + 2\gamma}\right),$$

where $\gamma = \gamma(\bar{x})$ is defined in (5.17). ■

Remark 5.1 (Use of line-search). Steps 3 and 4 can be augmented by a line-search of the objective function $F_w(x)$, without affecting the conclusion of Lemma 5.2. Because the projective transformation $g(\cdot)$ preserves directions from $\bar{x}$ one can perform the line-search in the space X directly. Specifically, one can replace the computation of α in Step 3 and all of Step 4 by finding a value of $\delta \geq 0$ for which $F_w\left(\bar{x} + \delta\bar{d}\right)$ is approximately maximized. As shown in Todd and Burrell [24], there will be only one local minimum of $F_w\left(\bar{x} + \delta\bar{d}\right)$ for $\delta \geq 0$. The search could be started with $\delta = \dfrac{\alpha}{1 + \alpha y^T\bar{d}}$, where α is given in Step 3, which corresponds to the value of α in (5.11).

Lemma 5.3 (Detecting Unboundedness in Algorithm WP).

(i) If Algorithm WP stops via (5.16), then P_w is unbounded.

(ii) If Algorithm WP stops via (5.18), then P_w is unbounded.

Proof: (i) If program EP of (5.14) has no solution, there exists a vector d for which $d^T\tilde{Q}\,d = 0$, $Md = 0$, and $-y^T d > 0$ (where $\tilde{Q}$ is defined in (5.15)).

Thus $\tilde{A}d = 0$, i.e., $Ad = \bar{s}y^T d$. If X is bounded, then $d = 0$, contradicting $-y^T d > 0$.

(ii) In this case, from Lemma 5.1 and Theorem 5.1 (i), program $\widetilde{CP}$ is unbounded from below, and so program CP is unbounded from below. ■

In the next section we will demonstrate the bounds of (5.19) and (5.20), and will prove that Algorithm WP is superlinearly convergent.

VI. Linear and Superlinear Convergence of Algorithm WP.

The purpose of this section is to establish the following four results regarding Algorithm WP for solving the w-center problem P_w .

Lemma 6.1 (Optimal Objective Value Bounds). At Step 2a of Algorithm WP ,

(i) If $\gamma < 1$, P_w has an optimal solution $\hat{x}$, and

$$F_w(\hat{x}) \le F_w(\bar{x}) + \gamma + \frac{\gamma^2}{2(1-\gamma)}$$

(ii) If $\gamma \le \frac{1}{8}$, then $F_w(\hat{x}) \le F_w(\bar{x}) + (0.82)\, r^2 \gamma^2$.

Note that Lemma 6.1 validates the upper bounds computed in (5.19) and (5.20) of the algorithm.

Lemma 6.2 (Local Improvement). At Step 2a of Algorithm WP ,

(i) If $\gamma \geq \frac{1}{8}$, $F_w(x_{NEW}) \geq F_w(\bar{x}) + (0.0069)\, r^2$

(ii) If $\gamma \leq \frac{1}{8}$, then $F_w(x_{NEW}) \geq F_w(\bar{x}) + (0.44)\, r^2 \gamma^2$.

Lemma 6.3 (Linear Convergence or Fixed Improvement). At each iteration of Algorithm WP , at least one of the following is true:

(i) $F_w(x_{NEW}) \geq F_w(\bar{x}) + (0.0069)\, r^2$

(ii) $F_w(\hat{x}) - F_w(x_{NEW}) \leq (0.46)\big(F_w(\hat{x}) - F_w(\bar{x})\big)$, where $\hat{x}$ is the w-center of X .

Lemma 6.3 states that each iteration achieves either constant improvement (i) or linear convergence (ii) with a convergence upper bound constant of 0.46. The next theorem states that this upper bound constant will go to zero in the limit, thus establishing superlinear convergence.

Theorem 6.1. If program P_w is bounded, then Algorithm WP exhibits superlinear convergence.

The proofs of these results will make use of the following functions, defined below for convenience.

$$k(\gamma) = 1 + \gamma - \sqrt{1 + 2\gamma}, \quad \gamma \geq 0 \tag{6.1}$$

$$j(\theta) = \frac{k(\theta)}{\theta^2}, \quad \theta > 0 \tag{6.2}$$

$$p(h) = \frac{h - \ln(1 + h)}{h^2} \tag{6.3}$$

$$q(h) = \frac{1}{2}\left(1 + h\,p(h) - \sqrt{1 + (h\,p(h))^2}\right), \quad h > 0 \tag{6.4}$$

$$v(h) = p(h) - \frac{(q(h))^2}{2(1 - q(h))}, \quad h > 0 \tag{6.5}$$

$$m(h) = k(q(h)) \tag{6.6}$$

$$n(h) = j(q(h)) \tag{6.7}$$

Inequalities relating to these functions can be found in Propositions A.4 – A.9 of the Appendix. We first will prove Lemma 6.1 (i). The proof of Lemma 6.1 (ii) is more involved.

Proof of Lemma 6.1 (i). Under the projective transformation $g(x) = g_{y,\bar{x}}(x)$ where $y = A^T \bar{S}^{-1} w$, $\bar{x}$ is the w-center of the system $Z = (\tilde{A}, \tilde{b}, M, g)$ and problem P_w (2.6) is transformed, as in Lemma 5.1, to the program

$$\widetilde{P}_w : \underset{z,t}{\text{maximize}} \ \widetilde{F}_w(z) = -\ln\left(1 + y^T(z - \bar{x})\right) + \sum_{i=1}^{m} w_i \ln\left(\widetilde{b}_i - \widetilde{A}_i z\right) \tag{6.8}$$

$$\begin{aligned} \text{s.t.} \qquad \widetilde{A}z + t &= \widetilde{b} \\ t &> 0 \\ Mz \quad &= g \\ -y^T z \quad &< 1 - y^T \bar{x} \,. \end{aligned}$$

Because $\bar{x}$ is the w-center of Z , then

$$\sum_{i=1}^{m} w_i \ln\left(\widetilde{b}_i - \widetilde{A}_i z\right) \le \sum_{i=1}^{m} w_i \ln\left(\bar{s}_i\right) = \widetilde{F}_w(\bar{x}) \,, \tag{6.9}$$

for all $z \in Z$. Also, because $\bar{x} + \bar{d}$ maximizes $-y^T z$ over the ellipsoid E_{IN} of the Z polytope (defined in Theorem 2.1), then $\bar{x} + \bar{d}R/r$ maximizes $-y^T z$ over the outer ellipsoid E_{OUT} of the Z polytope (also defined as in Theorem 2.1). Because $Z \subset E_{OUT}$, $-y^T z \le -y^T(\bar{x} + \bar{d}R/r)$ for all $z \in Z$. Put another way,

$$-y^T(z - \bar{x}) \le \gamma \text{ for all } z \in Z \,. \tag{6.10}$$

This follows because $-y^T\bar{d}R/r = -y^T\bar{d}\big/ r^2 = \gamma$.

$$\begin{aligned} \text{Therefore } \widetilde{F}_w(z) &= -\ln\left(1 + y^T(z - \bar{x})\right) + \sum_{i=1}^{m} w_i \ln\left(\widetilde{b}_i - \widetilde{A}_i z\right) \\ &\le -\ln\left(1 + y^T(z - \bar{x})\right) + \widetilde{F}_w(\bar{x}) \qquad \text{(from 6.9)} \end{aligned}$$

$$\leq \gamma + \frac{\gamma^2}{2(1-\gamma)} + \widetilde{F}_w(\bar{x}) \cdot \qquad \text{(from Proposition A.2)}$$

Therefore, from the equivalence of $\widetilde{P}_w$ and P_w under the projective transformation $g(x) = g_{y,\bar{x}}(x)$ and Lemma 5.1, $F_w(x) - F_w(\bar{x}) \leq \gamma + \frac{\gamma^2}{2(1-\gamma)}$ for all $x \in \operatorname{int} X$. ■

The proof of Lemma 6.1 (ii) will follow as a consequence of the following three lemmas.

Lemma 6.4. Let $h > 0$ be a given parameter. Suppose $\bar{x}$ is the w-center of X , let $\bar{s} = b - A\bar{x}$, and suppose $\hat{x} \in X$ satisfies:

$$(\hat{x} - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(\hat{x} - \bar{x}) = \beta^2 .$$

Then

$$\sum_{i=1}^{m} w_i \ln(b_i - A_i\hat{x}) - \sum_{i=1}^{m} w_i \ln(\bar{s}_i) \leq \begin{cases} -p(h)\beta^2 & \text{if } \beta \leq hr \\ -p(h)hr\beta & \text{if } \beta \geq hr \end{cases},$$

where r is defined in (2.8) and $p(h)$ is defined in (6.3).

Proof. First observe that

$$\sum_{i=1}^{m} w_i \ln(b_i - A_i\hat{x}) - \sum_{i=1}^{m} w_i \ln(\bar{s}_i) = \sum_{i=1}^{m} w_i \ln(1 + v_i), \text{ where}$$

$v = \bar{S}^{-1}A(\bar{x} - \hat{x})$. Then note that $w^T v = w^T\bar{S}^{-1}A(\bar{x} - \hat{x}) = \bar{\pi}^T M(\bar{x} - \hat{x}) = 0$, for some $\bar{\pi} \in R^k$, from (2.7d). Also $v^T Wv = \beta^2$. Therefore $(vr/\beta)^T W(vr/\beta) = r^2$, and so from Proposition 2.2, $|v_i r/\beta| \leq 1$, i.e.,

$$|v_i| \leq \beta/r, \; i=1,..,m. \tag{6.11}$$

We now prove the two cases of the Lemma.

Case 1. $(\beta \leq hr)$. In this case $|v_i| \leq h$. From Proposition A.7, $\ln(1 + v_i) \leq v_i - p(h)(v_i)^2$. Summing over i yields

$$\sum_{i=1}^{m} w_i \ln(1 + v_i) \leq w^T v - p(h)v^T Wv = -p(h)\beta^2 .$$

Case 2. $(\beta \geq hr)$. In this case, from proposition A.3 (with $a = (hr/\beta)$, $b = v_i$),

$$\sum_{i=1}^{m} w_i \ln(1 + v_i) \leq \left(\frac{\beta}{hr}\right) \sum_{i=1}^{m} w_i \ln\left(1 + \frac{rh}{\beta} v_i\right).$$

However, from (6.11), $\frac{rh}{\beta} v_i \leq h$, and so from Proposition A.7,

$$\sum_{i=1}^{m} w_i \ln(1 + v_i) \leq \left(\frac{\beta}{hr}\right)\left(\frac{rh}{\beta} w^T v - \frac{r^2h^2}{\beta^2} v^T Wv\, p(h)\right)$$

$$= -\frac{rh}{\beta}\beta^2 = -p(h)\,hr\,\beta . \quad \blacksquare$$

Lemma 6.5. Let $\bar{x}$ be the current iterate of Algorithm WP and let $\bar{s}$, y , $\tilde{b}$, $\tilde{A}$, $\tilde{Q}$, and γ be as defined in Steps 1, 2, and 3. Suppose $\hat{x}$ is the optimal solution to P_W , and let $\hat{z} = g(\hat{x}) = g_{y,\bar{x}}(\hat{x})$.

$$\text{Suppose } (\hat{z} - \bar{x})^T \tilde{Q} (\hat{z} - \bar{x}) = \beta^2 . \tag{6.12}$$

If $h > 0$ is a given parameter and $\gamma < 1$, then

$$F_w(\hat{x}) - F_w(\bar{x}) \le \begin{cases} -p(h)\beta^2 + \beta r\gamma + \dfrac{\beta^2 r^2 \gamma^2}{2(1-\gamma)} & \text{if} \quad \beta \le hr \\ -p(h)\,hr\,\beta + \beta r\gamma + \dfrac{\beta^2 r^2 \gamma^2}{2(1-\gamma)} & \text{if} \quad \beta \ge hr \end{cases} .$$

Proof. Let $\tilde{P}_w$ be the projectively transformed equivalent program of P_W , i.e., program (6.8). Then it suffices to show that $\tilde{F}_w(\hat{x}) - \tilde{F}_w(\bar{x})$ is less than or equal to the expressions in parentheses, by Lemma 5.1.

From Lemma 6.4.,

$$\sum_{i=1}^{m} w_i \ln\left(\tilde{b}_i - \tilde{A}_i \hat{z}\right) - \sum_{i=1}^{m} w_i \ln\left(\bar{s}_i\right) \le \begin{cases} -p(h)\beta^2 & \text{if} \quad \beta \le hr \\ -p(h)\,hr\,\beta & \text{if} \quad \beta \ge hr \end{cases} . \tag{6.13}$$

It thus remains to show that $-\ln\left(1 + y^T(\hat{z} - \bar{x})\right) \leq \beta r \gamma + \dfrac{\beta^2 r^2 \gamma^2}{2(1 - \gamma)}$. Let

$\hat{d} = \hat{z} - \bar{x}$. Then from (6.12), $\pm \hat{d}\, r/\beta$ satisfies the constraints of EP (5.14), so that

$$\pm y^T \hat{d}\, r/\beta \leq y^T \bar{d} = \gamma r^2 \quad \text{(from 5.17)},$$

and consequently

$$\left| y^T \hat{d} \right| \leq \gamma r \beta \,. \tag{6.14}$$

Furthermore, because of (6.12), from Theorem 2.1, we conclude that

$$\beta \leq R = 1/r\,, \quad \text{i.e.,} \quad r\beta \leq 1\,. \tag{6.15}$$

Finally, we obtain

$$\ln\left(1 + y^T(\hat{z} - \bar{x})\right) = \ln\left(1 + y^T \bar{d}\right) \geq \ln\left(1 - \gamma r \beta\right) \geq -\gamma r \beta - \frac{\gamma^2 r^2 \beta^2}{2(1 - \gamma)}\,,$$

from Proposition A.2, (6.14), and (6.15). ■

Lemma 6.6. Under the hypothesis of Lemma 6.5,

$$\text{if } \gamma \le q(h)\ , \text{ then } \beta \le hr\ ,$$

where $q(h)$ is defined in (6.4).

Proof. Suppose $\beta > hr$. Then from Lemma 6.5,

$$F_w(\hat{x}) - F_w(\bar{x}) \le f(\gamma, \beta)\ , \tag{6.16}$$

where $f(\gamma,\beta) = -p(h)hr\beta + \beta r\gamma + \dfrac{\beta^2 r^2 \gamma^2}{2(1-\gamma)}$.

Note that $f(\gamma,\beta)$ increases in γ for $\beta > 0$ and $0 \le \gamma < 1$. Straightforward calculation reveals that $f(\gamma,\beta) = 0$ if

$$\gamma = \frac{1 + hp(h) - \sqrt{1 + (hp(h))^2 - 2hp(h) + 2hp(h)\beta r}}{2 - r\beta}$$

$$> \frac{1 + hp(h) - \sqrt{1 + (hp(h))^2}}{2} = q(h)$$

because $0 \le r\beta \le 1$ from (6.15).

Thus if $\gamma \le q(h)$, $f(\gamma,\beta) < 0$, contradicting the optimality of $\hat{x}$ in (6.16).

Therefore if $\gamma \le q(h)$, $\beta \le hr$. ■

Proof of Lemma 6.1 (ii). We will actually prove a stronger result, namely:

$$\text{If } 0 < h \le 1 \text{, and } \gamma \le q(h) \text{, then } F_w(\hat{x}) - F_w(\bar{x}) \le \left(\frac{1}{4v(h)}\right) \gamma^2 r^2 , \tag{6.17}$$

where $v(h)$ is defined in (6.5).

Lemma 6.1 (ii) will follow by substituting $h = 0.93$. Then $q(h) \ge \frac{1}{8}$, and $\left(\frac{1}{4v(h)}\right) \le 0.82$.

To prove (6.17), observe that if $\gamma \le q(h)$, from Lemma 6.6, $\beta \le hr$, and so from Lemma 6.5,

$$\begin{aligned} F_w(\hat{x}) - F_w(\bar{x}) &\le -p(h)\beta^2 + \beta r \gamma + \frac{\beta^2 r^2 \gamma^2}{2(1-\gamma)} \\ &\le -p(h)\beta^2 + \beta r \gamma + \frac{\beta^2 \gamma^2}{2(1-\gamma)} \\ &= -\left(p(h) - \frac{\gamma^2}{2(1-\gamma)}\right)\beta^2 + \beta r \gamma . \end{aligned} \tag{6.18}$$

However, $\left(p(h) - \frac{\gamma^2}{2(1-\gamma)}\right) \ge v(h)$ because $\gamma \le q(h)$, and so from Proposition A.9, $\left(p(h) - \frac{\gamma^2}{2(1-\gamma)}\right) \ge v(h) > 0$ for $h \le 1$, so that the bound of (6.18) is a concave quadratic is β . The maximum possible value of the bound is then given by $\beta = \dfrac{r\gamma}{2p(h) - \dfrac{\gamma^2}{(1-\gamma)}}$, which yields from (6.18)

$$F_w(\hat{x}) - F_w(\bar{x}) \le \frac{r^2\gamma^2}{4p(h) - \dfrac{2\gamma^2}{(1-\gamma)}} \le \frac{r^2\gamma^2}{4v(h)} . \quad \blacksquare$$

Proof of Lemma 6.2. (i) We will actually prove a stronger result, namely:

If $0 < h \le 1$, and $\gamma \ge q(h)$, then $F_w(x_{NEW}) - F_w(\bar{x}) \ge m(h) r^2$ (6.19)

where $m(h)$ is defined in (6.6).

Lemma 6.2 (i) will follow by setting $h = 0.923$. Then $q(h) \le \frac{1}{8}$, and $m(h) \ge 0.0069$.

To prove (6.19), observe from Lemma 5.2 and Theorem 5.1 that

$$\begin{aligned} F_w(x_{NEW}) &\ge F_w(\bar{x}) + r^2(1 + \gamma - \sqrt{1 + 2\gamma}) && (6.20) \\ &= F_w(\bar{x}) + r^2 k(\gamma) && \text{(see (6.1))} \\ &\ge F_w(\bar{x}) + r^2 k(q(h)) = F_w(\bar{x}) + r^2 m(h) , \end{aligned}$$

from Proposition A.8.

(ii) We will prove a stronger result, namely:

If $0 < h \leq 1$, and $\gamma \leq q(h)$, $F_w(x_{NEW}) - F_w(\overline{x}) \geq n(h) r^2 \gamma^2$, (6.21)

where $n(h)$ is defined in (6.7).

Lemma 6.2 (ii) will follow by setting $h = 0.929$. Then $q(h) \geq \frac{1}{8}$ and $n(h) \geq (0.44)$.

To prove (6.21), observe from (6.20) that

$$F_w(x_{NEW}) \geq F_w(\overline{x}) + r^2 k(\gamma)$$

$$\geq F_w(\overline{x}) + r^2 j(\theta)\gamma^2 \text{ for } 0 \leq \gamma \leq \theta$$

(from Proposition A.5)

$$= F_w(\overline{x}) + r^2 j(q(h))\gamma^2$$

(substituting $q(h) = \theta$)

$$= F_w(\overline{x}) + n(h) r^2 \gamma^2 . \quad \blacksquare$$ (from (6.7))

Proof of Lemma 6.3. Let $h = 0.929$. (i) follows from (6.19) if $\gamma \geq q(h)$.

Now suppose $\gamma \leq q(h)$. Then from (6.17) and (6.21),

$$\frac{F_w(\hat{x}) - F_w(x_{NEW})}{F_w(\hat{x}) - F_w(\bar{x})} = 1 - \frac{F_w(x_{NEW}) - F_w(\bar{x})}{F_w(\hat{x}) - F_w(\bar{x})} \leq 1 - \frac{n(h)r^2\gamma^2}{\left(\frac{1}{4v(h)}\right)r^2\gamma^2}$$

$$= 1 - 4n(h)v(h) . \qquad (6.22)$$

$$\leq 0.46 . \quad \blacksquare$$

Proof of Theorem 6.1. It suffices to prove that as $h \to 0$, the convergence constant of (6.22) goes to zero. This constant is

$$1 - 4v(h)n(h) = 1 - 4\left(p(h) - \frac{q(h)^2}{2(1 - q(h))}\right)(j(q(h))) .$$

From Propositions A.4, A.6, and A.8, $p(h) \to \frac{1}{2}$, $q(h) \to 0$, and $j(q(h)) \to \frac{1}{2}$, as $h \to 0$. Thus

$$1 - 4v(h)n(h) \to 1 - 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 0 \text{ as } h \to 0 . \quad \blacksquare$$

Remark 6.1. (Alternative Convergence Constants). Lemma 6.3 asserts that at each iterate of the algorithm that we obtain a constant improvement of at least $0.0069r^2$, or linear convergence to the optimal objective value, with convergence constant 0.46 . The constants 0.0069 and 0.46 are derived by using the value of $h = .93$ in (6.17), (6.19), and (6.21). If instead of choosing $h = .93$, one chooses $h = 2$, for example, then by paralleling the methodology in Section 6, one obtains Lemma 6.3 with a constant improvement of at least $.013r^2$ or a linear convergence rate with convergence constant 0.65 . The choice of $h = .93$ was fairly arbitrary.

Remark 6.3. (Monotonic Values of γ). One natural question to ask regarding Algorithm WP is whether the values of γ generated at each iteration are monotonically decreasing. We have:

Proposition 6.1. Suppose γ_1 and γ_2 are two successive values of γ generated by Algorithm WP . Then if $\gamma_1 \leq 1/8$, $\gamma_2 \leq (0.92)\gamma_1$.

Proof. Let x_1 and x_2 be the successive iterates of Algorithm WP that generate the values of $\gamma = \gamma_1$ and $\gamma = \gamma_2$, respectively, and let x_3 be the iterate value of x after x_2 . Let $h = 0.93$. Then $\gamma_1 \leq q(h)$. Suppose γ_2 does not satisfy $\gamma_2 \leq q(h)$. Then from (6.17), (6.19), and (6.21),

$$F_w(x^3) \leq F_w(\hat{x}) \leq F_w(x^1) + \left(\frac{1}{4\,v(h)}\right)\gamma_1^2 r^2 \tag{6.23}$$

$$F_w(x^3) \geq F_w(x^2) \geq F_w(x^1) + n(h)\gamma_1^2 r^2 \tag{6.24}$$

$$F_w(\hat{x}) \geq F_w(x^3) \geq F_w(x^2) + m(h) r^2 .$$

Combining the above inequalities yields

$$\left(\frac{1}{4\,v(h)}\right)\gamma_1^2 \;\ge\; m(h) + n(h)\,\gamma_1^2,$$

i.e., $$\gamma_1 \;\ge\; \sqrt{\frac{m(h)}{4\,v(h) - n(h)}}\;.$$

However, $\gamma_1 \le q(h)$, which is a contradiction at $h = 0.93$. Thus $\gamma_2 \le q(h)$. This being the case, from (6.21) we obtain

$$F_w\left(x^3\right) - F_w\left(x^2\right) \;\ge\; n(h)\,\gamma_2^2 \tag{6.25}$$

which in combination with (6.23) and (6.24) yields

$$\gamma_2 \le \left(\sqrt{\frac{\frac{1}{4\,v(h)} - n(h)}{n(h)}}\right)\gamma_1 \le (0.92)\,\gamma_1\;. \quad \blacksquare$$

VII. The Improving Direction is the Newton Direction.

In this section, we show that the direction $\bar{d}$ of Step 2 of Algorithm WP is a positively scaled projected Newton direction. As a byproduct of this result, the computation of $\bar{d}$ in Step 2 can be carried out without solving equations involving the matrix $\tilde{Q} = \tilde{A}^T\bar{S}^{-1}W\bar{S}^{-1}\tilde{A}$, which will typically be extremely dense. Vaidya's algorithm for the center problem [26] corresponds to computing the Newton direction and performing an inexact line-search. Thus, Algorithm WP specializes to Vaidya's algorithm when the algorithm is augmented with a line-search, see Remark 5.1. Furthermore, this establishes that Vaidya's algorithm then will exhibit superlinear convergence.

Let $\bar{x}$ be the current iterate of Algorithm WP, let $\bar{s} = b - A\bar{x}$, and $y = A^T\bar{S}^{-1}w$ and $\tilde{A} = A - \bar{s}\,y^T$ as in Step 1 of Algorithm WP, and let $Q = A^T\bar{S}^{-1}W\bar{S}^{-1}A$, and $\tilde{Q} = \tilde{A}^T\bar{S}^{-1}W\bar{S}^{-1}\tilde{A}$. By assumption, A has full rank, so that Q is nonsingular and positive definite. Let $F_w(x)$ be the weighted logarithmic barrier objective function of P_w given in (2.6). Then the gradient of $F_w(\cdot)$ at $\bar{x}$ is given by $-y$, i.e., $\nabla F_w(\bar{x}) = -y$, and the Hessian of $F_w(\cdot)$ at $\bar{x}$ is given by $-Q$, i.e., $\nabla^2 F_w(\bar{x}) = -Q$. Thus the projected Newton direction d_N at $\bar{x}$ is the optimal solution to

$$\begin{array}{ll} \text{maximize} & -y^T d - (1/2)\,d^T Q d \\ \text{s.t.} & Md = 0 , \end{array} \tag{7.1}$$

and the Newton direction d_N together with Lagrange multipliers π_N is the unique solution to

$$\begin{aligned} Q d_N - M^T \pi_N &= -y \\ M d_N \qquad\quad &= 0 . \end{aligned} \tag{7.2}$$

Because Q has rank n and M has rank k, we can write the solution to (7.2) as

$$d_N = -Q^{-1}y + Q^{-1}M^T\pi_N \tag{7.3}$$

where $\qquad \pi_N = \left(M Q^{-1} M^T\right)^{-1} M Q^{-1} y$

Theorem 7.1 (Positive Scaled Newton Direction). Let d_N be the Newton direction given by the solution (7.3) to (7.2), and consider the scaled version of d_N :

$$\bar{d} = \frac{d_N r}{\sqrt{d_N^T \widetilde{Q} d_N}} , \tag{7.4}$$

where $\widetilde{Q}$ is given in (5.15), and r is given in (2.8).

(i) If the denominator of (7.4) is positive, $\bar{d}$ of (7.4) is the direction of Step 2 of Algorithm WP .

(ii) If the denominator of (7.4) is zero, d_N is a ray of X and program EP of Step 2 of Algorithm WP is unbounded from above, and hence so is P_w .

Proof. (i) Let π_N be as given in (7.3), $\bar{\pi} = \pi_N / \left(1 + y^T d_N\right)$, and $\bar{\beta} = \sqrt{d_N^T \widetilde{Q} d_N} / \left(2r\left(1 + y^T d_N\right)\right)$. Then $\bar{d}$, $\bar{\pi}$, $\bar{\beta}$ satisfy the K-K-T conditions of program EP , namely $\bar{d}^T \widetilde{Q} \bar{d} = r^2$, $M\bar{d} = 0$, and $-y = 2\bar{\beta}\widetilde{Q}\bar{d} - M^T\bar{\pi}$, $\bar{\beta} > 0$, so long as $1 + y^T d_N > 0$. It thus remains to show that $1 + y^T d_N > 0$. Note first that $\widetilde{Q} = Q - yy^T$, where $Q = A^T \bar{S}^{-1} W \bar{S}^{-1} A$. By hypothesis, we have

$$0 < d_N^T \widetilde{Q} d_N = d_N^T\left(Q - yy^T\right)d_N = d_N^T Q d_N - \left(y^T d_N\right)^2 = -y^T d_N - \left(y^T d_N\right)^2 , \tag{7.5}$$

which implies $y^T d_N > -1$, i.e., $1 + y^T d_N > 0$.

(ii) Suppose $d_N^T \widetilde{Q}\, d_N = 0$. In view of (7.5), we have $y^T d_N = -1$, and $d_N^T \widetilde{Q}\, d_N = 0$, and $M d_N = 0$. Thus program EP is unbounded, and as in the proof of Lemma 5.3, d_N is a ray of X . ■

Remark 7.1. (Simplified Computation of $\bar{d}$). Theorem 7.1 shows that $\bar{d}$ is just a positive scale of the Newton direction d_N . Thus in order to solve for $\bar{d}$, one need not solve a system involving the possibly-very-dense matrix $\widetilde{Q}$. Rather one need only solve the equations (7.2) for d_N and then compute $\bar{d} = d_N r / \sqrt{d_N^T \widetilde{Q}\, d_N}$.

Remark 7.2. (Relation of Algorithm WP to Vaidya's algorithm). Theorem 7.1 shows that $\bar{d}$ is just a positive scale of the Newton direction d_N . Suppose Algorithm WP is implemented with a line-search replacing Steps 3 and 4, as suggested by Remark 5.1. Then because the projective transformations $g(x)$ and $h(z)$ given by (3.8) and (3.9) preserve directions from $\bar{x}$, the algorithm's direction in the space X will be d_N . Therefore, when using a line-search, the algorithm is just searching in the Newton direction. This is precisely Vaidya's algorithm [26], when all weights w_i are identical. And because the complexity analysis of Sections V and VI carries through with or without a line-search, we see that Vaidya's algorithm exhibits superlinear convergence.

Remark 7.3. (An Extension of a Theorem of Bayer and Lagarias). In [4], Bayer and Lagarias have shown the following structural equivalence between Karmarkar's algorithm for linear programming and Newton's method: First one can projectively transform the problem of minimizing Karmarkar's potential function over a polyhedron X to finding the (unbounded) center of an unbounded polyhedron Z , where Z is the image of X under a projective transformation that sends the set of optimal solutions to the linear program to the hyperplane at infinity. Then the image of Karmarkar's algorithm (with a line-search) in the space Z corresponds to performing a line-search in the Newton direction for the center problem in the transformed space Z. Theorem 7.1 is in fact a generalization of this result. It states that if one is trying to find the center of any polyhedron X (bounded or not), then the direction generated at any iteration of the projective transformation method (i.e., Algorithm WP) is a positive scale of the Newton direction. Thus, if one determines step-lengths by a line-search of the objective function, then the projective transformation method corresponds to Newton's method with a line-search.

Another important relationship between directions generated by projective transformation methods and Newton's method can be found in Gill et al. [12]

Remark 7.4. (No Finite Termination of Algorithm WP). The solution $\hat{x}$ to the w-center program can have irrational components, and so Algorithm WP will not stop after finitely many iterations if the optimality tolerance ε is zero. Even if program P_w is unbounded, the algorithm may never detect unboundedness via (5.16) or (5.18). This is shown in an example of Section 4 of Bayer and Lagarias [4]. In that example, $X = \{x \in R^2 \mid x_1 \geq -1, x_1 \leq -1, x_2 \geq 0, \}$, $w = (1/3, 1/3, 1/3)$, and the starting point of the Algorithm WP is $\overset{\circ}{x} = (1/3, 2/3)$. They show that Newton's method (with a line-search) never produces a ray of X. As a consequence of Theorem 7.1, Algorithm WP (with a line-search) will never detect unboundedness for this example.

VIII. Inner and Outer Ellipsoids at an approximate w-center point $\bar{x}$.

One of the special features of the w-center $\hat{x}$ of a constraint system X is the fact that there exist ellipsoids E_{IN} and E_{OUT} , with center at $\hat{x}$, such that $E_{IN} \subset X \subset E_{OUT}$ and $E_{OUT} = (R/r)\, E_{IN}$, see Theorem 2.1. Although the iterates of Algorithm WP will converge to $\hat{x}$, there may not be finite termination, and in fact the solution $\hat{x}$ may involve irrational data. A natural question is whether one can construct good ellipsoids F_{IN} and F_{OUT} about points near $\hat{x}$, with the property that $F_{IN} \subset X \subset F_{OUT}$, and $F_{OUT} = c \cdot F_{IN}$, where $c = O(1/\overline{w})$. The main result of this section answers this question in the affirmative:

Theorem 8.1 (Inner and Outer Ellipses Near the w-center). If $\bar{x}$ is feasible for P_w , $\bar{s} = b - A\bar{x}$, and $\gamma = \gamma(\bar{x})$ of Algorithm WP satisfies $\gamma \le \frac{1}{8}$, then

$$F_{IN} = \left\{x \in R^n \mid (x - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(x - \bar{x}) \le \bar{r}^2 ,\ Mx = g\right\}$$

$$\text{and} \quad F_{OUT} = \left\{x \in R^n \mid (x - \bar{x})^T A^T \bar{S}^{-1} W \bar{S}^{-1} A(x - \bar{x}) \le \overline{R}^2 ,\ Mx = g\right\}$$

satisfy $F_{IN} \subset X \subset F_{OUT}$,

$$\text{where} \qquad \bar{r} = \sqrt{\overline{w}} \quad \text{and} \quad \overline{R} = \frac{1}{\sqrt{\overline{w}}}(1 + 14.6\gamma) \tag{8.1}$$

In particular, $\overline{R}/\bar{r} \le 2.9/\overline{w}$.

Remark 8.1. Note that if $w = (1/m)e$, then $\overline{R}/\bar{r} \le 2.9m$. Furthermore, as $\gamma \to 0$, $\overline{R}/\bar{r} \to (1/\overline{w}) = m$.

The proof of Theorem 8.1 will follow as a consequence of the following intermediate results.

Proposition 8.1. If $\gamma \le \frac{1}{8}$, and $h = 7.44\gamma$, then $\gamma \le q(h)$.

Proof. The function $q(h)$ is concave for $h > 0$, and $\lim_{h \to 0} q(h) = 0$, see Proposition A.8. Therefore, for any fixed value of $\bar{h} > 0$, $q(h) \ge \left(h\, q(\bar{h})\right)/\bar{h}$ for $h \in (0, \bar{h}]$. Now let $\bar{h} = 0.93$. For $\gamma \le \frac{1}{8}$ and $h = 7.44\gamma$, then $h \in (0, \bar{h}]$, and so $q(h) \ge h\, q(\bar{h})/\bar{h} = \gamma(7.44)\, q(\bar{h})/\bar{h} \ge \gamma$. ■

Lemma 8.1. If $\bar{x}$ is the current iterate of Algorithm WP , $\bar{s} = b - A\bar{x}$, $h > 0$ is given, $h \le 1$, and $\gamma \le q(h)$, then

$$(\hat{x} - \bar{x}) A^T \bar{S}^{-1} W \bar{S}^{-1} A (\hat{x} - \bar{x}) \le \frac{r^2 h^2 (1 + \gamma^2)}{(1 - h r \gamma)^2} , \tag{8.2}$$

where $\hat{x}$ is the w-center of $\mathbf{X}$.

Proof. Let $\hat{z} = g_{y,\bar{x}}(\hat{x})$, where $y = A^T \bar{S}^{-1} w$, i.e., $\hat{z}$ is the image of $\hat{x}$ under the w-centering projective transformation at $\bar{x}$. Let $\beta^2 = (\hat{z} - \bar{x})^T \tilde{A}^T \bar{S}^{-1} W \bar{S}^{-1} \tilde{A} (\hat{z} - \bar{x})$. Then from Lemma 6.6,

$$\beta \le hr . \tag{8.3}$$

Let $\hat{d} = \hat{z} - \bar{x}$, and from (6.14),

$$|y^T\hat{d}| \leq \gamma r \beta \,. \tag{8.4}$$

Let $Q = A^T\bar{S}^{-1}W\bar{S}^{-1}A$, $\tilde{Q} = \tilde{A}^T\bar{S}^{-1}W\bar{S}^{-1}\tilde{A}$, and note that $\tilde{Q} = Q - yy^T$. Then

$$(\hat{x} - \bar{x})^T A^T\bar{S}^{-1}W\bar{S}^{-1}A(\hat{x} - \bar{x}) = \frac{(\hat{z} - \bar{x})^T(\tilde{Q} + yy^T)(\hat{z} - \bar{x})}{\left(1 + y^T(\hat{z} - \bar{x})\right)^2} \qquad \text{(from (3.9))}$$

$$= \frac{\beta^2 + \left(\hat{d}^T y\right)^2}{\left(1 + y^T\hat{d}\right)^2} \leq \frac{\beta^2 + \gamma^2 r^2 \beta^2}{\left(1 - r\gamma\beta\right)^2} \qquad \text{(from (8.4))}$$

$$\leq \frac{h^2r^2 + h^2r^4\gamma^2}{\left(1 - hr^2\gamma\right)^2} \qquad \text{(from (8.3))}$$

$$\leq \frac{h^2r^2(1 + \gamma^2)}{(1 - hr\gamma)^2} \,. \quad \blacksquare$$

As demonstrated in Section VI, as $\gamma = \gamma(\bar{x}) \to 0$, the iterates $\bar{x}$ converge to the w-center $\hat{x}$ of the constraint system X . Therefore, if γ is "small", then $\bar{x}$ will be "close" to $\hat{x}$. The above phrase can be made mathematically precise, as follows. Let

$$\delta = \delta(\bar{x}) = \sqrt{(\hat{x} - \bar{x})^T A^T\bar{S}^{-1}W\bar{S}^{-1}A(\hat{x} - \bar{x})} \,. \tag{8.5}$$

Theorem 8.2. (Relationship of δ to γ). If $\bar{x}$ is feasible for P_w and $\gamma = \gamma(\bar{x})$ of Algorithm WP satisfies $\gamma \le \frac{1}{8}$, then $\delta = \delta(\bar{x})$ of (8.5) satisfies

$$\delta = \delta(\bar{x}) \le 8.5\gamma r .$$

Proof. Let $h = 7.44\gamma$. From Proposition 8.1, $\gamma \le q(h)$. Substituting in (8.2) and noting that $r \le 1$, we obtain $\delta = \delta(\bar{x}) \le \dfrac{r(7.44\gamma)\sqrt{1 + 1/64}}{(1 - 7.44/64)} \le 8.5\gamma r$. ■

Proof of Theorem 8.1. We first show that $F_{IN} \subset X$. Let $x \in F_{IN}$ and let $s = b - Ax$. It suffices to show that $s \ge 0$. Because $x \in F_{IN}$,
$(s - \bar{s})^T \bar{S}^{-1} W \bar{S}^{-1} (s - \bar{s}) \le \bar{r}^2 = \bar{w}$. Therefore, for $i = 1, \ldots, m$,
$(s_i - \bar{s}_i)^2 w_i / \bar{s}_i^2 \le \bar{w} \le w_i$. Therefore $s_i \ge 0$, $i = 1, \ldots, m$.

We next show that $X \subset F_{OUT}$. Let $\hat{s} = b - A\hat{x}$, where $\hat{x}$ is the w-center of X . Then from Theorem 8.2,

$$(\hat{s} - \bar{s})^T \bar{S}^{-1} W \bar{S}^{-1} (\hat{s} - \bar{s}) \le (8.5\gamma)^2 r^2 = \frac{(8.5\gamma)^2 \bar{w}}{1 - \bar{w}} , \tag{8.6}$$

which implies that

$$\frac{\hat{s}_i}{\bar{s}_i} \le 1 + \frac{8.5\gamma}{\sqrt{1 - \bar{w}}} , \qquad i = 1, \ldots, m . \tag{8.7}$$

For any $x \in X$, let $s = b - Ax$. Then

$$(s - \hat{s})\bar{S}^{-1} W \bar{S}^{-1} (s - \hat{s}) = (s - \hat{s})\left(\hat{S}^2 \bar{S}^{-2}\right)\hat{S}^{-1} W \hat{S}^{-1} (s - \hat{s})$$

$$\leq \quad \left(1 + \frac{8.5\gamma}{\sqrt{1 - \overline{w}}}\right)^2 (s - \hat{s})^T \hat{S}^{-1} W \hat{S}^{-1} (s - \hat{s}) \qquad \text{(from 8.7)}$$

$$\leq \quad \left(1 + \frac{8.5\gamma}{\sqrt{1 - \overline{w}}}\right)^2 R^2, \qquad (8.8)$$

from Theorem 2.1.

Using the triangle inequality on (8.6) and (8.8) yields

$$\sqrt{(x - \overline{x})^T A^T \overline{S}^{-1} W \overline{S}^{-1} A (x - \overline{x})} \leq R\left(1 + \frac{8.5\gamma}{\sqrt{1 - \overline{w}}}\right) + 8.5\gamma r$$

$$= \left(\sqrt{\frac{1 - \overline{w}}{\overline{w}}}\left(1 + \frac{8.5\gamma}{\sqrt{1 - \overline{w}}}\right) + 8.5\gamma\sqrt{\frac{\overline{w}}{1 - \overline{w}}}\right)$$

$$\leq \frac{1}{\sqrt{\overline{w}}}(1 + 8.5\gamma + 6.1\gamma)$$

$$= \frac{1}{\sqrt{\overline{w}}}(1 + 14.6\gamma)$$

because $\frac{1}{2} \geq \overline{w} \geq 0$ and $\sqrt{\frac{\overline{w}}{1 - \overline{w}}} \leq 1 \leq \frac{1}{\sqrt{2\overline{w}}}$.

Substituting $\gamma = \frac{1}{8}$ in (8.1) yields $\overline{R}/\overline{r} \leq 2.9/\overline{w}$. ■

Appendix – Inequalities Related to Logarithms

Proposition A.1. $\ln(1 + a) \leq a$.

Proof: Follows from the concavity of the logarithm function. ■

Proposition A.2. If $|\alpha| \leq \varepsilon < 1$, then $\ln(1 + \alpha) \geq \alpha - \frac{\alpha^2}{2(1 - \varepsilon)}$.

Proof: See, e.g., Todd and Ye [25] . ■

Proposition A.3. If $0 < a \leq 1$ and $|b| < 1$, then $\ln(1 + b) \leq \left(\frac{1}{a}\right) \ln(1 + ab)$.

Proof: $\ln(1 + ab) = \ln(a(1 + b) + (1 - a)(1)) \geq a \ln(1 + b) + (1 - a)\ln(1) = a \ln(1 + b)$, where the inequality follows from the concavity of the logarithm function. ■

Consider the functions $k(\gamma)$, $j(\theta)$, $p(h)$, $q(h)$, $v(h)$, $m(h)$, and $n(h)$ defined in (6.1) – (6.7).

Proposition A.4. (i) $j(\theta)$ is decreasing in θ .

(ii) $\lim_{\theta \to 0} j(\theta) = 1/2$. ■

Proposition A.5. (i) $k(\gamma) \geq j(\theta)\gamma^2$ for $0 \leq \gamma \leq \theta$,

Proof: (i) Follows from Proposition A.4 (i) . ■

Proposition A.6. (i) $p(h)$ is decreasing in h .

(ii) $\lim_{h \to 0} p(h) = 1/2$. ■

Proposition A.7. $\ln(1 + x) \leq x - p(h)x^2$ for $-1 < x \leq h$.

Proof: Follows from Proposition A.6 (i). ■

Proposition A.8. (i) $q(h)$ is increasing in h .

(ii) $\lim_{h \to 0} q(h) = 0$.

(iii) $0 < q(h) < 0.30$ for all $h > 0$. ■

(iv) $q(h)$ is a concave function.

Proposition A.9. (i) $v(h)$ is decreasing in h .

(ii) $v(h) > 0$ for $h \leq 1$.

Proof: (i) follows from Proposition A.6 (i), A.8 (i) and A.8 (iii). Assertion (ii) follows from (i) and direct substitution. ■

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